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Abstract

We investigate crossed products of Cuntz algebras by quasi-free actions of abelian groups. We prove that our algebras are AF-embeddable when actions satisfy a certain condition. We also give a necessary and sufficient condition that our algebras become simple and purely infinite, and consequently our algebras are either purely infinite or AF-embeddable when they are simple.

1 Introduction

There had been no examples of simple C^*-algebras which have both a finite projection and an infinite one until M. Rørdam found such a C^*-algebra recently [R]. However, we have found no examples of such simple C^*-algebras among nuclear ones, so far. Moreover we have not known examples of simple nuclear C^*-algebras which are not stably finite nor purely infinite. The property 'stable finiteness' has recently attracted much attention in connection with quasidiagonality and AF-embeddability. It is easy to see that AF-embeddability implies quasidiagonality and that quasidiagonality implies stable finiteness. It is still open whether or not stable finiteness implies AF-embeddability for nuclear C^*-algebras. On this topic, there is a nice survey [B3] written by N. P. Brown. Since M. Pimsner and D. Voiculescu showed that the irrational rotation algebras are AF-embeddable [PV], several authors have studied AF-embeddability of some classes of C^*-algebras. In particular, we can find many papers dealing with AF-embeddability of crossed products of finite C^*-algebras, for example, [Pu], [Pi1], [Pi2] for those of commutative C^*-algebras, and [V], [B1], [B2] for those of AF-algebras. On the other hand, the author has been unable to find any article related to AF-embeddability of crossed products of infinite C^*-algebras. We remark that it seems more difficult to show AF-embeddability of crossed products of infinite C^*-algebras by continuous groups than those of finite C^*-algebras. For crossed products of finite C^*-algebras, there is a method to derive AF-embeddability of crossed products by continuous groups from the discrete group case by using Green's imprimitivity theorem ([G], see also [B2]). However, for infinite C^*-algebras, we cannot use this method because their crossed products by discrete groups are never embedded into AF-algebras.

In this paper, we will deal with crossed products of Cuntz algebras by quasi-free actions of abelian groups, whose ideal structures were examined in our previous paper [Ka]. We will prove the AF-embeddability of our algebras under a certain condition for actions. To the author's knowledge, this is the first case to have succeeded in embedding crossed products of purely infinite C^*-algebras into AF-algebras except trivial cases. We will also show that our algebras are either purely infinite or AF-embeddable when they are simple.

This paper is organized as follows. After some preliminaries, we will show that the crossed products are AF-embeddable when actions satisfy a certain condition (Theorem 3.8). They were known to be stably finite in the case that the group is the real number group $\mathbb{R}$ [KK1]. In the case that the group is compact, this condition is also sufficient for the crossed products to be AF-embeddable, and moreover the crossed products become AF-algebras under this condition. For the general setting, we do not know whether our algebra is AF-embeddable or not when the action does not satisfy the condition (see Remark 3.10). In section 4, we will give a necessary and sufficient condition that our algebras become simple and purely infinite. Combining this characterization with our result on AF-embeddability, we can easily get the dichotomy which says that our algebras are either purely infinite or AF-embeddable when they are simple. In the last section, we will deal with crossed products of the Cuntz algebra $\mathcal{O}_\infty$, which is generated by infinitely many isometries, by the same type of actions of abelian groups. We will prove AF-embeddability of such algebras under a certain condition for actions, and give a necessary and sufficient condition for such algebras to be simple and purely infinite which will be shown to be equivalent to the property that they are simple.

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2 Preliminaries

In this section, we review some results and fix the notation. For $n = 2, 3, \dots,$ the Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by n isometries $S_1, S_2, \dots, S_n$, satisfying $\sum_{i=1}^n S_i S_i^* = 1$. For $k \in \mathbb{N} = \{0, 1, \dots\}$, we define the set $\mathcal{W}_n^{(k)}$ of k-tuples by $\mathcal{W}_n^{(0)} = \{\emptyset\}$ and

$$\mathcal{W}_n^{(k)} = \big\{(i_1, i_2, \dots, i_k) \bigm| i_j \in \{1, 2, \dots, n\}\big\}.$$

We set $\mathcal{W}_n = \bigcup_{k=0}^\infty \mathcal{W}_n^{(k)}$. For $\mu = (i_1, i_2, \dots, i_k) \in \mathcal{W}_n$, we denote its length k by $|\mu|$, and set $S_\mu = S_{i_1} S_{i_2} \cdots S_{i_k} \in \mathcal{O}_n$. Note that $|\emptyset| = 0$, $S_\emptyset = 1$. For $\mu = (i_1, i_2, \dots, i_k), \nu = (j_1, j_2, \dots, j_l) \in \mathcal{W}_n$, we define their product $\mu\nu \in \mathcal{W}_n$ by $\mu\nu = (i_1, i_2, \dots, i_k, j_1, j_2, \dots, j_l)$.

We fix a locally compact abelian group G whose dual group is denoted by Γ which is also a locally compact abelian group. We always use $+$ for multiplicative operations of abelian groups except for $\mathbb{T}$, which is the group of the unit circle in the complex plane $\mathbb{C}$. The pairing of $t \in G$ and $\gamma \in \Gamma$ is denoted by $\langle\, t \,|\, \gamma \,\rangle \in \mathbb{T}$.

Definition 2.1 Let $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \Gamma^n$ be given. We define the action $\alpha^\omega : G \curvearrowright \mathcal{O}_n$ by

$$\alpha_t^\omega(S_i) = \langle\, t \,|\, \omega_i \,\rangle S_i \quad (i = 1, 2, \dots, n,\ t \in G).$$

This type of action is called quasi-free (see [E] for quasi-free actions on the Cuntz algebras). Since the abelian group G is amenable, the reduced crossed product of the action $\alpha^\omega : G \curvearrowright \mathcal{O}_n$ coincides with the full crossed product of it. We will denote it by $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and call it the crossed product. The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ has a C^*-subalgebra $\mathbb{C}1 \rtimes_{\alpha^\omega} G$, which is isomorphic to $C_0(\Gamma)$. Throughout this paper, we always consider $C_0(\Gamma)$ as a C^*-subalgebra of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, and use $f, g, \dots$ for denoting elements of $C_0(\Gamma) \subset \mathcal{O}_n \rtimes_{\alpha^\omega} G$. The Cuntz algebra $\mathcal{O}_n$ is naturally embedded into the multiplier algebra $M(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. For each $\mu = (i_1, i_2, \dots, i_k)$ in $\mathcal{W}_n$, we define an element ω_μ of Γ by $\omega_\mu = \sum_{j=1}^k \omega_{i_j}$. For $\gamma_0 \in \Gamma$, we define a (reverse) shift automorphism $\sigma_{\gamma_0} : C_0(\Gamma) \to C_0(\Gamma)$ by $(\sigma_{\gamma_0} f)(\gamma) = f(\gamma + \gamma_0)$ for $f \in C_0(\Gamma)$. Once noting that $\alpha_t^\omega(S_\mu) = \langle\, t \,|\, \omega_\mu \,\rangle S_\mu$ for $\mu \in \mathcal{W}_n$, one can easily verify that $f S_\mu = S_\mu \sigma_{\omega_\mu} f$ for any $f \in C_0(\Gamma) \subset \mathcal{O}_n \rtimes_{\alpha^\omega} G$ and any $\mu \in \mathcal{W}_n$. The linear span of $\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n,\ f \in C_0(\Gamma)\}$ is dense in $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ (see [Ka]). We denote by $\mathbb{M}_k$ the C^*-algebra of $k \times k$ matrices for $k = 1, 2, \dots,$ and by $\mathbb{K}$ the C^*-algebra of compact operators of the infinite dimensional separable Hilbert space.

3 AF-embeddability of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$

A. Kishimoto and A. Kumjian showed that $\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}$ is stably projectionless if all the ω_i's have the same sign by using the KMS-state [KK1, Theorem 4.1]. Thus $\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}$ is stably finite in this case. In this section, we will show that $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ becomes AF-embeddable if ω satisfies a certain condition. This gives another proof of the stable finiteness of $\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}$ when all the ω_i's have the same sign. More precisely, we will prove that if $-\omega_i \notin \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$ for any $i \in \{1, 2, \ldots, n\}$, then $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is AF-embeddable (Theorem 3.8). Here we note that $\overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$ is the closed semigroup generated by $\omega_1, \omega_2, \ldots, \omega_n$.

Let us take a faithful representation $\mathcal{O}_n \hookrightarrow B(H)$ for some Hilbert space H. There exists a canonical embedding $\mathcal{O}_n \rtimes_{\alpha^\omega} G \hookrightarrow B(H \otimes L^2(G))$. Since $L^2(G)$ is isomorphic to $L^2(\Gamma)$ via the Fourier transform, we can consider $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ as a subalgebra of $B(H \otimes L^2(\Gamma))$. In this setting, an element of $C_0(\Gamma) \subset \mathcal{O}_n \rtimes_{\alpha^\omega} G$ acts by multiplication on $L^2(\Gamma)$ and as identity on H. Note that the weak closure of $C_0(\Gamma)$ in $B(H \otimes L^2(\Gamma))$ is $L^\infty(\Gamma)$.

Throughout this section, we fix $\omega \in \Gamma^n$ satisfying $-\omega_i \notin \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$ for any i. We also fix an open base $\{U_i\}_{i \in \mathbb{I}}$ of Γ such that for any $i \in \mathbb{I}$, $\overline{U_i}$ is compact and for any $i \in \mathbb{I}$ and $\mu \in \mathcal{W}_n$, there exists $j \in \mathbb{I}$ with $U_j = U_i - \omega_\mu$. Obviously such an open base exists, and we can take countable one when Γ satisfies the second countability axiom. For each $i \in \mathbb{I}$, let us consider the characteristic function χ_{U_i} of U_i which is an element of $L^\infty(\Gamma) \subset B(H \otimes L^2(\Gamma))$. Let $D_0(\Gamma)$ be the C^*-algebra generated by $\{\chi_{U_i}\}_{i \in \mathbb{I}}$. Let us denote by Λ the directed set of all finite subsets of $\mathbb{I}$ whose order is defined by the inclusion. For $\lambda = \{i_1, i_2, \ldots, i_k\} \in \Lambda$, the C^*-subalgebra D_λ of $D_0(\Gamma)$ is defined by the C^*-algebra generated by $\chi_{U_{i_1}}, \chi_{U_{i_2}}, \ldots, \chi_{U_{i_k}}$. One can easily verify the following.

Lemma 3.1 (i) *$C_0(\Gamma) \subset D_0(\Gamma)$.*

(ii) *We can define the shift $*$-homomorphism $\sigma_{\omega_\mu} : D_0(\Gamma) \to D_0(\Gamma)$ for any $\mu \in \mathcal{W}_n$.*

(iii) *$\varinjlim D_\lambda = D_0(\Gamma)$.*

(iv) *$D_0(\Gamma)$ is an AF-algebra.*

Define a subspace A of $B(H \otimes L^2(\Gamma))$ by

$$A = \overline{\mathrm{span}}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n,\ f \in D_0(\Gamma)\}.$$

By Lemma 3.1 (ii), A is a C^*-algebra and by Lemma 3.1 (i), A contains $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. We will show that A is an AF-algebra when $-\omega_i \notin \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$ for any $i \in \{1, 2, \ldots, n\}$, which implies that $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is AF-embeddable. We denote by A_λ the C^*-algebra generated by $\{S_\mu \chi_{U_i} S_\nu^* \mid \mu, \nu \in \mathcal{W}_n,\ i \in \lambda\}$. It is easy to see the following.

Lemma 3.2 *With the above notation, we have $A = \varinjlim A_\lambda$.*

By Lemma 3.2, to prove that A is an AF-algebra, it suffices to show that A_λ is an AF-algebra for any $\lambda \in \Lambda$. Let us take $\lambda \in \Lambda$ arbitrarily, and fix it. Let $p_1, p_2, \ldots, p_L$ be minimal projections of D_λ and $p = \sum_{l=1}^{L} p_l$ be its unit. Note that A_λ is generated by $\{S_\mu p_l S_\nu^* \mid \mu, \nu \in \mathcal{W}_n,\ l = 1, 2, \ldots, L\}$. Only in the next lemma, we use directly the assumption that ω satisfies $-\omega_i \notin \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$ for any $i \in \{1, 2, \ldots, n\}$, and this lemma implies all the following lemmas and the fact that A_λ is an AF-algebra.

Lemma 3.3 *There exists $K \in \mathbb{N}$ such that $pS_\mu p = 0$ for any $\mu \in \mathcal{W}_n$ with $|\mu| > K$.*

Proof. If we define a subset $U = \bigcup_{i \in \lambda} U_i$ of Γ, then p is the characteristic function of U. The closure of U is compact since $\overline{U_i}$ is compact for any $i \in \lambda$. To derive a contradiction, assume that for any $k \in \mathbb{N}$, there exists $\mu_k \in \mathcal{W}_n$ such that $|\mu_k| > k$ and $pS_{\mu_k} p \neq 0$. Then we have $S_{\mu_k}^* p S_{\mu_k} p \neq 0$. Since $S_{\mu_k}^* p S_{\mu_k}$ is the characteristic function of $U - \omega_{\mu_k}$, there exists $\gamma_k \in (U - \omega_{\mu_k}) \cap U$. We have $\omega_{\mu_k} = (\gamma_k + \omega_{\mu_k}) - \gamma_k \in U - U$ for any $k \in \mathbb{N}$. Since $\overline{U - U}$ is compact, there exists an increasing subsequence $k_1, k_2, \ldots, k_m, \ldots$ of $\mathbb{N}$ such that $\omega_{\mu_{k_m}}$ converges to some element $\gamma_0 \in \overline{U - U}$ when m goes to infinity. By replacing it by a subsequence of $\{k_m\}$ if necessary, we may assume that the number of i appearing in $\mu_{k_m} \in \mathcal{W}_n$ does not decrease for $i = 1, 2, \ldots, n$. Since $|\mu_{k_m}| \to \infty$ when $m \to \infty$, there exists $i_0 \in \{1, 2, \ldots, n\}$ such that the number of i_0 appearing in μ_{k_m} diverges to infinity when $m \to \infty$. By replacing it by a subsequence of

$\{k_m\}$ if necessary, we may assume that the number of i_0 appearing in μ_{k_m} increases strictly. Thus, we have $\omega_{\mu_{k_m}} - \omega_{\mu_{k_{m-1}}} - \omega_{i_0} \in \{\omega_\mu \mid \mu \in \mathcal{W}_n\}$ for any $m \in \mathbb{N}$. By

$$\lim_{m\to\infty} (\omega_{\mu_{k_m}} - \omega_{\mu_{k_{m-1}}} - \omega_{i_0}) = \gamma_0 - \gamma_0 - \omega_{i_0} = -\omega_{i_0},$$

we have $-\omega_{i_0} \in \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$. This is a contradiction. ∎

We fix a positive integer K satisfying the condition in Lemma 3.3. Before going further, we remark that $S_\mu f S_\mu^*$ and $S_\nu g S_\nu^*$ commute for any $\mu, \nu \in \mathcal{W}_n$ and any $f, g \in D_0(\Gamma)$. This fact will be used without further notice. For $k \in \mathbb{N}$, define a $*$-endomorphism ρ_k of A_λ by $\rho_k(x) = \sum_{\mu\in\mathcal{W}_n^{(k)}} S_\mu x S_\mu^*$. Set a projection q in A_λ by

$$q = \left(\prod_{k=1}^{K} \big(1 - \rho_k(p)\big)\right) p.$$

Note that $q \le p$ and $q \le 1 - \rho_k(p)$ for $k = 1, 2, \dots, K$.

Lemma 3.4 *For any $\mu, \nu \in \mathcal{W}_n$ with $\mu \ne \nu$, two projections $S_\mu q S_\mu^*$ and $S_\nu q S_\nu^*$ are orthogonal to each other.*

Proof. It suffices to show that $qS_\mu q = 0$ for any $\mu \in \mathcal{W}_n$ with $\mu \ne \emptyset$. When $1 \le |\mu| \le K$, since

$$\big(1 - \rho_{|\mu|}(p)\big) S_\mu p = (S_\mu - S_\mu p)p = 0,$$

we have $qS_\mu q = 0$. When $|\mu| > K$, $pS_\mu p = 0$ by Lemma 3.3, so $qS_\mu q = 0$. ∎

Denote a set $\{\mu \in \mathcal{W}_n \mid |\mu| \le K\}$ by $\mathcal{W}$.

Lemma 3.5 *We have $\sum_{\mu\in\mathcal{W}} S_\mu q S_\mu^* p = p$.*

Proof. For $l = 1, 2, \dots, K$, we have

$$\rho_l(q) = \left(\prod_{k=1}^{K} \big(1 - \rho_{l+k}(p)\big)\right) \rho_l(p).$$

Since $(1 - \rho_k(p))\, p = p$ for $k > K$ by Lemma 3.3, we have

$$\rho_l(q)p = \left(\prod_{k=l+1}^{K} \big(1 - \rho_k(p)\big)\right) \rho_l(p)p.$$

Hence

$$\sum_{\mu\in\mathcal{W}} S_\mu q S_\mu^* p = \sum_{l=0}^{K} \sum_{\mu\in\mathcal{W}_n^{(l)}} \rho_l(q)p = \sum_{l=0}^{K} \left(\prod_{k=l+1}^{K} \big(1 - \rho_k(p)\big)\right) \rho_l(p)p = p.$$

∎

Let us define a projection p_0 by $p_0 = 1 - p = 1 - \sum_{l=1}^{L} p_l$ where $p_1, p_2, \dots, p_L$ are the minimal projections of D_λ. Note that $p_0, p_1, \dots, p_L$ is a set of mutually orthogonal projections whose sum is 1. Let $\mathbb{J}'$ be a set of all maps from the set $\mathcal{W} = \{\mu \in \mathcal{W}_n \mid |\mu| \le K\}$ to the set $\{0, 1, 2, \dots, L\}$. For $\tau \in \mathbb{J}'$, we define a projection $q_\tau \in A_\lambda$ by

$$q_\tau = q \prod_{\mu\in\mathcal{W}} S_\mu^* p_{\tau(\mu)} S_\mu.$$

Set $\mathbb{J} = \{\tau \in \mathbb{J}' \mid q_\tau \ne 0\}$.

Lemma 3.6 (i) *$\{q_\tau\}_{\tau\in\mathbb{J}}$ is a set of mutually orthogonal non-zero projections.*

(ii) $\sum_{\tau\in\mathbb{J}} q_\tau = q$.

(iii) *For* $\mu \in \mathcal{W}$, $\tau \in \mathbb{J}$ *and* $l \in \{1, 2, \ldots, L\}$, *we have* $S_\mu q_\tau S_\mu^* p_l = \delta_{\tau(\mu),l} S_\mu q_\tau S_\mu^*$.

Proof.

(i) If $\tau_1 \neq \tau_2$, then $\tau_1(\mu) \neq \tau_2(\mu)$ for some $\mu \in \mathcal{W}$. Now $q_{\tau_1} q_{\tau_2} = 0$ follows from

$$(S_\mu^* p_{\tau_1(\mu)} S_\mu)(S_\mu^* p_{\tau_2(\mu)} S_\mu) = S_\mu^* S_\mu S_\mu^* p_{\tau_1(\mu)} p_{\tau_2(\mu)} S_\mu = 0,$$

since $q_{\tau_1} \leq S_\mu^* p_{\tau_1(\mu)} S_\mu$ and $q_{\tau_2} \leq S_\mu^* p_{\tau_2(\mu)} S_\mu$.

(ii) $\sum_{\tau\in\mathbb{J}} q_\tau = \sum_{\tau\in\mathbb{J}'} q_\tau = q \prod_{\mu\in\mathcal{W}} S_\mu^*(p_0 + p_1 + \cdots + p_L) S_\mu = q$.

(iii) It follows from the fact that

$$S_\mu(S_\mu^* p_{\tau(\mu)} S_\mu) S_\mu^* p_l = S_\mu S_\mu^* p_{\tau(\mu)} p_l S_\mu S_\mu^* = \delta_{\tau(\mu),l} S_\mu (S_\mu^* p_{\tau(\mu)} S_\mu) S_\mu^*.$$

∎

Proposition 3.7 *We have* $A_\lambda \cong \bigoplus_{\tau\in\mathbb{J}} \mathbb{K}$. *Hence,* A_λ *is an AF-algebra.*

Proof. For any $\tau_1, \tau_2 \in \mathbb{J}$ and $\mu, \nu \in \mathcal{W}_n$ with $\mu \neq \nu$, we have $(S_\mu q_{\tau_1} S_\mu^*)(S_\nu q_{\tau_2} S_\nu^*) = 0$ by Lemma 3.4. Thus, for $\tau_1, \tau_2 \in \mathbb{J}$ and $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{W}_n$, we get

$$\begin{aligned}(S_{\mu_1} q_{\tau_1} S_{\nu_1}^*)(S_{\mu_2} q_{\tau_2} S_{\nu_2}^*) &= \delta_{\nu_1,\mu_2} S_{\mu_1} q_{\tau_1} q_{\tau_2} S_{\nu_2}^* \\ &= \delta_{\nu_1,\mu_2} \delta_{\tau_1,\tau_2} S_{\mu_1} q_{\tau_1} S_{\nu_2}^*.\end{aligned}$$

For any $\tau \in \mathbb{J}$, the set $\{S_\mu q_\tau S_\nu^*\}_{\mu,\nu\in\mathcal{W}_n}$ satisfies the relation of matrix units, so the C^*-algebra generated by $\{S_\mu q_\tau S_\nu^*\}_{\mu,\nu\in\mathcal{W}_n}$ is isomorphic to $\mathbb{K}$. For any two elements τ_1, τ_2 in $\mathbb{J}$, the C^*-algebra generated by $\{S_\mu q_{\tau_1} S_\nu^*\}_{\mu,\nu\in\mathcal{W}_n}$ is orthogonal to the C^*-algebra generated by $\{S_\mu q_{\tau_2} S_\nu^*\}_{\mu,\nu\in\mathcal{W}_n}$. Therefore, the C^*-algebra generated by $\{S_\mu q_\tau S_\nu^* \mid \mu, \nu \in \mathcal{W}_n,\ \tau \in \mathbb{J}\}$ is isomorphic to $\bigoplus_{\tau\in\mathbb{J}} \mathbb{K}$.

Since $q_\tau \in A_\lambda$ for any $\tau \in \mathbb{J}$, the C^*-algebra generated by $\{S_\mu q_\tau S_\nu^* \mid \mu, \nu \in \mathcal{W}_n,\ \tau \in \mathbb{J}\}$ is contained in A_λ. Conversely, for $l = 1, 2, \ldots, L$,

$$\begin{aligned}p_l &= p p_l \\ &= \sum_{\mu\in\mathcal{W}} S_\mu q S_\mu^* p p_l && \text{(by Lemma 3.5)} \\ &= \sum_{\mu\in\mathcal{W}} S_\mu \left(\sum_{\tau\in\mathbb{J}} q_\tau\right) S_\mu^* p_l && \text{(by Lemma 3.6 (ii))} \\ &= \sum_{\mu\in\mathcal{W},\tau\in\mathbb{J}} S_\mu q_\tau S_\mu^* p_l \\ &= \sum_{\substack{\mu\in\mathcal{W},\tau\in\mathbb{J}, \\ \text{s.t. } \tau(\mu)=l}} S_\mu q_\tau S_\mu^* && \text{(by Lemma 3.6 (iii)).}\end{aligned}$$

Thus, for any $\mu, \nu \in \mathcal{W}_n$ and $l = 1, 2, \ldots, L$, the element $S_\mu p_l S_\nu^*$ is contained in the C^*-algebra generated by $\{S_\mu q_\tau S_\nu^* \mid \mu, \nu \in \mathcal{W}_n,\ \tau \in \mathbb{J}\}$. Therefore A_γ coincides with the C^*-algebra generated by $\{S_\mu q_\tau S_\nu^* \mid \mu, \nu \in \mathcal{W}_n,\ \tau \in \mathbb{J}\}$ which was proved to be isomorphic to $\bigoplus_{\tau\in\mathbb{J}} \mathbb{K}$. ∎

Now we can prove the main theorem.

Theorem 3.8 *If* ω *satisfies* $-\omega_i \notin \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$ *for any* $i \in \{1, 2, \ldots, n\}$, *then the crossed product* $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ *is AF-embeddable.*

Proof. The C^*-algebra A is an AF-algebra because it is an inductive limit of AF-algebras. Since the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is naturally embedded into A, it is AF-embeddable. ∎

Proposition 3.9 *When G is compact, the following are equivalent:*

(i) $-\omega_i \notin \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$ *for any* $i \in \{1, 2, \ldots, n\}$.

(ii) *The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is stably finite.*

(iii) *The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is AF-embeddable.*

(iv) *The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ itself is an AF-algebra.*

Proof. (i)⇒(iv): Note that Γ is discrete when G is compact. We can take $\{\{\gamma\}\}_{\gamma \in \Gamma}$ for an open base $\{U_i\}_{i \in \mathbb{I}}$. Then the C^*-algebra A which was proved to be an AF-algebra is $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ itself. Thus, $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is an AF-algebra.

(iv)⇒(iii)⇒(ii): Obvious.

(ii)⇒(i): If there exists $i \in \{1, 2, \ldots, n\}$ such that $-\omega_i \in \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$, then there exists $\mu' \in \mathcal{W}_n$ with $-\omega_i = \omega_{\mu'}$. Hence $\mu = i\mu' \in \mathcal{W}_n$ satisfies $|\mu| \geq 1$ and $\omega_\mu = 0$. Set $u = S_\mu \chi \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$ where $\chi \in C_0(\Gamma)$ is the characteristic function of $\{0\}$. We have $u^*u = \chi$ and $uu^* = S_\mu \chi S_\mu^*$. We get $\chi \neq S_\mu \chi S_\mu^*$ from $|\mu| \geq 1$, and $\chi(S_\mu \chi S_\mu^*) = S_\mu \chi S_\mu^*$ from $\omega_\mu = 0$. Therefore χ is an infinite projection. Thus $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is not stably finite. ∎

Remark 3.10 When $G = \mathbb{R}$, Theorem 3.8 implies that $\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}$ is AF-embeddable if all the ω_i's have the same sign. If there exist i, j such that $\omega_i < 0 < \omega_j$, then $\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}$ has infinite projections hence it is not AF-embeddable. We do not know whether $\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}$ is AF-embeddable or not if there exists $i \in \{1, 2, \ldots, n\}$ such that $\omega_i = 0$ and all the other ω_i's have the same sign, though it is not hard to see that it is stably finite.

4 Pure infiniteness of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$

In this section, we investigate for which $\omega \in \Gamma^n$ the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ becomes simple and purely infinite. Recall that a simple C^*-algebra is called purely infinite if any non-zero hereditary subalgebra has an infinite projection. An element x of a C^*-algebra is called a scaling element if $(x^*x)(xx^*) = xx^*$ and $x^*x \neq xx^*$. In [BC], B. E. Blackadar and J. Cuntz showed that if a simple stable C^*-algebra has a scaling element, then it has an infinite projection. One can omit the assumption of stability (Proposition 4.2). To do so, we need the following standard lemma.

Lemma 4.1 *Let A be a C^*-algebra, p a projection of A, and a a positive element of A. If there exist $x_1, x_2 \ldots, x_K$ and $y_1, y_2, \ldots, y_K$ in A with*

$$\left\| p - \sum_{k=1}^{K} x_k a y_k \right\| < \frac{1}{2},$$

then there exist $z_1, z_2 \ldots, z_{2K}$ in A such that

$$p = \sum_{k=1}^{2K} z_k^* a z_k.$$

In particular, if A is simple C^*-algebra, p is a projection of A, and a is a non-zero positive element of A, then there exist $x_1, x_2 \ldots, x_K$ in A such that $p = \sum_{k=1}^{K} x_k^* a x_k$.

Proof. See [D, Lemma V.5.4], for example. ∎

Proposition 4.2 *If a C^*-algebra A is simple and has a scaling element, then it has an infinite projection.*

Proof. If A has a scaling element, then A has mutually orthogonal, mutually equivalent, non-zero projections $\{p_k\}_{k=1}^{\infty}$ and a positive element a with $ap_k = p_k$ for any k [BC, Theorem 3.1]. Since A is simple, there exist $x_1, x_2 \ldots, x_K$ and $y_1, y_2, \ldots, y_K$ in A with

$$\left\| a - \sum_{k=1}^{K} x_k p_1 y_k \right\| < \frac{1}{2}.$$

Let us set $p = \sum_{k=1}^{2K+1} p_k$, which is a projection. Then we have

$$\left\| p - \sum_{k=1}^{K} x_k p_1 (y_k p) \right\| = \left\| (a - \sum_{k=1}^{K} x_k p_1 y_k) p \right\| < \frac{1}{2},$$

since $ap = p$. Hence there exist $z_1, z_2, \ldots, z_{2K}$ in A such that $p = \sum_{k=1}^{2K} z_k^* p_1 z_k$ by Lemma 4.1. For $k = 1, 2, \ldots, 2K$, let u_k be a partial isometry with $u_k^* u_k = p_1, u_k u_k^* = p_k$. Set $z = \sum_{k=1}^{2K} u_k z_k$. Then we have $z^* z = \sum_{k=1}^{2K} z_k^* p_1 z_k = p$. Since $zz^*(\sum_{k=1}^{2K} p_k) = zz^*$, we have $zz^* \leq \sum_{k=1}^{2K} p_k < p$. Therefore p is an infinite projection. ∎

A. Kishimoto and A. Kumjian proved that $\mathcal{O}_n \rtimes_{\alpha^\omega} \mathbb{R}$ is simple and purely infinite if and only if the closed semigroup generated by $\omega_1, \omega_2, \ldots, \omega_n$ is $\mathbb{R}$ in [KK2]. We will generalize their result for our setting by using the same technique as in [KK2]. Namely, we will prove that $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is simple and purely infinite if and only if the closed semigroup generated by $\omega_1, \omega_2, \ldots, \omega_n$ is Γ. When ω satisfies $\Gamma = \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$, the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is simple by [Ki, Theorem 4.4] (see also [Ka, Theorem 4.8]). First we will show that $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ has a scaling element and hence an infinite projection.

Lemma 4.3 *Suppose that ω satisfies $\Gamma = \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$. For any neighborhood U of $0 \in \Gamma$ and any positive integer K, there exist K elements $\mu_1, \mu_2, \cdots, \mu_K$ of $\mathcal{W}_n$ such that $\omega_{\mu_k} \in U$ for $k = 1, 2, \ldots, K$ and $S_{\mu_k}^* S_{\mu_l} = \delta_{k,l}$.*

Proof. We can find K elements $\nu_1, \nu_2, \ldots, \nu_K$ of $\mathcal{W}_n$ such that $S_{\nu_k}^* S_{\nu_l} = \delta_{k,l}$. For $k = 1, 2, \ldots, K$, there exists $\nu_k' \in \mathcal{W}_n$ with $\omega_{\nu_k'} \in U - \omega_{\nu_k}$ because $U - \omega_{\nu_k}$ is open and $\{\omega_\mu \mid \mu \in \mathcal{W}_n\}$ is dense in Γ. Set $\mu_k = \nu_k \nu_k'$ for $k = 1, 2, \ldots, K$. Then $S_{\mu_k}^* S_{\mu_l} = \delta_{k,l}$ and $\omega_{\mu_k} = \omega_{\nu_k} + \omega_{\nu_k'} \in U$ for $k = 1, 2, \ldots, K$. ∎

Lemma 4.4 *Suppose that ω satisfies $\Gamma = \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$. Let X be a compact neighborhood of $0 \in \Gamma$ that differs from Γ. Then, there exist positive functions $f_1, f_2, \ldots, f_K \in C_0(\Gamma)$ and $\mu_1, \mu_2, \ldots, \mu_K \in \mathcal{W}_n$ satisfying the following conditions:*

(i) $S_{\mu_k}^* S_{\mu_l} = \delta_{k,l}$.

(ii) $\sum_{k=1}^{K} f_k(\gamma) = 1$ *for any* $\gamma \in X$.

(iii) $\sum_{k=1}^{K} f_k(\gamma_0) \neq 0, 1$ *for some* $\gamma_0 \in \Gamma$.

(iv) *The support of* $\sigma_{-\omega_{\mu_k}} f_k$ *is contained in* X *for* $k = 1, 2, \ldots, K$.

Proof. Let us choose an open neighborhood U_1 of 0 such that the open neighborhood $U = U_1 + U_1$ of 0 is contained in X, and then choose an open neighborhood U_2 of 0 such that $\overline{U_2} \subset U_1$. For any $\gamma \in \Gamma$, there exists $\mu \in \mathcal{W}_n$ with $\omega_\mu \in U_2 + \gamma$ because $\{\omega_\mu \mid \mu \in \mathcal{W}_n\}$ is dense in Γ. Therefore $\bigcup_{\mu \in \mathcal{W}_n} (U_2 - \omega_\mu) = \Gamma$. Since X is compact, there exist finite elements $\nu_1, \nu_2, \ldots, \nu_K$ of $\mathcal{W}_n$ such that

$$X \subsetneqq \bigcup_{k=1}^{K} (U_2 - \omega_{\nu_k}).$$

By Lemma 4.3, there exist K elements $\nu'_1, \nu'_2, \ldots, \nu'_K \in \mathcal{W}_n$ such that $S^*_{\nu'_k} S_{\nu'_l} = \delta_{k,l}$ and $\omega_{\nu'_k} \in U_1$ for $k = 1, 2, \ldots, K$. Set $\mu_k = \nu'_k \nu_k$ for $k = 1, 2, \ldots, K$. Then $S^*_{\mu_k} S_{\mu_l} = \delta_{k,l}$. For $k = 1, 2, \ldots, K$, we get

$$\begin{aligned}\overline{U_2 - \omega_{\nu_k}} &\subset U_1 - \omega_{\nu_k} \\ &\subset U_1 + U_1 - \omega_{\nu_k} - \omega_{\nu'_k} \\ &= U - \omega_{\mu_k},\end{aligned}$$

since $\overline{U_2} \subset U_1$ and $\omega_{\nu'_k} \in U_1$. For $k = 1, 2, \ldots, K$, let $g_k \in C_0(\Gamma)$ be a function with $0 \le g_k \le 1$ such that $g_k(\gamma) = 1$ for $\gamma \in \overline{U_2 - \omega_{\nu_k}}$ and $g_k(\gamma) = 0$ for $\gamma \notin U - \omega_{\mu_k}$. Let us choose a continuous positive function F on Γ satisfying $F(\gamma) = 0$ for $\gamma \in X$ and $F(\gamma) = 1$ for $\gamma \notin \bigcup_{k=1}^K (U_2 - \omega_{\nu_k})$. Then the continuous function $G = F + \sum_{k=1}^K g_k$ on Γ satisfies $G(\gamma) \ge 1$ for any $\gamma \in \Gamma$ since $F, g_1, g_2, \ldots, g_K$ are positive functions, and $F(\gamma) = 1$ for $\gamma \notin \bigcup_{k=1}^K (U_2 - \omega_{\nu_k})$, and $g_k(\gamma) = 1$ for $\gamma \in U_2 - \omega_{\nu_k}$. Set $f_k = g_k / G$ for $k = 1, 2, \ldots, K$. Then for $k = 1, 2, \ldots, K$, the positive function $f_k \in C_0(\Gamma)$ satisfies $f_k(\gamma) = 0$ for any $\gamma \notin U - \omega_{\mu_k}$. For $\gamma \in X$, we have

$$\begin{aligned}\sum_{k=1}^K f_k(\gamma) &= \sum_{k=1}^K \frac{g_k(\gamma)}{G(\gamma)} \\ &= \frac{\sum_{k=1}^K g_k(\gamma)}{F(\gamma) + \sum_{k=1}^K g_k(\gamma)} \\ &= 1.\end{aligned}$$

Since $X \subsetneqq \bigcup_{k=1}^K (U_2 - \omega_{\nu_k})$, there exists $\gamma_0 \notin X$ that is an element of $U_2 - \omega_{\nu_{k_0}}$ for some $k_0 \in \{1, 2, \ldots, K\}$. Since $U_2 - \omega_{\nu_{k_0}}$ is open and X is closed, we can choose an open set O such that $\gamma_0 \in O \subset U_2 - \omega_{\nu_{k_0}}$ and $O \cap X = \emptyset$. Let us take a positive function f such that $f(\gamma) = 0$ for any $\gamma \notin O$ and $f(\gamma_0) + \sum_{k=1}^K f_k(\gamma_0)$ is neither 0 nor 1. Then $f'_{k_0} = f_{k_0} + f$ still satisfies that $f'_{k_0}(\gamma) = 0$ for any $\gamma \notin U - \omega_{\mu_k}$. We denote this new function f'_{k_0} by f_{k_0}. Then K functions $f_1, f_2, \ldots, f_K$ satisfy $\sum_{k=1}^K f_k(\gamma) = 1$ for $\gamma \in X$ and $\sum_{k=1}^K f_k(\gamma_0) \ne 0, 1$. For $k = 1, 2, \ldots, K$, since $\sigma_{-\omega_{\mu_k}} f_k(\gamma) = 0$ for any $\gamma \notin U \subset X$, the support of $\sigma_{-\omega_{\mu_k}} f_k$ is contained in X. We get desired elements $f_1, f_2, \ldots, f_K \in C_0(\Gamma)$ and $\mu_1, \mu_2, \ldots, \mu_K \in \mathcal{W}_n$. ∎

Proposition 4.5 *If ω satisfies that $\Gamma = \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$, then $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ has a scaling element.*

Proof. Let X be a compact neighborhood of $0 \in \Gamma$ that differs from Γ. Let us take positive functions $f_1, f_2, \ldots, f_K \in C_0(\Gamma)$ and $\mu_1, \mu_2, \ldots, \mu_K \in \mathcal{W}_n$ that satisfy the four conditions in Lemma 4.4. Let us define $x = \sum_{k=1}^K S_{\mu_k} f_k^{1/2} \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$. Since $S^*_{\mu_k} S_{\mu_l} = \delta_{k,l}$,

$$x^* x = \sum_{k,l=1}^K f_k^{1/2} S^*_{\mu_k} S_{\mu_l} f_l^{1/2} = \sum_{k=1}^K f_k.$$

On the other hand,

$$x x^* = \sum_{k,l=1}^K \left(S_{\mu_k} f_k^{1/2} f_l^{1/2} S^*_{\mu_l} \right) = \sum_{k,l=1}^K \left((\sigma_{-\omega_{\mu_k}} f_k^{1/2})(\sigma_{-\omega_{\mu_k}} f_l^{1/2}) S_{\mu_k} S^*_{\mu_l} \right).$$

Since the support of $\sigma_{-\omega_{\mu_k}} f_k^{1/2}$ is contained in X for any $k = 1, 2, \ldots, K$ and $\sum_{k=1}^K f_k(\gamma) = 1$ for $\gamma \in X$, we have $(x^* x)(x x^*) = x x^*$.

Finally we show $x^* x \ne x x^*$. If $x^* x = x x^*$, then $x^* x$ would become a projection. However, $x^* x = \sum_{k=1}^K f_k$ is not a projection, since there exists $\gamma_0 \in \Gamma$ with $\sum_{k=1}^K f_k(\gamma_0) \ne 0, 1$. Thus x is a scaling element. ∎

Since $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is simple, it has an infinite projection by Proposition 4.2 and Proposition 4.5. To prove that every non-zero hereditary subalgebra of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ has an infinite projection, we need the following lemma. In the proof of it, we use some computations done in [Ka] which is not difficult to see. Let $\beta : \mathbb{T} \curvearrowright \mathcal{O}_n \rtimes_{\alpha^\omega} G$ be the gauge action defined by $\beta_t(S_\mu f S_\nu^*) = t^{|\mu|-|\nu|} S_\mu f S_\nu^*$, and E be the faithful conditional expectation of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ defined by $E(x) = \int_{\mathbb{T}} \beta_t(x) dt$ where dt is the normalized Haar measure of $\mathbb{T}$.

Lemma 4.6 *Let y be a non-zero positive element of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, given as $y = \sum_{l=1}^{L} S_{\mu_l} f_l S_{\nu_l}^*$. Let C be a positive number with $1/\|E(y)\| < C^2$. Then, there exist $a \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$ with $\|a\| \le C$ and an open set O of Γ such that $a^* y a$ becomes an element of $C_0(\Gamma)$ which is 1 on O.*

Proof. Set $k = \max\{|\mu_l|, |\nu_l| \mid l = 1, 2, \dots, L\}$ and

$$\mathcal{F}_k = \operatorname{span}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_n^{(k)},\ f \in C_0(\Gamma)\}.$$

The C^*-algebra $\mathcal{F}_k$ is isomorphic to $C_0(\Gamma, \mathbb{M}_{n^k})$ and we will identify them. We can see that $E(y) = \sum_{|\mu_l|=|\nu_l|} S_{\mu_l} f_l S_{\nu_l}^*$ and $E(y) \in \mathcal{F}_k$. Set $u = \sum_{\mu \in \mathcal{W}_n^{(k)}} S_\mu S_1^k S_2 S_\mu^* \in \mathcal{O}_n \subset M(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$. Routine computation shows that u is an isometry and $u^* y u = \sigma_\gamma(E(y))$ where $\gamma = k\omega_1 + \omega_2$. Hence $u^* y u$ is a positive element of $\mathcal{F}_k$ whose norm is equal to $\|E(y)\|$. One can find $\gamma_0 \in \Gamma$ such that the norm of $(u^* y u)(\gamma_0) \in \mathbb{M}_{n^k}$ is $\|E(y)\|$. The C^*-subalgebra $\operatorname{span}\{S_\mu S_\nu^* \mid \mu, \nu \in \mathcal{W}_n^{(k)}\}$ of $\mathcal{O}_n \in M(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ is isomorphic to $\mathbb{M}_{n^k}$ and can be considered as the set of constant functions of $C_b(\Gamma, \mathbb{M}_{n^k}) \cong M(\mathcal{F}_k)$. Take an element μ in $\mathcal{W}_n^{(k)}$ arbitrarily. Then $S_\mu S_\mu^* \in M(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ is a minimal projection of $\mathbb{M}_{n^k}$. Since $u^* y u$ is positive, $(u^* y u)(\gamma_0)$ is a positive element of $\mathbb{M}_{n^k}$. Hence, there exists a partial isometry $v \in \operatorname{span}\{S_\mu S_\nu^* \mid \mu, \nu \in \mathcal{W}_n^{(k)}\}$ such that $v^* v = S_\mu S_\mu^*$ and

$$(v^* u^* y u v)(\gamma_0) = \|E(y)\| S_\mu S_\mu^*.$$

There exists a function $f \in C_0(\Gamma)$ with $v^* u^* y u v = S_\mu f S_\mu^*$, because the projection $S_\mu S_\mu^*$ is minimal. Since $f(\gamma_0) = \|E(y)\|$, there exists a positive function $g \in C_0(\Gamma)$ with $\|g\| \le C$ such that $f g^2 \in C_0(\Gamma)$ is 1 on some open neighborhood O of γ_0. If we set $a = u v S_\mu g \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$, then, we get $\|a\| \le C$ and $a^* y a = g f g$ becomes an element of $C_0(\Gamma)$ which is 1 on O. ∎

Theorem 4.7 *If ω satisfies that $\Gamma = \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$, then $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is simple and purely infinite.*

Proof. To prove that $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is purely infinite, it suffices to show that there exists an infinite projection in the hereditary subalgebra $\overline{x(\mathcal{O}_n \rtimes_{\alpha^\omega} G)x}$ generated by x for any non-zero positive element $x \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$. Take a non-zero positive element $x \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$ and a sufficiently small positive number $\varepsilon > 0$. There exists a positive element y with $\|x - y\| < \varepsilon$ that is a linear combination of elements of the form $S_\mu f S_\nu^*$. Since $\|E(x) - E(y)\| \le \|x - y\| < \varepsilon$, there exists a real number C with $1/\|E(y)\| < C^2$ which depends only on x. By Lemma 4.6, there exist $a \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$ with $\|a\| \le C$ and an open set O of Γ such that $a^* y a$ becomes an element of $C_0(\Gamma)$ which is 1 on O. Take an open subset O_1 of O and a neighborhood O_2 of $0 \in \Gamma$ with $O_1 + O_2 \subset O$. Let h be a non-zero positive function of $C_0(\Gamma)$ whose support is contained in O_1. The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ has an infinite projection p by Proposition 4.2 and Proposition 4.5. Since $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is simple and p is a projection, there exist $x_1, x_2, \dots, x_K \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$ satisfying $\sum_{k=1}^{K} x_k^* h x_k = p$ by Lemma 4.1. By Lemma 4.3, we can choose $\mu_1, \mu_2, \dots, \mu_K \in \mathcal{W}_n$ such that $S_{\mu_k}^* S_{\mu_l} = \delta_{k,l}$ and $\omega_{\mu_k} \in O_2$ for $k = 1, 2, \dots, K$. Set $b = \sum_{k=1}^{K} S_{\mu_k} h^{1/2} x_k$. We have

$$b^* b = \sum_{k,l=1}^{K} x_k^* h^{\frac{1}{2}} S_{\mu_k}^* S_{\mu_l} h^{\frac{1}{2}} x_l = \sum_{k=1}^{K} x_k^* h x_k = p.$$

Since the support of $\sigma_{-\omega_{\mu_k}}(h^{1/2})$ is contained in O for $k = 1, 2, \dots, K$, and the function $a^* y a \in C_0(\Gamma)$ is 1 on O, we have $(a^* y a) b = b$. Therefore, we get $b^* a^* y a b = p$. Thus $q = (y^{1/2} a b)(b^* a^* y^{1/2})$ is an infinite projection because it is equivalent to the infinite projection p. The hereditary subalgebra $\overline{x(\mathcal{O}_n \rtimes_{\alpha^\omega} G)x}$ has a positive element $c = x^{1/2}abb^*a^*x^{1/2}$ which is close to an infinite projection q. If we choose $\varepsilon > 0$ so small that $\|q - c\| < 1/2$, then we get a projection $q_0 = \chi(c)$ in $\overline{x(\mathcal{O}_n \rtimes_{\alpha^\omega} G)x}$ by the functional calculus where χ is a characteristic function of a certain neighborhood of 1. The projection q_0 of $\overline{x(\mathcal{O}_n \rtimes_{\alpha^\omega} G)x}$ is infinite since it is close to an infinite projection q. Therefore, $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is purely infinite. ∎

Once noting that $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is simple if and only if the closed semigroup generated by $\omega_1, \omega_2, \ldots, \omega_n$ and $-\omega_i$ is equal to Γ for any $i = 1, 2, \ldots, n$ (see [Ki, Theorem 4.4] or [Ka, Theorem 4.8]), we have the following corollaries.

Corollary 4.8 *The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is either purely infinite or AF-embeddable when it is simple.*

Corollary 4.9 *The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is simple and purely infinite if and only if $\Gamma = \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n\}}$.*

Remark 4.10 When the group G is compact, crossed products $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ are graph algebras [KP]. From this fact, one can easily prove Proposition 3.9 and two corollaries above when the group G is compact (see [BPRS], for example).

Remark 4.11 When the group G is discrete, crossed products $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ are never AF-embeddable and Corollary 4.8 implies that crossed products $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is purely infinite if it is simple. This fact was already proved in [KK2, Lemma 10].

5 AF-embeddability of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$

In this section, we deal with crossed products of the Cuntz algebra $\mathcal{O}_\infty$ which is the universal C^*-algebra generated by infinitely many isometries $S_1, S_2, \ldots$ satisfying $S_i^* S_j = \delta_{i,j}$. Let us denote by $\mathcal{W}_\infty$ the set of words whose letters are $\{1, 2, \ldots\}$, which is naturally identified with $\bigcup_{n=2}^\infty \mathcal{W}_n$. We can define an isometry $S_\mu \in \mathcal{O}_\infty$ for $\mu \in \mathcal{W}_\infty$. As in the case of $\mathcal{O}_n$, we define the action α^ω of abelian group G on $\mathcal{O}_\infty$ by

$$\alpha_t^\omega(S_i) = \langle t \,|\, \omega_i \rangle S_i \quad (i = 1, 2, \ldots,\ t \in G)$$

for $\omega = (\omega_1, \omega_2, \ldots) \in \Gamma^\infty$. The crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ has the C^*-algebra $\mathbb{C}1 \rtimes_{\alpha^\omega} G$ which is isomorphic to $C_0(\Gamma)$. One can easily see that $fS_\mu = S_\mu \sigma_{\omega_\mu} f$ for any $f \in C_0(\Gamma) \subset \mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and any $\mu \in \mathcal{W}_\infty$, and

$$\mathcal{O}_\infty \rtimes_{\alpha^\omega} G = \overline{\mathrm{span}}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_\infty,\ f \in C_0(\Gamma)\}.$$

Proposition 5.1 *If $\omega \in \Gamma^\infty$ satisfies $-\omega_i \notin \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n \subset \mathcal{W}_\infty\}}$ for any i and any $n \in \mathbb{N}$, then the crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is AF-embeddable.*

Proof. Fix an open base $\{U_i\}_{i \in \mathbb{I}}$ such that for any $i \in \mathbb{I}$, $\overline{U_i}$ is compact and for any $i \in \mathbb{I}$ and $\mu \in \mathcal{W}_\infty$, there exists $j \in \mathbb{I}$ with $U_j = U_i - \omega_\mu$. Let $D_0(\Gamma)$ be the C^*-algebra generated $\{\chi_{U_i}\}_{i \in \mathbb{I}}$ in $L^\infty(\Gamma)$ and define the C^*-subalgebra A of $B(H \otimes L^2(\Gamma))$ by

$$A = \overline{\mathrm{span}}\{S_\mu f S_\nu^* \mid \mu, \nu \in \mathcal{W}_\infty,\ f \in D_0(\Gamma)\}.$$

The crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ can be embedded into A. For a positive integer n and a finite set $\lambda \subset \mathbb{I}$, we denote by $A_{\lambda,n}$ the C^*-subalgebra of A generated by

$$\{S_\mu \chi_{U_i} S_\nu^* \mid \mu, \nu \in \mathcal{W}_n \subset \mathcal{W}_\infty,\ i \in \lambda\}.$$

One can easily see that $A = \varinjlim A_{\lambda,n}$. Take a positive integer n and a finite set $\lambda \subset \mathbb{I}$ and fix them. Since $-\omega_i \notin \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_n \subset \mathcal{W}_\infty\}}$ for any i, there exists $K \in \mathbb{N}$ such that $pS_\mu p = 0$ for any $\mu \in \mathcal{W}_n \subset \mathcal{W}_\infty$ with $|\mu| > K$ by Lemma 3.3, where p is the characteristic function of $\bigcup_{i \in \lambda} U_i$. Once fixing such an integer K, we can define the projection $q \in A_{\lambda,n}$ in the same manner as in Section 3 and prove the same statement as in Lemma 3.4 and Lemma 3.5. Hence as in a similar way to Proposition 3.7, we can prove that $A_{\lambda,n}$ is isomorphic to a direct product of finitely many $\mathbb{K}$. Hence A is an AF-algebra. Since the crossed product $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ can be embedded into A, it is AF-embeddable. ∎

In the case of $\mathcal{O}_n$, we have the dichotomy (Corollary 4.8). However in the case of $\mathcal{O}_\infty$, instead of dichotomy we have the following.

Proposition 5.2 *For $\omega \in \Gamma^\infty$, the following are equivalent:*

(i) $\Gamma = \overline{\{\omega_\mu \mid \mu \in \mathcal{W}_\infty\}}$.

(ii) $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ *is simple.*

(iii) $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ *is simple and purely infinite.*

Proof. The equivalence between (i) and (ii) was proved in [Ki]. Obviously (iii) implies (ii). One can prove the implication (i) $\Rightarrow$ (iii) in a similar way to arguments in Section 4, though we need more complicated computations to prove the proposition corresponding to Lemma 4.6. ∎

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