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DIRECT AND INVERSE INEQUALITIES FOR THE ISOTROPIC LAMÉ SYSTEM WITH VARIABLE COEFFICIENTS AND APPLICATIONS TO AN INVERSE SOURCE PROBLEM

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Abstract. In the linear theory of elasticity we consider a bounded, compressible, and isotropic body whose mechanical behavior is described by the Lamé system with density and Lamé coefficients depending on the space variables. Assuming null surface displacement on the whole boundary, we first prove an estimate of the surface traction in terms of the energy of the solution and the body force. Then, under suitable restrictions on the density and the Lamé coefficients, we show that, in the absence of body forces, the elastic energy can be controlled by the surface traction exerted on a suitable sub-boundary provided that the final observability time is sufficiently large. The latter condition is related with the density, the Lamé coefficients, and the geometry of the body. These inequalities are applied to an inverse source problem for the Lamé system.

§**1. Introduction.** Let us consider an elastic, compressible and isotropic body which occupies a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with boundary $\partial\Omega$ of class C^2, whose outward unit normal is indicated by ν. This body has mass density ρ and, referring to the linear elasticity theory (see, *e.g.*, [6]), its elastic behavior is characterized by the Lamé coefficients λ and μ. Here we assume that ρ, λ, and μ are smooth enough and depend on the space variable $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We now denote by $\mathbf{u}(x,t) = (u_1(x,t), \ldots, u_n(x,t))$ the displacement field with respect to a fixed unstrained state at point x, at time t, and we introduce the linear strain tensor and the stress-strain relationship. In order to do that, we first need some matrix notation, namely, if A is an $n \times n$ matrix, its trace is indicated by $\operatorname{Tr} A$, while $A^{\mathbf{T}}$ is the transposed matrix. Moreover, for any pair of $n \times n$ matrices A and B, we indicate by $A \cdot B$ the usual scalar

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matrix product and we set $|A| = \sqrt{(A \cdot A)}$. We use the same notation for vectors; while AB and Ay, $y \in \mathbb{R}^n$, will simply indicate the standard matrix and matrix-vector products, respectively.

The linear strain tensor is defined by

$$\mathrm{Sym}\, \nabla \mathbf{u} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathbf{T}}\right)$$

where $\nabla \mathbf{u}$ is the Jacobian matrix of $\mathbf{u}$. Then, the stress tensor $\sigma(\mathbf{u})$ is given by the constitutive law

$$\sigma(\mathbf{u}) = \lambda(\mathrm{Tr}\, \nabla \mathbf{u})\delta + 2\mu \mathrm{Sym}\, \nabla \mathbf{u}$$

being δ the Kronecker tensor.

Suppose now that the body is subject to a body force $\mathbf{F}$. Hence, the evolution of $\mathbf{u}$ over a time interval $(0, T)$, $T > 0$, is governed by the so-called Lamé system

$$\rho \mathbf{u}'' = L(\mathbf{u}) + \mathbf{F}, \qquad \text{in } \Omega \times (0, T) \tag{1.1}$$

where prime stands for the time derivative and L is the second-order linear differential operator given by

$$L(\mathbf{u}) = \nabla \cdot \sigma(\mathbf{u}) = (\lambda + \mu)\nabla(\mathrm{Tr}\, \nabla \mathbf{u}) + \mu \Delta \mathbf{u} + (\nabla \lambda)\mathrm{Tr}\, \nabla \mathbf{u} + 2(\mathrm{Sym}\, \nabla \mathbf{u})\nabla \mu. \tag{1.2}$$

Here $\nabla\cdot$ is the spatial divergence operator.

Supposing that the body displacement field is null over the whole boundary, we are led to associate with (1.1) the following initial and boundary conditions

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \mathbf{u}'(x, 0) = \mathbf{u}_1(x), \qquad x \in \Omega \tag{1.3}$$

$$\mathbf{u}(x, t) = \mathbf{0}, \qquad (x, t) \in \partial\Omega \times (0, T) \tag{1.4}$$

where $\mathbf{u}_0$ and $\mathbf{u}_1$ are given data.

If ρ, λ, and μ are smooth and strictly positive, then standard assumptions on $\mathbf{F}$ and on the initial data ensure that there is a unique solution $\mathbf{u}$ to (1.1) which satisfies (1.3) and (1.4).

We now introduce the elastic energy E associated with the Lamé system (1.1), namely

$$E(t) = \frac{1}{2}\int_\Omega \left\{2\mu(x)|\mathrm{Sym}\, \nabla \mathbf{u}(x, t)|^2 + \lambda(x)|\mathrm{Tr}\, \nabla \mathbf{u}(x, t)|^2 + \rho(x)|\mathbf{u}'(x, t)|^2\right\} dx. \tag{1.5}$$

Our main result is a pair of inequalities which relates the surface traction $\sigma(\mathbf{u})\nu$ to $E(0)$ and $\mathbf{F}$ (see [5]). More precisely, according to the terminology used in [12], the first (direct) inequality controls the L^2–norm of $\sigma(\mathbf{u})\nu$ on $\partial\Omega \times (0, T)$ by $E(0)$ and the L^2–norm of the body force $\mathbf{F}$. The second main result is an inverse inequality which allows us to estimate $E(0)$ with the L^2–norm of $\sigma(\mathbf{u})\nu$ on $\Gamma \times (0,T)$, provided that T is large enough, Γ is a subset of $\partial\Omega$ properly chosen, and ρ, λ, μ fulfill suitable restrictions. These inequalities can be considered as a first step towards the solution to an open problem in exact controllability formulated by J.-L. Lions (see [12, Ch. IV, Problème ouvert 6, p. 321]) in the case of Dirichlet-type action. Indeed, taking advantage of such inequalities one can implement the Hilbert Uniqueness Method (HUM) for the isotropic Lamé system with nonconstant coefficients (see [12]). In the case of constant coefficients, direct and inverse inequalities for the Lamé system have been proved in [12] (see also [8] and, for nonsmooth domains, [13]), while for anisotropic cases we refer to [2, 14, 15].

The present results generalize the ones obtained in [4] where μ was assumed to be constant (see [4, Lemma 2.3]). This generalization may appear trivial but, as we shall see, it requires some work. The argument is always based on the so-called multiplier method (see, *e.g.*, [7, 9, 11, 12]) which has to be suitably adapted to the Lamé system with variable coefficients (cf. also [3] for a related result obtained by different techniques).

In [4] (see also [16]) inverse and direct inequalities along with HUM techniques were applied to the following inverse source problem for the Lamé system. Suppose that the body force $\mathbf{F} : \Omega \times (0,T) \to \mathbb{R}^n$ has the form

$$\mathbf{F}(x,t) := \varphi(t)\mathbf{f}(x) \qquad (x,t) \in \Omega \times (0,T) \tag{1.6}$$

where $\varphi : (0,T) \to \mathbb{R}$ is a given and smooth function, while the spatial part $\mathbf{f} : \Omega \to \mathbb{R}^n$ is unknown. In this case our problem can be viewed as an approximated model for elastic wave generated from a point dislocation source (see, *e.g.*, [1, Ch. 4]). Then $\mathbf{f}$ has to be identified from the knowledge of the surface traction $\mathbf{g} = \sigma(\mathbf{u})\nu$ exerted on a portion Γ of the boundary $\partial\Omega$ over the time interval $(0,T)$, provided that $\mathbf{u}$ solves (1.1) and (1.3)-(1.4). As far as this problem is concerned, the main results obtained in [4] can be summarized as follows. The linear mapping $\mathbf{G} : \mathbf{f} \mapsto \mathbf{g}$ has a continuous inverse, $\mathbf{f}$ can be *reconstructed* by means of the eigenfunctions associated with the linear operator $-\rho^{-1}L$ with homogeneous Dirichlet boundary conditions, and the range of the adjoint operator $\mathbf{G}^\star$ can be partially characterized. We show that all these results can be extended to the present case.

The plan of the paper goes as follows. In the next Section 2 we present our inequalities and some related remarks. Then, Section 3 is devoted to the proofs of two energy-type integral identities which play a basic role in proving the direct and inverse inequalities. This will be done in Sections 4 and 5, respectively. The final Section 6 is concerned with the application to the inverse source problem described above.

§2. Main results.

Let us set

$$H = (L(\Omega))^n, \quad V = (H_0^1(\Omega))^n$$

and recall that $V^* \equiv (H^{-1}(\Omega))^n$, V^* being the dual space of V. Also, we set

$$W = V \cap (H^2(\Omega))^n.$$

Assume

$$\rho,\ \lambda,\ \mu \in C^1(\overline{\Omega}; (0, +\infty)) \tag{2.1}$$

and let

$$\rho_0 = \min_{x\in\overline{\Omega}} \rho(x), \quad \rho_1 = \max_{x\in\overline{\Omega}} \rho(x), \quad \mu_0 = \min_{x\in\overline{\Omega}} \mu(x). \tag{2.2}$$

The following proposition is well-known and can be proved by the Fourier method or the semigroup theory, as in the case of the wave equation (see, *e.g.*, [12, Chap. I, 3.2]).

Proposition 2.1. *Let (2.1) and*

$$\mathbf{u}_0 \in V, \quad \mathbf{u}_1 \in H, \quad \mathbf{F} \in L^1(0, T; H) \tag{2.3}$$

hold. Then there is a unique $\mathbf{u} \in C^0([0, T]; V) \cap C^1([0, T]; H)$ *which satisfies equation (1.1) in* V^**, for almost all* t *in* $(0, T)$ *and initial conditions (1.3).*

We now state the direct inequality which yields a so-called hidden regularity property of the solution $\mathbf{u}$. Indeed, we have

Theorem 2.2. *Let (2.1) and (2.3) hold. Then there is a positive constant* $C = C(\Omega, T, \rho, \lambda, \mu)$ *such that*

$$\|\sigma(\mathbf{u})\nu\|_{L^2(0,T;(L^2(\partial\Omega))^n)} \le C\left(\sqrt{E(0)} + \|\mathbf{F}\|_{L^1(0,T;H)}\right). \tag{2.4}$$

To introduce the inverse inequality, we choose a point $x_0 \in \mathbb{R}^n$ and set (see [12, Chap. I, Sec. 5] for more details)

$$\mathbf{m}(x) = x - x_0 \qquad \forall\, x \in \mathbb{R}^n.$$

Then we consider the subset of $\partial\Omega$:

$$\Gamma_+(x_0) = \{x \in \partial\Omega \ :\ \mathbf{m}(x) \cdot \nu(x) > 0\}$$

and we define

$$R_0 = R_0(x_0) = \|\mathbf{m}\|_{(L^\infty(\Omega))^n}.$$

The inverse inequality is given by

Theorem 2.3. *Let (2.1) and (2.3) hold with* $\mathbf{F} \equiv \mathbf{0}$. *Suppose there is a positive constant* γ_0 *such that*

$$\min_{\overline{\Omega}} \left\{1 + \rho^{-1}\nabla\rho \cdot \mathbf{m},\ 1 - \lambda^{-1}\nabla\lambda \cdot \mathbf{m},\ 1 - \mu^{-1}\nabla\mu \cdot \mathbf{m}\right\} \geq \gamma_0 \tag{2.5}$$

and let

$$T > T_0 := \frac{2\rho_1 R_0}{\gamma_0\sqrt{\rho_0\mu_0}}. \tag{2.6}$$

Then there exists a positive constant $C = C(\Omega, T, x_0, \rho, \lambda, \mu)$ *such that*

$$\sqrt{E(0)} \leq C\|\sigma(\mathbf{u})\nu\|_{L^2(0,T;(L^2(\Gamma_+(x_0))^n)}. \tag{2.7}$$

Remark 2.4. If ρ, λ, and μ are constants, we can take $\gamma_0 = 1$ and condition (2.6) reduces to

$$T > 2R_0\sqrt{\frac{\rho}{\mu}}$$

which says that T should be greater than twice the traveling time of the secondary wave (compare with [12, Chap. IV, Théorème 1.1] where $\rho = 1$).

Remark 2.5. If μ is constant as supposed in [4], then estimates similar to (2.4) and (2.7) hold, the only difference being that $\sigma(\mathbf{u})\nu$ is replaced by the normal derivative $(\nabla\mathbf{u})\nu$. This fact is related to the structure of the elastic energy which, in the present case, does not allow to control the L^2-norm of $(\nabla\mathbf{u})\nu$ directly (see (3.5) and Section 5 below). However, one can prove that the L^2-norm of $(\nabla\mathbf{u})\nu$ is equivalent to the L^2-norm of $\sigma(\mathbf{u})\nu$ whenever $\mathbf{u}$ satisfies (1.4) and μ as well as $n\lambda + 2\mu$ are strictly positive in $\overline{\Omega}$ (see Lemma 4.3).

§3. Integral identities.

Here we prove two integral identities. The former is the usual energy identity, while the latter is obtained via the multiplier method. However, we need to recall first the following regularity result about problem (1.1) and (1.3)-(1.4).

Proposition 3.1. *Let (2.1) and*

$$\mathbf{u}_0 \in W, \quad \mathbf{u}_1 \in V \tag{3.1}$$

$$\mathbf{F} \in W^{1,1}(0, T; H) \tag{3.2}$$

hold. Then there is a unique function $\mathbf{u} \in C^0([0, T]; W) \cap C^1([0, T]; V) \cap C^2([0, T]; H)$ *which satisfies equation (1.1) almost everywhere in* $\Omega \times (0, T)$ *and initial conditions (1.3).*

The energy identity is given by

Lemma 3.2. *Let (2.1) and suppose $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{F}$ sufficiently smooth to ensure the existence of a function $\mathbf{u} \in C^0([0,T];(H^2(\Omega)^n)) \cap C^1([0,T];(H^1(\Omega)^n)) \cap C^2([0,T];H)$ which satisfies (1.1) and (1.3). Then, for any $t \in [0,T]$,*

$$E(t) = E(0) + \int_0^t \int_{\partial\Omega} (\sigma(\mathbf{u}(x,t))\nu(x)) \cdot \mathbf{u}'(x,t)\, dSdt + \int_0^t \int_\Omega \mathbf{F}(x,t) \cdot \mathbf{u}'(x,t) dxdt. \tag{3.3}$$

Proof. Suppose that the initial data and $\mathbf{F}$ satisfy (3.1) and (3.2) so that, in particular, $\mathbf{u}' \in C^0([0,T];V)$. Then, choosing $\mathbf{u}'$ as test function, from (1.1) in V^* we deduce

$$\int_\Omega \rho \mathbf{u}'' \cdot \mathbf{u}' dx = \int_\Omega L(\mathbf{u}) \cdot \mathbf{u}' dx + \int_\Omega \mathbf{F} \cdot \mathbf{u}' dx$$

from which we deduce

$$\frac{d}{dt} E(t) = \int_{\partial\Omega} (\sigma(\mathbf{u})\nu) \cdot \mathbf{u}' dS + \int_\Omega \mathbf{F} \cdot \mathbf{u}' dx, \quad t \geq 0.$$

being dS the Lebesgue $(n-1)-$ dimensional surface measure. Thus, integrating with respect to time from 0 to t yields (3.3). A density argument completes the proof. ∎

Here is our basic integral identity.

Lemma 3.3. *Let (2.1) hold and suppose that $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{F}$ are sufficiently smooth to ensure the existence of a function $\mathbf{u} \in C^0([0,T];(H^2(\Omega)^n)) \cap C^1([0,T];(H^1(\Omega)^n)) \cap C^2([0,T];H)$ which satisfies (1.1) and (1.3). Then, for any given $\mathbf{h} \in C^2(\overline{\Omega};\mathbb{R}^n)$, we have*

$$\begin{aligned}
&\left[\int_\Omega \rho \mathbf{u}' \cdot (\nabla \mathbf{u})\mathbf{h} dx\right]_0^T + E(0)T \\
&+\frac{1}{2}\int_0^T \int_\Omega [\rho(\mathit{Tr}\,\nabla \mathbf{h} - 1)|\mathbf{u}'|^2 \\
&+\lambda(\mathit{Tr}\,\nabla \mathbf{u})\{\mathit{Tr}\,((\nabla \mathbf{u})(\nabla \mathbf{h})) - (\mathit{Tr}\,\nabla \mathbf{u})(\mathit{Tr}\,\nabla \mathbf{h})\} \\
&+2\mu\{\mathit{Sym}\,\nabla \mathbf{u} \cdot \mathit{Sym}\,((\nabla \mathbf{u})(\nabla \mathbf{h})) - |\mathit{Sym}\,\nabla \mathbf{u}|^2\,\mathit{Tr}\,\nabla \mathbf{h}\}]dxdt \\
&+\frac{1}{2}\int_0^T \int_\Omega \{(\nabla\rho \cdot \mathbf{h})|\mathbf{u}'|^2 - (\nabla\lambda \cdot \mathbf{h})|\mathit{Tr}\,\nabla \mathbf{u}|^2 - 2(\nabla\mu \cdot \mathbf{h})|\mathit{Sym}\,\nabla \mathbf{u}|^2\}dxdt \\
&+\frac{1}{2}\int_0^T \int_\Omega [\lambda(\mathit{Tr}\,\nabla \mathbf{u})\{\mathit{Tr}\,((\nabla \mathbf{u})(\nabla \mathbf{h})) - \mathit{Tr}\,\nabla \mathbf{u}\} \\
&+2\mu\{(\mathit{Sym}\,\nabla \mathbf{u}) \cdot \mathit{Sym}\,((\nabla \mathbf{u})(\nabla \mathbf{h})) - |\mathit{Sym}\,\nabla \mathbf{u}|^2\}]dxdt \\
&+\int_0^T \left(\int_0^t \int_\Omega \mathbf{F}(x,\eta) \cdot \mathbf{u}'(x,\eta) dxd\eta\right) dt - \int_0^T \int_\Omega \mathbf{F} \cdot (\nabla \mathbf{u})\mathbf{h} dxdt \\
&= -\int_0^T \left(\int_0^t \int_{\partial\Omega} \sigma(\mathbf{u})(x,\eta)\nu(x) \cdot \mathbf{u}'(x,\eta) dSd\eta\right) dt
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^T\int_{\partial\Omega}\{\rho|\mathbf{u}'|^2-\lambda|Tr\,\nabla\mathbf{u}|^2-2\mu|Sym\,\nabla\mathbf{u}|^2\}(\mathbf{h}\cdot\nu)\,dSdt\\
&+\int_0^T\int_{\partial\Omega}(\nabla\mathbf{u})\mathbf{h}\cdot\sigma(\mathbf{u})\nu\,dSdt.
\end{aligned}
\tag{3.4}
$$

Consequently, if (3.1)-(3.2) hold and $\mathbf{u}\in C^0([0,T];W)\cap C^1([0,T];V)\cap C^2([0,T];H)$ *is the unique solution to (1.1) and (1.3), then (3.4) reduces to*

$$
\begin{aligned}
&\left[\int_\Omega \rho\mathbf{u}'\cdot(\nabla\mathbf{u})\mathbf{h}dx\right]_0^T+E(0)T\\
&+\frac{1}{2}\int_0^T\int_\Omega[\rho(Tr\,\nabla\mathbf{h}-1)|\mathbf{u}'|^2\\
&+\lambda(Tr\,\nabla\mathbf{u})\{Tr\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-(Tr\,\nabla\mathbf{u})(Tr\,\nabla\mathbf{h})\}\\
&+2\mu\{Sym\,\nabla\mathbf{u}\cdot Sym\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-|Sym\,\nabla\mathbf{u}|^2\,Tr\,\nabla\mathbf{h}\}]dxdt\\
&+\frac{1}{2}\int_0^T\int_\Omega\{(\nabla\rho\cdot\mathbf{h})|\mathbf{u}'|^2-(\nabla\lambda\cdot\mathbf{h})|Tr\,\nabla\mathbf{u}|^2-2(\nabla\mu\cdot\mathbf{h})|Sym\,\nabla\mathbf{u}|^2\}dxdt\\
&+\frac{1}{2}\int_0^T\int_\Omega[\lambda(Tr\,\nabla\mathbf{u})\{Tr\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-Tr\,\nabla\mathbf{u}\}\\
&+2\mu\{(Sym\,\nabla\mathbf{u})\cdot Sym\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-|Sym\,\nabla\mathbf{u}|^2\}]dxdt\\
&+\int_0^T\left(\int_0^t\int_\Omega\mathbf{F}(x,\eta)\cdot\mathbf{u}'(x,\eta)dxd\eta\right)dt-\int_0^T\int_\Omega\mathbf{F}\cdot(\nabla\mathbf{u})\mathbf{h}dxdt\\
&=-\frac{1}{2}\int_0^T\int_{\partial\Omega}\{\lambda|Tr\,\nabla\mathbf{u}|^2+2\mu|Sym\,\nabla\mathbf{u}|^2\}(\mathbf{h}\cdot\nu)\,dSdt\\
&+\int_0^T\int_{\partial\Omega}(\nabla\mathbf{u})\mathbf{h}\cdot\sigma(\mathbf{u})\nu\,dSdt.
\end{aligned}
\tag{3.5}
$$

Proof. Scalar multiplication of (1.1) by $(\nabla\mathbf{u})\mathbf{h}$ yields

$$
\rho\mathbf{u}''\cdot(\nabla\mathbf{u})\mathbf{h}=L(\mathbf{u})\cdot(\nabla\mathbf{u})\mathbf{h}+\mathbf{F}\cdot(\nabla\mathbf{u})\mathbf{h}.
\tag{3.6}
$$

Some somewhat lengthy calculations are now in order.

Since $\nabla(\mathbf{w}\cdot\mathbf{v})=(\nabla\mathbf{w})^{\mathbf{T}}\mathbf{v}+(\nabla\mathbf{v})^{\mathbf{T}}\mathbf{w}$, we have $\nabla(|\mathbf{u}'|^2)=2(\nabla\mathbf{u}')^{\mathbf{T}}\mathbf{u}'$ and

$$
\begin{aligned}
\rho\mathbf{u}'\cdot(\nabla\mathbf{u}')\mathbf{h}=\rho(\nabla\mathbf{u}')^{\mathbf{T}}\mathbf{u}'\cdot\mathbf{h}&=\frac{1}{2}(\nabla|\mathbf{u}'|^2)\cdot\rho\mathbf{h}\\
&=\frac{1}{2}\mathrm{Tr}\,\nabla(|\mathbf{u}'|^2\rho\mathbf{h})-\frac{1}{2}\mathrm{Tr}\,\nabla(\rho\mathbf{h})|\mathbf{u}'|^2\\
&=\frac{1}{2}\mathrm{Tr}\,\nabla(|\mathbf{u}'|^2\rho\mathbf{h})-\frac{1}{2}\{\nabla\rho\cdot\mathbf{h}+\rho(\mathrm{Tr}\,\nabla\mathbf{h})\}|\mathbf{u}'|^2.
\end{aligned}
$$

Then, noting that

$$\rho\mathbf{u}''\cdot(\nabla\mathbf{u})\mathbf{h}=\frac{\partial}{\partial t}(\rho\mathbf{u}'\cdot(\nabla\mathbf{u})\mathbf{h})-\rho\mathbf{u}'\cdot(\nabla\mathbf{u}')\mathbf{h},$$

we deduce

$$\begin{aligned}\rho\mathbf{u}''\cdot(\nabla\mathbf{u})\mathbf{h}=&\frac{\partial}{\partial t}(\rho\mathbf{u}'\cdot(\nabla\mathbf{u})\mathbf{h})\\ &-\frac{1}{2}\mathrm{Tr}\,\nabla(|\mathbf{u}'|^2\rho\mathbf{h})+\frac{1}{2}\{\nabla\rho\cdot\mathbf{h}+\rho(\mathrm{Tr}\,\nabla\mathbf{h})\}|\mathbf{u}'|^2.\end{aligned}\tag{3.7}$$

Henceforth, $[\dots]_{ij}$ and $[\dots]_j$ stand for the (ij)–component and the $j-$ component of the matrix and the vector under consideration, respectively.

On the other hand, recalling the symmetry of $\sigma(\mathbf{u})$, observe that (cf. (1.2))

$$\begin{aligned}L(\mathbf{u})\cdot(\nabla\mathbf{u})\mathbf{h}=&\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}[\sigma(\mathbf{u})]_{ij}\cdot[(\nabla\mathbf{u})\mathbf{h}]_i\\ =&\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}([\sigma(\mathbf{u})]_{ij}[(\nabla\mathbf{u})\mathbf{h}]_i)-\sum_{i,j=1}^{n}[\sigma(\mathbf{u})]_{ij}\frac{\partial}{\partial x_j}([(\nabla\mathbf{u})\mathbf{h}]_i),\end{aligned}\tag{3.8}$$

thus we obtain

$$L(\mathbf{u})\cdot(\nabla\mathbf{u})\mathbf{h}=\mathrm{Tr}\,\nabla(\sigma(\mathbf{u})(\nabla\mathbf{u})\mathbf{h})-\sigma(\mathbf{u})\cdot\nabla\{(\nabla\mathbf{u})\mathbf{h}\}.\tag{3.9}$$

Since

$$[\nabla((\nabla\mathbf{u})\mathbf{h})]_{ij}=\sum_{k=1}^{n}\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_k}h_k\right)=\nabla([\nabla\mathbf{u}]_{ij})\cdot\mathbf{h}+[(\nabla\mathbf{u})(\nabla\mathbf{h})]_{ij},$$

we deduce

$$\mathrm{Tr}\,\nabla((\nabla\mathbf{u})\mathbf{h})=\nabla(\mathrm{Tr}\,\nabla\mathbf{u})\cdot\mathbf{h}+\nabla\mathbf{u}\cdot(\nabla\mathbf{h})^{\mathbf{T}}$$

and

$$[\mathrm{Sym}\,\nabla((\nabla\mathbf{u})\mathbf{h})]_{ij}=\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left([\mathrm{Sym}\,\nabla\mathbf{u}]_{ij}\right)h_k+[\mathrm{Sym}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))]_{ij}.$$

Therefore, taking advantage of the symmetry of $\sigma(\mathbf{u})$, we have

$$\begin{aligned}&\sigma(\mathbf{u})\cdot\nabla((\nabla\mathbf{u})\mathbf{h})\\ =&\sigma(\mathbf{u})\cdot\mathrm{Sym}\,\nabla((\nabla\mathbf{u})\mathbf{h})\\ =&\lambda(\mathrm{Tr}\,\nabla\mathbf{u})\mathrm{Tr}\,[\mathrm{Sym}\,\nabla((\nabla\mathbf{u})\mathbf{h})]+2\mu(\mathrm{Sym}\,\nabla\mathbf{u})\cdot\mathrm{Sym}\,\nabla((\nabla\mathbf{u})\mathbf{h})\\ =&\lambda(\mathrm{Tr}\,\nabla\mathbf{u})\{\nabla(\mathrm{Tr}\,\nabla\mathbf{u})\cdot\mathbf{h}+\mathrm{Tr}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))\}\end{aligned}$$

$$
\begin{aligned}
&+2\mu\sum_{i,j,k=1}^{n}[\mathrm{Sym}\,\nabla\mathbf{u}]_{ij}\frac{\partial}{\partial x_k}[\mathrm{Sym}\,\nabla\mathbf{u}]_{ij}h_k+2\mu(\mathrm{Sym}\,\nabla\mathbf{u})\cdot\mathrm{Sym}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))\\
&=\lambda(\mathrm{Tr}\,\nabla\mathbf{u})\mathrm{Tr}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))+2\mu(\mathrm{Sym}\,\nabla\mathbf{u})\cdot\mathrm{Sym}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))\\
&+\lambda(\mathrm{Tr}\,\nabla\mathbf{u})\nabla(\mathrm{Tr}\,\nabla\mathbf{u})\cdot\mathbf{h}+\mu\sum_{i,j,k=1}^{n}\frac{\partial}{\partial x_k}([\mathrm{Sym}\,\nabla\mathbf{u}]_{ij})^2h_k\\
&=\lambda(\mathrm{Tr}\,\nabla\mathbf{u})\mathrm{Tr}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))+2\mu(\mathrm{Sym}\,\nabla\mathbf{u})\cdot\mathrm{Sym}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))\\
&+\frac{1}{2}\lambda\nabla(|\mathrm{Tr}\,\nabla\mathbf{u}|^2)\cdot\mathbf{h}+\mu\nabla(|\mathrm{Sym}\,\nabla\mathbf{u}|^2)\cdot\mathbf{h}.
\end{aligned}
$$

Observe now that

$$
\begin{aligned}
\frac{1}{2}\lambda\nabla(|\mathrm{Tr}\,\nabla\mathbf{u}|^2)\cdot\mathbf{h}&=\frac{1}{2}\mathrm{Tr}\,\nabla(\lambda|\mathrm{Tr}\,\nabla\mathbf{u}|^2\mathbf{h})-\frac{1}{2}|\mathrm{Tr}\,\nabla\mathbf{u}|^2\mathrm{Tr}\,\nabla(\lambda\mathbf{h})\\
&=\frac{1}{2}\mathrm{Tr}\,\nabla(\lambda|\mathrm{Tr}\,\nabla\mathbf{u}|^2\mathbf{h})-\frac{1}{2}|\mathrm{Tr}\,\nabla\mathbf{u}|^2(\nabla\lambda\cdot\mathbf{h}+\lambda\mathrm{Tr}\,\nabla\mathbf{h})
\end{aligned}
$$

and

$$
\mu\nabla(|\mathrm{Sym}\,\nabla\mathbf{u}|^2)\cdot\mathbf{h}=\mathrm{Tr}\,\nabla(\mu|\mathrm{Sym}\,\nabla\mathbf{u}|^2\mathbf{h})-|\mathrm{Sym}\,\nabla\mathbf{u}|^2(\nabla\mu\cdot\mathbf{h}+\mu\mathrm{Tr}\,\nabla\mathbf{h}).
$$

Then we infer

$$
\begin{aligned}
\sigma(\mathbf{u})\cdot(\nabla(\nabla\mathbf{u})\mathbf{h})=&\frac{1}{2}\{\lambda(\mathrm{Tr}\,\nabla\mathbf{u})\mathrm{Tr}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))+2\mu(\mathrm{Sym}\,\nabla\mathbf{u})\cdot\mathrm{Sym}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))\}\\
&+\frac{1}{2}\mathrm{Tr}\,\nabla\{(\lambda|\mathrm{Tr}\,\nabla\mathbf{u}|^2+2\mu|\mathrm{Sym}\,\nabla\mathbf{u}|^2)\mathbf{h}\}\\
&-\frac{1}{2}\{(\nabla\lambda\cdot\mathbf{h})|\mathrm{Tr}\,\nabla\mathbf{u}|^2+2(\nabla\mu\cdot\mathbf{h})|\mathrm{Sym}\,\nabla\mathbf{u}|^2\}\\
&+\frac{1}{2}\{\lambda(\mathrm{Tr}\,\nabla\mathbf{u})[\mathrm{Tr}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-(\mathrm{Tr}\,\nabla\mathbf{u})(\mathrm{Tr}\,\nabla\mathbf{h})]\\
&+2\mu[(\mathrm{Sym}\,\nabla\mathbf{u})\cdot\mathrm{Sym}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-|\mathrm{Sym}\,\nabla\mathbf{u}|^2\mathrm{Tr}\,\nabla\mathbf{h}]\}.
\end{aligned}
\tag{3.10}
$$

We now integrate (3.7) over $\Omega\times(0,T)$. Using the divergence theorem, we obtain

$$
\begin{aligned}
&\int_0^T\int_\Omega\rho\mathbf{u}''\cdot(\nabla\mathbf{u})\mathbf{h}dxdt\\
&=\left[\int_\Omega\rho\mathbf{u}'\cdot(\nabla\mathbf{u})\mathbf{h}dx\right]_0^T-\frac{1}{2}\int_0^T\int_{\partial\Omega}\rho|\mathbf{u}'|^2\mathbf{h}\cdot\nu\,dSdt\\
&+\frac{1}{2}\int_0^T\int_\Omega(\nabla\rho\cdot\mathbf{h})|\mathbf{u}'|^2dxdt+\frac{1}{2}\int_0^T\int_\Omega\rho(\mathrm{Tr}\,\nabla\mathbf{h}-1)|\mathbf{u}'|^2dxdt\\
&+\frac{1}{2}\int_0^T\int_\Omega\rho|\mathbf{u}'|^2dxdt.
\end{aligned}
\tag{3.11}
$$

On the other hand, on account of (3.9)-(3.10), a further use of the divergence theorem allows us to deduce

$$
\begin{aligned}
&\int_0^T\int_\Omega L(\mathbf{u})\cdot(\nabla\mathbf{u})\mathbf{h}dxdt+\int_0^T\int_\Omega \mathbf{F}\cdot(\nabla\mathbf{u})\mathbf{h}dxdt\\
&=\int_0^T\int_{\partial\Omega}(\sigma(\mathbf{u})(\nabla\mathbf{u})\mathbf{h})\cdot\nu dSdt\\
&-\frac{1}{2}\int_0^T\int_\Omega(\lambda|\mathrm{Tr}\,\nabla\mathbf{u}|^2+2\mu|\mathrm{Sym}\,\nabla\mathbf{u}|^2)dxdt\\
&-\frac{1}{2}\int_0^T\int_\Omega \lambda(\mathrm{Tr}\,\nabla\mathbf{u})\{\mathrm{Tr}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-\mathrm{Tr}\,\nabla\mathbf{u}\}dxdt\\
&-\frac{1}{2}\int_0^T\int_\Omega 2\mu\{(\mathrm{Sym}\,\nabla\mathbf{u})\cdot\mathrm{Sym}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-|\mathrm{Sym}\,\nabla\mathbf{u}|^2\}dxdt\\
&-\frac{1}{2}\int_0^T\int_{\partial\Omega}(\lambda|\mathrm{Tr}\,\nabla\mathbf{u}|^2+2\mu|\mathrm{Sym}\,\nabla\mathbf{u}|^2)(\mathbf{h}\cdot\nu)dSdt\\
&+\frac{1}{2}\int_0^T\int_\Omega\{(\nabla\lambda\cdot\mathbf{h})|\mathrm{Tr}\,\nabla\mathbf{u}|^2+2(\nabla\mu\cdot\mathbf{h})|\mathrm{Sym}\,\nabla\mathbf{u}|^2\}dxdt\\
&-\frac{1}{2}\int_0^T\int_\Omega \lambda(\mathrm{Tr}\,\nabla\mathbf{u})\{\mathrm{Tr}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-(\mathrm{Tr}\,\nabla\mathbf{u})(\mathrm{Tr}\,\nabla\mathbf{h})\}dxdt\\
&-\frac{1}{2}\int_0^T\int_\Omega 2\mu\{(\mathrm{Sym}\,\nabla\mathbf{u})\cdot\mathrm{Sym}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-|\mathrm{Sym}\,\nabla\mathbf{u}|^2\mathrm{Tr}\,\nabla\mathbf{h}\}dxdt\\
&+\int_0^T\int_\Omega \mathbf{F}\cdot(\nabla\mathbf{u})\mathbf{h}dxdt.
\end{aligned}
\tag{3.12}
$$

A combination of (3.6) with (3.11) and (3.12) leads us to the following identity

$$
\begin{aligned}
&\left[\int_\Omega \rho\mathbf{u}'\cdot(\nabla\mathbf{u})\mathbf{h}dx\right]_0^T+\frac{1}{2}\int_0^T\int_\Omega(\rho|\mathbf{u}'|^2+\lambda|\mathrm{Tr}\,\nabla\mathbf{u}|^2+2\mu|\mathrm{Sym}\,\nabla\mathbf{u}|^2)dxdt\\
&+\frac{1}{2}\int_0^T\int_\Omega\{(\nabla\rho\cdot\mathbf{h})|\mathbf{u}'|^2-(\nabla\lambda\cdot\mathbf{h})|\mathrm{Tr}\,\nabla\mathbf{u}|^2-2(\nabla\mu\cdot\mathbf{h})|\mathrm{Sym}\,\nabla\mathbf{u}|^2\}dxdt\\
&+\frac{1}{2}\int_0^T\int_\Omega[\rho(\mathrm{Tr}\,\nabla\mathbf{h}-1)|\mathbf{u}'|^2+\lambda(\mathrm{Tr}\,\nabla\mathbf{u})\{\mathrm{Tr}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-(\mathrm{Tr}\,\nabla\mathbf{u})(\mathrm{Tr}\,\nabla\mathbf{h})\}\\
&+2\mu\{(\mathrm{Sym}\,\nabla\mathbf{u})\cdot\mathrm{Sym}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-|\mathrm{Sym}\,\nabla\mathbf{u}|^2\mathrm{Tr}\,\nabla\mathbf{h}\}]dxdt\\
&+\frac{1}{2}\int_0^T\int_\Omega \lambda(\mathrm{Tr}\,\nabla\mathbf{u})\{\mathrm{Tr}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-\mathrm{Tr}\,\nabla\mathbf{u}\}dxdt\\
&+\frac{1}{2}\int_0^T\int_\Omega 2\mu\{(\mathrm{Sym}\,\nabla\mathbf{u})\cdot\mathrm{Sym}\,((\nabla\mathbf{u})(\nabla\mathbf{h}))-|\mathrm{Sym}\,\nabla\mathbf{u}|^2\}dxdt\\
&-\int_0^T\int_\Omega \mathbf{F}\cdot(\nabla\mathbf{u})\mathbf{h}dxdt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^T\int_{\partial\Omega}(\rho|\mathbf{u}'|^2 - \lambda|\mathrm{Tr}\,\nabla\mathbf{u}|^2 - 2\mu|\mathrm{Sym}\,\nabla\mathbf{u}|^2)(\mathbf{h}\cdot\nu)dSdt \\
&+ \int_0^T\int_{\partial\Omega}((\nabla\mathbf{u})\mathbf{h})\cdot\sigma(\mathbf{u})\nu\, dSdt.
\end{aligned} \tag{3.13}
$$

Recalling (1.5) and Lemma 3.2, we observe that

$$
\begin{aligned}
&[\text{the second term on the left hand side of (3.13)}] = \int_0^T E(t)dt \\
&= E(0)T + \int_0^T\left(\int_0^t\int_{\partial\Omega}\sigma(\mathbf{u})\nu\cdot\mathbf{u}'dSd\eta\right)dt + \int_0^T\left(\int_0^t\int_{\Omega}\mathbf{F}\cdot\mathbf{u}'dxd\eta\right)dt.
\end{aligned} \tag{3.14}
$$

Finally, identity (3.4) follows from substituting (3.14) into (3.13). Clearly, (3.5) follows directly from (3.4) since $\mathbf{u}$ satisfies (1.4). ■

§**4. Proof of Theorem 2.2.** Let us suppose for the moment that (3.1) and (3.2) hold so that $\mathbf{u} \in C^2([0,T];W)\cap C^1([0,T];V)\cap C^2([0,T];H)$ by Proposition 3.1.

Let us recall the following (see, for example, [9, pp. 18-19] or [12, pp. 29-30]).

Lemma 4.1. *If $\partial\Omega$ is of class C^2, then there exists $\mathbf{h}_0 \in C^1(\overline{\Omega};\mathbb{R}^n)$ such that*

$$
\mathbf{h}_0(x) = \nu(x) \qquad \text{on } \partial\Omega.
$$

The proof of Theorem 2.2 is achieved by setting $\mathbf{h} = \mathbf{h}_0$ in identity (3.5), but this procedure requires some work which is distributed in the following three lemmas.

First, to estimate the last boundary integral in (3.5), we need to find a convenient expression for $(\nabla\mathbf{u})\nu\cdot\sigma(\mathbf{u})\nu$.

Lemma 4.2. *Let (2.1) and (3.1)-(3.2) hold. Then*

$$
(\nabla\mathbf{u})\nu\cdot\sigma(\mathbf{u})\nu = \lambda|\mathit{Tr}\,\nabla\mathbf{u}|^2 + 2\mu|\mathit{Sym}\,\nabla\mathbf{u}|^2 \qquad \text{on } \partial\Omega\times(0,T). \tag{4.1}
$$

Before stating our second technical lemma, we introduce the tensor product $\mathbf{a}\otimes\mathbf{b}$ between two vectors $\mathbf{a} = (a_1,\dots,a_n)^{\mathbf{T}}$ and $\mathbf{b} = (b_1,\dots,b_n)^{\mathbf{T}}$, that is

$$
\mathbf{a}\otimes\mathbf{b} = (a_ib_j)_{1\le i,j\le n}. \tag{4.2}
$$

Note that $\mathbf{a}\otimes\mathbf{b}$ is an $n\times n$ matrix.

Lemma 4.3. *Let (2.1) hold. Define an $n\times n$ matrix $B(x)$ for any $x\in\partial\Omega$ by setting*

$$
B(x)\mathbf{a} = \lambda(x)(\mathbf{a}\cdot\nu(x))\nu(x) + 2\mu(x)\{\mathit{Sym}\,(\mathbf{a}\otimes\nu(x))\}\nu(x), \tag{4.3}
$$

for any $\mathbf{a} \in \mathbb{R}^n$ *and* $x \in \partial\Omega$*. Then*

$$B^{-1}(x) \text{ exists for any } x \in \partial\Omega \tag{4.4}$$

$$B(x)\mathbf{a} \cdot \mathbf{a} \geq \frac{c_0}{2}|\mathbf{a}|^2, \qquad \forall\, \mathbf{a} \in \mathbb{R}^n, \quad \forall\, x \in \partial\Omega \tag{4.5}$$

and

$$B((\nabla \mathbf{u})\nu) = \sigma(\mathbf{u})\nu \qquad \text{on } \partial\Omega \times (0,T). \tag{4.6}$$

Proof of Lemma 4.2. Thanks to (1.4), we have

$$\nabla \mathbf{u} = \{(\nabla \mathbf{u})\nu\} \otimes \nu \quad \text{a.e. on } \partial\Omega \times (0,T) \tag{4.7}$$

and

$$\operatorname{Tr} \nabla \mathbf{u} = (\nabla \mathbf{u})\nu \cdot \nu \quad \text{a.e. on } \partial\Omega \times (0,T). \tag{4.8}$$

In fact, setting $\mathbf{u} = (u_1, ..., u_n)^{\mathbf{T}}$ and $\nu = (\nu_1, ..., \nu_n)^{\mathbf{T}}$, we see that boundary condition (1.4) implies

$$\nabla u_i = \begin{bmatrix} \frac{\partial u_i}{\partial x_1} \\ \vdots \\ \frac{\partial u_i}{\partial x_n} \end{bmatrix} = (\nabla u_i \cdot \nu)\nu = \left[\frac{\partial u_i}{\partial \nu}\right]\nu, \quad 1 \leq i \leq n$$

almost everywhere on $\partial\Omega \times (0,T)$. Therefore we have

$$\nabla \mathbf{u} = \begin{bmatrix} (\nabla u_1)^{\mathbf{T}} \\ \vdots \\ (\nabla u_n)^{\mathbf{T}} \end{bmatrix} = \begin{bmatrix} (\nabla u_1 \cdot \nu)\nu^{\mathbf{T}} \\ \vdots \\ (\nabla u_n \cdot \nu)\nu^{\mathbf{T}} \end{bmatrix} = [(\nabla u_i \cdot \nu)\nu_j]_{1 \leq i,j \leq n} \tag{4.9}$$

almost everywhere on $\partial\Omega \times (0,T)$. On account of (4.2), this gives (4.7). Moreover, since $\nu\nu^{\mathbf{T}} = 1$, we have

$$(\nabla \mathbf{u})\nu = \begin{bmatrix} (\nabla u_1 \cdot \nu)\nu^{\mathbf{T}} \\ \vdots \\ (\nabla u_n \cdot \nu)\nu^{\mathbf{T}} \end{bmatrix} \nu = \begin{bmatrix} (\nabla u_1 \cdot \nu) \\ \vdots \\ (\nabla u_n \cdot \nu) \end{bmatrix},$$

so that, owing to (4.9),

$$(\nabla \mathbf{u})\nu \cdot \nu = (\nabla u_1 \cdot \nu)\nu_1 + \cdots + (\nabla u_n \cdot \nu)\nu_n = \operatorname{Tr} \nabla \mathbf{u}$$

almost everywhere on $\partial\Omega \times (0, T)$. Thus, (4.7) and (4.8) follow.

Recalling the definition of $\sigma(\mathbf{u})$ and (4.8), we obtain

$$\begin{aligned} &(\nabla \mathbf{u})\nu \cdot \sigma(\mathbf{u})\nu = (\nabla \mathbf{u})\nu \cdot \{\lambda(\mathrm{Tr}\,\nabla \mathbf{u})\nu + 2\mu(\mathrm{Sym}\,\nabla \mathbf{u})\nu\} \\ =&\lambda(\mathrm{Tr}\,\nabla \mathbf{u})((\nabla \mathbf{u})\nu \cdot \nu) + 2\mu(\nabla \mathbf{u})\nu \cdot (\mathrm{Sym}\,\nabla \mathbf{u})\nu \\ =&\lambda|\mathrm{Tr}\,\nabla \mathbf{u}|^2 + 2\mu(\nabla \mathbf{u})\nu \cdot (\mathrm{Sym}\,\nabla \mathbf{u})\nu \end{aligned} \tag{4.10}$$

almost everywhere on $\partial\Omega \times (0, T)$.

On the other hand, by (4.7), noting that $(\mathbf{a} \otimes \mathbf{b}) \cdot A = \mathbf{a} \cdot A\mathbf{b}$, we have

$$\begin{aligned} &|\mathrm{Sym}\,\nabla \mathbf{u}|^2 = \frac{1}{4}|\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathbf{T}}|^2 = \frac{1}{2}(|\nabla \mathbf{u}|^2 + (\nabla \mathbf{u})^{\mathbf{T}} \cdot (\nabla \mathbf{u})) \\ =&\frac{1}{2}((\nabla \mathbf{u})\nu \otimes \nu) \cdot \nabla \mathbf{u} + \frac{1}{2}(\nu \otimes (\nabla \mathbf{u})\nu) \cdot \nabla \mathbf{u} \\ =&\frac{1}{2}((\nabla \mathbf{u})\nu \cdot (\nabla \mathbf{u})\nu) + \frac{1}{2}((\nabla \mathbf{u})\nu \cdot (\nabla \mathbf{u})^{\mathbf{T}}\nu) \\ =&\frac{1}{2}(\nabla \mathbf{u})\nu \cdot (\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathbf{T}})\nu = (\nabla \mathbf{u})\nu \cdot (\mathrm{Sym}\,\nabla \mathbf{u})\nu, \end{aligned}$$

which combined with (4.10) yields (4.1). ■

Proof of Lemma 4.3. Let I_n be the $n \times n$ identity matrix. Since

$$(\mathrm{Sym}\,(\mathbf{a} \otimes \nu))\nu \cdot \mathbf{a} = (\mathrm{Sym}\,(\mathbf{a} \otimes \nu)) \cdot (\mathbf{a} \otimes \nu) = |\mathrm{Sym}\,(\mathbf{a} \otimes \nu)|^2$$

by direct calculations, we obtain

$$B\mathbf{a} \cdot \mathbf{a} = \lambda|\mathbf{a} \cdot \nu|^2 + 2\mu|\mathrm{Sym}\,(\mathbf{a} \otimes \nu)|^2. \tag{4.11}$$

Let us set

$$A = \mathrm{Sym}\,(\mathbf{a} \otimes \nu)$$

and

$$D = A - \frac{1}{n}(\mathrm{Tr}\,A)I_n \tag{4.12}$$

Then $\mathrm{Tr}\,D = 0$, so that

$$D \cdot I_n = 0 \tag{4.13}$$

by the identity $D \cdot I_n = \mathrm{Tr}\,D$. Therefore (4.11)-(4.13) imply

$$\begin{aligned} &B\mathbf{a} \cdot \mathbf{a} = \lambda|\mathrm{Tr}\,A|^2 + 2\mu\left|\frac{1}{n}(\mathrm{Tr}\,A)I_n + D\right|^2 \\ =&\lambda|\mathrm{Tr}\,A|^2 + 2\mu\left(\left|\frac{1}{n}(\mathrm{Tr}\,A)I_n\right|^2 + |D|^2 + \frac{2}{n}(\mathrm{Tr}\,A)I_n \cdot D\right) \\ =&(n\lambda + 2\mu)\frac{1}{n}|\mathrm{Tr}\,A|^2 + 2\mu|D|^2 \geq \frac{c_0}{n}|\mathrm{Tr}\,A|^2 + c_0|D|^2. \end{aligned}$$

Note that in the last inequality, we have used (2.1). Recalling (4.12), we have $A = D + \frac{1}{n}(\mathrm{Tr}\, A)I_n$, so that $c_0|A|^2 = \frac{c_0}{n}|\mathrm{Tr}\, A|^2 + c_0|D|^2$ by (4.13). Therefore

$$B\mathbf{a}\cdot\mathbf{a} \geq c_0|\mathrm{Sym}\,(\mathbf{a}\otimes\nu)|^2.$$

On the other hand, we have

$$\begin{aligned}|\mathrm{Sym}\,(\mathbf{a}\otimes\nu)|^2 &= \frac{1}{4}(|\mathbf{a}\otimes\nu|^2 + 2(\mathbf{a}\otimes\nu)\cdot(\nu\otimes\mathbf{a}) + |\nu\otimes\mathbf{a}|^2)\\ &= \frac{1}{4}(|\mathbf{a}|^2 + 2|\mathbf{a}\cdot\nu|^2 + |\mathbf{a}|^2) \geq \frac{1}{2}|\mathbf{a}|^2.\end{aligned}$$

Thus (4.5) holds.

Direct calculations verify that $B\mathbf{a}\cdot\mathbf{b} = B\mathbf{b}\cdot\mathbf{a}$ for any $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$. This means that B is a symmetric matrix. Consequently, (4.5) implies (4.4).

Finally, by (4.7) and (4.8) we deduce

$$\begin{aligned}&B((\nabla\mathbf{u})\nu) = \lambda\{(\nabla\mathbf{u})\nu\cdot\nu\}\nu + 2\mu\{\mathrm{Sym}\,((\nabla\mathbf{u})\nu\otimes\nu)\}\nu\\ =&\lambda(\mathrm{Tr}\,\nabla\mathbf{u})\nu + 2\mu\{\mathrm{Sym}\,(\nabla\mathbf{u})\}\nu = \sigma(\mathbf{u})\nu \quad \text{a.e. on } \partial\Omega\times(0,T)\end{aligned}$$

and the proof of Lemma 4.3 is complete. ■

The next technical lemma is concerned with the control of E by a simpler energy-type functional.

Lemma 4.4. *Let (2.1) and (2.3) hold. Set*

$$G(t) = \frac{1}{2}\int_\Omega\{\mu_0|\nabla\mathbf{u}(x,t)|^2 + (\lambda(x)+\mu_0)|Tr\,\nabla\mathbf{u}(x,t)|^2 + \rho(x)|\mathbf{u}'(x,t)|^2\}dx \tag{4.14}$$

for $\mathbf{u}$ *satisfying (1.4). Then there exists a constant* $C = C(\mu) > 0$ *such that*

$$G(t) \leq E(t) \leq C(\mu)G(t), \qquad 0\leq t\leq T. \tag{4.15}$$

Proof. Since

$$2|\mathrm{Sym}\,\nabla\mathbf{u}|^2 = |\nabla\mathbf{u}|^2 + |\mathrm{Tr}\,\nabla\mathbf{u}|^2 + \mathrm{Tr}\,\nabla\{(\nabla\mathbf{u})\mathbf{u} - (\mathrm{Tr}\,\nabla\mathbf{u})\mathbf{u}\}, \tag{4.16}$$

we have (cf. (2.2))

$$\begin{aligned}&\int_\Omega 2\mu|\mathrm{Sym}\,\nabla\mathbf{u}|^2dx \geq \int_\Omega 2\mu_0|\mathrm{Sym}\,\nabla\mathbf{u}|^2dx\\ =&\int_\Omega(\mu_0|\nabla\mathbf{u}|^2 + \mu_0|\mathrm{Tr}\,\nabla\mathbf{u}|^2)dx + \int_\Omega\mu_0\mathrm{Tr}\,\nabla\{(\nabla\mathbf{u})\mathbf{u} - (\mathrm{Tr}\,\nabla\mathbf{u})\mathbf{u}\}dx\\ =&\int_\Omega(\mu_0|\nabla\mathbf{u}|^2 + \mu_0|\mathrm{Tr}\,\nabla\mathbf{u}|^2)dx.\end{aligned}$$

In the last equality we have used the divergence theorem and (1.4). Hence we have $E(t) \geq G(t)$, $0 \leq t \leq T$. On the other hand,

$$\int_\Omega 2\mu|\mathrm{Sym}\,\nabla\mathbf{u}|^2 dx \leq C(\mu)\int_\Omega |\nabla\mathbf{u}|^2 dx,$$

and (4.15) follows. ■

We are now able to conclude the proof of Theorem 2.2. Recalling Lemma 4.1, we set $\mathbf{h} = \mathbf{h}_0$ in (3.5). Then, using Lemma 4.2 and Lemma 4.3, we obtain

$$\begin{aligned}
&[\text{the right hand side of (3.5)}] = \frac{1}{2}\int_0^T\int_{\partial\Omega} \sigma(\mathbf{u})\nu\cdot(\nabla\mathbf{u})\nu dSdt \\
=&\frac{1}{2}\int_0^T\int_{\partial\Omega} B((\nabla\mathbf{u})\nu)\cdot(\nabla\mathbf{u})\nu dSdt \\
\geq&\frac{c_0}{4}\int_0^T\int_{\partial\Omega} |(\nabla\mathbf{u})\nu|^2 dSdt \geq C\int_0^T\int_{\partial\Omega}|\sigma(\mathbf{u})\nu|^2 dSdt. \qquad (4.17)
\end{aligned}$$

Here and henceforth $C > 0$ denotes a generic constant depending only on Ω, T, λ, μ, ρ, but independent of $\mathbf{u}$.

On the other hand, by Lemma 3.2, for any fixed $\delta > 0$, we have

$$\begin{aligned}
E(t) &\leq E(0) + \|\mathbf{F}\|_{L^1(0,T;H)}\|\mathbf{u}'\|_{L^\infty(0,T;H)} \\
&\leq E(0) + \frac{1}{2\delta}\|\mathbf{F}\|^2_{L^1(0,T;H)} + \frac{\delta}{2}\|\mathbf{u}'\|^2_{L^\infty(0,T;H)} \\
&\leq E(0) + \frac{1}{2\delta}\|\mathbf{F}\|^2_{L^1(0,T;H)} + C\delta \sup_{0\leq t\leq T} E(t). \qquad (4.18)
\end{aligned}$$

Then, taking for instance $\delta = \frac{1}{2C}$, we infer

$$\sup_{0\leq t\leq T} E(t) \leq C\|\mathbf{F}\|^2_{L^1(0,T;H)} + CE(0). \qquad (4.19)$$

Hence Lemma 4.4 yields

$$\sup_{0\leq t\leq T} G(t) \leq C\|\mathbf{F}\|^2_{L^1(0,T;H)} + CE(0), \qquad (4.20)$$

namely, using the Poincaré inequality,

$$\|\mathbf{u}'\|^2_{L^\infty(0,T;H)} + \|\mathbf{u}\|^2_{L^\infty(0,T;V)} \leq C\|\mathbf{f}\|^2_{L^1(0,T;H)} + CE(0). \qquad (4.21)$$

Consequently, (2.1) and (4.21) entail

$$[\text{the left hand side of (3.5)}] \leq C(\|\mathbf{f}\|^2_{L^1(0,T;L^2(\Omega)^n)} + E(0)). \qquad (4.23)$$

Thus, a combination of (4.17) and (4.23) gives (2.4) under the regularity assumptions (3.1)-(3.2). A density argument completes the proof. ■

Remark 4.5. From the proof of Lemma 4.4 we deduce that if μ is constant as in [4], then $G \equiv E$.

§**5. Proof of Theorem 2.3.** Let us suppose for the moment that (3.1) holds so that $\mathbf{u} \in C^0([0,T];W) \cap C^1([0,T];V) \cap C^2([0,T];H)$. Take $\mathbf{h}(x) = \mathbf{m}(x) = x - x_0$ in (3.5). Then, on account of Lemma 4.2, we obtain

$$\begin{aligned}
&\left[\int_\Omega \rho\mathbf{u}' \cdot (\nabla\mathbf{u})\mathbf{m} dx\right]_0^T \\
&+\frac{1}{2}\int_0^T\int_\Omega \{n\rho|\mathbf{u}'|^2 + (2-n)(\lambda|\mathrm{Tr}\,\nabla\mathbf{u}|^2 + 2\mu|\mathrm{Sym}\,\nabla\mathbf{u}|^2)\} dxdt \\
&+\frac{1}{2}\int_0^T\int_\Omega \{(\nabla\rho\cdot\mathbf{m})|\mathbf{u}'|^2 - (\nabla\lambda\cdot\mathbf{m})|\mathrm{Tr}\,\nabla\mathbf{u}|^2 - 2(\nabla\mu\cdot\mathbf{m})|\mathrm{Sym}\,\nabla\mathbf{u}|^2\} dxdt \\
&=\frac{1}{2}\int_0^T\int_{\partial\Omega} ((\nabla\mathbf{u})\nu\cdot\sigma(\mathbf{u})\nu)(\mathbf{m}\cdot\nu) dSdt.
\end{aligned} \tag{5.1}$$

Here we have used the identity

$$(\mathbf{m}\cdot\nu)(\nabla\mathbf{u})\nu = (\nabla\mathbf{u})\mathbf{m}$$

which follows from substitution of (4.7) in $(\nabla\mathbf{u})\mathbf{m}$.

Moreover, multiplying both the hand sides of (1.1) by $\frac{n-1}{2}\mathbf{u}$, and integrating over $\Omega \times (0,T)$, we deduce (recall that $\mathbf{F} \equiv \mathbf{0}$)

$$0 = \int_0^T\int_\Omega \frac{n-1}{2}\rho\mathbf{u}''\cdot\mathbf{u} dxdt - \int_0^T\int_\Omega \frac{n-1}{2}L(\mathbf{u})\cdot\mathbf{u} dxdt.$$

Thus, integrating by parts with respect to time the first term and using the divergence theorem in the second one (cf. also (1.4)), we infer

$$\begin{aligned}
0 = &\left[\int_\Omega \rho\mathbf{u}' \cdot \frac{n-1}{2}\mathbf{u} dx\right]_0^T + \frac{1}{2}\int_0^T\int_\Omega (1-n)\rho|\mathbf{u}'|^2 dxdt \\
&+\frac{1}{2}\int_0^T\int_\Omega \{\lambda(n-1)|\nabla^T\mathbf{u}|^2 + 2\mu(n-1)|\mathrm{Sym}\,\nabla\mathbf{u}|^2\} dxdt.
\end{aligned}$$

Adding this identity to (5.1), we obtain

$$\begin{aligned}
&\left[\int_\Omega \rho\mathbf{u}' \cdot M(\mathbf{u}) dx\right]_0^T + \frac{1}{2}\int_0^T\int_\Omega (\rho|\mathbf{u}'|^2 + \lambda|\mathrm{Tr}\,\nabla\mathbf{u}|^2 + 2\mu|\mathrm{Sym}\,\nabla\mathbf{u}|^2) dxdt \\
&+\frac{1}{2}\int_0^T\int_\Omega \{(\nabla\rho\cdot\mathbf{m})|\mathbf{u}'|^2 - (\nabla\lambda\cdot\mathbf{m})|\mathrm{Tr}\,\nabla\mathbf{u}|^2 - 2(\nabla\mu\cdot\mathbf{m})|\mathrm{Sym}\,\nabla\mathbf{u}|^2\} dxdt \\
&=\frac{1}{2}\int_0^T\int_{\partial\Omega} ((\nabla\mathbf{u})\nu\cdot\sigma(\mathbf{u})\nu)(\mathbf{m}\cdot\nu) dSdt
\end{aligned} \tag{5.2}$$

where we have set

$$M(\mathbf{u}) = (\nabla \mathbf{u})\mathbf{m} + \frac{n-1}{2}\mathbf{u}.$$

Arguing as in [9, Lemma 3.2], we can prove the inequality (cf. (2.2))

$$\left| \int_\Omega \rho \mathbf{u}'(x,t) \cdot M(\mathbf{u})(x,t)dx \right| \le \frac{R_0\rho_1}{2\sqrt{\rho_0\mu_0}} \int_\Omega (\rho_0|\mathbf{u}'(x,t)|^2 + \mu_0|\nabla \mathbf{u}(x,t)|^2)dx \tag{5.3}$$

for $0 \le t \le T$, where we recall that $R_0 = \|\mathbf{m}\|_{(L^\infty(\Omega))^n}$.

To obtain (5.3) we first show the following

$$\|M(\mathbf{u})(t)\|_H \le \|(\nabla \mathbf{u}(t))\mathbf{m}\|_H. \tag{5.4}$$

Indeed, on account of (1.4), Green's formula yields

$$\begin{aligned}
&\|M(\mathbf{u})(t)\|_H^2 - \|(\nabla \mathbf{u}(t))\mathbf{m}\|_H^2 \\
&= \left\| (\nabla \mathbf{u}(t))\mathbf{m} + \frac{n-1}{2}\mathbf{u}(t) \right\|_H^2 - \|(\nabla \mathbf{u}(t))\mathbf{m}\|_H^2 \\
&= \int_\Omega \left(\sum_{i=1}^n \left| \sum_{j=1}^n \frac{\partial u_i(x,t)}{\partial x_j} m_j + \frac{n-1}{2} u_i(x,t) \right|^2 - \sum_{i=1}^n \left| \sum_{j=1}^n \frac{\partial u_i(x,t)}{\partial x_j} m_j \right|^2 \right) dx \\
&= \int_\Omega \left\{ \sum_{i,j=1}^n (n-1) \frac{\partial u_i(x,t)}{\partial x_j} m_j u_i + \sum_{i=1}^n \frac{(n-1)^2}{4} (u_i(x,t))^2 \right\} dx \\
&= \int_\Omega \left\{ \sum_{i,j=1}^n \frac{n-1}{2} m_j \frac{\partial (u_i(x,t))^2}{\partial x_j} + \sum_{i=1}^n \frac{(n-1)^2}{4} (u_i(x,t))^2 \right\} dx \\
&= \int_\Omega \sum_{i=1}^n \left(-\frac{n(n-1)}{2} + \frac{(n-1)^2}{4} \right) (u_i(x,t))^2 dx = \frac{1-n^2}{4} \int_\Omega |\mathbf{u}(x,t)|^2 dx \le 0.
\end{aligned}$$

Thus, using (5.4), we have

$$\begin{aligned}
&\left| \int_\Omega \rho \mathbf{u}'(t) \cdot M(\mathbf{u})(t)dx \right| \le \rho_1 \|\mathbf{u}'(t)\|_H \|M(\mathbf{u})(t)\|_H \\
\le & \frac{\rho_1}{\sqrt{\rho_0\mu_0}} (\sqrt{\rho_0}\|\mathbf{u}'(t)\|_H)(\sqrt{\mu_0}\|(\nabla \mathbf{u}(t))\mathbf{m}\|_H) \\
\le & \frac{\rho_1}{\sqrt{\rho_0\mu_0}} \left(\frac{R_0}{2}\rho_0 \|\mathbf{u}'(t)\|_H^2 + \frac{\mu_0}{2R_0} \|(\nabla \mathbf{u}(t))\mathbf{m}\|_H^2 \right)
\end{aligned}$$

which entails (5.3).

Observe that (3.3) implies $E(t) = E(0)$ since $\mathbf{F} \equiv \mathbf{0}$ and (1.4) holds. Then, thanks to Lemma 4.4 and (5.3), we deduce from (5.3)

$$\left| \int_\Omega \rho \mathbf{u}'(x,t) \cdot M(\mathbf{u})(x,t)dx \right| \le \frac{\rho_1}{\sqrt{\rho_0\mu_0}} R_0 E(0), \quad 0 \le t \le T. \tag{5.5}$$

Consider now (5.2) and recall that $(\mathbf{m}\cdot\nu) \le 0$ on $\partial\Omega\setminus\Gamma_+(x_0)$. Using (2.5) and (5.5), we derive

$$\begin{aligned}&\frac{1}{2}\gamma_0\int_0^T\int_\Omega(\rho|\mathbf{u}'|^2+\lambda|\mathrm{Tr}\,\nabla\mathbf{u}|^2+2\mu|\mathrm{Sym}\,\nabla\mathbf{u}|^2)dxdt-\frac{2\rho_1R_0}{\sqrt{\rho_0\mu_0}}E(0)\\ &\le C\int_0^T\int_{\Gamma_+(x_0)}((\nabla\mathbf{u})\nu\cdot\sigma(\mathbf{u})\nu)(\mathbf{m}\cdot\nu)dSdt. \qquad (5.6)\end{aligned}$$

Finally, owing to Lemma 3.2 with $\mathbf{F}=0$ (cf. also (1.4)) and Lemma 4.3, we have

$$\left(\gamma_0T-\frac{2\rho_1R_0}{\sqrt{\rho_0\mu_0}}\right)E(0)\le C\int_0^T\int_{\Gamma_+(x_0)}|\sigma(\mathbf{u})\nu|^2dSdt. \qquad (5.7)$$

and the proof follows from (2.6) provided that (3.1) hold. A standard density argument shows that (2.7) still holds when (3.1) is replaced by (2.3) with $\mathbf{F}\equiv\mathbf{0}$.

§**6. Applications to an inverse source problem.** Here Theorems 2.2 and 2.3 are applied to the inverse source problem described in the Introduction. We need first to recall the following result which can be easily deduced from Proposition 2.1 and Theorem 2.2 (see also [4] and [16]).

Proposition 6.1. *Let (2.1) and*

$$\mathbf{u}_0\in V,\quad L\mathbf{u}_0\in H,\quad \mathbf{u}_1\in V \qquad (6.1)$$

$$\mathbf{F}\in W^{1,1}(0,T;H) \qquad (6.2)$$

hold. Then there is a unique function $\mathbf{u}\in C^1([0,T];V)\cap C^2([0,T];H)$ *which satisfies equation (1.1) almost everywhere in* $\Omega\times(0,T)$ *and initial conditions (1.3). Moreover, there is a positive constant* $C=C(\Omega,T,\rho,\lambda,\mu)$ *such that*

$$\begin{aligned}&\|\sigma(\mathbf{u})\nu\|_{H^1(0,T;(L^2(\partial\Omega))^n)}\\ &\le C\left(\|\mathbf{u}_0\|_V+\|L\mathbf{u}_0+\mathbf{F}(0)\|_H+\|\mathbf{u}_1\|_V+\|\mathbf{F}\|_{W^{1,1}(0,T;H)}\right). \qquad (6.3)\end{aligned}$$

As a consequence, if $\mathbf{F}$ has the form (1.6) with, for instance,

$$\varphi\in C^1([0,T]), \qquad (6.4)$$

then, for any $\mathbf{f}\in H$, there is a unique $\mathbf{u}=\mathbf{u}(\mathbf{f})\in C^1([0,T];V)\cap C^2([0,T];H))$ which solves equation (1.1) with null initial conditions and satisfies (1.4).

Hence, for any given $\Gamma\subseteq\partial\Omega$ of positive Lebesgue measure, we can define a linear mapping $\mathbf{G}:H\to H^1(0,T;(L^2(\Gamma))^n)$ by setting

$$\mathbf{G}(\mathbf{f})=\sigma(\mathbf{u})\nu \qquad \text{a.e. on } \Gamma\times(0,T), \qquad (6.5)$$

which is continuous, thanks to Proposition 6.1. Suppose now that the surface traction is measured on some $\Gamma_0\subseteq\partial\Omega$ over a time interval $[0,T]$ and let $\mathbf{g}=\sigma(\mathbf{u})\nu$ on $\Gamma_0\times(0,T)$. The following result shows that $\mathbf{f}$ is uniquely determined by $\mathbf{g}$, provided that Γ_0 is suitably chosen and T is sufficiently large.

Theorem 6.2. *Pick a point* $x_0 \in \mathbb{R}^n$ *and consider* $\Gamma_+(x_0) \subseteq \partial\Omega$. *Suppose that (2.5) holds for some* $\gamma_0 > 0$ *and let* $\mathbf{u}_0 = \mathbf{u}_1 \equiv \mathbf{0}$. *Assume in addition that* $\mathbf{F}$ *has the form (1.6) with* $\mathbf{f} \in H$ *and* φ *satisfying (6.4) and*

$$\varphi(0) \neq 0. \tag{6.6}$$

If

$$T > \frac{T_0}{2}, \tag{6.7}$$

then there exists a positive constant $C = C(\Omega, T, x_0, \rho, \lambda, \mu, \varphi)$ *such that*

$$\|\mathbf{f}\|_H \leq C\|\sigma(\mathbf{u}'(\mathbf{f}))\nu\|_{L^2(0,T;(L^2(\Gamma_+(x_0)))^n)}. \tag{6.8}$$

Remark 6.3. Here $T_0 > 0$ is defined in (2.6). As is seen from the proof, the reverse inequality to (6.8) holds. This means that the norm on the right hand side of (6.8) is the best possible for estimating $\|\mathbf{f}\|_H$.

Proof. On account of Proposition 2.1, we let $\mathbf{v} \in C^0([0,T];V) \cap C^1([0,T];H)$ be the unique solution to the homogeneous Lamé system with null initial displacement and $\mathbf{v}'(0) = \mathbf{f}$. Then, arguing as in [4] (see also [16]), we set

$$\widetilde{\mathbf{u}} = \varphi * \mathbf{v}, \qquad \text{a.e. in } \Omega \times (0,T),$$

where $*$ stands for the usual time convolution product over $(0,t)$, $t \in [0,T]$. Of course, due to (6.4), $\tilde{\mathbf{u}} \in C^1([0,T];V) \cap C^2([0,T];H)$. Also, by the uniqueness of the solution, it is easy to realize that $\mathbf{u}(\mathbf{f}) \equiv \widetilde{\mathbf{u}}$. Hence

$$\sigma(\mathbf{u}(\mathbf{f}))\nu = \varphi * \sigma(\mathbf{v})\nu, \qquad \text{a.e. on } \partial\Omega \times (0,T),$$

so that we obtain the Volterra integral equation of the second kind (cf. (6.4) and (6.6))

$$\sigma(\mathbf{u}'(\mathbf{f}))\nu = \varphi(0)\sigma(\mathbf{v})\nu + \varphi' * \sigma(\mathbf{v})\nu, \qquad \text{a.e. on } \partial\Omega \times (0,T). \tag{6.9}$$

Thus, due to (6.6), from (6.9) we can find a positive constant $C = C(\varphi)$ such that, for any $\Gamma \subseteq \partial\Omega$,

$$\|\sigma(\mathbf{v})\nu\|_{L^2(0,T;(L^2(\Gamma))^n)} \leq C\|\sigma(\mathbf{u}'(\mathbf{f}))\nu\|_{L^2(0,T;(L^2(\Gamma))^n)}. \tag{6.10}$$

On the other hand, Theorem 2.3 yields

$$\|\mathbf{f}\|_H \leq C\|\sigma(\mathbf{v})\nu\|_{L^2(0,T;(L^2(\Gamma_+(x_0)))^n)}, \tag{6.11}$$

provided that T satisfies (6.7). Indeed, since $\mathbf{v}(0) = \mathbf{0}$, in (5.2) written for $\mathbf{v}$ we just need the estimate (cf. (5.5))

$$\left| \left[\int_\Omega \rho \mathbf{v}' \cdot M(\mathbf{v}) dx \right]_0^T \right| \leq \frac{\rho_1}{\sqrt{\rho_0 \mu_0}} R_0 E(0),$$

so that in place of (5.7) we have

$$\left(\gamma_0 T - \frac{\rho_1 R_0}{\sqrt{\rho_0 \mu_0}} \right) E(0) \leq C \int_0^T \int_{\Gamma_+(x_0)} |\sigma(\mathbf{v})\nu|^2 dS dt.$$

where

$$E(0) = \frac{1}{2} \int_\Omega \rho(x) |\mathbf{f}(x)|^2 dx.$$

Therefore (6.8) follows from (6.10) and (6.11). ∎

Theorem 6.2 entails

Corollary 6.4. *Under the assumptions of Theorem 6.2, let $\Gamma_+(x_0) \subseteq \Gamma_0$. Then, for any $\mathbf{g} \in H^1(0, T; (L^2(\Gamma_0))^n)$, there is at most one $\mathbf{f} \in H$ such that $\mathbf{G}(\mathbf{f}) = \mathbf{g}$ provided that T satisfies (6.7).*

We now want to obtain a representation formula for $\mathbf{f}$ similar to [4, (3.18)] by using a slightly different approach. We first recall that the positive, linear, unbounded, and self-adjoint operator $-\mathbf{L} : D(-\mathbf{L}) = W \subset H \to H$ defined through L (cf. (1.2)) generates a strongly continuous cosine operator $\mathbf{C}(t)$ on H, $t \in \mathbb{R}$, and the corresponding sine operator is defined by

$$\mathbf{S}(t)\mathbf{v} = \int_0^t \mathbf{C}(\tau)\mathbf{v} \, d\tau \tag{6.12}$$

for any $\mathbf{v} \in H$ (see, e.g., [10, p. 171] and references therein).

Henceforth, by letting $\rho = 1$ for the sake of simplicity, thanks to [10, (3.5), (a) and (c)], $\mathbf{u}(\mathbf{f})$ admits the explicit representation

$$\mathbf{u}(\mathbf{f})(t) = \int_0^t \mathbf{S}(t - \tau)\varphi(\tau)\mathbf{f} \, d\tau \qquad t \in [0, T]. \tag{6.13}$$

Fix now $\Gamma_0 \subset \partial\Omega$ of positive Lebesgue surface measure and introduce the Dirichlet map $\mathbf{D} : \theta \to \mathbf{z}$ where $\mathbf{z}$ solves the Dirichlet problem

$$-\nabla \cdot \sigma(\mathbf{z}) = 0 \qquad \text{in } \Omega$$

$$\mathbf{z}|_{\partial\Omega} = \begin{cases} \theta & \text{on } \Gamma_0 \\ \mathbf{0} & \text{on } \Gamma_0^c. \end{cases}$$

Arguing as in [10, pp. 171-172], one can prove that $\mathbf{D} : (L^2(\Gamma_0))^n \to (H^{1/2-2\alpha}(\Omega))^n \equiv D((-\mathbf{L})^{1/4-\alpha})$ is continuous for any $\alpha \in (0, \frac{1}{4}]$, and that the mapping $\Sigma : D(-\mathbf{L}) \to (L^2(\Gamma_0))^n$ defined by

$$\Sigma \mathbf{z} = \sigma(\mathbf{z})\nu \qquad \text{a.e. on } \Gamma_0$$

can be represented as

$$\Sigma = \mathbf{D}^\star \mathbf{L}^\star$$

where the superscript $\star$ denotes the adjoint operator.

Taking advantage of Σ, we thus have an explicit representation of $\sigma(\mathbf{u})\nu$ (see [10, proof of Thm. 3.7, p. 178])

$$\sigma(\mathbf{u}(\mathbf{f})(t))\nu = \mathbf{D}^\star \mathbf{L}^\star \int_0^t \mathbf{S}(t-\tau)\varphi(\tau)\mathbf{f}\, d\tau \qquad t \in [0,T]. \tag{6.14}$$

Thanks to Theorems 2.2-2.3 and the HUM techniques, we know that in the Cauchy problem for the Lamé system (1.1) with $\mathbf{F} = \mathbf{0}$, by a Dirichlet-type action, we have partially exact controllability (contrôlabilité exacte elargie according to [12, Chap. I, §9]), provided that T satisfies (6.7) (see [4, Thm. 2.5] and [12, Chap. IV]). This amounts to say that, for any $\widetilde{\mathbf{z}}^0 \in H$, there exists a unique $\widetilde{\mathbf{w}} \in L^2(0,T;(L^2(\Gamma_0))^n)$, provided that $\Gamma_+(x_0) \subseteq \Gamma_0$, such that the solution $\widetilde{\mathbf{v}}$ to the Lamé system (1.1) with final and boundary conditions

$$\widetilde{\mathbf{v}}(T) = \mathbf{0} \qquad \text{a.e. in } \Omega, \quad \widetilde{\mathbf{v}}'(T) = \mathbf{0} \qquad \text{in } V^*$$

$$\widetilde{\mathbf{v}}|_{\partial\Omega} = \begin{cases} \widetilde{\mathbf{w}} & \text{on } \Gamma_0 \\ \mathbf{0} & \text{on } \Gamma_0^c \end{cases}$$

fulfills

$$\widetilde{\mathbf{v}}(0) = \widetilde{\mathbf{z}}^0 \qquad \text{a.e. in } \Omega.$$

Therefore, whenever $\Gamma_0 \subseteq \Gamma_+(x_0)$ and T satisfies (6.7), we can define an operator $\Pi : H \to L^2(0,T;(L^2(\Gamma_0))^n)$ by setting

$$\Pi \widetilde{\mathbf{z}}^0 = \widetilde{\mathbf{w}}. \tag{6.15}$$

Then, let us consider the set $\{\mathbf{z}_k\}_{k\in\mathbb{N}}$ of eigenfunctions associated with the operator $-\mathbf{L}$ and the corresponding Dirichlet controls

$$\mathbf{w}_k = \Pi \mathbf{z}_k.$$

We note that $\{\mathbf{z}_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in H. The controlled solution $\widetilde{\mathbf{v}}_k$ can be represented as (cf. [10, (3.5), (a) and (b), p. 172])

$$\widetilde{\mathbf{v}}_k(t) = \mathbf{L} \int_t^T \mathbf{S}(\tau - t)\mathbf{D}\mathbf{w}_k(\tau) d\tau,$$

so that

$$\mathbf{z}_k = J^*\mathbf{w}_k$$

where

$$J^*\mathbf{w}_k = \mathbf{L}\int_0^T \mathbf{S}(\tau)\mathbf{D}\mathbf{w}_k(\tau)d\tau \tag{6.16}$$

is the adjoint operator of the linear operator $J : H \to (L^2(\partial\Omega))^n$ (see [10, Corollaries 3.1 and 3.2, p. 173])

$$J(\mathbf{y})(t) = \mathbf{D}^*\mathbf{L}^*\mathbf{S}^*(t)\mathbf{y} \qquad \forall\, \mathbf{y} \in H. \tag{6.17}$$

Observe now that $\mathbf{S}(t)$ is self-adjoint for any $t \geq 0$ since $\mathbf{L}$ is so. Hence, on account of (6.12), (6.16) and (6.17), we have

$$\begin{aligned}
&\int_\Omega \mathbf{f}\cdot\mathbf{z}_k\,dx\\
&= \int_\Omega J^*\mathbf{w}_k\cdot\mathbf{f}\,dx\\
&= \int_0^T\int_{\Gamma_0} \mathbf{w}_k(\tau)\cdot\mathbf{D}^*\mathbf{L}^*\mathbf{S}^*(\tau)\mathbf{f}\,dSd\tau\\
&= \int_0^T\int_{\Gamma_0} \mathbf{w}_k(\tau)\cdot\mathbf{D}^*\mathbf{L}^*\left(\int_0^\tau \mathbf{C}(\eta)\mathbf{f}d\eta\right)dSd\tau\\
&= \int_0^T\int_{\Gamma_0}\left(\int_\tau^T \mathbf{w}_k(\eta)d\eta\right)\cdot\mathbf{D}^*\mathbf{L}^*\mathbf{C}(\tau)\mathbf{f}\,dSd\tau.
\end{aligned} \tag{6.18}$$

At the last equality, we used integration by parts in τ. Therefore, if we are able to find, for any $k \in \mathbb{N}$, a unique θ_k such that

$$\int_\tau^T \varphi(t-\tau)\theta_k(t)dt = \int_\tau^T \mathbf{w}_k(\eta)d\eta \qquad \text{a.e. on } \Gamma_0,\ \forall\,\tau\in[0,T] \tag{6.19}$$

then, from (6.18) we deduce (cf. also (6.12) and (6.14))

$$\begin{aligned}
&\int_\Omega \mathbf{f}\cdot\mathbf{z}_k dx\\
&= \int_0^T\int_{\Gamma_0}\left(\int_\tau^T \varphi(t-\tau)\theta_k(t)dt\right)\cdot\mathbf{D}^*\mathbf{L}^*\mathbf{C}(\tau)\mathbf{f}\,dSd\tau\\
&= \int_0^T\int_{\Gamma_0}\theta_k(t)\cdot\left(\mathbf{D}^*\mathbf{L}^*\int_0^t \mathbf{C}(\tau)\varphi(t-\tau)\mathbf{f}d\tau\right)dSdt\\
&= \int_0^T\int_{\Gamma_0}\theta_k(t)\cdot\sigma(\mathbf{u}'(\mathbf{f})(t))\nu\,dSdt.
\end{aligned} \tag{6.20}$$

Hence we can compute the Fourier coefficients of $\mathbf{f}$ with respect to the orthonormal system $\{\mathbf{z}_k\}_{k\in\mathbb{N}}$. To this aim, let $\mathbf{w} \in L^2(0,T;(L^2(\Gamma_0))^n)$ and consider the integral equation

$$\int_\tau^T \varphi(t-\tau)\theta(t)dt = \int_\tau^T \mathbf{w}(\eta)d\eta \qquad \text{a.e. on } \Gamma_0,\ \forall\,\tau \in [0,T]. \tag{6.21}$$

Differentiating the both members with respect to τ, we obtain (cf. (6.4))

$$-\varphi(0)\theta(\tau) - \int_\tau^T \varphi'(t-\tau)\theta(t)dt = -\mathbf{w}(\tau) \qquad \text{a.e. on } \Gamma_0,\ \forall\,\tau \in [0,T].$$

Then, due to (6.4) and (6.6), we deduce that there is a unique

$$\theta \in L^2(0,T;(L^2(\Gamma_0))^n)$$

which solves equation (6.21). We thus set $\theta = K\mathbf{w}$ and we obtain a linear and continuous operator K from $L^2(0,T;(L^2(\Gamma_0))^n)$ to itself. In particular, we have

$$\theta_k = K\mathbf{w}_k \qquad \forall\, k \in \mathbb{N}. \tag{6.22}$$

Recalling now (6.19), we finally obtain, owing to (6.15),

$$\begin{aligned}&\int_\Omega \mathbf{f}\cdot\mathbf{z}_k dx\\ &= \int_0^T\int_{\Gamma_0} K\Pi\mathbf{z}_k\cdot\sigma(\mathbf{u}'(\mathbf{f})(t))\nu\, dSdt.\end{aligned}$$

Summing up, we have thus proved the following

Theorem 6.5. *Under the assumptions of Theorem 6.2, let $\rho = 1$ and $\Gamma_+(x_0) \subseteq \Gamma_0$. Suppose moreover that there exists $\mathbf{f} \in H$ such that $\mathbf{G}(\mathbf{f}) = \mathbf{g}$ for some $\mathbf{g} \in H^1(0,T;(L^2(\Gamma_0))^n)$ with T satisfying (6.7). Then*

$$\mathbf{f} = \sum_{k=0}^{+\infty} \phi_k\mathbf{z}_k$$

where

$$\phi_k = \int_0^T\int_{\Gamma_0} K\Pi\mathbf{z}_k\cdot\mathbf{g}'\, dSdt \qquad \forall\, k\in\mathbb{N},$$

$\{\mathbf{z}_k\}_{k\in\mathbb{N}}$ being the eigenfunctions of the operator $-\mathbf{L}$.

To conclude, we mention that the same arguments used in [4, Sec. 6] lead us to

Theorem 6.6. *Under the assumptions of Theorem 6.2, let* $\Gamma_+(x_0) \subseteq \Gamma_0$*. Then, regarding* $\mathbf{G}$ *as a linear operator from* H *to* $L^2(0,T;((L^2(\Gamma))^n)$*, we have*

$$V \subset \text{Range }(\mathbf{G}^\star) \subset (H^{1/2}(\Omega))^n.$$

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