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OBSERVABILITY INEQUALITIES BY INTERNAL OBSERVATION AND ITS APPLICATIONS

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Abstract

By means of Carleman estimate, we obtain several observability inequalities by internal observation for hyperbolic equations with lower order terms. Also, we apply our observability results to exact controllability problems and an inverse wave sourse problem.

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Key Words: Observability inequality, wave equation, Carleman estimate, exact controllability, inverse problem.

1 Introduction

Let us consider the following wave equation:

$$\begin{cases} \Box y(t,x) = a_1(t,x)y + a_2(t,x)y_t + \langle\, a_3(t,x), \nabla y\,\rangle, & (t,x)\in Q,\\ y(t,x)=0, & (t,x)\in\Sigma,\\ y(0,x)=y_0(x), \quad y_t(0,x)=y_1(x), & x\in\Omega. \end{cases} \tag{1.1}$$

Here $x=(x_1,\cdots,x_n)$, $\Box y = y_{tt}-\Delta y$, $Q \overset{\triangle}{=} (0,T)\times\Omega$, $\Sigma=(0,T)\times\partial\Omega$, $T>0$, $\Omega\subset\mathbb{R}^n$ is a bounded domain which is either convex or of class $C^{1,1}$. Let G be a subdomain of Ω. In this paper, we will establish observability estimate for (1.1) by internal observation in the subdomain G. By this we mean an estimate of $|(y_0,y_1)|_{H_0^1(\Omega)\times L^2(\Omega)}$ by a suitable norm of $y|_{(0,T)\times G}$.

Moreover we consider a special case of (1.1):

$$\begin{cases} \Box w(t,x) = a(t,x)w, & (t,x)\in Q,\\ w(t,x)=0, & (t,x)\in\Sigma,\\ w(0,x)=w_0(x), \quad w_t(0,x)=w_1(x), & x\in\Omega. \end{cases} \tag{1.2}$$

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Concerning (1.2), under some further conditions, we can sharpen the observability inequality and even establish the observability inequality by weaker norms.

We remark that observability inequality is essential for the exact controllability by the duality argument (e.g. Lions [9]). Moreover, this kind of inequality is very closely related to some inverse problems (*see* Puel and Yamamoto [13], Yamamoto [14], [15], [16]).

As for the observability by means of boundary data, there are many papers and we are obliged to refer to a very restricted list of papers: Bardos, Lebeau and Rauch [1], Ho [4], Komornik [6], [9] and the readers can further consult the references therein. However, for the observability inequality by internal observation, to our best knowledge, there are only a few papers ([9], Liu [10], Liu and Yamamoto [11] and Zhang [17])considering this problem for some special cases of equation (1.1) and subdomain G.

By the character of equation (1.1) as hyperbolic equation, T and G must satisfy some conditions for the observability and such conditions are characterized as the geometric optics condition ([1]). However, the geometric optics condition in [1] (see also Burq [2] for further development) requires some extra regularity conditions on the coefficients in (1.1) and the domain Ω. Such extra regularity is not suitable for further discussions on controllability for semilinear wave equation and an inverse problem of determining non-smooth functions (*see* Section 3).

Hence some geometrical sufficient condition for G and T which is valid in the non-smooth case, is very desirable. As so oriented work, we refer to [9] and [11], where the main ingredient is a classical multiplier method. Furthermore, we refer to Liu [10] as for a different method. In this paper, we will use a Carleman-type estimate, which is due to Lavrentiev, Romanov and Shishat.skiĭ [8], and apply some argument in Cheng, Isakov, Yamamoto and Zhou [3], Kazemi and Klibanov [5], Lasiecka, Triggiani and Zhang [7] to derive the desired observability inequality.

The rest of this paper is organized as follows. In Section 2, we state our main results. In Section 3, we apply our observability inequalities to exact controllability problem of linear and semilinear wave equatins and an inverse wave sourse problem. Some preliminaries for the proof of our main results are listed in Section 4. The next 3 sections, Sections 5–7, are devoted to prove our main results.

2 Main Results

First of all, let us introduce some notations. For any $M \in \mathbb{R}^n$ and $\varepsilon > 0$, put $\mathcal{O}_\varepsilon(M) = \{y \in \mathbb{R}^n \mid |y - x| < \varepsilon \text{ for some } x \in M\}$. For some fixed $x_0^j \in \mathbb{R}^n$ and domains $\Omega^j \subset \Omega$, $1 \le j \le J$, put

$$\begin{cases} \Gamma_j = \partial\Omega^j, \\ \Gamma_{j0} = \{x \in \Gamma_j \mid (x - x_0^j) \cdot \nu^j(x) > 0\}, \end{cases} \tag{2.1}$$

where $\nu^j(x)$ is the unit normal vector to Γ_j at x pointing towards the exterior of Ω^j. Throughout this paper, C denotes an generic positive constant depending only on T, Ω, $a(\cdot)$ and/or $a_i(\cdot)$ $(i = 1, 2, 3)$, which may change from line to line.

Next, let us pose the following assumption:

(H) *Suppose* $G \subseteq \Omega$ *satisfies*

$$G \supseteq \Omega \cap \mathcal{O}_\delta\Big(\cup_{j=1}^J \Gamma_{j0} \cup (\Omega \setminus \cup_{j=1}^J \Omega^j)\Big) \tag{2.2}$$

for some fixed $\delta > 0$, domains $\Omega^j \subseteq \Omega$ with Lipschitz boundary Γ_j satisfying

$$\overline{\Omega^i} \cap \overline{\Omega^j} = \emptyset, \ 1 \le i < j \le J, \tag{2.3}$$

$J \in \mathbb{N}$ and points $x_0^j \in \mathbb{R}^n$, $1 \le j \le J$.

Also, we put

$$\begin{cases} S \stackrel{\triangle}{=} \cup_{j=1}^J \Gamma_{j0} \cup \left(\Omega \setminus \cup_{j=1}^J \Omega^j\right), \\ R_1^j \stackrel{\triangle}{=} \sup_{x \in \Omega^j \setminus \mathcal{O}_\delta(S)} |x - x_0^j|. \end{cases} \tag{2.4}$$

Our main results on observability inequalities are the following.

Theorem 2.1 *Let (H) hold and $T > 2\max\{R_1^j \mid 1 \le j \le J\}$. Let $a_1(\cdot) \in L^{n+1}(Q)$, $a_2(\cdot) \in L^\infty(Q)$ and $a_3(\cdot) \in L^\infty(Q;\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that the weak solution $y(\cdot) \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ of system (1.1) satisfies*

$$\begin{aligned} |y_0|^2_{H_0^1(\Omega)} + |y_1|^2_{L^2(\Omega)} \le \ & C \int_0^T \int_G (y^2 + y_t^2) dxdt, \\ & \forall \ (y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega). \end{aligned} \tag{2.5}$$

Theorem 2.2 *Let (H) hold and $T > 2\max\{R_1^j \mid 1 \le j \le J\}$. Let $a(\cdot) = a(t,x) \equiv a(x) \in L^\infty(\Omega)$ with $a(x) \le 0$. Then there exists a constant $C > 0$ such that the weak solution $w(\cdot) \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ of system (1.2) satisfies*

$$\begin{aligned} |w_0|^2_{H_0^1(\Omega)} + |w_1|^2_{L^2(\Omega)} \le \ & C \int_0^T \int_G w_t^2 dxdt, \\ & \forall \ (w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega). \end{aligned} \tag{2.6}$$

Theorem 2.3 *Let (H) hold and $T > 2\max\{R_1^j \mid 1 \le j \le J\}$. Let $a(\cdot) \in L^\infty(Q)$. Then there exists a constant $C > 0$ such that the weak solution $w(\cdot) \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$ of system (1.2) satisfies*

$$\begin{aligned} |w_0|^2_{L^2(\Omega)} + |w_1|^2_{H^{-1}(\Omega)} \le \ & C \int_0^T \int_G w^2 dxdt, \\ & \forall \ (w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega). \end{aligned} \tag{2.7}$$

The proof of Theorems 2.1–2.3 will be given in Sections 5–7. Now several remarks are in order.

Remark 2.1 *We remark that (2.6) does not hold for the the general case, even if $a_0(\neq 0)$ is a constant and $G = \Omega$. For example, take μ as the eigenvalue of the following problem*

$$\begin{cases} -\Delta \xi = \mu \xi \quad in \ \ \Omega, \\ \xi|_{\partial\Omega} = 0, \end{cases} \tag{2.8}$$

where $\xi(\neq 0)$ is the corresponding eigenvector. Then, one sees easily that ξ solves system (1.2) with the initial data (w_0, w_1) and $a(\cdot)$ replaced by $(\xi, 0)$ and μ respectively. However (2.6) is false for this ξ.

Remark 2.2 *The constant C in (2.5)–(2.7) can be estimated explicitly with respect to $a(\cdot)$ and/or $a_i(\cdot)$ $(i = 1, 2, 3)$, in the style of [17].*

Remark 2.3 *Theorems 2.2–2.3 cover the main results in [11] and [17].*

3 Application to exact controllability problems and inverse source problem

We denote by $1\!\!1_G$ the characteristic function of G. First of all, applying the duality argument (*see* [9] and [18]) and using Theorem 2.3, one obtains immediately the following two exact controllability results.

Theorem 3.1 *Let (H) hold and $T > 2\max\{R_1^j \mid 1 \le j \le J\}$. Let $a(\cdot) \in L^\infty(Q)$. Then for any (y_0, y_1), $(z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there is a control $u(\cdot) \in L^2(Q)$ such that the weak solution of the following equation*

$$\begin{cases} \Box y(t,x) = a(t,x)y + 1\!\!1_G(x)u(t,x), & (t,x) \in Q, \\ y(t,x) = 0, & (t,x) \in \Sigma, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), & x \in \Omega \end{cases} \tag{3.1}$$

satisfies

$$y(T,x) = z_0(x), \quad y_t(T,x) = z_1(x), \quad x \in \Omega. \tag{3.2}$$

Theorem 3.2 *Let (H) hold and $T > 2\max\{R_1^j \mid 1 \le j \le J\}$. Let $f(\cdot) \in C^1(\mathbb{R}^1)$ with $f'(\cdot) \in L^\infty(\mathbb{R}^1)$. Then for any (y_0, y_1), $(z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there is a control $u(\cdot) \in L^2(Q)$ such that the weak solution of the following equation*

$$\begin{cases} \Box y(t,x) = f(y) + 1\!\!1_G(x)u(t,x), & (t,x) \in Q, \\ y(t,x) = 0, & (t,x) \in \Sigma, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), & x \in \Omega \end{cases} \tag{3.3}$$

satisfies

$$y(T,x) = z_0(x), \quad y_t(T,x) = z_1(x), \quad x \in \Omega. \tag{3.4}$$

Next let us consider an inverse problem of determining a wave sourse term. More precisely, for any fixed

$$\begin{cases} \lambda \in C^1[0,T] \text{ and } \lambda(0) \neq 0; \\ a \in L^\infty(\Omega), \end{cases} \tag{3.5}$$

we consider the following equation:

$$\begin{cases} \Box y(t,x) = a(x)y + \lambda(t)f(x), & (t,x) \in Q, \\ y(t,x) = 0, & (t,x) \in \Sigma, \\ y(0,x) = y_t(0,x) = 0, & x \in \Omega, \end{cases} \tag{3.6}$$

where $f \in L^2(\Omega)$ is unknown. The sourse term $\lambda(t)f(x)$ is assumed to cause the vibration and we hope to determine $f = f(x)$ from the interior observation $y|_{(0,T)\times G}$. Similar inverse problems are discussed in Puel and Yamamoto [13], Yamamoto [14], [15], [16] where boundary observations are used. By Theorem 2.3 and similar to [14], one gets easily the following stability result for the above inverse source problem.

Theorem 3.3 *Let (H) hold and $T > \max\{R_1^j \mid 1 \le j \le J\}$. Let λ and a satisfy (3.5). Then there exists a constant $C = C(T, \Omega, G, a, \lambda) > 0$ such that*

$$C^{-1}|f|^2_{L^2(\Omega)} \le \int_0^T \int_G y_{tt}^2 dxdt \le C|f|^2_{L^2(\Omega)}, \quad \forall\, f \in L^2(\Omega), \tag{3.7}$$

where $y(=y(f)) \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; H_0^1(\Omega)) \cap C^2([0,T]; L^2(\Omega))$ is the solution of system (3.6).

Remark 3.1 *The condition $\lambda(0) \neq 0$ in (3.5) can be replaced by $\frac{d^k\lambda}{dt^k}(0) \neq 0$ with some $k \in \mathbb{N}$; but we do not treat this case in this paper (see Yamamoto [15] in the case of boundary observations).*

4 Some Preliminaries

In order to prove Theorems 2.1–2.3, we need some preliminaries.

First of all, the following lemma is a special case of LEMMA 1 on pp. 124 in [8].

Lemma 4.1 *Let $\lambda > 0$, $\alpha \in (0,1)$, and*

$$\begin{cases} \varphi = \varphi(t,s,x) = |x - x_0|^2 - \alpha(t - T/2)^2 - \alpha(s - T/2)^2, \\ \eta = \frac{\lambda}{2}\varphi, \ \theta = e^\eta, \\ \Psi = (n - 1 + \alpha)\lambda. \end{cases} \tag{4.1}$$

Let $v = v(t,s,x) \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n)$. Then

$$\begin{array}{l} \theta^2(v_{tt} + v_{ss} - \Delta v)^2 \\ \ge 2(1-\alpha)\lambda\theta^2(v_t^2 + v_s^2 + \sum_i v_{x_i}^2) + \theta^2 B v^2 \\ \quad + \Big\{2\theta^2\Big[\eta_t(v_t^2 + v_s^2 - \sum_j v_{x_j}^2) - 2v_t(\eta_t v_t + \eta_s v_s - \sum_j \eta_{x_j} v_{x_j}) \\ \quad + (\Psi - 2A)v_t v + (A_t - 2\eta_t A)v^2\Big]\Big\}_t \\ \quad + \Big\{2\theta^2\Big[\eta_s(v_t^2 + v_s^2 - \sum_j v_{x_j}^2) - 2v_s(\eta_t v_t + \eta_s v_s - \sum_j \eta_{x_j} v_{x_j}) \\ \quad + (\Psi - 2A)v_s v + (A_s - 2\eta_s A)v^2\Big]\Big\}_s \\ \quad - \sum_i \Big\{2\theta^2\Big[\eta_{x_i}(v_t^2 + v_s^2 - \sum_j v_{x_j}^2) - 2v_{x_i}(\eta_t v_t + \eta_s v_s - \sum_j \eta_{x_j} v_{x_j}) \\ \quad + (\Psi - 2A)v_{x_i} v + (A_{x_i} - 2\eta_{x_i} A)v^2\Big]\Big\}_{x_i}, \end{array} \tag{4.2}$$

where

$$A = \lambda^2\Big[\alpha^2(t - T/2)^2 + \alpha^2(s - T/2)^2 - |x - x_0|^2\Big] \tag{4.3}$$

and

$$\begin{array}{rl} B = & 4(1+\alpha)\lambda^3\Big[|x - x_0|^2 - \alpha^2(t - T/2)^2 - \alpha^2(s - T/2)^2\Big] \\ & -\Big[8\alpha^2 + 4n + (n + \alpha - 1)^2\Big]\lambda^2. \end{array} \tag{4.4}$$

□

Furthermore, we need the following simple result (*see* for example [17]).

Lemma 4.2 *Let $0 \leq s_1 < s_2 < t_2 < t_1 \leq T$.*

(1) If $a_1(\cdot) \in L^{n+1}(Q)$, $a_2(\cdot) \in L^\infty(Q)$ and $a_3(\cdot) \in L^\infty(Q; \mathbb{R}^n)$, then

$$\int_{s_2}^{t_2} |y(t,\cdot)|^2_{H_0^1(\Omega)} dt \leq C \int_{s_1}^{t_1} \Big[|y(t,\cdot)|^2_{L^2(\Omega)} + |y_t(t,\cdot)|^2_{L^2(\Omega)} \Big] dt \tag{4.5}$$

for some constant $C = C(|a_1|_{L^{n+1}(Q)}, |(a_2, a_3)|_{L^\infty(Q;\mathbb{R}^{n+1})}, T, s_1, s_2, t_1, t_2)$, where $y(\cdot)$ is the weak solution of system (1.1).

(2) If $a(\cdot) \in L^\infty(Q)$, then

$$\int_{s_2}^{t_2} |w_t(t,\cdot)|^2_{H^{-1}(\Omega)} dt \leq C \int_{s_1}^{t_1} |w(t,\cdot)|^2_{L^2(\Omega)} dt \tag{4.6}$$

for some constant $C = C(|a|_{L^\infty(Q)}, T, s_1, s_2, t_1, t_2)$, where $w(\cdot)$ is the weak solution of system (1.2). □

Finnally, denote

$$\mathcal{E}(t) \triangleq \frac{1}{2}\Big[|y_t(t,\cdot)|^2_{L^2(\Omega)} + |y(t,\cdot)|^2_{H_0^1(\Omega)}\Big], \quad E(t) \triangleq \frac{1}{2}\Big[|w_t(t,\cdot)|^2_{H^{-1}(\Omega)} + |w(t,\cdot)|^2_{L^2(\Omega)}\Big] \tag{4.7}$$

where $y(\cdot)$ and $w(\cdot)$ are the weak solution of systems (1.1) and (1.2), respectively. The usual energy estimate yields that

Lemma 4.3 *It holds*

$$\mathcal{E}(t) \leq C\mathcal{E}(s), \qquad E(t) \leq CE(s), \qquad \forall\, t, s \in [0, T] \tag{4.8}$$

for some constant $C > 0$ depending on a_i $(i = 1, 2, 3)$ and a, respectively. □

5 Proof of Theorem 2.1

By the density argument, it suffices to prove the theorem for $y \in C^2(\overline{Q})$. We take $\mu^j \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \mu^j \leq 1$,

$$\begin{cases} \mu^j = 1 & \text{on } \overline{\Omega^j} \setminus P_0, \\ \mu^j = 0 & \text{in } P_1. \end{cases} \tag{5.1}$$

Here (recall (2.4) for S)

$$P_0 \triangleq \mathcal{O}_{\delta_0}(S) \supset P_1 \triangleq \mathcal{O}_{\delta_1}(S) \tag{5.2}$$

where $0 < \delta_1 < \delta_0 < \delta$. Then (recall assumption (H) for G)

$$G \supset P_0 \cap \Omega \supset P_1 \cap \Omega. \tag{5.3}$$

For any fixed $\varepsilon > 0$, we choose $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$ such that

$$\chi(t) = \begin{cases} 0, & 0 \leq t \leq \frac{\varepsilon}{2},\ T - \frac{\varepsilon}{2} \leq t \leq T, \\ 1, & \varepsilon \leq t \leq T - \varepsilon. \end{cases} \tag{5.4}$$

Set

$$y_j^1 = \chi\mu^j y, \qquad y_j^0 = (1 - \chi)\mu^j y, \tag{5.5}$$

where y is the weak solution of (1.1). Noting that $\Gamma_{j0} \subset P_1$ (recall (2.1) for Γ_{j0}), we see that

$$\frac{\partial y_j^1}{\partial \nu^j} = 0 \quad \text{on } (0,T) \times \Gamma_{j0}. \tag{5.6}$$

For any fixed $c > 0$, we set

$$Q_j(c) = \{(t,x) \in (0,T) \times \Omega^j; \varphi_j(t,x) \geq c\}, \tag{5.7}$$

where

$$\varphi_j(t,x) \stackrel{\triangle}{=} |x - x_0^j|^2 - \beta(t - T/2)^2 \tag{5.8}$$

with $0 < \beta < 1$. By our assumption on T, we may choose an $\beta \in (0,1)$ such that

$$\varphi_j(0,x) = \varphi_j(T,x) < 0, \qquad 1 \leq j \leq J,\ x \in \overline{\Omega^j \setminus \mathcal{O}_\delta(S)}. \tag{5.9}$$

Now, let us use inequality (4.2) in Lemma 4.1 with $\varphi = \varphi(t,s,x)$ replaced by $\varphi_j = \varphi_j(t,x)$ and v replaced by $y_j^1 = y_j^1(t,x)$. Note that in this case both φ and v do not depend on s. Integrating (4.2) on $Q_j(c)$, using integration by parts, and by (5.1), (5.4), (5.5), (5.6) and (5.9), we get

$$\lambda \int_{Q_j(c)} \theta^2[(y_j^1)_t^2 + \sum_i (y_j^1)_{x_i}^2] dxdt + \lambda^3 \int_{Q_j(c)} \theta^2 (y_j^1)^2 dxdt \leq C \int_{Q_j(c)} (\theta \Box y_j^1)^2 dxdt \tag{5.10}$$

for large $\lambda > 0$, where $\theta = e^{\lambda \varphi_j/2}$.

By the first equation of (1.1), we see that y_j^1 satisfies

$$\Box y_j^1 = \tilde{a}_1(t,x) y + \tilde{a}_2(t,x) y_t + \langle\, \tilde{a}_3(t,x), \nabla y \,\rangle, \tag{5.11}$$

where

$$\tilde{a}_1 = \mu^j \chi a_1 + \mu^j \chi_{tt} - \chi \Delta \mu^j, \quad \tilde{a}_2 = \mu^j \chi a_2 + 2\mu^j \chi_t, \quad \tilde{a}_3 = \mu^j \chi a_3 - 2\chi \nabla \mu^j. \tag{5.12}$$

By our assumption on a_i $(i = 1,2,3)$, it is easy to see that

$$\tilde{a}_1 \in L^{n+1}(Q), \quad \tilde{a}_2 \in L^\infty(Q), \quad \tilde{a}_3 \in L^\infty(Q; \mathbb{R}^n). \tag{5.13}$$

Now, by (5.11) and (5.13), using Hölder inequality and Sobolev embedding theorem, one gets

$$\int_{Q_j(c)} (\theta \Box y_j^1)^2 dxdt \leq C\Big[\int_{Q_j(c)} \theta^2 (y_t^2 + \sum_i y_{x_i}^2) dxdt + \lambda^2 \int_{Q_j(c)} \theta^2 y^2 dxdt\Big]. \tag{5.14}$$

Adding the both sides of (5.10) by

$$\lambda \int_{Q_j(c)} \theta^2[(y_j^0)_t^2 + \sum_i (y_j^0)_{x_i}^2] dxdt + \lambda^3 \int_{Q_j(c)} \theta^2 (y_j^0)^2 dxdt,$$

noting $\mu^j y = y_j^0 + y_j^1$ and using (5.14), we obtain

$$\begin{aligned} &\lambda \textstyle\int_{Q_j(c)} \theta^2[(\mu^j y)_t^2 + \sum_i (\mu^j y)_{x_i}^2] dxdt + \lambda^3 \int_{Q_j(c)} \theta^2 (\mu^j y)^2 dxdt \\ &\leq C\Big[\textstyle\int_{Q_j(c)} \theta^2 (y_t^2 + \sum_i y_{x_i}^2) dxdt + \lambda^2 \int_{Q_j(c)} \theta^2 y^2 dxdt \\ &\quad + \lambda \textstyle\int_{Q_j(c)} \theta^2[(y_j^0)_t^2 + \sum_i (y_j^0)_{x_i}^2] dxdt + \lambda^3 \int_{Q_j(c)} \theta^2 (y_j^0)^2 dxdt\Big]. \end{aligned} \tag{5.15}$$

By (5.1) and (5.3), from (5.15), we get

$$\begin{aligned}&\lambda \int_{Q_j(c)} \theta^2(y_t^2+\sum_i y_{x_i}^2)dxdt+\lambda^3\int_{Q_j(c)}\theta^2 y^2 dxdt\\ &\le C\Big[\int_{Q_j(c)} \theta^2(y_t^2+\sum_i y_{x_i}^2)dxdt+\lambda^2\int_{Q_j(c)}\theta^2 y^2 dxdt\\ &\quad +\lambda\int_0^T\int_G \theta^2(y_t^2+\sum_i y_{x_i}^2)dxdt+\lambda^3\int_0^T\int_G\theta^2 y^2 dxdt\\ &\quad +\lambda\int_{Q_j(c)}\theta^2[(y_j^0)_t^2+\sum_i (y_j^0)_{x_i}^2]dxdt+\lambda^3\int_{Q_j(c)}\theta^2(y_j^0)^2dxdt\Big].\end{aligned} \tag{5.16}$$

Taking $\lambda>0$ large, we can absorb the first term at the right side into the left side to obtain

$$\begin{aligned}&\int_{Q_j(c)} \theta^2(y_t^2+\sum_i y_{x_i}^2)dxdt\\ &\le C\Big[\int_0^T\int_G \theta^2(y_t^2+\sum_i y_{x_i}^2)dxdt+\lambda^2\int_0^T\int_G\theta^2 y^2 dxdt\\ &\quad +\int_{Q_j(c)}\theta^2[(y_j^0)_t^2+\sum_i (y_j^0)_{x_i}^2]dxdt+\lambda^2\int_{Q_j(c)}\theta^2(y_j^0)^2dxdt\Big].\end{aligned} \tag{5.17}$$

for large $\lambda>0$. By (5.4) we have $y_j^0(t,x)=0$ for $(t,x)\in(\varepsilon,T-\varepsilon)\times\Omega$, and hence (5.17) gives

$$\begin{aligned}&\int_{Q_j(c)} \theta^2(y_t^2+\sum_i y_{x_i}^2)dxdt\\ &\le C\Big[\int_0^T\int_G \theta^2(y_t^2+\sum_i y_{x_i}^2)dxdt+\lambda^2\int_0^T\int_G\theta^2 y^2 dxdt\\ &\quad +\int_0^\varepsilon\int_{\Omega^j}\theta^2[(y_j^0)_t^2+\sum_i (y_j^0)_{x_i}^2]dxdt+\lambda^2\int_0^\varepsilon\int_{\Omega^j}\theta^2(y_j^0)^2dxdt\\ &\quad +\int_{T-\varepsilon}^T\int_{\Omega^j}\theta^2[(y_j^0)_t^2+\sum_i (y_j^0)_{x_i}^2]dxdt+\lambda^2\int_{T-\varepsilon}^T\int_{\Omega^j}\theta^2(y_j^0)^2dxdt\Big].\end{aligned} \tag{5.18}$$

For simplicity, we assume that $x_0^j\in\mathbb{R}^n\setminus\overline{\Omega_j}$ $(j=1,2,\cdots,J)$. For the case $x_0^j\in\overline{\Omega_j}$ for some $j\in\{1,2,\cdots,n\}$, we can modify an argument in [12] (*see* case 2 in the proof of Theorem 5.1 in [12]) and adjust the argument to our case; hence, we do not give the details. Now, by $x_0^j\notin\overline{\Omega_j}$, we have

$$\varphi_j(x,T/2)>0,\qquad x\in\overline{\Omega_j},\ 1\le j\le J.$$

Hence by (5.9) we can choose small $\varepsilon>0$ such that

$$\varphi_j<-\varepsilon\quad \text{ on } \Big((0,\varepsilon)\cup(T-\varepsilon,T)\Big)\times\overline{\Omega^j\setminus\mathcal{O}_\delta(S)} \tag{5.19}$$

and

$$\varphi_j>\varepsilon\quad \text{ on } (-\varepsilon+T/2,\varepsilon+T/2)\times\overline{\Omega^j\setminus\mathcal{O}_\delta(S)} \tag{5.20}$$

for $1\le j\le J$.

We set $c=\varepsilon$. Then

$$(-\varepsilon+T/2,\varepsilon+T/2)\times\overline{\Omega^j\setminus\mathcal{O}_\delta(S)}\subset\overline{Q_j(\varepsilon)\setminus\mathcal{O}_\delta(S)}.$$

Therefore

$$\int_{Q_j(c)}\theta^2(y_t^2+\sum_i y_{x_i}^2)dxdt\ge\int_{-\varepsilon+T/2}^{\varepsilon+T/2}\int_{\Omega_j\setminus\mathcal{O}_\delta(S)}\theta^2(y_t^2+\sum_i y_{x_i}^2)dxdt \tag{5.21}$$

Hence, in terms of (5.19)–(5.21), the inequality (5.18) yields

$$\begin{aligned}&e^{\lambda\varepsilon}\int_{-\varepsilon+T/2}^{\varepsilon+T/2}\int_{\Omega_j\setminus\mathcal{O}_\delta(S)}(y_t^2+\sum_i y_{x_i}^2)dxdt\\ &\le C\Big[\int_0^T\int_G \theta^2(y_t^2+\sum_i y_{x_i}^2)dxdt+\lambda^2\int_0^T\int_G\theta^2 y^2 dxdt\\ &\quad +e^{-\lambda\varepsilon}\Big(\int_0^\varepsilon\int_\Omega(y_t^2+\sum_i y_{x_i}^2)dxdt+\lambda^2\int_0^\varepsilon\int_\Omega y^2dxdt\\ &\quad +\int_{T-\varepsilon}^T\int_\Omega(y_t^2+\sum_i y_{x_i}^2)dxdt+\lambda^2\int_{T-\varepsilon}^T\int_\Omega y^2dxdt\Big)\Big].\end{aligned} \tag{5.22}$$

Summing up over $j=1,...,J$ and noting that $\cup_{j=1}^J\Omega^j\cup G\supset\Omega$, we have

$$\begin{array}{ll} e^{\lambda\varepsilon}\int_{-\varepsilon+T/2}^{\varepsilon+T/2}\int_\Omega(y_t^2+\sum_i y_{x_i}^2)dxdt \\ \le C\Big[\sum_{j=1}^J\Big(\int_0^T\int_G\theta^2(y_t^2+\sum_i y_{x_i}^2)dxdt+\lambda^2\int_0^T\int_G\theta^2y^2dxdt\Big) \\ \quad +e^{-\lambda\varepsilon}\Big(\int_0^\varepsilon\int_\Omega(y_t^2+\sum_i y_{x_i}^2)dxdt+\lambda^2\int_0^\varepsilon\int_\Omega y^2dxdt \\ \quad +\int_{T-\varepsilon}^T\int_\Omega(y_t^2+\sum_i y_{x_i}^2)dxdt+\lambda^2\int_{T-\varepsilon}^T\int_\Omega y^2dxdt\Big)\Big]. \end{array} \tag{5.23}$$

By Lemma 4.3 and (5.23), we obtain

$$\begin{array}{ll} e^{\lambda\varepsilon}\mathcal{E}(0)\le & C\Big[\sum_{j=1}^J\Big(\int_0^T\int_G\theta^2(y_t^2+\sum_i y_{x_i}^2)dxdt+\lambda^2\int_0^T\int_G\theta^2y^2dxdt\Big) \\ & +(1+\lambda^2)e^{-\lambda\varepsilon}\mathcal{E}(0)\Big]. \end{array} \tag{5.24}$$

Taking $\lambda>0$ large, we conclude from (5.24) that

$$\mathcal{E}(0)\le C\int_0^T\int_G(y^2+y_t^2+|\nabla y|^2)dxdt. \tag{5.25}$$

We note that, from the above argument, it is easy to see that for any sufficiently small $\tilde\varepsilon$, (5.25) can be sharpened as

$$\mathcal{E}(0)\le C\int_{\tilde\varepsilon}^{T-\tilde\varepsilon}\int_{\tilde G}(y^2+y_t^2+|\nabla y|^2)dxdt, \tag{5.26}$$

where $\tilde G\stackrel{\triangle}{=}\Omega\cap\mathcal{O}_{\tilde\delta}(S)$ with any fixed $\tilde\delta\in(\delta_0,\delta)$.

Now, we choose a function $h=h(x)\in C^\infty(\mathbb{R}^n;[0,1])$ such that

$$\left\{\begin{array}{ll} h(x)\equiv 1, & x\in\tilde G, \\ h(x)\equiv 0, & x\in\Omega\setminus G. \end{array}\right. \tag{5.27}$$

Put

$$\zeta=\zeta(t,x)\stackrel{\triangle}{=}t(T-t). \tag{5.28}$$

By (1.1) and (5.27)–(5.28), we get

$$\begin{array}{ll} \int_0^T\int_G h\zeta y[a_1y+a_2y_t+\langle a_3,\nabla y\rangle]=\int_0^T\int_G h\zeta y\Box ydxdt \\ =-\int_0^T\int_G y_t(h\zeta_ty+h\zeta y_t))dxdt \\ \quad +\int_0^T\int_G h\zeta|\nabla y|^2dxdt+\int_0^T\int_G\zeta(\nabla y)\cdot(\nabla h)ydxdt \\ \ge -\int_0^T\int_G y_t(h\zeta_ty+h\zeta y_t)dxdt \\ \quad +\frac12\int_0^T\int_G h\zeta|\nabla y|^2dxdt-C\int_0^T\int_G y^2dxdt. \end{array} \tag{5.29}$$

However

$$\begin{array}{ll} \int_0^T\int_G h\zeta y[a_1y+a_2y_t+\langle a_3,\nabla y\rangle] \\ \le\frac14\int_0^T\int_G h\zeta|\nabla y|^2dxdt+C\int_0^T\int_G(y^2+y_t^2)dxdt. \end{array} \tag{5.30}$$

Thus, by (5.27)–(5.30), we get

$$\int_{\tilde\varepsilon}^{T-\tilde\varepsilon}\int_{\tilde G}|\nabla y|^2dxdt\le C\int_0^T\int_G(y^2+y_t^2)dxdt. \tag{5.31}$$

Thus, combining (5.26) and (5.31), one obtains the desired inequality immediately. $\square$

6 Proof of Theorem 2.3

For simplicity, we assume that $x_0^j \in \mathbb{R}^n \setminus \overline{\Omega^j}$ $(j = 1, 2, \cdots, J)$. For the case $x_0^j \in \overline{\Omega^j}$ for some $j \in \{1, 2, \cdots, n\}$, we can modify an argument in [12] (*see* case 2 in the proof of Theorem 5.1 in [12]) and adjust the argument to our case; hence, we do not give the details. We divide the proof into several steps.

Step 1. Let us introduce some notations and some transformations. For simplicity, we assume that $J = 2$. Recall assumption (H) for δ and (2.4) for S and R_1^j. By $T > 2\max_{1\leq j\leq J} R_1^j$, one can find a $\tilde{\delta} \in (0, \delta)$ (close to δ) such that

$$T > 2\tilde{R}_1^j \triangleq \max_{1\leq j\leq J} \max_{x\in\Omega^j\setminus\mathcal{O}_{\tilde{\delta}}(S)} |x - x_0^j|. \tag{6.1}$$

Denote

$$\begin{cases} \tilde{\Omega}^j \triangleq \Omega^j \setminus \mathcal{O}_{\tilde{\delta}}(S), \\ R_0^j \triangleq \min_{x\in\tilde{\Omega}^j} |x - x_0^j|, \quad R_0 = \min_{1\leq j\leq J} R_0^j, \\ \mathcal{Q} \triangleq (0, T) \times (0, T) \times \Omega, \quad \mathcal{S} \triangleq (0, T) \times (0, T) \times \partial\Omega, \\ \mathcal{Q}^j \triangleq (0, T) \times (0, T) \times \tilde{\Omega}^j, \quad \mathcal{S}^j \triangleq (0, T) \times (0, T) \times \partial\tilde{\Omega}^j, \\ T_i \triangleq T/2 - \varepsilon_i T, \quad T_i' \triangleq T/2 + \varepsilon_i T, \\ \mathcal{Q}_i^j \triangleq (T_i, T_i') \times (T_i, T_i') \times \tilde{\Omega}^j, \quad \mathcal{S}_i^j \triangleq (T_i, T_i') \times (T_i, T_i') \times \partial\tilde{\Omega}^j. \end{cases} \tag{6.2}$$

where $j \in \{1, 2\}$, $i \in \{1, 2, 3\}$, and $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < 1/2$ will be given later.

Since $x_0^j \in \mathbb{R}^n \setminus \overline{\Omega^j}$, we see that $R_0 > 0$. By (6.1), we can choose a sufficiently small $c \in (0, R_0)$ and an $\alpha \in (0, 1)$ such that

$$(\tilde{R}_1^j)^2 < c^2 + \alpha(T/2)^2. \tag{6.3}$$

For any $b > 0$, set

$$\begin{cases} \varphi_j = \varphi_j(t, s, x) = |x - x_0^j|^2 - \alpha(t - T/2)^2 - \alpha(s - T/2)^2, \\ \mathcal{Q}^j(b) = \{(t, s, x) \in (-\infty, \infty) \times (-\infty, \infty) \times \tilde{\Omega}^j \mid \varphi_j(t, s, x) > b^2\}. \end{cases} \tag{6.4}$$

Now take $\varepsilon_2 \in (0, 1/2)$ sufficiently close to $1/2$ such that (recall $c \in (0, R_0)$)

$$\mathcal{Q}^j(c) \subset \mathcal{Q}_2^j. \tag{6.5}$$

Noting that $\{T/2\} \times \{T/2\} \times \tilde{\Omega}^j \subset \mathcal{Q}^j(c)$, thus for any small $\varepsilon > 0$, there is an $\varepsilon_1 \in (0, \varepsilon_2)$ such that (recall (6.2))

$$\mathcal{Q}_1^j \subset \mathcal{Q}^j(c + 2\varepsilon) \subset \mathcal{Q}^j(c + \varepsilon) \subset \mathcal{Q}^j(c). \tag{6.6}$$

Now, take $0 < \delta_1 < \delta_0 < \tilde{\delta}$ and $\xi \in C_0^\infty(\mathbb{R}^n; [0, 1])$ such that

$$\begin{cases} \xi \equiv 1 & \text{on } \tilde{\Omega}^1 \setminus \mathcal{O}_{\delta_0}(S); \\ \xi \equiv 0 & \text{on } \tilde{\Omega}^1 \cap \mathcal{O}_{\delta_1}(S). \end{cases} \tag{6.7}$$

Denote

$$G_1 \triangleq \tilde{\Omega}^1 \cap \mathcal{O}_{\delta_1}(S). \tag{6.8}$$

Then it is easy to see that

$$G_1 \subset G. \tag{6.9}$$

Set

$$p = p(t,x) = \xi(x)w(t,x), \quad (t,x) \in Q. \tag{6.10}$$

where w is the weak solution of (1.2). Then it is easy to see that $(j = 1,2)$

$$\begin{cases} \Box p = ap - (\Delta\xi)w - (\nabla\xi)\cdot(\nabla w), & \text{in } (0,T)\times\tilde\Omega^1 \\ p = 0, & \text{on } (0,T)\times\partial\tilde\Omega^1 \\ p \equiv 0, & \text{in } (0,T)\times(\tilde\Omega^1\cap\mathcal{O}_{\delta_1}(S)) \end{cases} \tag{6.11}$$

We need the following simple transformation. Put

$$\begin{cases} z(t,s,x) \overset{\triangle}{=} \int_s^t p(\tau,x)d\tau, & \forall\ (t,s,x)\in\mathcal{Q}, \\ \phi(t,s,x) \overset{\triangle}{=} \int_s^t w(\tau,x)d\tau, & \forall\ (t,s,x)\in\mathcal{Q}. \end{cases} \tag{6.12}$$

Then $z(\cdot)$ satisfies

$$\begin{cases} z_{tt} + z_{ss} - \Delta z = & \int_s^t a(\tau,x)p(\tau,x)d\tau - (\Delta\xi(x))\phi - (\nabla\xi(x))\cdot(\nabla\phi) \ \text{ in } \mathcal{Q}^1 \\ z = 0 & \text{on } \ \mathcal{S}^1 \\ z \equiv 0 & \text{in } (0,T)\times(0,T)\times(\tilde\Omega^1\cap\mathcal{O}_{\delta_1}(S)). \end{cases} \tag{6.13}$$

Finally, choose $\chi(\cdot)\in C^\infty(\mathbb{R}^{n+2};[0,1])$ such that

$$\chi(t,s,x) = \begin{cases} 1, & \text{for } (t,s,x)\in\mathcal{Q}^1(c+2\varepsilon), \\ 0, & \text{for } (t,s,x)\in\mathcal{Q}^1_2\setminus\mathcal{Q}^1(c+\varepsilon). \end{cases} \tag{6.14}$$

Put

$$v(t,s,x) = \chi(t,s,x)z(t,s,x), \qquad (t,s,x)\in\mathcal{Q}^1, \tag{6.15}$$

where $\chi(\cdot)$ is defined in (6.14), $z(\cdot)$ is defined in (6.12). By (6.13)–(6.14) and (6.5), we see that $v(\cdot)$ satisfies

$$\begin{cases} v_{tt} + v_{ss} - \Delta v = F(t,s,x) & \text{in } \ \mathcal{Q}^1_2, \\ v = 0 & \text{on } \ \mathcal{S}^1_2, \\ v \equiv 0 & \text{in } \ (0,T)\times(0,T)\times(\tilde\Omega^1\cap\mathcal{O}_{\delta_1}(S)) \end{cases} \tag{6.16}$$

where

$$\begin{aligned} F(t,s,x) = \ & \chi(t,s,x)\Big(\int_s^t a(\tau,x)p(\tau,x)d\tau \\ & -(\Delta\xi(x))\phi(t,s,x) - (\nabla\xi(x))\cdot(\nabla\phi(t,s,x))\Big) \\ & +\Big[\chi_{tt}(t,s,x) + \chi_{ss}(t,s,x) - \Delta\chi(t,s,x)\Big]z \\ & +2\chi_t(t,s,x)z_t + 2\chi_s(t,s,x)z_s - 2\nabla\chi(t,s,x)\cdot\nabla z. \end{aligned} \tag{6.17}$$

Step 2. Let us use inequality (4.2) in Lemma 4.1 with φ replaced by φ_1 (recall (6.4)) and v given by (6.15). Integrating (4.2) on $\mathcal{Q}^1_2$ (recall (6.2) for $\mathcal{Q}^1_2$), by (6.5), (6.14) and (6.16), and using integration by parts, we get

$$\begin{aligned} & 2(1-\alpha)\lambda\int_{\mathcal{Q}^1(c)}\theta^2(v_t^2 + v_s^2 + \textstyle\sum_i v_{x_i}^2)dxdtds + \int_{\mathcal{Q}^1(c)}\theta^2 Bv^2 dxdtds \\ & \le \int_{\mathcal{Q}^1_2}\theta^2|F(t,s,x)|^2 dxdtds, \end{aligned} \tag{6.18}$$

where θ and B are given in (4.1) and (4.4) (with φ replaced by φ_1) respectively, $F(t,s,x)$ is defined by (6.17). However, by (6.7) and (6.14) we see that

$$\begin{cases} \xi_{x_i} = \xi_{x_ix_i} = 0 \quad \text{in } \tilde\Omega^1\setminus G_1, \ i = 1,2,\cdots,n; \\ \chi_t = \chi_s = \chi_{x_i} = \chi_{tt} = \chi_{ss} = \chi_{x_ix_i} = 0 \quad \text{in } \mathcal{Q}^1(c+2\varepsilon), \ i = 1,2,\cdots,n. \end{cases} \tag{6.19}$$

Thus, by (6.17) and (6.19), noting that $p(\tau,x)=z_t(\tau,s,x)=-z_s(t,\tau,x)$ (recalling (6.12)), we get

$$\begin{array}{ll}\int_{\mathcal{Q}_2^1}\theta^2|F(t,s,x)|^2dxdtds\\ \le C\int_{\mathcal{Q}_2^1}\theta^2\Big|\int_s^t a(\tau,x)z_t(\tau,s,x)d\tau\Big|^2dxdtds+Ce^{C\lambda}\int_{T_2}^{T_2'}\int_{T_2}^{T_2'}\int_{G_1}(\phi^2+|\nabla\phi|^2)dxdtds\\ \quad +Ce^{(c+2\varepsilon)^2\lambda}\int_{\mathcal{Q}_2^1}(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\\ =C\int_{\mathcal{Q}_2^1}\theta^2\Big|\int_{T/2}^t a(\tau)z_t(\tau)d\tau+\int_{T/2}^s a(\tau)z_s(\tau)d\tau\Big|^2dxdtds\\ \quad +Ce^{C\lambda}\int_{T_2}^{T_2'}\int_{T_2}^{T_2'}\int_{G_1}(\phi^2+|\nabla\phi|^2)dxdtds\\ \quad +Ce^{(c+2\varepsilon)^2\lambda}\int_{\mathcal{Q}_2^1}(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\\ \le C\Big\{\int_{\mathcal{Q}_2^1}\theta^2\Big[\Big|\int_{T/2}^t z_t^2(\tau)d\tau\Big|+\Big|\int_{T/2}^s z_s^2(\tau)d\tau\Big|\Big]dxdtds\\ \quad +e^{C\lambda}\int_{T_2}^{T_2'}\int_{T_2}^{T_2'}\int_{G_1}(\phi^2+|\nabla\phi|^2)dxdtds\\ \quad +e^{(c+2\varepsilon)^2\lambda}\int_{\mathcal{Q}_2^1}(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\Big\}.\end{array}\tag{6.20}$$

However, similar to [17], one get

$$\int_{\mathcal{Q}_2^1}\theta^2\Big[\Big|\int_{T/2}^t z_t^2(\tau)d\tau\Big|+\Big|\int_{T/2}^s z_s^2(\tau)d\tau\Big|\Big]dxdtds\le C\int_{\mathcal{Q}_2^1}\theta^2(z_t^2+z_s^2)dxdtds.\tag{6.21}$$

Now, combining (6.20)–(6.21), we get (recall (6.3 for T_2 and T_2')

$$\begin{array}{ll}\int_{\mathcal{Q}_2^1}\theta^2|F^z|^2dxdtds\\ \le C\Big[\int_{\mathcal{Q}_2^1}\theta^2(z_t^2+z_s^2)dxdtds+e^{C\lambda}\int_{T_2}^{T_2'}\int_{T_2}^{T_2'}\int_{G_1}(\phi^2+|\nabla\phi|^2)dxdtds\\ \quad +e^{(c+2\varepsilon)^2\lambda}\int_{\mathcal{Q}_2^1}(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\\ =C\Big[\int_{\mathcal{Q}^1(c+2\varepsilon)}+\int_{\mathcal{Q}_2^1\setminus\mathcal{Q}^1(c+2\varepsilon)}\Big)\theta^2(z_t^2+z_s^2)dxdtds\\ \quad +e^{C\lambda}\int_{T_2}^{T_2'}\int_{T_2}^{T_2'}\int_{G_1}(\phi^2+|\nabla\phi|^2)dxdtds\\ \quad +e^{(c+2\varepsilon)^2\lambda}\int_{\mathcal{Q}_2^1}(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\Big]\\ \le C\Big[\int_{\mathcal{Q}^1(c+2\varepsilon)}\theta^2(z_t^2+z_s^2)dxdtds+e^{C\lambda}\int_{T_2}^{T_2'}\int_{T_2}^{T_2'}\int_{G_1}(\phi^2+|\nabla\phi|^2)dxdtds\\ \quad +e^{(c+2\varepsilon)^2\lambda}\int_{\mathcal{Q}_2^1}(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\Big].\end{array}\tag{6.22}$$

Note that

$$\begin{array}{ll}B&=4(1+\alpha)\lambda^3\Big[|x-x_0^1|^2-\alpha^2(t-T/2)^2-\alpha^2(s-T/2)^2\Big]\\ &\quad -\Big[8\alpha^2+4n+(n+\alpha-1)^2\Big]\lambda^2\\ &>4(1+\alpha)c^2\lambda^3-\Big[8\alpha^2+4n+(n+\alpha-1)^2\Big]\lambda^2,\qquad\forall\ (t,s,x)\in\mathcal{Q}^1(c).\end{array}\tag{6.23}$$

Thus, by (6.14) and (6.23), and taking $\lambda>0$ large enough, we can find a constant $c_0>0$ such that

$$\begin{array}{ll}2(1-\alpha)\lambda\int_{\mathcal{Q}^1(c)}\theta^2(v_t^2+v_s^2+\sum_i v_{x_i}^2)dxdtds+\int_{\mathcal{Q}^1(c)}\theta^2Bv^2dxdtds\\ \ge 2(1-\alpha)\lambda\int_{\mathcal{Q}^1(c)}\theta^2(v_t^2+v_s^2+\sum_i v_{x_i}^2)dxdtds+c_0\lambda^3\int_{\mathcal{Q}^1(c)}\theta^2v^2dxdtds\\ \ge 2(1-\alpha)\lambda\int_{\mathcal{Q}^1(c+2\varepsilon)}\theta^2(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds.\end{array}\tag{6.24}$$

Now, combining (6.18), (6.22) and (6.24), we arrive at

$$\begin{array}{ll}\lambda\int_{\mathcal{Q}^1(c+2\varepsilon)}\theta^2(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\\ \le C\Big[e^{C\lambda}\int_{T_2}^{T_2'}\int_{T_2}^{T_2'}\int_{G_1}(\phi^2+|\nabla\phi|^2)dxdtds+\int_{\mathcal{Q}^1(c+2\varepsilon)}\theta^2(z_t^2+z_s^2)dxdtds\\ \quad +e^{(c+2\varepsilon)^2\lambda}\int_{\mathcal{Q}_2^1}(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\Big].\end{array}\tag{6.25}$$

Taking λ large enough, we get

$$\begin{array}{l}\lambda\int_{\mathcal{Q}^1(c+2\varepsilon)}\theta^2(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\\ \le C\Big[e^{C\lambda}\int_{T_2}^{T_2'}\int_{T_2}^{T_2'}\int_{G_1}(\phi^2+|\nabla\phi|^2)dxdtds+e^{(c+2\varepsilon)^2\lambda}\int_{\mathcal{Q}_2^1}(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\Big].\end{array}\tag{6.26}$$

However, by (4.1), (6.4) and (6.6), we have

$$\begin{array}{ll}\int_{\mathcal{Q}^1(c+2\varepsilon)}\theta^2(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds & \ge e^{(c+2\varepsilon)^2\lambda}\int_{\mathcal{Q}^1(c+2\varepsilon)}(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\\ & \ge e^{(c+2\varepsilon)^2\lambda}\int_{\mathcal{Q}_1^1}(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds.\end{array}\tag{6.27}$$

Thus, by (6.26)–(6.27), we arrive at

$$\begin{array}{l}\lambda\int_{\mathcal{Q}_1^1}(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\\ \le C\Big[e^{C\lambda}\int_{T_2}^{T_2'}\int_{T_2}^{T_2'}\int_{G_1}(\phi^2+|\nabla\phi|^2)dxdtds+\int_{\mathcal{Q}_2^1}(z_t^2+z_s^2+\sum_i z_{x_i}^2)dxdtds\Big].\end{array}\tag{6.28}$$

Step 3. We now estimate "$\int_{\mathcal{Q}_2^1}\sum_i z_{x_i}^2 dxdtds$". Let us fix a ε_3 satisfying $\varepsilon_2<\varepsilon_3<1/2$. Denote

$$\eta=\eta(t,s)\stackrel{\triangle}{=}(t-T_3)(T_3'-t)(s-T_3)(T_3'-s).\tag{6.29}$$

By (6.13), we get

$$\begin{array}{l}\int_{\mathcal{Q}_3^1}\eta z\Big[\int_s^t a(\tau,x)p(\tau,x)d\tau-(\Delta\xi(x))\phi-(\nabla\xi(x))\cdot(\nabla\phi)\Big]dxdtds\\ =\int_{\mathcal{Q}_3^1}\eta z(z_{tt}+z_{ss}-\Delta z)dxdtds\\ =-\int_{\mathcal{Q}_3^1}\Big[z_t(\eta_t z+\eta z_t)+z_s(\eta_s z+\eta z_s)\Big]dxdtds\\ \quad+\int_{\mathcal{Q}_3^1}\eta|\nabla z|^2dxdtds+\int_{\mathcal{Q}_3^1}(\nabla z)\cdot(\nabla\eta)zdxdtds\\ \ge-\int_{\mathcal{Q}_3^1}\Big[z_t(\eta_t z+\eta z_t)+z_s(\eta_s z+\eta z_s)\Big]dxdtds\\ \quad+\frac12\int_{\mathcal{Q}_3^1}\eta|\nabla z|^2dxdtds-C\int_{\mathcal{Q}_3^1}z^2dxdtds\\ \ge-\int_{\mathcal{Q}_3^1}\Big[z_t(\eta_t z+\eta z_t)+z_s(\eta_s z+\eta z_s)\Big]dxdtds\\ \quad+\frac{C}{2}\int_{\mathcal{Q}_2^1}\sum_i z_{x_i}^2dxdtds-C\int_{\mathcal{Q}_3^1}z^2dxdtds.\end{array}\tag{6.30}$$

Thus by (6.30) and (6.19) and noting that $p(\tau)=z_t(\tau)$, we get

$$\begin{array}{ll}\int_{\mathcal{Q}_2^1}\sum_i z_{x_i}^2dxdtds\le & C\Big[\int_{\mathcal{Q}_3^1}(z_t^2+z_s^2+z^2)dxdtds\\ & +\int_{T_3}^{T_3'}\int_{T_3}^{T_3'}\int_{G_1}(\phi^2+|\nabla\phi|^2)dxdtds\Big].\end{array}\tag{6.31}$$

Therefore by (6.28) and (6.31), we end up with

$$\begin{array}{l}\lambda\int_{\mathcal{Q}_1^1}(z_t^2+z_s^2)dxdtds\\ \le C\Big[e^{C\lambda}\int_{T_3}^{T_3'}\int_{T_3}^{T_3'}\int_{G_1}(\phi^2+|\nabla\phi|^2)dxdtds+\int_{\mathcal{Q}_3^1}(z_t^2+z_s^2+z^2)dxdtds\Big].\end{array}\tag{6.32}$$

Step 4. We now estimate "$\int_{T_3}^{T_3'}\int_{T_3}^{T_3'}\int_{G_1}|\nabla\phi|^2dxdtds$". For this purpose, we choose a function $h=h(x)\in C^\infty(\mathbb{R}^n;[0,1])$ such that

$$\begin{cases}h(x)\equiv 1, & x\in G_1,\\ h(x)\equiv 0, & x\in\Omega\setminus G.\end{cases}\tag{6.33}$$

Denote

$$\zeta = \zeta(t,x) \stackrel{\triangle}{=} t(T-t)s(T-s). \tag{6.34}$$

Note that by (6.12), we see that ϕ satisfies

$$\begin{cases} \phi_{tt} + \phi_{ss} - \Delta\phi = \int_s^t a(\tau,x)\phi_t(\tau,x)d\tau, & (t,s,x) \in \mathcal{Q}, \\ \phi = 0 & \text{on } \ \mathcal{S}. \end{cases} \tag{6.35}$$

Thus, by (6.33)–(6.35), we get

$$\begin{array}{ll} \int_0^T \int_0^T \int_G h\zeta\phi \int_s^t a(\tau)\phi_t(\tau)d\tau dxdtds \\ = \int_0^T \int_0^T \int_G h\zeta\phi(\phi_{tt} + \phi_{ss} - \Delta\phi)dxdtds \\ = -\int_0^T \int_0^T \int_G \Big[\phi_t(h\zeta_t\phi + h\zeta\phi_t) + \phi_s(h\zeta_s\phi + h\zeta\phi_s)\Big]dxdtds \\ \quad + \int_0^T \int_0^T \int_G h\zeta|\nabla\phi|^2 dxdtds + \int_0^T \int_0^T \int_G \zeta(\nabla\phi)\cdot(\nabla h)\phi dxdtds \\ \geq -\int_0^T \int_0^T \int_G \Big[\phi_t(h\zeta_t\phi + h\zeta\phi_t) + \phi_s(h\zeta_s\phi + h\zeta\phi_s)\Big]dxdtds \\ \quad + \frac{1}{2}\int_0^T \int_0^T \int_G h\zeta|\nabla\phi|^2 dxdtds - C\int_0^T \int_0^T \int_G \phi^2 dxdtds \\ \geq -\int_0^T \int_0^T \int_G \Big[\phi_t(h\zeta_t\phi + h\zeta\phi_t) + \phi_s(h\zeta_s\phi + h\zeta\phi_s)\Big]dxdtds \\ \quad + \frac{C}{2}\int_{T_3}^{T_3'} \int_{T_3}^{T_3'} \int_{G_1} |\nabla\phi|^2 dxdtds - C\int_0^T \int_0^T \int_G \phi^2 dxdtds. \end{array} \tag{6.36}$$

Thus

$$\int_{T_3}^{T_3'} \int_{T_3}^{T_3'} \int_{G_1} |\nabla\phi|^2 dxdtds \leq C\int_0^T \int_0^T \int_G (\phi^2 + \phi_t^2 + \phi_s^2)dxdtds. \tag{6.37}$$

Combining (6.32) and (6.37), we conclude that

$$\begin{array}{ll} \lambda \int_{\mathcal{Q}_1^1} (z_t^2 + z_s^2)dxdtds \\ \leq C\Big[e^{C\lambda}\int_0^T \int_0^T \int_G (\phi^2 + \phi_t^2 + \phi_s^2)dxdtds + \int_{\mathcal{Q}_3^1}(z_t^2 + z_s^2 + z^2)dxdtds\Big]. \end{array} \tag{6.38}$$

Step 5. Let us return to the function "w". By (6.12), we get

$$\lambda \int_{\mathcal{Q}_1^1} [p^2(t,x) + p^2(s,x)]dxdtds \leq C\Big[e^{C\lambda}\int_0^T \int_G w^2 dxdt + \int_Q p^2 dxdt\Big]. \tag{6.39}$$

Howevr, by (6.7)–(6.10) and (6.39), we have

$$\lambda \int_{T_1}^{T_1'} \int_{\Omega_1 \setminus G} w^2 dxdt \leq C\Big[e^{C\lambda}\int_0^T \int_G w^2 dxdt + \int_Q w^2 dxdt\Big]. \tag{6.40}$$

Similarly, we can prove that

$$\lambda \int_{T_1}^{T_1'} \int_{\Omega_2 \setminus G} w^2 dxdt \leq C\Big[e^{C\lambda}\int_0^T \int_G w^2 dxdt + \int_Q w^2 dxdt\Big]. \tag{6.41}$$

By (6.40)–(6.41), we get

$$\lambda \int_{T_1}^{T_1'} \int_{(\Omega_1 \cup \Omega_2) \setminus G} w^2 dxdt \leq C\Big[e^{C\lambda}\int_0^T \int_G w^2 dxdt + \int_Q w^2 dxdt\Big]. \tag{6.42}$$

Now, adding both sides of (6.42) by $\lambda\int_0^T \int_G w^2 dxdt$, one end up with

$$\lambda \int_{T_1}^{T_1'} \int_{\Omega} w^2 dxdt \leq C\Big[e^{C\lambda}\int_0^T \int_G w^2 dxdt + \int_Q w^2 dxdt\Big]. \tag{6.43}$$

On the other hand, choose $s_0 \in (T_1, T/2)$ and $s_0' \in (T/2, T_1')$. By (4.6) (in Lemma 4.2) and (6.43), we obtain that

$$\lambda \int_{s_0}^{s_0'} \Big[|w_t(t,\cdot)|^2_{H^{-1}(\Omega)} + |w(t,\cdot)|^2_{L^2(\Omega)}\Big]dt \leq C\Big[e^{C\lambda}\int_0^T \int_G w^2 dxdt + \int_Q w^2 dxdt\Big]. \tag{6.44}$$

Thus

$$\lambda \int_{s_0}^{s_0'} E(t)dt \leq C\Big[e^{C\lambda}\int_0^T \int_G w^2 dxdt + \int_0^T E(t)dt\Big]. \tag{6.45}$$

where $E(\cdot)$ is defined by (4.7). Finally, by (4.8) (in Lemma 4.3), we see that

$$\lambda E(0) \leq C\Big[e^{C\lambda}\int_0^T \int_G w^2 dxdt + E(0)\Big]. \tag{6.46}$$

Consequently, if we take

$$\lambda > 1 + C, \tag{6.47}$$

where C is the constant appeared in (6.46), we get

$$\lambda E(0) \leq Ce^{C\lambda}\int_0^T \int_G w^2 dxdt, \quad \forall\, \lambda > 1 + C. \tag{6.48}$$

(6.48) is exactly the desired estimate. Thus the proof of Theorem 2.3 is completed. □

7 Proof of Theorem 2.2

Denote

$$v \stackrel{\triangle}{=} w_t \tag{7.1}$$

Then, by equation (1.2) and noting that $a(t,x) \equiv a(x)$ (i.e. a depending only on x), we get

$$\begin{cases} \Box v(t,x) = a(x)v, & (t,x) \in Q \\ v(t,x) = 0, & (t,x) \in \Sigma \\ v(0,x) = w_1(x), \quad v_t(0,x) = \Delta w_0 + a(x)w_0, & x \in \Omega \end{cases} \tag{7.2}$$

Thus, by Theorem 2.3 and (7.1)–(7.2), we get

$$\begin{array}{rl} |w_1|^2_{L^2(\Omega)} + |\Delta w_0 + aw_0|^2_{H^{-1}(\Omega)} \leq & C\int_0^T \int_G w_t^2 dxdt, \\ & \forall\ (w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega) \end{array} \tag{7.3}$$

However, by our assumption on a, we see easily that

$$\begin{array}{l} |\Delta w_0 + aw_0|^2_{H^{-1}(\Omega)} \\ = \sup\Big\{\frac{\int_\Omega (\Delta w_0 + aw_0) f dx}{|f|_{H_0^1(\Omega)}}\ \Big|\, 0 \neq f \in H_0^1(\Omega)\Big\} \\ \geq \frac{-\int_\Omega (\Delta w_0 + aw_0) w_0 dx}{|w_0|_{H_0^1(\Omega)}} \geq |w_0|_{H_0^1(\Omega)}. \end{array} \tag{7.4}$$

Now, combining (7.3)–(7.4), we obtain the desired result immediately. Thus the proof of Theorem 2.2 is completed. □

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