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Classifying involutions fixing $\mathbb{RP}^{odd} \sqcup P(h,i)$ up to equivariant cobordism

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CLASSIFYING INVOLUTIONS FIXING $\mathbb{RP}^{\mathbf{odd}} \sqcup P(h,i)$ UP TO EQUIVARIANT COBORDISM

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1. Introduction

The objective of this paper is to classify up to equivariant cobordism the smooth involutions fixing the disjoint union of an odd-dimensional real projective space $\mathbb{RP}^j$ and a Dold manifold $P(h,i)$ with $h>0$ and $i>0$, where $P(h,i)$ is defined as $S^h \times \mathbb{CP}^i/-1\times$ (conjugation), see [Do]. The speical cases $j=1,3$ have been considered in [Gu] and [L-L]. Here we deal with the general case of j. Note that since $P(h,0)=\mathbb{RP}^h$ and $P(0,i)=\mathbb{CP}^i$, and the involutions fixing $\mathbb{RP}^j \sqcup \mathbb{RP}^h$ (resp. $\mathbb{RP}^j \sqcup \mathbb{CP}^i$) has been known well (see [Re] and [St2]), one will exclude the case $h=0$ or $i=0$ in this paper.

Suppose (M^m,T) is a closed manifold with involution fixing a disjoint union of $\mathbb{RP}^j$ with normal bundle ν^{m-j} and $P(h,i)$ with normal bundle ν^k, so $m=h+2i+k$. In order to avoid that (M^m,T) is cobordant to an involution fixing only either $\mathbb{RP}^j$ or $P(h,i)$, one may assume that $(\mathbb{RP}^j,\nu^{m-j})$ is nonbounding, and thus $w(\nu^{m-j})=(1+\alpha)^q$ with q odd where $H^*(\mathbb{RP}^j;\mathbb{Z}_2)=\mathbb{Z}_2[\alpha]/(\alpha^{j+1}=0)$ and $\alpha \in H^1(\mathbb{RP}^j;\mathbb{Z}_2)$. In fact, since $w_1(\nu^{m-j})=q\alpha \neq 0$ one has $m>j$. Since $(\mathbb{RP}^j,\nu^{m-j})$ is nonbounding and every involution fixing $\mathbb{RP}^j$ bounds, the component of M containing $\mathbb{RP}^j$ must contain $P(h,i)$, so $m>h+2i$ or $k>0$. Also, $(P(h,i),\nu^k)$ must be nonbounding, for if not (M,T) is cobordant to an involution fixing $(\mathbb{RP}^j,\nu^{m-j})$. Here one uses convention that $(\mathbb{RP}^j,\nu^{m-j})$ *is nonbounding*, and thus (M^m,T) does not bound equivariantly if (M^m,T) exists.

Letting $2^p<j<2^{p+1}$, q is only determined modulo 2^{p+1}, so it is assumed that $q<2^{p+1}$.

The mod 2 cohomology of the Dold manifold is given by

$$H^*(P(h,i);\mathbb{Z}_2)=\mathbb{Z}_2[c,d]/(c^{h+1}=d^{i+1}=0)$$

where $c\in H^1(P(h,i);\mathbb{Z}_2)$ and $d\in H^2(P(h,i);\mathbb{Z}_2)$. According to the recent work of Stong [St1], one may write the total Stiefel-Whitney class of ν^k in the form

$$w(\nu^k)=(1+c)^a(1+c+d)^b w(\rho)^\varepsilon$$

where $\varepsilon=0$ or 1 and $w(\rho)=1+$ terms of dimension at least 4 is an exotic class ($\varepsilon=0$ except for $h=2,4,5$, or 6).

Now form the class

$$w[r]=\frac{w(\mathbb{RP}(\nu))}{(1+e)^{m-h-2i-r}}$$

where e is the characteristic class of the double cover of $\mathbb{RP}(\nu)$ by the sphere bundle of ν, so that each $w[r]_x$ is a polynomial in $w_y(\mathbb{RP}(\nu))$ and e. Then

$$w[r]=\begin{cases}(1+c)^h(1+c+d)^{i+1}\left\{(1+e)^r+(a+b)c(1+e)^{r-1}+\cdots\right\} & \text{on } P(h,i)\\ (1+\alpha)^{j+1}\left\{(1+e)^{h+2i+r-j}+q\alpha(1+e)^{h+2i+r-j-1}+\cdots\right\} & \text{on } \mathbb{RP}^j.\end{cases} \tag{1.1}$$

According to Conner and Floyd [C-F], $\mathbb{RP}(\nu^k)$ and $\mathbb{RP}(\nu^{m-j})$ are cobordant in $B\mathbb{Z}_2$, and thus the characteristic numbers

$$w[r_1]_{\omega_1}\cdots w[r_s]_{\omega_s}e^{m-1-|\omega_1|-\cdots-|\omega_s|}[\mathbb{RP}(\nu^k)] = w[r_1]_{\omega_1}\cdots w[r_s]_{\omega_s}e^{m-1-|\omega_1|-\cdots-|\omega_s|}[\mathbb{RP}(\nu^{m-j})]$$

where each $\omega = (i_1, ..., i_t)$ is a partition of $|\omega| = i_1 + \cdots + i_t$. This provides a method of studying involutions fixing $\mathbb{RP}^j \sqcup P(h, i)$. Such a method was first used by Pergher and Stong to study involutions fixing a disjoint union of a point and a closed manifold (see [P-S]).

Throughout this paper, the coefficient group is $\mathbb{Z}_2$. w denotes the total Stiefel-Whitney class and w_s denotes the s-th Stiefel-Whitney class.

2. Examples for which involutions exist

Now let us bulid some involutions fixing $\mathbb{RP}^j \sqcup P(j, i)$ with j odd.

Write $i = 2^u(2v + 1)$ and let

$$k_0 = \begin{cases} 2^u + 1 & \text{if } u = 1 \\ 2^u & \text{if } u \neq 1. \end{cases}$$

From [P-S] and [St2], there is an involution (N^{i+l}, T_l) with $1 \le l \le k_0$ having fixed point set $*\sqcup \mathbb{RP}^i$ with the normal bundle of $\mathbb{RP}^i$ in N^{i+l} being $\iota \oplus (l-1)\mathbb{R}$, with ι the nontrivial line bundle, where $*$ denotes a point. This is constructed by applying the operation Γ $l-1$ times to the involution $(\mathbb{RP}^{i+1}, T_1)$ defined by

$$T_1[x_0, x_1, ..., x_{i+1}] = [-x_0, x_1, ..., x_{i+1}]$$

which fixes $\mathbb{RP}^0 \sqcup \mathbb{RP}^i$ with the normal bundle ι of $\mathbb{RP}^i$ and cobording away various bounding fixed components (see Royster [Ro]).

Consider the involution $T_{N^{i+l}}$ on

$$P(j, N^{i+l}) = \frac{S^j \times N^{i+l} \times N^{i+l}}{-1 \times \text{twist}}$$

induced by $1 \times T_l \times T_l$. The fixed point set of this involution is

(1). $\dfrac{S^j \times \text{point} \times \text{point}}{-1 \times \text{twist}} = \mathbb{RP}^j$ and the normal bundle is formed by $\dfrac{S^j \times \mathbb{R}^{i+l} \times \mathbb{R}^{i+l}}{-1 \times \text{twist}}$, so is $(i+l)\iota \oplus (i+l)\mathbb{R}$.

(2). $\dfrac{S^j \times ((\mathbb{RP}^i \times \text{point}) \sqcup (\text{point} \times \mathbb{RP}^i))}{-1 \times \text{twist}}$ and the twist exchanges the two copies of $\mathbb{RP}^i$, so the quotient is $S^j \times (\mathbb{RP}^i \times \text{ point})$ with normal bundle $S^j \times$(normal bundle of $\mathbb{RP}^i \times$ point). Since S^j bounds, this component bounds away.

(3). $\dfrac{S^j \times \mathbb{RP}^i \times \mathbb{RP}^i}{-1 \times \text{twist}}$ with the normal bundle $\dfrac{S^j \times ((\iota \oplus (l-1)\mathbb{R}) \times (\iota \oplus (l-1)\mathbb{R}))}{-1 \times \text{twist}}$ and this is cobordant to $\dfrac{S^j \times \mathbb{CP}^i}{-1 \times \text{conjugation}} = P(j, i)$ with the normal bundle $\eta \oplus (l-1)\xi \oplus (l-1)\mathbb{R}$, where ξ induced by ι is a 1-plane bundle over $P(j, i)$, and η is a 2-plane bundle over $P(j, i)$. Note that $w(\xi) = 1 + c$ and $w(\eta) = 1 + c + d$ (see [Do], [Uc]).

This produces an involution $(P(j, N^{i+l}), T_{N^{i+l}})$ fixing $\mathbb{RP}^j$ with the normal bundle ν^{2i+2l} having $w(\nu^{2i+2l}) = (1 + \alpha)^{i+l}$ and $P(j, i)$ with the normal bundle ν^{2l} having $w(\nu^{2l}) = (1 + c)^{l-1}(1 + c + d)$. The normal bundle to the fixed point set has $l - 1$ sections. Thus, there exist the involutions (M^{j+2i+k}, T) with $l + 1 \le k \le 2l$ which has the same fixed information as $(P(j, N^{i+l}), T_{N^{i+l}})$ such that M^{j+2i+k} bounds for $k < 2l$. On the other hand, since the normal bundle to the fixed point set has $l - 1$ sections, then one may apply the inverse operation Γ^{-1} $l-1$ times to $(P(j, N^{i+l}), T_{N^{i+l}})$, so that (M^{j+2i+k}, T) is cobordant to $\Gamma^{k-2l}(P(j, N^{i+l}), T_{N^{i+l}})$ for $l + 1 \le k \le 2l$.

Now let us look at $P(j, N^{i+l})$. One has

Lemma 2.1. *For $1 \le l < k_0$, $P(j, N^{i+l})$ bounds.*

Proof. When $1 \le l < k_0$, N^{i+l} bounds. Furthermore, one has that $(N^{i+l} \times N^{i+l},$ twist) fixing N^{i+l} with the normal bundle μ bounds equivariantly, and thus the bundle $(N^{i+l}, \mu \oplus s\mathbb{R})$ bounds for any $s \ge 0$. So, $\mathbb{RP}(\mu \oplus (s+1)\mathbb{R})$ bounds. On the other hand, consider the involution on $P(j, N^{i+l})$ induced by $T' \times 1 \times 1$ on $S^j \times N^{i+l} \times N^{i+l}$ where

$$T'(x_0, x_1, \cdots, x_j) = (-x_0, x_1, \cdots, x_j).$$

It is easy to see that the fixed data is $(N^{i+l}, \mu \oplus j\mathbb{R}) \sqcup (P(j-1, N^{i+l}), \xi^1)$ where ξ^1 is a real line bundle over $P(j-1, N^{i+l})$. Therefore, by [C-F] one obtains that the cobordism class $\{P(j, N^{i+l})\} = \{\mathbb{RP}(\mu \oplus (j+1)\mathbb{R})\} + \{\mathbb{RP}(\xi^1 \oplus \mathbb{R})\} = 0$. □

If $l < k_0$, by Lemma 2.1 and applying the operation Γ to $(P(j, N^{i+l}), T_{N^{i+l}})$, then the resulting involutions $\Gamma^x(P(j, N^{i+l}), T_{N^{i+l}})$ denoted by $(M^{j+2i+2l+x}, T)$ have the following properties:
(i). There is an integer x_0 such that for $x < x_0$, $M^{j+2i+2l+x}$ bounds, but $M^{j+2i+2l+x_0}$ does not bound.
(ii). For $x \le x_0$, $(M^{j+2i+2l+x}, T)$ has the same fixed information as $(P(j, N^{i+l}), T_{N^{i+l}})$.

This gives the following result.

Proposition 2.1. *Let $l < k_0$. There exist involutions (M^{j+2i+k}, T) fixing $\mathbb{RP}^j \sqcup P(j,i)$ with $l+1 \le k \le 2l + x_0$ such that*
(i). (M^{j+2i+k}, T) is cobordant to $\Gamma^{k-2l}(P(j, N^{i+l}), T_{N^{i+l}})$ for each k;
(ii). M^{j+2i+k} bounds for $k < 2l + x_0$, but not for $k = 2l + x_0$.

Note that if $l = k_0$ then N^{i+k_0} does not bound. It will be proved later that $P(j, N^{i+k_0})$ must be non-bounding, so $\Gamma(P(j, N^{i+k_0}), T_{N^{i+k_0}})$ does not have the same fixed information as $(P(j, N^{i+k_0}), T_{N^{i+k_0}})$.

3. The case in which h is odd

Following the notations of the section 1, one first discusses the case in which h is odd. From (1.1) one then has

$$w[0]_1 = \begin{cases} (h+i+1+a+b)c & \text{on } P(h,i) \\ \alpha & \text{on } \mathbb{RP}^j. \end{cases}$$

So

$$w[0]_1^j e^{m-1-j}[\mathbb{RP}(\nu^{m-j})] = \alpha^j e^{m-j}[\mathbb{RP}(\nu^{m-j})] = \alpha^j [\mathbb{RP}^j] \neq 0$$

and

$$0 \neq w[0]_1^j e^{m-1-j}[\mathbb{RP}(\nu^k)] = (h+i+1+a+b)c^j e^{m-1-j}[\mathbb{RP}(\nu^k)]$$

which implies that $h+i+1+a+b \not\equiv 0 \bmod 2$ and $c^j \neq 0$, so $h \ge j$.

Now, there are certain operations in the bordism of $B\mathbb{Z}_2$. For $x = e, w_1$, or $w_1 + e$, one may dualize any power of x, giving homomorphisms

$$(\text{dual } x^t) : \mathfrak{N}_n(B\mathbb{Z}_2) \longrightarrow \mathfrak{N}_{n-t}(B\mathbb{Z}_2).$$

Dualizing e is the Smith homomorphism of Conner and Floyd [C-F]. Dualizing w_1 and w_1^2 was used by C.T.C. Wall [Wa] in studying oriented bordism.

Consider the operation

$$(\text{dual } w[0]_1^2) = (\text{dual } (w_1 + (m-h-2i)e)^2 : \mathfrak{N}_{m-1}(B\mathbb{Z}_2) \longrightarrow \mathfrak{N}_{m-3}(B\mathbb{Z}_2).$$

When applied to $\mathbb{RP}(\nu^{m-j})$, $w[0]_1 = \alpha$ and the dual is $\mathbb{RP}(\nu^{m-j}|_{\mathbb{RP}^{j-2}})$ which is the projective space bundle of ν^{m-j} with $w(\nu^{m-j}) = (1+\alpha)^q$ over $\mathbb{RP}^{j-2}$. When applied to $\mathbb{RP}(\nu^k)$, $w[0]_1 = c$, and the dual is $\mathbb{RP}(\nu^k|_{P(h-2,i)})$ which is the projective space bundle of $\nu^k|_{P(h-2,i)}$ over $P(h-2,i)$. Since $\mathbb{RP}(\nu^{m-j})$ is cobordant to $\mathbb{RP}(\nu^k)$, the duals will be cobordant in $B\mathbb{Z}_2$, and one has

Proposition 3.1. *There is an involution (M^{m-2}, T) fixing $\mathbb{RP}^{j-2}$ with $w(\nu^{m-j}) = (1+\alpha)^q$ and $P(h-2, i)$ with normal bundle $\nu^k|_{P(h-2,i)}$.*

Note. When restricted to $P(0,i)=\mathbb{CP}^i$, $w(\nu^k)$ becomes $(1+d)^b$ and b does not change under restriction since i is unchanged. The values of a and q may reduce to smaller equivalent values.

By iterating this procedure, one may reduce j to 1 and quote resulrs of Guo [Gu] ($j=1$). Since Guo assumes $w(\nu^k)=(1+c)^a(1+c+d)^b$ which is not valid, we will not use her results. (In fact, there is an error in Guo's results.)

So, by iteration one may consider the case $j=1$ with h odd (so $h\geq 1$ obviously).

Theorem 3.2. *Suppose (M^{h+2i+k},T) fixes $\mathbb{RP}^1\sqcup P(h,i)$. Then*
(1). $q=h=b=1, a=\varepsilon=0$, *and i is even, and*
(2). *let $i=2^u(2v+1)$ with $u>0$,*

$$2\leq k\leq\begin{cases}2^{u+1}+2 & \text{if } u=1\\ 2^{u+1}-1 & \text{if } u>1.\end{cases}$$

Further, (M^{1+2i+k},T) fixing $\mathbb{RP}^1\sqcup P(1,i)$ is cobordant to $\Gamma^{k-2}(P(1,N^{i+1}),T_{N^{i+1}})$.

Proof. Obviously, $q=1$ holds since $j=1$. Now one computes the values of $w[1]_2$. On $P(h,i)$,

$$\begin{aligned}w[1] &= \left\{1+(h+i+1)c+(\binom{h+i+1}{2}c^2+(i+1)d)+\cdots\right\}\\ &\quad\times\left\{1+e+(a+b)c+(\binom{a+b}{2}c^2+bd)(1+e)^{-1}+\cdots\right\}\end{aligned}$$

so

$$\begin{aligned}w[1]_2 &= \left\{\binom{h+i+1}{2}c^2+(i+1)d\right\}+(h+i+1)c\{e+(a+b)c\}+\binom{a+b}{2}c^2+bd\\ &= (h+i+1)ce+(i+1+b)d+\binom{h+i+1+a+b}{2}c^2.\end{aligned}$$

On $\mathbb{RP}^1$,

$$w[1]=(1+e)^{h+2i}+\alpha(1+e)^{h+2i-1}$$

so

$$w[1]_2=\binom{h+2i}{2}e^2.$$

Form the class

$$\begin{aligned}\hat{w}_2 &= w[1]_2+(h+i+1)w[0]_1e+\binom{h+i+1+a+b}{2}w[0]_1^2\\ &= \begin{cases}(i+1+b)d & \text{on } P(h,i)\\ \binom{h+2i}{2}e^2+(h+i+1)\alpha e & \text{on } \mathbb{RP}^1.\end{cases}\end{aligned}$$

If i is odd, then $P(h,i)$ bounds, for

$$w(P(h,i))=(1+c)^h(1+c+d)^{i+1}$$

has only even powers of d. Since $(P(h,i),\nu^k)$ is nonbounding,

$$w(\nu^k)=(1+c)^a(1+c+d)^b w(\rho)^\varepsilon$$

must have some term with an odd power of d. Since h is odd, the only exotic class occurs for $h=5$ and then by [St1]

$$w(\rho)=1+\frac{c^4d^2}{(1+d)^2}$$

which has no odd powers of d. Thus b is odd. This gives

$$\hat{w}_2 = \begin{cases} d & \text{on } P(h,i) \\ \binom{h+2i}{2}e^2 + \alpha e & \text{on } \mathbb{RP}^1. \end{cases}$$

Then

$$w[0]_1^h \hat{w}_2^i e^{k-1}[\mathbb{RP}(\nu^k)] = c^h d^i e^{k-1}[\mathbb{RP}(\nu^k)] \neq 0$$

and so

$$\begin{aligned} w[0]_1^h \hat{w}_2^i e^{k-1} &= \alpha^h \left\{ \binom{h+2i}{2} e^2 + \alpha e \right\}^i e^{k-1} \\ &= \alpha^h \binom{h+2i}{2} e^{k+2i-1} \end{aligned}$$

must be nonzero on $\mathbb{RP}^1$. This implies $h = 1$ and then $\binom{h+2i}{2} = \binom{2i+1}{2} = 1$.

Since $w_2(\nu^k) = bd + \binom{a+b}{2}c^2 \neq 0$, $k \geq 2$ and $2(i+1) < 1 + 2i + 2 \leq h + 2i + k = m$. Further, one has that

$$\hat{w}_2^{i+1} e^{m-1-2(i+1)}[\mathbb{RP}(\nu^k)] = d^{i+1} e^{m-1-2(i+1)}[\mathbb{RP}(\nu^k)] = 0$$

but

$$\begin{aligned} \hat{w}_2^{i+1} e^{m-1-2(i+1)}[\mathbb{RP}(\nu^{m-1})] &= (e^2 + \alpha e)^{i+1} e^{m-1-2(i+1)}[\mathbb{RP}(\nu^{m-1})] \\ &= e^{m-1}[\mathbb{RP}(\nu^{m-1})] \\ &= \text{coefficient of } \alpha \text{ in } \frac{1}{(1+\alpha)^q} \\ &\neq 0 \end{aligned}$$

which is a contradiction.

Thus, i *is even.*

If b is even, then

$$\hat{w}_2 = \begin{cases} d & \text{on } P(h,i) \\ \binom{h+2i}{2}e^2 & \text{on } \mathbb{RP}^1 \end{cases}$$

and $w[0]_1^h \hat{w}_2^i e^{k-1} = c^h d e^{k-1} \neq 0$ for $P(h,i)$, and for $\mathbb{RP}^1$ this is $\binom{h+2i}{2}\alpha e^{2i+k-1}$. Since this is nonzero, one must have $h = 1$, but then $\binom{h+2i}{2} = \binom{2i+1}{2} = 0$ since $2i + 1 \equiv 1 \bmod 4$.

Thus, b *is odd.* Moreover, a *is even* since $h + i + 1 + a + b \not\equiv 0 \bmod 2$.

For b odd,

$$\hat{w}_2 = \begin{cases} 0 & \text{on } P(h,i) \\ \binom{h+2i}{2}e^2 & \text{on } \mathbb{RP}^1. \end{cases}$$

gives $\hat{w}_2 e^{m-3} = 0$ on $P(h,i)$, but on $\mathbb{RP}^1$ this is $\binom{h+2i}{2}e^{m-1}$ with the value of e^{m-1} on $\mathbb{RP}(\nu^{m-1})$ being the coefficient of α in $\frac{1}{(1+\alpha)^q} = \frac{1}{1+\alpha}$, which is 1. Thus $\binom{h+2i}{2} = 0$ which says $h \equiv 1 \bmod 4$.

If $h > 1$, dualizing $w[0]_1^h$ gives an involution (M^{2i+k}, T) fixing $P(0,i) = \mathbb{CP}^i$ with $k > 0$ and with normal bundle $\nu^k|_{\mathbb{CP}^i}$. The involutions fixing $\mathbb{CP}^i$ are well-known and one has $k = 2i$ and $b = i + 1$ with (M^{2i+k}, T) being cobordant to $(\mathbb{CP}^i \times \mathbb{CP}^i, \text{twist})$.

Now, let us find $w[2]_4$. One has that on $P(h,i)$,

$$w[2] = (1 + w_1 + w_2 + \cdots)\{(1+e)^2 + u_1(1+e) + u_2 + u_3(1+e)^{-1} + u_4(1+e)^{-2} + \cdots\}$$

where $w_s = w_s(P(h,i))$ and $u_t = w_t(\nu^k)$ from which

$$w[2]_4 = w_2 e^2 + x_3 e + x_4 \quad (\dim x_3 = 3, \dim x_4 = 4)$$

and

$$w_2(P(h,i)) = (i+1)d + \binom{h+i+1}{2}c^2 = d + \binom{h+i+1}{2}c^2$$

so

$$w[2]_4 = de^2 + \binom{h+i+1}{2}c^2e^2 + x_3e + x_4$$

and on $\mathbb{RP}^1$

$$w[2] = (1+e)^{h+2i+1} + \alpha(1+e)^{h+2i}$$

so

$$w[2]_4 = \binom{h+2i+1}{4}e^4 + \alpha\binom{h+2i}{3}e^3 = \binom{h+2i+1}{4}e^4$$

since $h \equiv 1 \bmod 4$. Then

$$\hat{w}_4 = w[2]_4 + \binom{h+i+1}{2}w[0]_1^2e^2 = \begin{cases} de^2 + x_3e + x_4 & \text{on } P(h,i) \\ \binom{h+2i+1}{4}e^4 & \text{on } \mathbb{RP}^1 \end{cases}$$

so

$$\begin{aligned} w[0]_1^{h-1}\hat{w}_4^ie^{k-2i} &= c^{h-1}(de^2+x_3e+x_4)^ie^{k-2i} \\ &= c^{h-1}d^ie^{2i}e^{k-2i} \\ &= c^{h-1}d^ie^k \end{aligned}$$

on $P(h,i)$ for since i is even all other terms have dimension more than $h+2i$ in c and d. The value of this on $\mathbb{RP}(\nu^k)$ is

$$\begin{aligned} c^{h-1}d^i\bar{w}_1(\nu^k)[P(h,i)] &= c^{h-1}d^iw_1(\nu^k)[P(h,i)] \\ &= c^{h-1}d^i(a+b)c[P(h,i)] \\ &= a+b \end{aligned}$$

and $a+b$ is odd. Thus, this is nonzero. However,

$$w[0]_1^{h-1}\hat{w}_4^ie^{k-2i} = \alpha^{h-1}\binom{h+2i+1}{4}e^{k+2i} = 0$$

on $\mathbb{RP}^1$ since $h-1$ is even and positive.

Thus, $h=1$. Moreover, $a=0$ and the exotic class cannot occur, so $w(\nu^k) = (1+c+d)^b$.

Now dualizing $w[0]_1$ gives an involution (M^{2i+k}, T) fixing a point $= \mathbb{RP}^0$ and $P(0,i) = \mathbb{CP}^i$ with the normal bundle of $\mathbb{CP}^i$ being $\nu^k|_{\mathbb{CP}^i}$ with $w(\nu^k) = (1+d)^b$. Royster's argument for involutions fixing (point) $\sqcup\, \mathbb{RP}^{\text{even}}$ also works for fixing (point) $\sqcup\, \mathbb{CP}^{\text{even}}$ to give

$$b = 1.$$

Furthermore, one knows the possible values of k. (See [St2]). Writing $i = 2^u(2v+1)$ with $u > 0$ one has

$$2 \le k \le \begin{cases} 2^{u+1}+2 & \text{if } u = 1 \\ 2^{u+1} & \text{if } u > 1. \end{cases}$$

Next, it suffices only to show that $k = 2^{u+1}$ with $u \neq 1$ is impossible.

If $k = 2^{u+1}$ with $u > 1$, then $m = 1 + 2^{u+1}(2v+1) + 2^{u+1} = 1 + 2^{u+2}(v+1)$, and by direct computations, one has that on $P(1, 2^u(2v+1))$

$$w[1]_4 = cde + de^2 + d^2$$

and on $\mathbb{RP}^1$

$$w[1]_4 = 0.$$

Then

$$w[1]_4^{2^u(v+1)}[\mathbb{RP}(\nu^{m-1})] = 0$$

but

$$\begin{aligned} w[1]_4^{2^u(v+1)}[\mathbb{RP}(\nu^{2^{u+1}})] &= (cde + de^2 + d^2)^{2^u(v+1)}[\mathbb{RP}(\nu^{2^{u+1}})] \\ &= \frac{(cd + d + d^2)^{2^u(v+1)}}{w(\nu^{2^{u+1}})}[P(1, 2^u(2v+1))] \\ &= \frac{(cd + d + d^2)^{2^u(v+1)}}{1 + c + d}[P(1, 2^u(2v+1))] \\ &= d^{2^u(v+1)}(1 + c + d)^{2^u(v+1)-1}[P(1, 2^u(2v+1))] \\ &= \binom{2^u(v+1) - 1}{2^u v}\binom{2^u - 1}{1} cd^{2^u(2v+1)}[P(1, 2^u(2v+1))] \\ &= \binom{2^u v + 2^u - 1}{2^u v} \\ &= 1 \end{aligned}$$

which is a contradiction.

Finally, let us observe the involution $(P(1, N^{i+l}), T_{N^{i+l}})$ with $i = 2^u(2v+1)$ even. Taking $l = 1$, one sees that $(P(1, N^{i+1}), T_{N^{i+1}})$ has the fixed date $\mathbb{RP}^1$ with $w(\nu^{2i+2}) = 1 + \alpha$ and $P(1, i)$ with $w(\nu^2) = 1 + c + d$. If $u = 1$, choosing $l = 2^u + 1 = 3$ one has that $(P(1, N^{i+3}), T_{N^{i+3}})$ also fixes $\mathbb{RP}^1$ with $w(\nu^{2i+6}) = 1 + \alpha$ and $P(1, i)$ with $w(\nu^6) = 1 + c + d$. Hence, for $2 \le k \le 2^{u+1} + 2$ with $u = 1$, (M^{1+2i+k}, T) fixing $\mathbb{RP}^1 \sqcup P(1, i)$ exists and then (M^{1+2i+k}, T) is cobordant to $\Gamma^{k-2}(P(1, N^{i+1}), T_{N^{i+1}})$. If $u > 1$, taking $l = 2^u - 1$, it is easy to see that $(P(1, N^{i+2^u-1}), T_{N^{i+2^u-1}})$ has the same fixed information as $(P(1, N^{i+1}), T_{N^{i+1}})$. Since $l = 2^u - 1 < 2^u$, $P(1, N^{i+2^u-1})$ bounds, and one may apply the operation Γ one time to $(P(1, N^{i+2^u-1}), T_{N^{i+2^u-1}})$, so that $\Gamma(P(1, N^{i+2^u-1}), T_{N^{i+2^u-1}})$ has the same fixed information as $(P(1, N^{i+1}), T_{N^{i+1}})$. Thus, for $2 \le k \le 2^{u+1} - 1$ with $u > 1$, (M^{1+2i+k}, T) fixing $\mathbb{RP}^1 \sqcup P(1, i)$ exists, so is cobordant to $\Gamma^{k-2}(P(1, N^{i+1}), T_{N^{i+1}})$. □

Note. In her paper [Gu], Guo showed that when $u = 1$, there exists an involution with $k = 7$. This is false.

Returning to the general case of j one has

Lemma 3.1. *Suppose that (M^{h+2i+k}, T) fixes $\mathbb{RP}^j \sqcup P(h, i)$ with h odd. Then*
(i) $h = j$;
(ii) i is even;
(iii) $b = 1$, and a is even;
(iv) For $i = 2^u(2v+1)$, one has

$$2 \le k \le \begin{cases} 2^{u+1} + 2 & \text{if } u = 1 \\ 2^{u+1} - 1 & \text{if } u > 1; \end{cases}$$

(v) Exotic characteristic classes do not occur in the bundle ν^k. Thus $w(\nu^k) = (1+c)^a(1+c+d)$.

Proof. (i)-(iv) follow applying to Proposition 3.1 and Theorem 3.2. It suffices only to show that (v) holds. For this, one need only consider the case $j = 5$ and suppose $w(\nu^k) = (1+c)^a(1+c+d)(1 + \dfrac{c^4 d^2}{(1+d)^2})$.

Then

$$\begin{aligned} w(\nu^k) &= (1+c)^a(1+c+d)(1+\frac{c^4d^2}{(1+d)^2}) \\ &= (1+c)^{a-4}(1+c+d)(1+\frac{c^4}{(1+d)^2}) \\ &= (1+c)^{a-4}(1+c+d)^{-1}(1+c^2+d^2+c^4) \\ &= (1+c)^{a-4}(1+c+d)+\frac{c^4(1+c)^{a-4}}{1+c+d} \\ &= (1+c)^{a-4}(1+c+d)+\frac{c^4}{1+c+d} \end{aligned}$$

since a is even. Now

$$\frac{1}{1+c+d}=\frac{1}{1+c}\cdot\frac{1}{1+\frac{d}{1+c}}=\frac{1}{1+c}\left\{1+\frac{d}{1+c}+\frac{d^2}{(1+c)^2}+\cdots+\frac{d^i}{(1+c)^i}\right\}$$

so

$$\frac{c^4}{1+c+d}=c^4\left\{\frac{1}{1+c}+\frac{d}{(1+c)^2}+\frac{d^2}{(1+c)^3}+\cdots+\frac{d^i}{(1+c)^{i+1}}\right\}$$

and

$$\frac{c^4}{(1+c)^{2s}}=c^4,\quad \frac{c^4}{(1+c)^{2s+1}}=c^4+c^5.$$

Thus

$$w(\nu^k)=\{(1+c)^{a-3}+c^4+c^5\}+d\{(1+c)^{a-4}+c^4\}+d^2(c^4+c^5)+d^3c^4+\cdots+d^i(c^4+c^5).$$

Further, it follows that $w_{2i+5}(\nu^k)\neq 0$ and so $k\geq 2i+5$. However, k never exceeds $2i+2$ since

$$2\leq k\leq\begin{cases} 2^{u+1}+2 & \text{if } u=1\\ 2^{u+1}-1 & \text{if } u>1.\end{cases}$$

Hence the exotic class cannot occur. □

Note. (1). From the results for $j=3$ [L-L], one finds that there exist examples with

$u=1$ for $q=1, a=2$ and $4\leq k\leq 6$
$q=3, a=0$ and $2\leq k\leq 4$, and

$u\neq 1$ for $q=1, a=0$ and $2\leq k\leq 2^{u+1}-3$
$q=3, a=2$ and $4\leq k\leq 2^{u+1}-1$.

One sees that given a pair (q,a), there are $k_{\min}$ and $k_{\max}$ with

$$2\leq k_{\min},k_{\max}\leq\begin{cases} 2^{u+1}+2 & \text{if } u=1\\ 2^{u+1}-1 & \text{if } u>1.\end{cases}$$

such that $k_{\min}\leq k\leq k_{\max}$, but $k_{\min}$ may not be equal to 2, and $k_{\max}$ may not be equal to

$$\begin{cases} 2^{u+1}+2 & \text{if } u=1\\ 2^{u+1}-1 & \text{if } u>1.\end{cases}$$

However, for $j=1$, $k_{\min}=2$ and

$$k_{\max}=\begin{cases} 2^{u+1}+2 & \text{if } u=1\\ 2^{u+1}-1 & \text{if } u>1.\end{cases}$$

This is because (q,a) has only a choice, i.e., $(q,a)=(1,0)$ for $j=1$, but not for $j\geq 3$. Thus, Lemma 3.1 does not provide the complete information for the general case of j, and the argument is not finished yet.

(2). It is known that $P(j,N^{i+l})$ bounds if $l<k_0$. Let $l=k_0$. One claims that $P(j,N^{i+k_0})$ does not bound. If $P(j,N^{i+k_0})$ bounds, then one may apply the operation Γ one time to $(P(j,N^{i+k_0}),T_{N^{i+k_0}})$,

so that the resulting involution $\Gamma(P(j,N^{i+k_0}),T_{N^{i+k_0}})$ fixes $\mathbb{RP}^j$ with $w(\nu^{2i+2k_0+1}) = (1+\alpha)^{i+k_0}$ and $P(j,N^{i+k_0})$ with $w(\nu^{2k_0+1}) = (1+c)^{k_0-1}(1+c+d)$, and has dimension $j+2i+2k_0+1$. However,

$$2k_0+1 > \begin{cases} 2^{u+1}+2 & \text{if } u=1 \\ 2^{u+1}-1 & \text{if } u>1 \end{cases}$$

gives a contradiction.

Recall that $2^p < j < 2^{p+1}$ and $q < 2^{p+1}$. Since $j = h$, a is only determined by modulo 2^{p+1} too and it is assumed that $a < 2^{p+1}$. Throughout the following discussions, (M^m,T) fixing $\mathbb{RP}^j \sqcup P(h,i)$ is always assumed to satisfy (i)-(v) stated in Lemma 3.3.

Lemma 3.2. *Suppose (M^m,T) fixes $\mathbb{RP}^j \sqcup P(j,i)$. Then $q \equiv a+i+1 \mod 2^{p+1}$.*

Proof. One first claims that $m > q$. If $q \le j$ then $w_q(\nu^{m-j}) = \binom{q}{q}\alpha^q = \alpha^q \neq 0$, so $m \ge j+q > q$. If $2^p < j < q < 2^{p+1}$, then $w_{2^p}(\nu^{m-j}) = \binom{q}{2^p}\alpha^{2^p} \neq 0$ so $m \ge j + 2^p > 2^{p+1} > q$.

Now let $x \equiv a+i+1 \mod 2^{p+1}$. One claims again that $m > x$. If $i \ge 2^p$ then $m = j+2i+k > 2i \ge 2^{p+1} > x$. If $i < 2^p$ and $a \ge 2^p$, then $w_{2^p+2}(\nu^k) = \binom{a+1}{2^p+2}c^{2^p+2} + c^{2^p}d \neq 0$, so $k \ge 2^p+2$ and $m > j+k > j+2^p > 2^{p+1} > x$. If $i < 2^p$ and $a < 2^p$, then $x = a+i+1$ and $w_{a+2}(\nu^k) = \binom{a}{a}c^a d = c^a d \neq 0$, so $k \ge a+2$ and $m > i+k > i+a+1 = x$.

From (1.1) one has that

$$w[1]_1 = \begin{cases} e+c & \text{on } P(j,i) \\ e+\alpha & \text{on } \mathbb{RP}^j. \end{cases}$$

The argument proceeds as follows.

(i). If $x > q$ then $x-(q+1) \ge 0$. When $0 \le x-(q+1) < j$, one has

$$\begin{aligned} w[1]_1^{x-1}e^{m-x}[\mathbb{RP}(\nu^k)] &= (e+c)^{x-1}e^{m-x}[\mathbb{RP}(\nu^k)] \\ &= \frac{(1+c)^{x-1}}{(1+c)^a(1+c+d)}[P(j,i)] \\ &= \frac{(1+c)^{x-1}}{(1+c)^{a+1}} \cdot \frac{1}{1+\frac{d}{1+c}}[P(j,i)] \\ &= \frac{(1+c)^{x-1}}{(1+c)^{a+1}}\left\{1+\frac{d}{1+c}+\cdots+\frac{d^i}{(1+c)^i}\right\}[P(j,i)] \\ &= \frac{d^i}{1+c}[P(j,i)] \\ &= c^j d^i[P(j,i)] \\ &= 1 \end{aligned}$$

but

$$\begin{aligned} w[1]_1^{x-1}e^{m-x}[\mathbb{RP}(\nu^{m-j})] &= (e+\alpha)^{x-1}e^{m-x}[\mathbb{RP}(\nu^{m-j})] \\ &= \frac{(1+\alpha)^{x-1}}{(1+\alpha)^q}[\mathbb{RP}^j] \\ &= (1+\alpha)^{x-q-1}[\mathbb{RP}^j] \\ &= 0 \end{aligned}$$

since $x-q-1<j$, which leads to a contradiction. When $j \le x-(q+1) < 2^{p+1}$, one has

$$\begin{aligned} w[1]_1^{q-1} e^{m-q}[\mathbb{RP}(\nu^k)] &= (e+c)^{q-1} e^{m-q}[\mathbb{RP}(\nu^k)] \\ &= \frac{(1+c)^{q-1}}{(1+c)^a(1+c+d)}[P(j,i)] \\ &= \frac{1}{(1+c)^{x+1-q}} d^i [P(j,i)] \\ &= (1+c)^{2^{p+1}-1-x+q} d^i [P(j,i)] \\ &= \binom{2^{p+1}-1-x+q}{j} \\ &= 0 \end{aligned}$$

since $2^{p+1}-1-x+q = 2^{p+1}-2-(x-q-1) \le 2^{p+1}-2-j < j$, but

$$\begin{aligned} w[1]_1^{q-1} e^{m-q}[\mathbb{RP}(\nu^{m-j})] &= (e+\alpha)^{q-1} e^{m-q}[\mathbb{RP}(\nu^{m-j})] \\ &= \frac{(1+\alpha)^{q-1}}{(1+\alpha)^q}[\mathbb{RP}^j] \\ &= \frac{1}{1+\alpha}[\mathbb{RP}^j] \\ &= 1. \end{aligned}$$

Thus, $x>q$ is impossible.

(ii). If $x<q$, in a similar way as (i), one may obtain that this is also impossible.

Combining (i) and (ii), x must be equal to q. □

Since the case $j=1$ is understood well (see Theorem 3.2), one always assumes $j \ge 3$ in the following discussions. Now one divides the argument into two cases: (I). $u=1$; (II). $u>1$.

Case (I): $u=1$**.**

For $u=1$ one has $i=4v+2$. Suppose $(M^{j+4v+2+k},T)$ fixes $\mathbb{RP}^j \sqcup P(j,4v+2)$. The argument proceeds as follows.

First, one cannot have $a>6$. For $a \ge 8$, one must have $j \ge 8$ (else a is taken mod 8) and a must have a power of 2 which at least 8 and less than j in its 2-adic expansion. Then there is at least a nonzero term $w_s(\nu^k)$ with $s>6$ in $w(\nu^k)$, and ν^k cannot be realized by a bundle of dimension less than or equal to 6.

For $a=6$, one cannot have $j \ge 7$, for then $\binom{6}{6}c^6 d \ne 0$ making $k \ge 8$. Thus $a=6$ can occur only for $j=5$, and one must have $k=6$ and $q \equiv 4v+1 \mod 8$. In particular, $q=1$ if v is even, and $q=5$ if v is odd.

Claim. $a=6$ is impossible.

Proof. One computes the values of $w[1]_4$ and $w[1]_{8v+6}$. On $P(5,4v+2)$, one has

$$\begin{aligned} w[1] \;=\; & (1+c)^5(1+c+d)^{4v+3}\{1+e+c+(c^2+d)(1+e)^{-1}+c^3(1+e)^{-2}+(c^4+c^2d)(1+e)^{-3} \\ & +c^5(1+e)^{-4}+c^4d(1+e)^{-5}\} \end{aligned}$$

so

$$w[1]_4 = cde+c^2e^2+de^2+c^4+\binom{v}{1}c^4 = \begin{cases} cde+c^2e^2+de^2+c^4 & \text{if } v \text{ is even} \\ cde+c^2e^2+de^2 & \text{if } v \text{ is odd.} \end{cases}$$

and

$$w[1]_{8v+6} = d^{4v+2}(ce+e^2) + \text{terms of degree less than } 4v+2 \text{ in } d.$$

On $\mathbb{RP}^5$,

$$\begin{aligned} w[1] &= (1+\alpha)^6 \left\{(1+e)^{8v+5} + \alpha(1+e)^{8v+4} + \binom{q}{4}\alpha^4(1+e)^{8v+1} + \binom{q}{5}\alpha^5(1+e)^{8v}\right\} \\ &= (1+\alpha)^6 \left\{(1+e)^5 + \alpha(1+e)^4 + \binom{q}{4}\alpha^4(1+e) + \binom{q}{5}\alpha^5\right\}(1+e)^{8v} \end{aligned}$$

so

$$w[1]_4 = \begin{cases} \alpha^4 + e^4 & \text{if } q=1 \\ e^4 & \text{if } q=5. \end{cases}$$

and $w[1]_{8v+6} = \alpha^2 e^{8v+4}$ whichever $q=1$ or $q=5$.

If v is even, then

$$w[1]_4 + w[1]_1^4 = \begin{cases} cde + c^2e^2 + de^2 + e^4 & \text{on } P(5,4v+2) \\ 0 & \text{on } \mathbb{RP}^5 \end{cases}$$

with $w[1]_1$ and $w[1]_{8v+6}$ together gives

$$w[1]_1^3(w[1]_4 + w[1]_1^4)w[1]_{8v+6}e[\mathbb{RP}(\nu^{8v+10})] = 0$$

but

$$\begin{aligned} & w[1]_1^3(w[1]_4 + w[1]_1^4)w[1]_{8v+6}e[\mathbb{RP}(\nu^6)] \\ = & \{(e+c)^3(cde + c^2e^2 + de^2 + e^4) \\ & \times(cd^{4v+2}e + d^{4v+2}e^2 + \text{terms of degree less than } 4v+2 \text{ in } d)\}e[\mathbb{RP}(\nu^6)] \\ = & \frac{(1+c)^3(1+cd+c^2+d)(cd^{4v+2} + d^{4v+2} + \text{terms of degree less than } 4v+2 \text{ in } d)}{(1+c)^6(1+c+d)}[P(5,4v+2)] \\ = & \frac{(1+c+d+c+c^2+cd)(d^{4v+2}(1+c) + \text{terms of degree less than } 4v+2 \text{ in } d)}{(1+c)^3(1+c+d)}[P(5,4v+2)] \\ = & \frac{(1+c)(1+c+d)(d^{4v+2}(1+c) + \text{terms of degree less than } 4v+2 \text{ in } d)}{(1+c)^3(1+c+d)}[P(5,4v+2)] \\ = & \frac{d^{4v+2}(1+c) + \text{terms of degree less than } 4v+2 \text{ in } d}{(1+c)^2}[P(5,4v+2)] \\ = & \frac{d^{4v+2}}{1+c}[P(5,4v+2)] + \frac{\text{terms of degree less than } 4v+2 \text{ in } d}{(1+c)^2}[P(5,4v+2)] \\ = & 1+0 \\ = & 1. \end{aligned}$$

If v is odd, in a similar way as above, then

$$w[1]_4 + w[1]_1^4 + w[0]_1^4 = \begin{cases} cde + c^2e^2 + de^2 + e^4 & \text{on } P(5,4v+2) \\ 0 & \text{on } \mathbb{RP}^5 \end{cases}$$

with $w[1]_1$ and $w[1]_{8v+6}$ together gives

$$w[1]_1^3(w[1]_4 + w[1]_1^4 + w[0]_1^4)w[1]_{8v+6}e[\mathbb{RP}(\nu^{8v+10})] = 0$$

but

$$w[1]_1^3(w[1]_4 + w[1]_1^4 + w[0]_1^4)w[1]_{8v+6}e[\mathbb{RP}(\nu^6)] = 1.$$

Therefore, $a=6$ is impossible. □

For $a = 4$, $\nu^6 = 4\iota \oplus \eta$ provides a suitable ν^k, and, of course $k = 6$ is the only possibility. However, dualizing $w[0]_1^{j-3}$ may change this case into the case $j = 3$ with $a = 0$, and the range of the values of k must lie in $2 \le k \le 4$. Therefore, $a = 4$ is impossible.

For $a = 2$, one has $q \equiv 4v + 5 \mod 2^{p+1}$. Dualizing $w[0]_1^{j-3}$ changes this case into the case $j = 3$ with $q = 1$, thus one has that $4 \le k \le 6$. Taking $l = 3$ in the involution $(P(j, N^{4v+2+l}), T_{N^{4v+2+l}})$, then for each $4 \le k \le 6$, $\Gamma^{k-6}(P(j, N^{4v+5}), T_{N^{4v+5}})$ fixes $\mathbb{RP}^j$ with $w(\nu^{8v+4+k}) = (1 + \alpha)^q$ and $P(j, 4v + 2)$ with $w(\nu^k) = (1 + c)^2(1 + c + d)$. Hence, $(M^{j+4v+2+k}, T)$ is cobordant to $\Gamma^{k-6}(P(j, N^{4v+5}), T_{N^{4v+5}})$ for $4 \le k \le 6$.

For $a = 0$, one has $q \equiv 4v + 3 \mod 2^{p+1}$. Now dualizing $w[0]_1^{j-3}$ changes the general case j into the case $j = 3$, and one knows that $2 \le k \le 4$. Proposition 2.1 provides the examples of the involutions of this type. $\Gamma^{k-2}(P(j, N^{4v+3}, T_{N^{4v+3}})$ belongs to the involution of this type for $2 \le k \le 2 + x_0$, so x_0 must be less than or equal to 2, and $x_0 \ge 1$ since $P(j, N^{4v+3})$ bounds. There is a sufficient reason for which $x_0 = 1$ is impossible since the involutions of the case $a = 0$ possess completely the analogous structures as those of the case $a = 2$. Also, this can be seen clearly when $j = 3$ (see [L-L]). So it should be certain that $x_0 = 2$, and the proof is omitted.

Combining the above arguments, one has

Theorem 3.3. *Suppose that $(M^{j+4v+2+k}, T)$ fixes $\mathbb{RP}^j \sqcup P(j, 4v + 2)$ with $j \ge 3$. Then either*
(1) $a = 0$, $q \equiv 4v + 3 \mod 2^{p+1}$ and $2 \le k \le 4$. Further, $(M^{j+4v+2+k}, T)$ is cobordant to $\Gamma^{k-2}(P(j, N^{4v+3}), T_{N^{4v+3}})$; or
(2) $a = 2$, $q \equiv 4v + 5 \mod 2^{p+1}$ and $4 \le k \le 6$. Further, $(M^{j+4v+2+k}, T)$ is cobordant to $\Gamma^{k-6}(P(j, N^{4v+5}), T_{N^{4v+5}})$.

Case (II): $u > 1$.

Lemma 3.3. *If $u > 1$, then $k \ge 2 + a$.*

Proof. One sees that $w(\nu^k) = (1 + c)^a(1 + c + d)$ is a product of the classes $1 + c$ and $1 + c + d$. Since one has assumed that $a < 2^{p+1}$, there does not exist the integer a' less than a such that $(1 + c)^a = (1 + c)^{a'}$ and $(1 + c)^{a+1} = (1 + c)^{a'+1}$ (note that a is even). Also, since the exotic classes cannot occur in $w(\nu^k)$, ν^k may be expressed by $a\xi \oplus \eta \oplus s\mathbb{R}$ which has dimension $a + 2 + s \ge a + 2$ where $s \ge 0$, so k must be more than or equal to $a + 2$. $\square$

From the case $u = 1$, one sees that $a \le 2^u = 2$, so that the involutions may correspond to those examples constructed in the section 2. Also, for the special cases $j = 1, 3$ with $u > 1$, one has $a < 2^u$. Now one considers the general cases with $u > 1$.

Lemma 3.4. *If $u > 1$ then $a < 2^u$.*

Proof. First, one computes the values of $w[1]_4$. On $P(j, i)$,

$$\begin{aligned} w[1] &= (1+c)^j(1+c+d)^{i+1}\{1+e+c+\frac{\binom{a+1}{2}c^2+d}{1+e}+\frac{\binom{a+1}{3}c^3}{(1+e)^2}+\frac{\binom{a+1}{4}c^4+\binom{a}{2}c^2d}{(1+e)^3}+\cdots\} \\ &= \{(1+c)^{i+j+1}+(1+c)^{i+j}d+\binom{i+1}{2}(1+c)^{i+j-1}d^2+\cdots\}\times\{1+e+c+\binom{a+1}{2}c^2+d \\ &\quad +\binom{a+1}{2}c^2e+de+\binom{a+1}{2}c^2e^2+de^2+\binom{a+1}{3}c^3+\binom{a+1}{4}c^4+\binom{a}{2}c^2d+\cdots\} \\ &= \{1+\binom{i+j+1}{2}c^2+\binom{i+j+1}{4}c^4+d+cd+\binom{i+j}{2}c^2d+\cdots\}\times\{1+e+c+\binom{a+1}{2}c^2 \\ &\quad +d+\binom{a+1}{2}c^2e+de+\binom{a+1}{2}c^2e^2+de^2+\binom{a+1}{3}c^3+\binom{a+1}{4}c^4+\binom{a}{2}c^2d+\cdots\} \end{aligned}$$

(note that $i+j+1$ is even and $\binom{i+1}{2} = 0$) so

$$\begin{aligned} w[1]_4 &= \binom{a+1}{2}c^2e^2 + de^2 + \binom{a+1}{4}c^4 + \binom{a}{2}c^2d + \binom{i+j+1}{2}\binom{a+1}{2}c^4 + \binom{i+j+1}{2}c^2d \\ &\quad + \binom{a+1}{2}c^2d + d^2 + c^2d + cde + \binom{i+j+1}{4}c^4 + \binom{i+j}{2}c^2d \\ &= \binom{a+1}{2}c^2e^2 + \left\{\binom{a+1}{4} + \binom{i+j+1}{2}\binom{a+1}{2} + \binom{i+j+1}{4}\right\}c^4 + de^2 + d^2 + cde. \end{aligned}$$

Note that $\binom{a}{2} + \binom{a+1}{2} = 0$ and $\binom{i+j+1}{2} + \binom{i+j}{2} = 1$. On $\mathbb{RP}^j$, one has

$$\begin{aligned} w[1] &= (1+\alpha)^{j+1}\{(1+e)^{2i+1} + \alpha(1+e)^{2i} + \binom{q}{2}\alpha^2(1+e)^{2i-1} + \binom{q}{3}\alpha^3(1+e)^{2i-2} \\ &\quad + \binom{q}{4}\alpha^4(1+e)^{2i-3} + \cdots\} \\ &= \{1 + \binom{j+1}{2}\alpha^2 + \binom{j+1}{4}\alpha^4 + \cdots\} \\ &\quad \times\{1 + e + \alpha + \binom{q}{2}\alpha^2 + \binom{q}{2}\alpha^2 e + \binom{q}{2}\alpha^2e^2 + \binom{q}{2}\alpha^3 + \binom{q}{4}\alpha^4 + \cdots\} \end{aligned}$$

so

$$w[1]_4 = \binom{q}{2}\alpha^2e^2 + \binom{q}{4}\alpha^4 + \binom{j+1}{2}\binom{q}{2}\alpha^4 + \binom{j+1}{4}\alpha^4.$$

Form the class

$$\begin{aligned} \hat{w}_4 &= w[1]_4 + \binom{q}{2}w[0]_1^2(w[0]_1 + w[1]_1)^2 + \left\{\binom{q}{4} + \binom{j+1}{2}\binom{q}{2} + \binom{j+1}{4}\right\}w[0]_1^4 \\ &= \begin{cases} de^2 + cde + d^2 & \text{on } P(j,i) \\ 0 & \text{on } \mathbb{RP}^j \end{cases} \end{aligned}$$

for since $\binom{q}{2} + \binom{a+1}{2} = \binom{j+1}{2} + \binom{i+j+1}{2} = 0$ and

$$\binom{q}{4} + \binom{j+1}{4} = \binom{a+1}{4} + \binom{i+j+1}{4}$$

by Lemma 3.2.

Now suppose that $a \geq 2^u$. Then $2^{p+1} > a \geq 2^u$ so $p \geq u$ (this happens in the case $j \geq 5$). By Lemma 3.3, one knows $k \geq a+2$. If $a < j$ then

$$a - 2^u + 2^{u+2}(v+1) = a - 2^u + 2i + 2^{u+1} = a + 2i + 2^u < j + 2i + (a+2) \leq m.$$

If $a > j$ then $a > 2^p$ so $w_{2^p+2}(\nu^k) = \binom{a+1}{2^p+2}c^{2^p+2} + c^{2^p}d \neq 0$. Thus $2^p + 2 \leq k \leq 2^{u+1} - 1$. This implies that $u \geq p$, so $u = p$ and

$$a - 2^u + 2^{u+2}(v+1) = 2^p + 2i + a < j + 2i + k = m.$$

Further, one has that

$$\begin{aligned} w[1]_1^{a-2^u}\hat{w}_4^{2^u(v+1)}e^{m-1-a-2i-2^u}[\mathbb{RP}(\nu^k)] &= (e+c)^{a-2^u}(de^2+cde+d^2)^{2^u(v+1)}e^{m-1-a-2i-2^u}[\mathbb{RP}(\nu^k)] \\ &= \frac{(1+c)^{a-2^u}(d+cd+d^2)^{2^u(v+1)}}{(1+c)^a(1+c+d)}[P(j,i)] \\ &= \frac{d^{2^u(v+1)}(1+c+d)^{2^u(v+1)}}{(1+c)^{2^u}}[P(j,i)] \\ &= \binom{2^u(v+1)-1}{2^uv}\frac{d^{2^u(2v+1)}}{1+c}[P(j,i)] \\ &= c^jd^i[P(j,i)] \\ &= 1 \end{aligned}$$

but

$$w[1]_1^{a-2^u}\hat{w}_4^{2^u(v+1)}e^{m-1-a-2i-2^u}[\mathbb{RP}(\nu^{m-j})] = 0$$

which is a contradiction. Therefore, $a \geq 2^u$ is impossible. $\square$

Up to now, if (M^{j+2i+k}, T) fixes $\mathbb{RP}^j \sqcup P(j,i)$ with $u > 1$ and with those properties indicated in Lemmas 3.1, 3.2, 3.3, and 3.4, then Proposition 2.1 shows that (M^{j+2i+k}, T) exists when k is restricted to a range of values, and, taking $l = a+1$ in the involution $(P(j, N^{i+l}), T_{N^{i+l}})$, (M^{j+2i+k}, T) is cobordant to $\Gamma^{k-2}(P(j, N^{i+a+1}), T_{N^{i+a+1}})$ for $a+2 \leq k \leq 2a+2+x_0$. However, x_0 is only a unknown number. We wish to know the certain value x_0. This is equivalent to determine the upper bound of k.

Now let us estimate the maximum k value for realizing the Stiefel-Whitney class $w(\nu^k) = (1+c)^a(1+c+d)$.

Let $E(s)$ denote the set formed by all $i_1, ..., i_t$ in the dyadic decomposition $2^{i_1} + \cdots + 2^{i_t}$ of an integer $s > 0$, and $E(0)$ is defined as an empty set. We will use Lucas Theorem, which states that for two integers $s_1, s_2 \geq 0$, $\binom{s_1}{s_2} \equiv 1 \mod 2$ if and only if $E(s_2) \subset E(s_1)$ (see [Si]).

When $p \geq u$, one has $j > 2^p \geq 2^u > a$ (this only happens in the case $j \geq 5$). If $k > 2^u + a + 1$, then

$$j - 2^u + a + 1 + 2^{u+2}(v+1) = j + 2i + 2^u + a + 1 < j + 2i + k = m.$$

Using the class $\hat{w}_4$ in the proof of Lemma 3.4, one has

$$\begin{aligned} 0 &= w[1]_1^{j-2^u+a+1}\hat{w}_4^{2^u(v+1)}e^{m-j-2i-2^u-a-2}([\mathbb{RP}(\nu^k)] + [\mathbb{RP}(\nu^{m-j})]) \\ &= (e+c)^{j-2^u+a+1}(de^2+cde+d^2)^{2^u(v+1)}e^{m-j-2i-2^u-a-2}[\mathbb{RP}(\nu^k)] + 0 \\ &= (1+c)^{j-2^u+1}d^{2^u(v+1)}(1+c+d)^{2^u(v+1)-1}[P(j,i)] \\ &= \binom{2^u(v+1)-1}{2^uv}d^{2^u(2v+1)}(1+c)^j[P(j,i)] \\ &= c^jd^i[P(j,i)] \\ &= 1 \end{aligned}$$

which is impossible. Thus k must be less than or equal to $2^u + a + 1$.

When $p < u$, one has $q = a + 1$ by Lemma 3.2. Let a_0 be an integer having the property that $E(a_0) = E(j) \cap E(a)$. It is easy to see that a_0 is even, and $a_0 < j$ and $a_0 \leq a$. In particular, $E(j) \subset E(2^u - 1 - a + a_0)$. If $k > 2^{u+1} - j + a_0$, then

$$a_0 + 2^{u+2}(v+1) = a_0 + 2i + 2^{u+1} < j + 2i + k = m.$$

Using the class $\hat{w}_4$ in the proof of Lemma 3.4, one has

$$\begin{aligned}
0 &= w[1]_1^{a_0}\hat{w}_4^{2^u(v+1)}e^{m-1-a_0-2^{u+2}(v+1)}([\mathbb{RP}(\nu^k)]+[\mathbb{RP}(\nu^{m-j})]) \\
&= (e+c)^{a_0}(de^2+cde+d^2)^{2^u(v+1)}e^{m-1-a_0-2^{u+2}(v+1)}[\mathbb{RP}(\nu^k)]+0 \\
&= (1+c)^{a_0-a}d^{2^u(v+1)}(1+c+d)^{2^u(v+1)-1}[P(j,i)] \\
&= d^{2^u(2v+1)}(1+c)^{2^u-1-a+a_0}[P(j,i)] \\
&= \binom{2^u-1-a+a_0}{j}c^jd^i[P(j,i)] \\
&= 1
\end{aligned}$$

which is a contrary equation. Thus k must be less than or equal to $2^{u+1}-j+a_0$.

Observation. The upper bound of k estimated as above is attainable in some special cases. For example, when $a=2^u-2$, the above arguments show that if $p\geq u$ then the upper bound of k should be $2^u+a+1=2^{u+1}-1$, and if $u\geq p+1$ then $a_0=j-1$ so the upper bound of k should be $2^{u+1}-j+a_0=2^{u+1}-1$. The examples in the section 2 make sure that $2^{u+1}-1$ with $a=2^u-2$ can become the upper bound of k. In fact, if $u>1$, then $P(j,N^{i+2^u-1})$ bounds, and thus one may apply the operation Γ just one time to $(P(j,N^{i+2^u-1}),T_{N^{i+2-1}})$ such that the resulting involution $\Gamma(P(j,N^{i+2^u-1}),T_{N^{i+2^u-1}})$ has the same fixed information as $(P(j,N^{i+2^u-1}),T_{N^{i+2^u-1}})$ and has dimension $j+2i+2^{u+1}-1$. Also, if $u>1$ and $j=1$, or 3, then $u\geq p+1$ must be satisfied. It is easy to see that $a_0=a$ when $j=1,3$, so the upper bound of k should be $2^{u+1}-j+a$. This just corresponds to those results showed in Theorem 3.2 and [L-L, Theorem 5.1]. For the general case, the proof for which the upper bound of k estimated as above is attainable seems to be a difficult thing. We try to give a proof, but nothing conclusion.

It is extremely tempting to conjecture that the upper bound of k is 2^u+a+1 if $p\geq u$, and $2^{u+1}-j+a_0$ if $u\geq p+1$. In other words, x_0 should be 2^u-a-1 if $p\geq u$, and $2^{u+1}-j-2a-2+a_0$ if $u\geq p+1$.

Now suppose (M^m,T) fixes $\mathbb{RP}^j\sqcup P(h,j)$ with h odd. Then (M^m,T) possesses those properties indicated in Lemmas 3.1, 3.2, 3.3, and 3.4. If the above conjecture is true, then the result with $u>1$ may be stated as follows.

Theorem 3.4. *If $u>1$ then (M^{j+2i+k},T) fixing $\mathbb{RP}^j\sqcup P(j,i)$ is cobordant to $\Gamma^{k-2}(P(j,N^{i+a+1}),T_{N^{i+a+1}})$ for which $a+2\leq k\leq 2^u+a+1$ if $p\geq u$ and $a+2\leq k\leq 2^{u+1}-j+a_0$ if $u\geq p+1$ where a_0 is an integer having the property $E(a_0)=E(j)\cap E(a)$.*

4. The case in which h is even

In this section, one considers the involution (M^m,T) fixing $\mathbb{RP}^j\sqcup P(h,i)$ with h even. First, let us prove some lemmas.

Lemma 4.1. *If h is even, then $h\geq q-1$ and $j+1\geq 2i+k$.*

Proof. From (1.1) one then has

$$w[0]_1=\begin{cases}(i+1+a+b)c & \text{on } P(h,i)\\ e+\alpha & \text{on } \mathbb{RP}^j.\end{cases}$$

Since $m>q$ (see the proof of Lemma 3.2), one may form the characteristic number for

$$w[0]_1^{q-1}e^{m-1-(q-1)}=(e+\alpha)^{q-1}e^{m-q}$$

which has value on $\mathbb{RP}(\nu^{m-j})$ equal to the coefficient of α^j in $\frac{(1+\alpha)^{q-1}}{(1+\alpha)^q}=\frac{1}{1+\alpha}$ and that coefficient is nonzero. On $\mathbb{RP}(\nu^k)$,

$$w[0]_1^{q-1}e^{m-1-(q-1)}=(i+1+a+b)c^{q-1}e^{m-q}$$

and the value of this on $\mathbb{RP}(\nu^k)$ must be nonzero. Thus, one has that if $q>1$, then $i+1+a+b$ must be odd and $h\geq q-1$. If $q=1$, it is obvious that $h\geq q-1$.

Now for $h < t \le m-1$, one has that on $\mathbb{RP}(\nu^k)$

$$w[0]_1^t e^{m-1-t} = (i+1+a+b)c^t e^{m-1-t} = 0$$

and the coefficient of α^j in $\frac{(1+\alpha)^t}{(1+\alpha)^q}$ is zero. If one writes

$$\frac{(1+\alpha)^{h+1}}{(1+\alpha)^q} = 1 + \cdots + \alpha^{s_0}$$

where s_0 is the degree of the highest term, $0 \le s_0 \le j$, and s_0 is even since $h+1$ and q are odd so $s_0 < j$. Further,

$$\frac{(1+\alpha)^{h+1+(j-s_0)}}{(1+\alpha)^q} = (1+\alpha)^{j-s_0}(1+\cdots+\alpha^{s_0})$$

has the coefficient of α^j being 1. Since $h+1+j-s_0 > h$, this makes $h+1+j-s_0 \ge m = h+2i+k$ so $j+1 \ge s_0+2i+k \ge 2i+k$. This completes the proof. □

Lemma 4.2. *If $m \ne j+q$, then*
(1). *$i+1+a+b$ is odd;*
(2). *$h \ge 2i+k$;*
(3). *The exotic classes cannot occur in $w(\nu^k)$.*

Proof. If $m < j+q$ then q must be more than j, so $q-1 \ge j+1$ and $q > 1$. By Lemma 4.1, one has

$$h \ge q-1 \ge j+1 \ge s_0+2i+k \ge 2i+k$$

and $i+1+a+b$ is odd. If $m > j+q$, then the characteristic number for

$$w[0]_1^{j+q} e^{m-1-j-q} = (e+\alpha)^{j+q} e^{m-1-j-q}$$

has value on $\mathbb{RP}^j$ equal to the coefficient of α^j in $\frac{(1+\alpha)^{j+q}}{(1+\alpha)^q} = (1+\alpha)^j$, which is nonzero. On $P(h,i)$,

$$w[0]_1^{j+q} e^{m-1-j-q} = (i+1+a+b)c^{j+q} e^{m-1-j-q}$$

and the value of this on $P(h,i)$ must be nonzero. Thus, $i+1+a+b$ is odd and

$$h \ge j+q.$$

Further, by Lemma 4.1 one has

$$h \ge j+q = (j+1)+(q-1) \ge 2i+k+(q-1) \ge 2i+k.$$

If the exotic classes occur in $w(\nu^k)$, then generally $k \ge 4$ since $\dim\rho \ge 4$, so $h \ge 2i+k \ge 6$, and the only possibility for which the exotic classes may occur is that $h=6, i=1$, and $j=5$. However, Stong's argument [St1] shows that if the exotic class occurs when $h=6$, then $w(\rho) = 1+c^6d$ with $\dim\rho = 8$, so $k \ge 8$ and

$$6 = h \ge 2i+k \ge 10$$

which leads to a contradiction. Thus, the exotic classes cannot occur in $w(\nu^k)$. □

Letting $2^A \le h < 2^{A+1}$ and $2^B \le i < 2^{B+1}$, one may assume that $a < 2^{A+1}$ and $b < 2^{B+1}$ since a (resp. b) is only determined by modulo 2^{A+1} (resp. 2^{B+1}). Let $C = \max\{A+1, B+1\}$.

Lemma 4.3. *If $m \ne j+q$, then*
(1). *$b < 2^B \le i$ and further, $k \ge 2b$.*
(2). $\frac{(1+\alpha)^h}{(1+\alpha)^q}[\mathbb{RP}^j] = 1$.
(3). *$k > 2i+4b$ for $a \ge h$.*

Proof. By Lemma 4.2, one can write $w(\nu^k) = (1+c)^a(1+c+d)^b$.

(1). Since

$$w[0]_1^{q-1}e^{m-q}[\mathbb{RP}(\nu^{m-j})] = \frac{1}{1+\alpha}[\mathbb{RP}^j] = 1,$$

one has that

$$\begin{aligned} w[0]_1^{q-1}e^{m-q}[\mathbb{RP}(\nu^k)] &= \frac{c^{q-1}}{(1+c)^a(1+c+d)^b}[P(h,i)] \\ &= \binom{2^C-b}{i} d^i c^{q-1}(1+c)^{2^C+2^{A+1}-a-b-i}[P(h,i)] \\ &= \binom{2^C-b}{i}\binom{2^C+2^{A+1}-a-b-i}{h-q+1} \end{aligned}$$

is nonzero, so

$$\binom{2^C-b}{i} = 1. \tag{4.1}$$

Since $\binom{2^C-b}{i} = \binom{2^{B+1}-b}{i} = 1$, one has that $b < 2^B$, for if not $2^{B+1}-b$ is less than $2^B(\leq i)$ so $\binom{2^{B+1}-b}{i} = 0$, but this is a contradiction. Furthermore, it follows that $k \geq 2b$ since there exists the nonzero term d^b in $w(\nu^k) = (1+c)^a(1+c+d)^b$.

(2). The characteristic number for

$$w[0]_1^h e^{m-1-h} = c^h e^{m-1-h}$$

has value on $\mathbb{RP}(\nu^k)$ equal to $\binom{2^C-b}{i}$, which is 1 by (4.1), thus on $\mathbb{RP}(\nu^{m-j})$,

$$w[0]_1^h e^{m-1-h}[\mathbb{RP}(\nu^{m-j})] = \frac{(1+\alpha)^h}{(1+\alpha)^q}[\mathbb{RP}^j]$$

must be nonzero.

(3). If $a \geq h$, then $a \geq 2^A$ so the coefficient of the term $a^{2^A}d^b$ is nonzero in $w(\nu^k) = (1+c)^a(1+c+d)^b$. Thus

$$k \geq 2^A + 2b. \tag{4.2}$$

On the other hand, by Lemma 4.2(2), one has that $2^{A+1} > h \geq 2i+k$ so

$$2^A > \frac{2i+k}{2}. \tag{4.3}$$

From (4.2) and (4.3), it follows that

$$k \geq 2^A + 2b > \frac{2i+k}{2} + 2b$$

and thus

$$k > 2i + 4b.$$

This completes the proof. □

Proposition 4.1. *If (M^m, T) fixes $\mathbb{RP}^j \sqcup P(h,i)$ with h even, then $m = j + q$.*

Proof. Suppose that $m \neq j+q$. The argument proceeds as follows.

(I). The case in which i is odd.

If i is odd, then one has b is odd since $\binom{2^C-b}{i}=1$ by (4.1), and further a is even. Now one computes the values of $w[1]_1$ and $w[1]_2$. On $P(h,i)$,

$$\begin{aligned} w[1] &= (1+c)^h(1+c+d)^{i+1}\{1+e+c+\binom{a+b}{2}c^2+d+\text{terms of degree more than 2}\} \\ &= \{1+\binom{h+i+1}{2}c^2+\text{terms of degree more than 2}\} \\ &\quad \times\{1+e+c+\binom{a+b}{2}c^2+d+\text{terms of degree more than 2}\} \end{aligned}$$

so $w[1]_1=e+c$ and

$$w[1]_2=\binom{a+b}{2}c^2+\binom{h+i+1}{2}c^2+d$$

and on $\mathbb{RP}^j$

$$\begin{aligned} w[1] &= (1+\alpha)^{j+1}\{(1+e)^{h+2i+1-j}+\alpha(1+e)^{h+2i-j}+\binom{q}{2}\alpha^2(1+e)^{h+2i-1-j}+\cdots\} \\ &= \{1+\binom{j+1}{2}\alpha^2+\cdots\}\{1+\alpha+\alpha e+\binom{h+2i+1-j}{2}e^2+\binom{q}{2}\alpha^2+\cdots\} \end{aligned}$$

so $w[1]_1=\alpha$ and

$$w[1]_2=\alpha e+\binom{j+1}{2}\alpha^2+\binom{h+2i+1-j}{2}e^2+\binom{q}{2}\alpha^2.$$

Form the class

$$x_2=w[1]_2+w[1]_1(w[0]_1+w[1]_1)+(\binom{j+1}{2}+\binom{q}{2})w[1]_1^2+\binom{h+2i+1-j}{2}(w[0]_1+w[1]_1)^2,$$

then on $\mathbb{RP}^j$, $x_2=0$, and on $P(h,i)$, $w[0]_1^h x_2$ is either $c^h d$ or $c^h(e^2+d)$.

When $w[0]_1^h x_2=c^h d$ on $P(h,i)$, one has that

$$w[0]_1^h x_2^i e^{m-1-h-2i}=\begin{cases} c^h d^i e^{k-1} & \text{on } P(h,i) \\ 0 & \text{on } \mathbb{RP}^j \end{cases}$$

gives a nonzero value on $\mathbb{RP}(\nu^k)$, but the value of this on $\mathbb{RP}(\nu^{m-j})$ is zero. This is a contradiction.

When $w[0]_1^h x_2=c^h(e^2+d)$ on $P(h,i)$, one has that

$$w[0]_1^h x_2^{i+b} e^{m-1-h-2(i+b)}[\mathbb{RP}(\nu^{m-j})]=0$$

but

$$\begin{aligned} w[0]_1^h x_2^{i+b} e^{m-1-h-2(i+b)}[\mathbb{RP}(\nu^k)] &= c^h(e^2+d)^{i+b}e^{m-1-h-2(i+b)}[\mathbb{RP}(\nu^k)] \\ &= \frac{c^h(1+d)^{i+b}}{(1+c)^a(1+c+d)^b}[P(h,i)] \\ &= c^h(1+d)^i[P(h,i)] \\ &= 1 \end{aligned}$$

which leads to a contradiction. Note that $m-1-h-2(i+b)=k-1-2b\geq 0$ by Lemma 4.3(1).

Thus, there does not exist the case in which i is odd.

(II). The case in which i is even.

Let i be even. Then $a+b$ is even by Lemma 4.1 and

$$\chi(M^m) = \chi(\mathbb{RP}^j) + \chi(P(h,i)) = 0 + \chi(P(h,i)) = (h+1)(i+1)$$

is nonzero modulo 2 where $\chi(\cdot)$ denotes the Euler characteristic number, and thus m must be even since the Euler characteristic number of any odd-dimensional manifold is always zero. Further, k is also even.

By direct computions, one has that on $P(h,i)$, $w[1]_1 = e + c$ and

$$w[0]_2 = \binom{a+b}{2}c^2 + bd + \binom{h+i+1}{2}c^2 + d$$

and

$$\begin{aligned} w[0]_4 \;=\; & \binom{a+b}{2}c^2e^2 + bde^2 + bcde + \binom{a+b}{4}c^4 + \binom{a+b-1}{2}bc^2d + \binom{b}{2}d^2 + bc^2d \\ & +(\binom{h+i+1}{2}c^2 + d)(\binom{a+b}{2}c^2 + bd) + \binom{h+i+1}{4}c^4 + \binom{h+i}{2}c^2d + \binom{i+1}{2}d^2 \end{aligned}$$

and on $\mathbb{RP}^j$, $w[1]_1 = \alpha$ and

$$w[0]_2 = \binom{h+2i-j}{2}e^2 + \binom{q}{2}\alpha^2 + \binom{j+1}{2}\alpha^2 = \binom{h-j}{2}e^2 + \binom{q}{2}\alpha^2 + \binom{j+1}{2}\alpha^2$$

and

$$\begin{aligned} w[0]_4 \;=\; & \binom{h+2i-j}{4}e^4 + \binom{q}{2}\binom{h+2i-2-j}{2}\alpha^2e^2 + \binom{q}{4}\alpha^4 + \binom{j+1}{2}\binom{h+2i-j}{2}\alpha^2e^2 \\ & +\binom{j+1}{2}\binom{q}{2}\alpha^4 + \binom{j+1}{4}\alpha^4. \end{aligned}$$

(a). If b is even, form the class

$$\hat{x}_2 = w[0]_2 + \binom{h-j}{2}(w[0]_1 + w[1]_1)^2 + (\binom{q}{2} + \binom{j+1}{2})w[1]_1^2,$$

then on $P(h,i)$, $w[0]_1^h\hat{x}_2$ is c^hd or $c^h(e^2+d)$, and on $\mathbb{RP}^j$, $\hat{x}_2 = 0$. Similarly to the argument as the case (I), one can obtain that there does not exist the case in which b is even if i is even.

(b). If b is odd, form the class

$$\begin{aligned} x_4 \;=\; & w[0]_4 + \binom{h+2i-j}{4}(w[0]_1 + w[1]_1)^4 + (\binom{q}{4} + \binom{j+1}{2}\binom{q}{2} + \binom{j+1}{4})w[1]_1^4 \\ & +(\binom{q}{2}\binom{h+2i-2-j}{2} + \binom{j+1}{2}\binom{h+2i-j}{2})w[1]_1^2(w[0]_1 + w[1]_1)^2, \end{aligned}$$

then one has that on $\mathbb{RP}^j$,

$$x_4 = 0$$

and on $P(h,i)$,

$$w[0]_1^hx_4 = \begin{cases} c^hde^2 \text{ or } c^h(e^4 + de^2) & \text{if } \binom{b}{2} + \binom{i+1}{2} = 1 \\ c^h(de^2 + d^2) \text{ or } c^h(e^4 + de^2 + d^2) & \text{if } \binom{b}{2} + \binom{i+1}{2} = 0. \end{cases}$$

One knows from Lemma 4.3(1) that $k > 2b$ so $2i + k > 4b$. The argument is divided into the following four cases.

(i). When $w[0]_1^h x_4 = c^h de^2$ on $P(h,i)$, one has that

$$\begin{aligned} w[0]_1^h(x_4 + w[0]_1^4 + w[1]_1^4)^b e^{m-1-h-4b}[\mathbb{RP}(\nu^k)] &= c^h(de^2+e^4)^b e^{m-1-h-4b}[\mathbb{RP}(\nu^k)] \\ &= \frac{c^h(1+d)^b}{(1+c)^a(1+c+d)^b}[P(h,i)] \\ &= c^h[P(h,i)] \\ &= 0 \end{aligned}$$

but

$$\begin{aligned} w[0]_1^h(x_4 + w[0]_1^4 + w[1]_1^4)^b e^{m-1-h-4b}[\mathbb{RP}(\nu^{m-j})] &= (e+\alpha)^h e^{m-1-h}[\mathbb{RP}(\nu^{m-j})] \\ &= \frac{(1+\alpha)^h}{(1+\alpha)^q}[\mathbb{RP}^j] \\ &= 1 \end{aligned}$$

by Lemma 4.3(2). This is impossible.

(ii). When $w[0]_1^h x_4 = c^h(e^4 + de^2)$ on $P(h,i)$, if $b > 1$ then one has that

$$w[0]_1^h x_4^{b-1} e^{m-1-h-4(b-1)}[\mathbb{RP}(\nu^k)] = \frac{c^h(1+d)^{b-1}}{(1+c)^a(1+c+d)^b}[P(h,i)] = \frac{c^h}{1+d}[P(h,i)] = 1$$

but

$$w[0]_1^h x_4^{b-1} e^{m-1-h-4(b-1)}[\mathbb{RP}(\nu^{m-j})] = 0.$$

If $b = 1$ and $a < h$, then the top nonzero Stiefel-Whitney class in $w(\nu^k) = (1+c)^a(1+c+d)$ is $c^a d$ so $k > a+2$ (note that a is odd and k is even). Thus, one has that

$$w[1]_1^j e^{m-1-j}[\mathbb{RP}(\nu^{m-j})] = \frac{\alpha^j}{(1+\alpha)^q}[\mathbb{RP}^j] = 1$$

but

$$\begin{aligned} w[1]_1^j e^{m-1-j}[\mathbb{RP}(\nu^k)] &= (e+\alpha)^j e^{m-1-j}[\mathbb{RP}(\nu^k)] \\ &= \frac{(1+c)^j}{(1+c)^a(1+c+d)}[P(h,i)] \\ &= \frac{(1+c)^j}{(1+c)^{a+1}} \cdot \frac{1}{1+\frac{d}{1+c}}[P(h,i)] \\ &= (1+c)^{j-a-1}\{1 + \frac{d}{1+c} + \cdots + \frac{d^i}{(1+c)^i}\}[P(h,i)] \\ &= (1+c)^{j-a-1-i}d^i[P(h,i)] \\ &= 0 \end{aligned}$$

since $a+1+i < 2i+k \le j+1 \le h$ by the proof of Lemma 4.2. If $b = 1$ and $a \ge h$, by Lemma 4.3(3) one knows that $k > 2i+4$ so $m-1 = h+2i+k-1 \ge h+4i+5$ for since k is even. Now

$$w[0]_1^h x_4^{i+1} e^{m-1-h-4i-4} = \begin{cases} c^h(e^4+de^2)^{i+1}e^{m-1-h-4i-4} & \text{on } P(h,i) \\ 0 & \text{on } \mathbb{RP}^j \end{cases}$$

has a nonzero value on $\mathbb{RP}(\nu^k)$, but the value of this on $\mathbb{RP}(\nu^{m-j})$ is zero, which gives to a contradiction.

(iii). When $w[0]_1^h x_4 = c^h(de^2 + d^2)$ on $P(h,i)$, if $b > 1$ then

$$w[0]_1^h x_4^{b-1} e^{m-1-h-4(b-1)} = \begin{cases} c^h(de^2+d^2)^{b-1}e^{m-1-h-4(b-1)} & \text{on } P(h,i) \\ 0 & \text{on } \mathbb{RP}^j \end{cases}$$

gives a nonzero value on $\mathbb{RP}(\nu^k)$ but not on $\mathbb{RP}(\nu^{m-j})$, which leads to a contradiction. If $b=1$, in a similar way as the case (ii), one may conclude that $b=1$ is impossible.

(iv). When $w[0]_1^h x_4 = c^h(e^4+de^2+d^2)$ on $P(h,i)$, one has that

$$w[0]_1^h(x_4+w[0]_1^4+w[1]_1^4)^b e^{m-1-h-4b}[\mathbb{RP}(\nu^k)] = \frac{c^h d^b(1+d)^b}{(1+c)^a(1+c+d)^b}[P(h,i)] = c^h d^b[P(h,i)] = 0$$

but

$$\begin{aligned} w[0]_1^h(x_4+w[0]_1^4+w[1]_1^4)^b e^{m-1-h-4b}[\mathbb{RP}(\nu^{m-j})] &= (e+\alpha)^h e^{m-1-h}[\mathbb{RP}(\nu^{m-j})] \\ &= \frac{(1+\alpha)^h}{(1+\alpha)^q}[\mathbb{RP}^j] \\ &= 1 \end{aligned}$$

by Lemma 4.3(2).

Thus, there does not exist the case in which i is even.

Combining the above arguments, one completes the proof. □

For the case $m=j+q$, consider the involution T_q on $\mathbb{RP}^{j+q}$ defined by

$$T_q([x_0,\cdots,x_j,x_{j+1},\cdots,x_{j+q}]) = [x_0,\cdots,x_j,-x_{j+1},\cdots,-x_{j+q}]$$

fixing $\mathbb{RP}^j$ with normal bundle $\nu^q = q\iota$ having $w(\nu^q)=(1+\alpha)^q$ and $\mathbb{RP}^{q-1}$ with normal bundle $\nu^{j+1} = (j+1)\iota$ having $w(\nu^{j+1}) = (1+\alpha)^{j+1}$, forming the union $(M^m,T)\sqcup(\mathbb{RP}^{j+q},T_q)$ one obtains an involution $(\bar{M}^{j+q},\bar{T})$ fixing $\mathbb{RP}^{q-1}$ with $w(\nu^{j+1})=(1+\alpha)^{j+1}$ and $P(h,i)$ with nornal bundle ν^k, with $h\geq q-1$.

Observation. Finding involutions fixing $\mathbb{RP}^j$ and $P(h,i)$ with h even reduces to a problem about finding involutions that fix $\mathbb{RP}^{q-1}$ and $P(h,i)$, which is the problem for *even* projective spaces. This means that in order to understand the problem with h even one is going to be forced to study the problem of involutions fixing $\mathbb{RP}^{\text{even}}\sqcup P(h,i)$. Classifying involutions fixing $\mathbb{RP}^{\text{even}}\sqcup P(h,i)$ up to cobordism does not belong to the purpose of this paper. The problem will be discussed in the other paper.

Finally, one points out that there exist the examples for the case $m=j+q$. For $h=q-1$, there is an obvious way to get an involution fixing $\mathbb{RP}^{q-1}$ and $P(h,i)$, which is to begin with the involution on $P(q-1,i+1)$ induced by $T_1([z_0,\cdots,z_i,z_{i+1}]) = [z_0,\cdots,z_i,-z_{i+1}]$. This fixes $P(q-1,i)$ with normal bundle η and $P(q-1,0)=\mathbb{RP}^{q-1}$ with normal bundle $(i+1)\eta = (i+1)\iota+(i+1)\mathbb{R}$. In order that the normal bundle of $\mathbb{RP}^{q-1}$ having dimension $j+1$, one needs $2(i+1)=j+1$ or $i=\frac{j+1}{2}-1=\frac{j-1}{2}$. The normal bundle of $\mathbb{RP}^{q-1}$ has $w(\nu^{2(i+1)}) = (1+\alpha)^{i+1} = (1+\alpha)^{\frac{j+1}{2}}$ and one wants it to have $w(\nu^{j+1})=(1+\alpha)^{j+1}$. This occurs only for $(1+\alpha)^{\frac{j+1}{2}}=1$ which means $\frac{j+1}{2}=2^u(v+1)$ with $2^u>q-1$. Thus $j=2^{u+}(2v+1)-1$ and $2^u\geq q$, and $i=\frac{j+1}{2}-1=2^u(2v+1)-1$. Thus one has

Proposition 4.2. *For $j=2^{u+1}(2v+1)-1$ and $q\leq 2^u$, there is an involution (M^{j+q},T) fixing $\mathbb{RP}^j$ with $w(\nu^q)=(1+\alpha)^q$ and $P(q-1,\frac{j-1}{2})$ with normal bundle η where $w(\eta)=1+c+d$.*

Note. For $j=3=2^2-1$ this gives $q\leq 2$ so $q=1$ and $P(q-1,\frac{j-1}{2})=P(0,1)$ which was excluded since $h=0$. Thus, this involution doesn't *occur* for $j=3$. For $q>1$, this is a valid involution.

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