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ANNIHILATORS OF GENERALIZED VERMA MODULES OF THE SCALAR TYPE FOR CLASSICAL LIE ALGEBRAS

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Abstract. We construct a generator system of the annihilator of a generalized Verma module of a classical reductive Lie algebra induced from a character of a parabolic subalgebra as an analogue of the minimal polynomial of a matrix. In a classical limit it gives a generator system of the defining ideal of any semisimple co-adjoint orbit of the Lie algebra.

1. Introduction

In [O3] generalized Capelli operators are defined in the universal enveloping algebra of $GL(n, \mathbb{R})$ and it is shown that they characterize the differential equations satisfied by the functions in degenerate principal series representations of $GL(n, \mathbb{R})$. The operators are used to formulate boundary value problems for various boundaries of the symmetric $GL(n, \mathbb{R})/O(n)$ and to construct generalized hypergeometric equations related to Radon transformations on Grassmannian manifolds. In [O4] using these operators we construct a generator system of the annihilator of the generalized Verma module for $\mathfrak{gl}_n(\mathbb{C})$ induced from any character of any parabolic subalgebra.

In this paper the generator system is constructed for any classical Lie algebra. It is different from the one constructed in [O4]. In the case of $\mathfrak{gl}_n(\mathbb{C})$ the generator system in [O4] is an analogue of minors and elementary divisors. The generator system here is an analogue of the minimal polynomial of a matrix. For the generator system of the center of the universal enveloping algebra the former corresponds to Capelli identity in [C1] and [C2] and the latter to the trace of the power of a matrix with components in the Lie algebra which is presented by [Ge].

In §2 we define a matrix F with components in $\mathfrak{g}$ or the universal enveloping algebra $U(\mathfrak{g})$ associated to a finite dimensional representation of a Lie algebra $\mathfrak{g}$ and define a *minimal polynomial* of F with respect to a $\mathfrak{g}$-module (cf. Definition 2.4).

In §3 we calculate the Harish-Chandra homomorphism of certain polynomials of F. It is a little complicated but elementary. Owing to this calculation, in §4 we introduce some polynomials of F and study their action on the generalized Verma module.

Then we construct a two-sided ideal of $U(\mathfrak{g})$ generated by the components $q(F)_{ij}$ for the minimal polynomial $q(x)$ of F and prove Theorem 4.4, which is the main result in this paper. It says that the ideal describes the *gap* between the generalized Verma module and the usual Verma module (cf. (5.1) and (5.7)) if at least the infinitesimal character is regular. The main motivation to write this paper is to construct a two-sided ideal with this property originated in the problem in [O1].

It follows from this theorem that the ideal equals the annihilator of the generalized Verma module induced from the character of the parabolic subalgebra of the classical Lie algebra if at least the infinitesimal character is regular and dominant (cf. Corollary 4.6). But the assumption that the infinitesimal character is dominant may be unnecessary (cf. Conjecture 2 and 3 in §6).

We will use the homogenized universal enveloping algebra $U^\epsilon(\mathfrak{g})$ introduced in [O4] so that we can compare the generator system of a co-adjoint orbit in the dual of $\mathfrak{g}$. As a classical limit we get the generator system of any semisimple co-adjoint orbit for a classical Lie algebra, which is described in Theorem 4.11 (cf. Remark 4.12).

In §5 we show some applications of our two-sided ideals to integral transformations of sections of a line bundle over a generalized flag manifold. For example, Theorem 5.1 is a typical application, which shows that the system of differential equations defined by our two-sided ideal characterizes the image of the Poisson transform of the functions on any boundary of the Riemannian symmetric space of the non-compact type.

In §6 we discuss the infinitesimal character which is excluded in the results in §4 and present some conjectures.

In the subsequent paper [OO] we will give a simple explicit formula of minimal polynomials of generalized Verma modules of the scalar type for any reductive Lie algebra and study the same problem as in this paper.

In order to explain our idea, suppose $G = GL(2n,\mathbb{C})$ and put $A = \begin{pmatrix} \lambda I_n & 0 \\ B & \mu I_n \end{pmatrix} \in \mathfrak{g} = \mathrm{Lie}(G)$. Here λ, $\mu \in \mathbb{C}$ and $B \in M(n,\mathbb{C})$ is a generic element. Note that A is conjugate to $\lambda I_n \oplus \mu I_n$ if $\lambda \neq \mu$ and to $\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$ otherwise. We will identify $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ by the symmetric bilinear form $\langle X, Y\rangle = \mathrm{Trace}\, XY$. Let $I_\Theta (\subset S(\mathfrak{g}))$ be the defining ideal of the closure $\bar{V}_\Theta$ of the conjugacy class $V_\Theta = \sum_{g\in G} \mathrm{Ad}(g)A$ with $\mathrm{Ad}(g)X = gXg^{-1}$.

Note that $I_\Theta = I_\Theta^0$ by denoting

$$\begin{cases} I_\Theta^\epsilon = \displaystyle\bigcap_{g\in G} \mathrm{Ad}(g) J_\Theta^\epsilon, \\ J_\Theta^\epsilon = \displaystyle\sum_{X_1,\, X_2,\, X_3 \in M(n,\mathbb{C})} U^\epsilon(\mathfrak{g}) \Big(\begin{pmatrix} X_1 & 0 \\ X_3 & X_2 \end{pmatrix} - \lambda\, \mathrm{Trace}\, X_1 - \mu\, \mathrm{Trace}\, X_2 \Big). \end{cases}$$

Here $U^\epsilon(\mathfrak{g})$ is the quotient of the tensor algebra of $\mathfrak{g}$ by the two sided ideal generated by elements of the form $X \otimes Y - Y \otimes X - \epsilon[X,Y]$. Then $U^0(\mathfrak{g})$ is the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$ and $U^1(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. We call a generalization of I_Θ^ϵ a quantization of I_Θ^0 and the quantization I_Θ^1 is nothing but the annihilator of the generalized Verma module $U(\mathfrak{g})/J_\Theta^1$.

Since $\mathrm{rank}(X - \lambda I_{2n}) \le n$ and $\mathrm{rank}(X - \mu I_{2n}) \le n$ for $X \in \bar{V}_\Theta$, the $(n+1)$-minors ($\in S(\mathfrak{g})$) of $((E_{ij}) - \lambda I_{2n})$ and $((E_{ij}) - \mu I_{2n})$ are in I_Θ. On the contrary, they generate I_Θ if $\lambda \neq \mu$. The quantizations of the minors are generalized Capelli operators and studied by [O3].

If $\lambda = \mu$, the derivatives of $(n+1)$-minors of $((E_{ij}) - xI_{2n})$ at $x = \lambda$ are also in I_Θ and in general the generators are described by using the elementary divisors. In [O4], we define their quantizations, namely, we explicitly construct the corresponding generators for any generalized Verma module of the scalar type for $\mathfrak{gl}(n,\mathbb{C})$ using quantized elementary divisors. Moreover in [O4] we determine the condition that the annihilator determines the gap between the generalized Verma module and the Verma module. In the example here, the equality

$$J_\Theta^\epsilon = I_\Theta^\epsilon + \sum_{i>j} U^\epsilon(\mathfrak{g}) E_{ij} + \sum_{i=1}^{n} U^\epsilon(\mathfrak{g})(E_{ii} - \lambda) + \sum_{i=n+1}^{2n} U^\epsilon(\mathfrak{g})(E_{ii} - \mu) \tag{1.1}$$

holds if and only if $\lambda - \mu \notin \{\epsilon, \dots, (n-1)\epsilon\}$. When $\epsilon = 1$, this condition is also equivalent to the fact that $U(\mathfrak{g})/J^\epsilon_\Theta$ has a regular infinitesimal character. If (1.1) holds, the quantized generators are considered to be the differential equations which characterize the representations of the group G related to the generalized Verma module. Hence they are important and the motivation of our study in this note is this fact.

On the other hand, since $(x-\lambda)(x-\mu)$ is the minimal polynomial of A, all the components of $((E_{ij}) - \lambda I_{2n})((E_{ij}) - \mu I_{2n})$ are in J^0_Θ. They generate I^0_Θ together with $\sum_{i=1}^{2n} E_{ii} - n\lambda - n\mu$ if $\lambda \neq \mu$. We can quantize this minimal polynomial and the quantized minimal polynomial in this example equals $q^\epsilon(x) = (x-\lambda)(x-\mu-n\epsilon)$. We can show that the $4n^2$ components of the matrix $q^\epsilon((E_{ij})) \in M(2n, U^\epsilon(\mathfrak{g}))$ and the element $\sum_{i=1}^{2n} E_{ii} - n\lambda - n\mu$ generate I^ϵ_Θ if $\lambda - \mu \notin \{\epsilon, \dots, (n-1)\epsilon\}$.

The main topic in this paper is to construct the elements in $U(\mathfrak{g})$ which kills the generalized Verma module of the scalar type for the classical Lie algebra by using the quantized minimal polynomial.

The author expresses his sincere gratitude to Mittag-Leffler Institute since the result for $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ was obtained when the author was invited there from September until November in 1995 and the result is reported in [O2].

2. Minimal Polynomials

For a module $\mathfrak{A}$ and positive integers N and N', we denote by $M(N, N', \mathfrak{A})$ the set of matrices of size $N \times N'$ with components in $\mathfrak{A}$. If $N = N'$, we simply denote it by $M(N, \mathfrak{A})$ and then $M(N, \mathfrak{A})$ is naturally an associative algebra if so is $\mathfrak{A}$.

We use the standard notation $\mathfrak{gl}_n$, $\mathfrak{o}_n$ and $\mathfrak{sp}_n$ for classical Lie algebras over $\mathbb{C}$. For a Lie algebra $\mathfrak{g}$ we denote by $U(\mathfrak{g})$ and $S(\mathfrak{g})$ the universal enveloping algebra and the symmetric algebra of $\mathfrak{g}$, respectively. For a non-negative integer k let $S(\mathfrak{g})^{(k)}$ be the subspace of $S(\mathfrak{g})$ formed by elements of degree at most k. If we fix a Poincare-Birkhoff-Witt base of $U(\mathfrak{g})$, we can identify $U(\mathfrak{g})$ and $S(\mathfrak{g})$ as vector spaces and we denote by $U(\mathfrak{g})^{(k)}$ the subspace of $U(\mathfrak{g})$ corresponding to $S(\mathfrak{g})^{(k)}$.

The Lie algebra $\mathfrak{gl}_N$ is identified with $M(N,\mathbb{C}) \simeq \mathrm{End}(\mathbb{C}^N)$ by $[X,Y] = XY - YX$. Let $E_{ij} = \Bigl(\delta_{\mu i}\delta_{\nu j}\Bigr)_{\substack{1\le\mu\le N\\ 1\le\nu\le N}} \in M(N,\mathbb{C})$ be the standard matrix units. Note that the symmetric bilinear form

$$\langle X, Y\rangle = \mathrm{Trace}\, XY \quad \text{for} \quad X, Y \in \mathfrak{gl}_N \tag{2.1}$$

on $\mathfrak{gl}_N$ is non-degenerate and satisfies

$$\begin{aligned}\langle E_{ij}, E_{\mu\nu}\rangle &= \delta_{i\nu}\delta_{j\mu},\\ X &= \sum_{i,j}\langle X, E_{ji}\rangle E_{ij},\\ \langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle &= \langle X, Y\rangle \quad \textit{for } X,\ Y \in \mathfrak{gl}_N \text{ and } g \in GL(N,\mathbb{C}).\end{aligned}$$

Lemma 2.1. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and let $(\pi, \mathbb{C}^N)$ be a representation of $\mathfrak{g}$. We denote by $U(\pi(\mathfrak{g}))$ the subalgebra of the universal enveloping algebra $U(\mathfrak{gl}_N)$ of $\mathfrak{gl}_N$ generated by $\pi(\mathfrak{g})$. Let p be a linear map of $\mathfrak{gl}_N$ to $U(\pi(\mathfrak{g}))$ satisfying*

$$p([X,Y]) = [X, p(Y)] \quad \textit{for } X \in \pi(\mathfrak{g}) \textit{ and } Y \in \mathfrak{gl}_N, \tag{2.2}$$

that is, $p \in \mathrm{Hom}_{\pi(\mathfrak{g})}(\mathfrak{gl}_N, U(\pi(\mathfrak{g})))$.

Fix $f(x) \in \mathbb{C}[x]$ and put

$$\begin{cases} p(E) = \Bigl(p(E_{ij})\Bigr)_{\substack{1\le i\le N\\ 1\le j\le N}} \in M\bigl(N, U(\pi(\mathfrak{g}))\bigr),\\ \Bigl(F_{ij}\Bigr)_{\substack{1\le i\le N\\ 1\le j\le N}} = f(p(E)) \in M\bigl(N, U(\pi(\mathfrak{g}))\bigr).\end{cases} \tag{2.3}$$

Then

$$[X, F_{ij}] = \sum_{\nu=1}^{n} X_{\nu i} F_{\nu j} - \sum_{\nu=1}^{n} X_{j\nu} F_{i\nu} \tag{2.4}$$
$$= \sum_{\mu=1}^{n} \langle X, E_{i\nu}\rangle F_{\nu j} - \sum_{\nu=1}^{n} F_{i\nu}\langle X, E_{\nu j}\rangle \quad \textit{for } X = \Big(X_{ij}\Big)_{\substack{1\le i\le N\\ 1\le j\le N}} \in \pi(\mathfrak{g})$$

with $X_{ij} \in \mathbb{C}$.

Proof. Fix $X \in \pi(\mathfrak{g})$. Since

$$[X, E_{ij}] = [\sum_{\mu,\nu} X_{\mu\nu} E_{\mu\nu}, E_{ij}] = \sum_{\mu=1}^{n} X_{\mu i} E_{\mu j} - \sum_{\nu=1}^{n} X_{j\nu} E_{i\nu},$$

we have (2.4) for $f(x) = x$ by (2.2).

Suppose $\Big(F^1_{ij}\Big)$ and $\Big(F^2_{ij}\Big) \in M\big(N, U(\pi(\mathfrak{g})\big)$ satisfy (2.4). Put $F_{ij} = \sum_{k=1}^{n} F^1_{ik} F^2_{kj}$ in $U(\pi(\mathfrak{g}))$. Then

$$\begin{aligned}
[X, F_{ij}] &= \sum_{k=1}^{n} [X, F^1_{ik}] F^2_{kj} + \sum_{k=1}^{n} F^1_{ik} [X, F^2_{kj}] \\
&= \sum_{k=1}^{n} \left(\sum_{\mu=1}^{n} X_{\mu i} F^1_{\mu k} F^2_{kj} - \sum_{\nu=1}^{n} X_{k\nu} F^1_{i\nu} F^2_{kj} \right) \\
&\quad + \sum_{k=1}^{n} \left(\sum_{\mu=1}^{n} F^1_{ik} X_{\mu k} F^2_{\mu j} - \sum_{\nu=1}^{n} F^1_{ik} X_{j\nu} F^2_{k\nu} \right) \\
&= \sum_{\mu=1}^{n} X_{\mu i} F_{\mu j} - \sum_{\nu=1}^{n} X_{j\nu} F_{i\nu}
\end{aligned}$$

and therefore the elements $\Big(F_{ij}\Big)$ of $M\big(N, U(\pi(\mathfrak{g}))\big)$ satisfying (2.4) form a subalgebra of $M\big(N, U(\pi(\mathfrak{g}))\big)$. □

Definition 2.2. If the symmetric bilinear form (2.1) is non-degenerate on $\pi(\mathfrak{g})$, the orthogonal projection of $\mathfrak{gl}_N$ onto $\pi(\mathfrak{g})$ satisfies the assumption for p in Lemma 2.1, which we call the *canonical projection* of $\mathfrak{gl}_N$ to $\pi(\mathfrak{g})$.

Remark 2.3. Assume that $\mathfrak{g}$ is reductive and that the finite dimensional representation (π, V) in Lemma 2.1 is completely reducible. Let G be a connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$ and let G_U be a maximal compact subgroup of G. Moreover let $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be a triangular decomposition of $\mathfrak{g}$ such that $\exp \mathfrak{a} \cap G_U$ is a maximal torus of G_U. Let $\Sigma(\mathfrak{a})$ and $\Sigma(\mathfrak{a})^+$ be the sets of the roots for the pair $(\mathfrak{g}, \mathfrak{a})$ and $(\mathfrak{n}, \mathfrak{a})$, respectively, and let $\Psi(\mathfrak{a})$ denote the fundamental system of $\Sigma(\mathfrak{a})^+$. We fix a Hermitian inner product on V so that π is a unitary representation of G_U. Let $\{v_1, \ldots, v_N\}$ be an orthonormal basis of V such that v_j is a weight vector of a weight ϖ_j with respect to the Cartan subalgebra $\mathfrak{a}$. We may assume that $\varpi_i - \varpi_j \in \Sigma(\mathfrak{a})^+$ means $i < j$. Hence ϖ_1 is the lowest weight and ϖ_N is the highest weight of the representation π. Under this basis we identify $\pi(X) = \big(\pi(X)_{ij}\big) \in M(N, \mathbb{C}) \simeq \mathrm{End}(\mathbb{C}^N) \simeq \mathfrak{gl}_N$ for $X \in \mathfrak{g}$ by $\pi(X)v_j = \sum_{i=1}^{N} \pi(X)_{ij} v_i$. Note that $\pi(\mathfrak{a}) \subset \mathfrak{a}_N$, $\pi(\mathfrak{n}) \subset \mathfrak{n}_N$ and $\pi(\bar{\mathfrak{n}}) \subset \bar{\mathfrak{n}}_N$ by denoting

$$\mathfrak{a}_N = \sum_{j=1}^{N} \mathbb{C}E_{ii},\ \mathfrak{n}_N = \sum_{1\le j<i\le N} \mathbb{C}E_{ij} \ \text{ and } \ \bar{\mathfrak{n}}_N = \sum_{1\le i<j\le N} \mathbb{C}E_{ij}. \tag{2.5}$$

Since $\pi(X)$ is skew Hermitian for the element X in the Lie algebra $\mathfrak{g}_U$ of G_U and $\mathbb{C}\pi(\mathfrak{g}_U) + \pi(\mathfrak{a}) = \pi(\mathfrak{g})$, we have ${}^t\pi(\mathfrak{g}) = \pi(\mathfrak{g})$. Hence the symmetric bilinear form (2.1) is non-degenerate on $\pi(\mathfrak{g})$ and there exists the canonical projection of $\mathfrak{gl}_N$ to $\pi(\mathfrak{g})$.

Definition 2.4 (Characteristic polynomials and minimal polynomials)**.** Given a Lie algebra $\mathfrak{g}$, a faithful finite dimensional representation $(\pi, \mathbb{C}^N)$ and a $\mathfrak{g}$-homomorphism p of $\mathrm{End}(\mathbb{C}^N) \simeq \mathfrak{gl}_N$ to $U(\mathfrak{g})$. Here we identify $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}_N$ through π. Let $U(\mathfrak{g})$ and $U(\mathfrak{g})^G$ be the universal enveloping algebra of $\mathfrak{g}$ and the center of $U(\mathfrak{g})$, respectively. Put $F = \Bigl(p(E_{ij})\Bigr) \in M(N, U(\mathfrak{g}))$. We say $q_F(x) \in U(\mathfrak{g})^G[x]$ is the *characteristic polynomial* of F if it is a non-zero polynomial of x satisfying

$$q_F(F) = 0$$

with the minimal degree.

Suppose moreover a $\mathfrak{g}$-module M is given. Then we call $q_{F,M}(x) \in \mathbb{C}[x]$ is the *minimal polynomial* of F with respect to M if it is the monic polynomial with the minimal degree which satisfies

$$q_{F,M}(F)M = 0.$$

If p is the canonical projection in Definition 2.2, we sometimes denote F_π, q_π and $q_{\pi,M}$ in place of F, q_F and $q_{F,M}$, respectively.

Remark 2.5. i) After the results in this paper was obtained, the author was informed that [Go2] studied the characteristic polynomial of F_π for the irreducible representation π of the reductive Lie algebra.

ii) If $\mathfrak{g}$ is reductive, the characteristic polynomial is uniquely determined by (π, p) up to a constant multiple of the element of $U(\mathfrak{g})^G$ since $U(\mathfrak{g})^G$ is an integral domain.

iii) If $\mathfrak{g}$ is reductive and M has an infinitesimal character χ, that is, χ is an algebra homomorphism of $U(\mathfrak{g})^G$ to $\mathbb{C}$ with $(D - \chi(D))M = 0$ for $D \in U(\mathfrak{g})$, then $\chi(q_F(x)) \in \mathbb{C}[x]q_{F,M}(x)$.

iv) The characteristic polynomial and minimal polynomial of a matrix in the linear algebra can be regarded as a classical limit of our definition. See the proof of Proposition 4.16.

Theorem 2.6. *Let $\mathfrak{g}$ be a reductive Lie algebra and let F be a matrix of $U(\mathfrak{g})$ defined from a representation of π under* Definition 2.4.

i) *There exists the characteristic polynomial $q_F(x)$ whose degree is not larger than $\sum_\omega m_\pi(\varpi)^2$. Here ϖ runs through the weights of π and $m_\pi(\varpi)$ denotes the multiplicity of the generalized weight ϖ in π.*

ii) *The minimal polynomial $q_{F,M}(x)$ exists if a $\mathfrak{g}$-module M has a finite length or an infinitesimal character. Its degree is not larger than that of the characteristic polynomial $q_F(x)$ if M has an infinitesimal character.*

Proof. Let $\hat{U}(\mathfrak{g})^G$ denote the quotient field of $U(\mathfrak{g})^G$ and put $\hat{U}(\mathfrak{g}) = \hat{U}(\mathfrak{g})^G \otimes_{U(\mathfrak{g})^G} U(\mathfrak{g})$. Owing to [Ko] it is known that $U(\mathfrak{g}) = \Lambda(H(\mathfrak{g})) \otimes U(\mathfrak{g})^G$, where $H(\mathfrak{g})$ is the space of $\mathfrak{g}$-harmonic polynomials of $S(\mathfrak{g})$ and Λ is the map of the symmetrization of $S(\mathfrak{g})$ onto $U(\mathfrak{g})$. It is also known that $H(\mathfrak{g}) \simeq \sum_{\tau \in \hat{\mathfrak{g}}_f} m_\tau(0)\tau$ as a representation space of $\mathfrak{g}$ by denoting $\hat{\mathfrak{g}}_f$ the equivalence classes of the finite dimensional irreducible representations of $\mathfrak{g}$.

Hence the dimension of the $\mathfrak{g}$-homomorphisms of $\pi \otimes \pi^*$ to $\hat{U}(\mathfrak{g})$ over the field $\hat{U}(\mathfrak{g})^G$ is not larger than $\sum_{\tau \in \hat{\mathfrak{g}}_f} [\pi \otimes \pi^*, \tau] m_\tau(0)$. Here $[\pi \otimes \pi^*, \tau]$ is the multiplicity of τ appeared in $[\pi \otimes \pi^*]$ in the sense of the Grothendieck group. Moreover it is clear that $\sum_{\tau \in \hat{\mathfrak{g}}_f} [\pi \otimes \pi^*, \tau] m_\tau(0) = m_{\pi\otimes\pi^*}(0) = \sum_\varpi m_\pi(\varpi)^2$. On the other hand Lemma 2.1 says that the space $V_k = \sum_{i,j} \mathbb{C}F_{ij}^k$ is naturally a subrepresentation of the representation of $\mathfrak{g}$ which is realized in $M(N,\mathbb{C})$ and belongs to $\pi\otimes\pi^*$ and that the map $T_k : E_{ij} \mapsto F^k_{ij}$ defines a $\mathfrak{g}$-homomorphism of $M(N,\mathbb{C})$ to $U(\mathfrak{g})$. Hence $T_1,\dots,T_m$ are linearly dependent over $\hat{U}(\mathfrak{g})^G$ if $m > \sum_{\varpi} m_\pi(\varpi)^2$. Thus we have proved the existence of the characteristic polynomial with the required degree.

For the existence of the minimal polynomial it is sufficient to prove the existence of a non-zero polynomial $f(x)$ with $f(F)M = 0$. Considering the irreducible subquotients of M in Definition 2.4, we may assume M has an infinitesimal character λ. Let $q_F(x)$ be the characteristic polynomial. We can choose $\mu\in\mathfrak{a}^*$ so that $\bar\omega(q_F(x))(\lambda+\mu t)\in\mathbb{C}[x,t]$ is not zero. We can find a non-negative integer k such that $f(x,t) = t^{-k}\bar\omega(q_F(x))\in\mathbb{C}[x,t]$ and $f(x,0)$ is not zero. Put $I_\lambda = \sum_{Z\in U(\mathfrak{g})^G} U(\mathfrak{g})\bigl(Z-\bar\omega(Z)(\lambda)\bigr)$. We define $h(t)\in M(N,H(\mathfrak{g})\otimes\mathbb{C}[t])$ so that $f(F,t)-\Lambda(h(t))\in M(N,I_{\lambda+\mu t})$. Since $d_F(F)(\lambda+\mu t)\in M(N,I_{\lambda+\mu t})$, $h(t)=0$ for $t\neq 0$ and hence $h=0$ and therefore $f(F,0)\bigl(U(\mathfrak{g})/I_\lambda\bigr)=0$. Hence $f(F,0)M=0$ because $\operatorname{Ann}(M)\supset I_\lambda$. □

Hereafter in this note we assume

$$\begin{cases}\pi \text{ is injective,}\\ p(\mathfrak{gl}_N)\subset\mathfrak{g},\\ p(X)=CX \quad\text{for}\quad X\in\mathfrak{g}\end{cases}\tag{2.6}$$

in Lemma 2.1 with a suitable non-zero constant C. Then we have the following.

Remark 2.7. i) Since π is faithful, $\mathfrak{g}$ is identified with the Lie subalgebra $\pi(\mathfrak{g})$ of $\mathfrak{gl}_N$ and $U(\pi(\mathfrak{g}))$ is identified with the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. We note that the existence of p with (2.6) is equivalent to the existence of a $\mathfrak{g}$-invariant subspace of $\mathfrak{gl}_N$ complementary to $\mathfrak{g}$.

ii) Fix $g\in GL(N,\mathbb{C})$. If we replace X by gXg^{-1} for $X\in\pi(\mathfrak{g})$ in Lemma 2.1, $\bigl(F_{ij}\bigr)$ naturally changes into $g\bigl(F_{ij}\bigr)g^{-1}$ and therefore the corresponding characteristic polynomial and minimal polynomial does not depend on the realization of the representation π.

iii) Suppose $\mathfrak{g}$ is semisimple. Then the existence of p is clear because any finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ is completely reducible.

iv) Let σ be an involutive automorphism of $\mathfrak{gl}_N$. Put

$$\mathfrak{g}=\{X\in\mathfrak{gl}_N;\ \sigma(X)=X\}.$$

Let π be the inclusion map of $\mathfrak{g}\subset\mathfrak{gl}_N$. Since $\mathfrak{q}=\{X\in\mathfrak{gl}_N;\ \sigma(X)=-X\}$ is $\mathfrak{g}$-stable, we may put $p(X)=\frac{X+\sigma(X)}{2}$ in Lemma 2.1, which is the canonical projection with respect to the bilinear form of $\mathfrak{gl}_N$.

v) For a positive integer k and complex numbers $\lambda_1,\dots,\lambda_k$, the vector space spanned by the N^2 components of the matrix $(p(E)-\lambda_1 I_N)\cdots(p(E)-\lambda_k I_N)$ is $\mathfrak{g}$-invariant. Moreover the trace of the matrix is a central element of $U(\mathfrak{g})$, which is clear from Lemma 2.1 and studied by [Ge] and [Go1] etc.

3. Projection to the Cartan subalgebra

Now we consider the natural realization of classical simple Lie algebras. Denoting

$$\tilde I_n=\bigl(\delta_{i,n+1-j}\bigr)_{\substack{1\le i\le n\\1\le j\le n}}=\begin{pmatrix}&&1\\&\iddots&\\1&&\end{pmatrix}\text{ and }\tilde J_n=\begin{pmatrix}&\tilde I_n\\-\tilde I_n&\end{pmatrix},$$

we naturally identify

$$\begin{aligned}\mathfrak{o}_n&=\{X\in\mathfrak{gl}_n;\,\sigma_{\mathfrak{o}_n}(X)=X\}\quad\text{with } \sigma_{\mathfrak{o}_n}(X)=-\tilde I_n{}^tX\tilde I_n\\ \mathfrak{sp}_n&=\{X\in\mathfrak{gl}_{2n};\,\sigma_{\mathfrak{sp}_n}(X)=X\}\quad\text{with } \sigma_{\mathfrak{sp}_n}(X)=-\tilde J_n{}^tX\tilde J_n.\end{aligned}\tag{3.1}$$

Definition 3.1. Let $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{o}_{2n}$ or $\mathfrak{o}_{2n+1}$ or $\mathfrak{sp}_n$ and put $N = n$ or $2n$ or $2n+1$ or $2n$, respectively, so that $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}_N$. Put

$$\bar{i} = N + 1 - i \tag{3.2}$$

for any integer i and define

$$\epsilon_i = \begin{cases} 0 & \text{if} \quad \mathfrak{g} = \mathfrak{gl}_n, \\ 1 & \text{if} \quad \mathfrak{g} = \mathfrak{o}_N, \\ 1 & \text{if} \quad \mathfrak{g} = \mathfrak{sp}_n \quad \text{and} \quad i \le n, \\ -1 & \text{if} \quad \mathfrak{g} = \mathfrak{sp}_n \quad \text{and} \quad i > n. \end{cases} \tag{3.3}$$

Then the involutions $\sigma_{\mathfrak{g}}$ of $\mathfrak{gl}_N$ defining $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{o}_N$ and $\mathfrak{sp}_n$ satisfy

$$\sigma_{\mathfrak{g}}(E_{ij}) = -\epsilon_i \epsilon_j E_{\bar{j}\bar{i}}.$$

We moreover define

$$F = \Bigl(F_{ij}\Bigr)_{\substack{1\le i\le N\\ 1\le j\le N}} = \Bigl(E_{ij} - \epsilon_i\epsilon_j E_{\bar{j}\bar{i}}\Bigr)_{\substack{1\le i\le N\\ 1\le j\le N}}. \tag{3.4}$$

This definition of F means $C = 2$ in (2.6) if $\mathfrak{g} = \mathfrak{o}_N$ or $\mathfrak{sp}_n$. We will denote F_i in place of F_{ii} for simplicity. Then $\mathfrak{g} = \sum_{i,j} \mathbb{C}F_{ij}$ and

$$[X, F_{ij}] = \sum_{\nu=1}^{N} \bigl(X_{\nu i}F_{\nu j} - X_{j\nu}F_{i\nu}\bigr) \quad \text{for} \quad X = \Bigl(X_{ij}\Bigr) \in \mathfrak{g} \subset M(N, \mathbb{C}) \tag{3.5}$$

by Lemma 2.1.

Use the notation (2.5) and define $\mathfrak{a} = \mathfrak{g} \cap \mathfrak{a}_N$, $\mathfrak{n} = \mathfrak{g} \cap \mathfrak{n}_N$ and $\bar{\mathfrak{n}} = \mathfrak{g} \cap \bar{\mathfrak{n}}_N$. Then

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{a} \oplus \mathfrak{n} \tag{3.6}$$

is a triangular decomposition of $\mathfrak{g}$.

Definition 3.2. For a positive integer k and complex numbers $\lambda_1, \ldots, \lambda_k$ put

$$F^k(\lambda_1, \ldots, \lambda_k) = (F - \lambda_1 I_N) \cdots (F - \lambda_k I_N)$$

and define an element $\bar{F}^k(\lambda_1, \ldots, \lambda_k)$ in $M(N, U(\mathfrak{a}))$ by

$$F^k(\lambda_1, \ldots, \lambda_k) \equiv \bar{F}^k(\lambda_1, \ldots, \lambda_k) \mod M(N, \bar{\mathfrak{n}}U(g) + U(\mathfrak{g})\mathfrak{n}) \tag{3.7}$$

In this section we will study the image $\bar{F}^k(\lambda_1, \ldots, \lambda_k)$ of $F^k(\lambda_1, \ldots, \lambda_k)$ under the Harish-Chandra homomorphism with respect to (3.6). First we note that if

$$F_{ij} \in \begin{cases} \bar{\mathfrak{n}} & \text{if } i < j, \\ \mathfrak{a} & \text{if } i = j, \\ \mathfrak{n} & \text{if } i > j, \end{cases} \tag{3.8}$$

we have

(3.9)

$$\begin{aligned}F^k_{ij}(\lambda_1,\ldots,\lambda_k) &\equiv \sum_{\mu=1}^{j} F^{k-1}_{i\mu}(\lambda_1,\ldots,\lambda_{k-1})(F_{\mu j}-\lambda_k\delta_{\mu j}) \mod U(\mathfrak{g})\mathfrak{n}\\
&= F^{k-1}_{ij}(\lambda_1,\ldots,\lambda_{k-1})(F_j-\lambda_k)\\
&\quad+\sum_{\mu=1}^{j-1}\Big(F_{\mu j}F^{k-1}_{i\mu}(\lambda_1,\ldots,\lambda_{k-1})-[F_{\mu j},F^{k-1}_{i\mu}(\lambda_1,\ldots,\lambda_{k-1})]\Big)\\
&= F^{k-1}_{ij}(\lambda_1,\ldots,\lambda_{k-1})(F_j-\lambda_k)+\sum_{\mu=1}^{j-1}\Big(F_{\mu j}F^{k-1}_{i\mu}(\lambda_1,\ldots,\lambda_{k-1})\\
&\quad-\sum_{\nu=1}^{i-1}\langle F_{\mu j},E_{i\nu}\rangle F^{k-1}_{\nu\mu}(\lambda_1,\ldots,\lambda_{k-1})+\sum_{\nu=\mu+1}^{N}\langle F_{\mu j},E_{\nu\mu}\rangle F^{k-1}_{i\nu}(\lambda_1,\ldots,\lambda_{k-1})\Big)\end{aligned}$$

by Lemma 2.1.

The following is clear by the induction on k.

Remark 3.3. i) The highest homogeneous part of $\bar F^k(\lambda_1,\ldots,\lambda_k)$ with the degree k is given by

$$\bar F^k(\lambda_1,\ldots,\lambda_k)\equiv\Big(\delta_{ij}F^k_{ii}\Big)_{\substack{1\le i\le N\\1\le j\le N}} \mod M(N,U(\mathfrak{a})^{(k-1)}).$$

ii) If $\mathfrak{g}=\mathfrak{gl}_n$ or $\mathfrak{o}_{2n+1}$ or $\mathfrak{sp}_n$ and π is the natural representation of $\mathfrak{g}$, it is clear that $\operatorname{Trace}F^k_\pi$ for $k=1,2,\ldots,n$ or $k=2,4,\ldots,2n$ or $2,4,\ldots,2n$, respectively, generate $U(\mathfrak{g})^G$ as an algebra. In particular for any $D\in U(\mathfrak{g})^G$ there uniquely exists a polynomial $f(x)$ with $\operatorname{Trace}f(F)=D$. In the case when $\mathfrak{g}=\mathfrak{o}_{2n}$ we use both the natural representation π and the half-spin representation π' of $\mathfrak{g}$ and then $\operatorname{Trace}F^k_\pi$ for $k=2,4,\ldots,2(n-1)$ and $\operatorname{Trace}F^n_{\pi'}$ generate $U(\mathfrak{g})^G$.

iii) The Killing form of $\mathfrak{g}$ is a positive constant multiple of the restriction of the bilinear form (2.1) to $\mathfrak{g}$ if $\mathfrak{g}$ is simple.

Hereafter suppose that $\mathfrak{g}=\mathfrak{gl}_n$ or $\mathfrak{o}_{2n}$ or $\mathfrak{o}_{2n+1}$ or $\mathfrak{sp}_n$ and that F is given by (3.4). Then (3.4) means

$$\langle F_{\mu j},E_{i\nu}\rangle=\delta_{ij}\delta_{\mu\nu}-\epsilon_\mu\epsilon_j\delta_{\bar\mu i}\delta_{\bar j\nu}\quad\text{and}\quad\langle F_{\mu j},E_{\mu\nu}\rangle=\delta_{j\nu}-\epsilon_\mu\epsilon_j\delta_{\bar\mu\nu}\delta_{\bar j\mu}$$

and therefore it follows from (3.9) that

$$\begin{aligned}F^k_{ij}(\lambda_1,\ldots,\lambda_k)&\equiv F^{k-1}_{ij}(\lambda_1,\ldots,\lambda_{k-1})(F_j-\lambda_k+j-1)\\
&\quad+\sum_{\mu=1}^{j-1}\Big(F_{\mu j}F^{k-1}_{i\mu}(\lambda_1,\ldots,\lambda_{k-1})\\
&\quad-\delta_{ij}F^{k-1}_{\mu\mu}(\lambda_1,\ldots,\lambda_{k-1})+\epsilon_\mu\epsilon_j\delta_{\mu\bar i}F^{k-1}_{\bar j\bar i}(\lambda_1,\ldots,\lambda_{k-1})\\
&\quad-\epsilon_\mu\epsilon_j\delta_{\mu\bar j}F^{k-1}_{ij}(\lambda_1,\ldots,\lambda_{k-1})\Big)\mod U(\mathfrak{g})\mathfrak{n}.\end{aligned}\tag{3.10}$$

Since $[U(\mathfrak{g})\mathfrak{n},U(\mathfrak{a})]\subset U(\mathfrak{g})\mathfrak{n}$ and since $F_{ij}\in\mathfrak{n}$ and $F_{\bar j\bar i}\in\mathfrak{n}$ for $i>j$, the equation (3.10) shows $F^k_{ij}(\lambda_1,\ldots,\lambda_k)\equiv 0 \mod U(\mathfrak{g})\mathfrak{n}$ for $i>j$ by the induction on k.

Similarly we have $F^k_{\mu\nu}(\lambda_1,\dots,\lambda_k)\in\bar{\mathfrak n}U(\mathfrak g)$ if $i<j$. Hence by denoting

$$\omega_i=\begin{cases}0 & \text{if}\quad i\le n,\\ \epsilon_i & \text{if}\quad i>n,\end{cases}\tag{3.11}$$
$$\omega'_j=\begin{cases}0 & \text{if}\quad j\le n\quad\text{or}\quad \bar{j}\ge j,\\ \epsilon_j & \text{if}\quad j>n\quad\text{and}\quad \bar{j}<j,\end{cases}$$

we have

$$\begin{aligned}&F^k_{ii}(\lambda_1,\dots,\lambda_k)\\ &\quad\equiv F^{k-1}_{ii}(\lambda_1,\dots,\lambda_{k-1})(F_i-\lambda_k+i-1-\omega_i)+\omega_i F^{k-1}_{\bar{i}\bar{i}}(\lambda_1,\dots,\lambda_{k-1})\\ &\qquad-\sum_{\mu=1}^{i-1}F^{k-1}_{\mu\mu}(\lambda_1,\dots,\lambda_{k-1})\mod U(\mathfrak g)\mathfrak n,\end{aligned}\tag{3.12}$$

$$\begin{aligned}&F^k_{ii+1}(\lambda_1,\dots,\lambda_k)\\ &\quad\equiv F^{k-1}_{ii+1}(\lambda_1,\dots,\lambda_{k-1})(F_{i+1}-\lambda_k+i-\omega'_{i+1})+\omega_i F^{k-1}_{\overline{i-1}\bar{i}}(\lambda_1,\dots,\lambda_{k-1})\\ &\qquad+F_{ii+1}F^{k-1}_{ii}(\lambda_1,\dots,\lambda_{k-1})\mod U(\mathfrak g)\mathfrak n.\end{aligned}\tag{3.13}$$

Now we give the main result in this section:

Proposition 3.4. *Suppose that $\mathfrak g=\mathfrak{gl}_n$ or $\mathfrak o_{2n}$ or $\mathfrak o_{2n+1}$ or $\mathfrak{sp}_n$ and that F is given by (3.4). Let $\Theta=\{n_1<n_2<\cdots<n_L=n\}$ be a sequence of positive integers. Put $n'_\nu=n_\nu-n_{\nu-1}$ for $\nu=1,\dots,L$ with $n_0=0$ and fix k with $1\le k\le L$. Let $\lambda_1,\dots,\lambda_k$ are complex numbers. Put $n_0=0$ and $n_\nu=n$ for $\nu>L$ and define*

$$\iota_\Theta(\nu)=p\quad\textit{if}\quad n_{p-1}<\nu\le n_p,$$
$$\tilde J(\lambda)_i=U(\mathfrak g)\mathfrak n+\sum_{\nu=1}^{i}U(\mathfrak g)(F_\nu-\lambda_{\iota_\Theta(\nu)}+n_{\iota_\Theta(\nu)-1}).$$

If $\mathfrak g=\mathfrak{gl}_n$, we put $H(\Theta,\lambda_1,\dots,\lambda_L)=F^L(\lambda_1,\dots,\lambda_L)$.
If $\mathfrak g=\mathfrak{sp}_n$ or $\mathfrak o_{2n}$, we put

$$\begin{aligned}H(\Theta,\lambda_1,\dots,\lambda_L)=F^{2L}(&\lambda_1,\dots,\lambda_L,\\ &-\lambda_1-n'_1+2n+\delta,\dots,-\lambda_L-n'_L+2n+\delta).\end{aligned}$$

If $\mathfrak g=\mathfrak o_{2n+1}$, we put

$$\begin{aligned}H(\Theta,\lambda_1,\dots,\lambda_L)=F^{2L+1}(&\lambda_1,\dots,\lambda_L,n,\\ &-\lambda_1-n'_1+2n,\dots,-\lambda_L-n'_{L-1}+2n).\end{aligned}$$

Moreover we define

$$\begin{aligned}\tilde H(\Theta,\lambda_1,\dots,\lambda_{L-1})=F^{2L-1}(&\lambda_1,\dots,\lambda_{L-1},n_{L-1},\\ &-\lambda_1-n'_1+2n+\delta,\dots,-\lambda_{L-1}-n'_{L-1}+2n+\delta).\end{aligned}$$

Here

$$\delta=\begin{cases}1 & \textit{if}\quad \mathfrak g=\mathfrak{sp}_n,\\ 0 & \textit{if}\quad \mathfrak g=\mathfrak o_{2n+1}\quad\textit{or}\quad\mathfrak{gl}_n,\\ -1 & \textit{if}\quad \mathfrak g=\mathfrak o_{2n}.\end{cases}\tag{3.14}$$

i) *The off-diagonal elements of $F^k(\lambda_1,\dots,\lambda_k)$ satisfy*

$$F^k_{ij}(\lambda_1,\dots,\lambda_k)\equiv 0\mod U(\mathfrak g)\mathfrak n\quad\textit{if}\quad i>j,$$
$$F^k_{ij}(\lambda_1,\dots,\lambda_k)\equiv 0\mod \bar{\mathfrak n}U(\mathfrak g)\quad\textit{if}\quad i<j.$$

ii) *If $i \le n$, then*

$$F_{ii}^k(\lambda_1,\ldots,\lambda_k) \equiv \begin{cases} 0 & \mod \tilde J(\lambda)_i \quad \text{if} \quad i\le n_k,\\ \prod_{\nu=1}^k(\lambda_{k+1}-\lambda_\nu-n'_\nu) & \mod \tilde J(\lambda)_i \quad \text{if} \quad n_k<i\le n_{k+1}.\end{cases}$$

iii) *If $i<n$, then*

$$F_{ii+1}^k(\lambda_1,\ldots,\lambda_k) \equiv \Bigl(\prod_{\nu=1}^{\ell-1}(\lambda_\ell-\lambda_\nu-n'_\nu-n_{\ell-1}+i)\prod_{\nu=\ell+1}^{k}(\lambda_\ell-\lambda_\nu-n_{\ell-1}+i)\Bigr)F_{ii+1}$$
$$\mod \tilde J(\lambda)_i \quad \text{if} \quad n_{\ell-1}<i<n_\ell \quad \text{and} \quad k\ge\ell.$$

iv) *Suppose $\mathfrak g=\mathfrak o_{2n}$ or $\mathfrak o_{2n+1}$ or $\mathfrak{sp}_n$. Then*

$$H_{ii}(\Theta,\lambda_1,\ldots,\lambda_L)\equiv 0 \mod \tilde J(\lambda)_n \quad \text{for} \quad i=1,\ldots,N.$$

In particular, if $\lambda_L=n_{L-1}$, then

$$\tilde H_{ii}(\Theta,\lambda_1,\ldots,\lambda_{L-1})\equiv 0 \mod \tilde J(\lambda)_n \quad \text{for} \quad i=1,\ldots,N$$

and

$$\tilde H_{nn+1}(\Theta,\lambda_1,\ldots,\lambda_{L-1}) \equiv (-1)^{L-1}\Bigl(\prod_{\nu=1}^{L-1}(\lambda_\nu+n'_\nu-n)(\lambda_\nu+n'_\nu-n-\delta)\Bigr)F_{nn+1} \mod U(\mathfrak g)\tilde J(\lambda)_n.$$

Proof. Put $F^k(\lambda)=F_k(\lambda_1,\ldots,\lambda_k)$ for simplicity. If $i<n$, it follows from (3.12) that

$$F_{i+1i+1}^k(\lambda)-F_{ii}^k(\lambda)\equiv F_{i+1i+1}^{k-1}(\lambda)(F_{i+1}-\lambda_k+i)-F_{ii}^{k-1}(\lambda)(F_i-\lambda_k+i) \mod U(\mathfrak g)\mathfrak n$$

and therefore by the induction on k we have

$$F_{ii}^k(\lambda)\equiv F_{i+1i+1}^k(\lambda) \mod U(\mathfrak g)\mathfrak n+U(\mathfrak g)(F_{i+1}-F_i). \tag{3.15}$$

Here we note that $F_{\nu+1}-F_\nu\in\tilde J(\lambda)_{n_\ell}$ if $n_{\ell-1}<\nu<n_\ell$. Hence we have

$$F_{ii}^k(\lambda)+\tilde J(\lambda)_{n_\ell}=F_{n_\ell n_\ell}^k(\lambda)+\tilde J(\lambda)_{n_\ell} \quad \text{for} \quad n_{\ell-1}<i\le n_\ell \quad \text{and} \quad 1\le\ell\le L.$$

Put $s_\nu=n_\nu-n_{\nu-1}$ and introduce polynomials $f(k,\ell)$ of $(\lambda_1,\ldots,\lambda_L,s_1,\ldots,s_L)$ with $\ell\le n$ so that

$$F_{n_\ell n_\ell}^k(\lambda_1,\ldots,\lambda_k)\equiv f(k,\ell) \mod \tilde J(\lambda)_{n_\ell}. \tag{3.16}$$

Similarly for i with $n_{\ell-1}<i<n_\ell$, we put $t=\lambda_\ell-n_{\ell-1}+i$ and define polynomials $g(k,\ell)$ of $(t,\lambda_1,\ldots,\lambda_L,s_1,\ldots,s_L)$ so that

$$F_{ii+1}^k(\lambda_1,\ldots,\lambda_k)\equiv g(k,\ell)E_{ii+1} \mod \tilde J(\lambda)_i. \tag{3.17}$$

Then we have

$$\begin{aligned} f(k,\ell)&=\begin{cases}1 & \text{if} \quad k=0,\\ f(k-1,\ell)(\lambda_\ell-\lambda_k)-\sum_{\nu=1}^{\ell-1}s_\nu f(k-1,\nu) & \text{if} \quad k\ge1,\end{cases}\\ g(k,\ell)&=\begin{cases}1 & \text{if} \quad k=1,\\ g(k-1,\ell)(t-\lambda_k)+f(k-1,\ell) & \text{if} \quad k>1.\end{cases}\end{aligned} \tag{3.18}$$

We will first prove $f(k,\ell)=0$ if $k\ge\ell$ by the induction on ℓ. Putting $\ell=1$ in (3.18), we have $f(k,1)=f(k-1,1)(\lambda_1-\lambda_k)$ and $f(1,1)=0$ and therefore

$f(k,1)=0$ for $k\geq 1$. Then if $k\geq \ell+1$, we have $f(k,\ell+1)=f(k-1,\ell+1)(\lambda_{\ell+1}-\lambda_k)-\sum_{\nu=1}^{\ell}s_\nu f(k-1,\nu)=f(k-1,\ell+1)(\lambda_{\ell+1}-\lambda_k)$ by the hypothesis of the induction. Hence we have $f(k,\ell+1)=0$ for $k\geq \ell+1$ by the induction on k.

Putting $\lambda_\ell=\lambda_{\ell-1}+s_{\ell-1}$ in (3.18), we have $f(k,\ell)-f(k,\ell-1)=f(k-1,\ell)-f(k-1,\ell)=\cdots=0$ and therefore $f(\ell-1,\ell)|_{\lambda_\ell=\lambda_{\ell-1}+s_{\ell-1}}=0$. Hence there exist polynomials $h(\ell)$ with $f(\ell-1,\ell)=h(\ell)(\lambda_\ell-\lambda_{\ell-1}-s_{\ell-1})$. Then (3.18) shows

$$h(\ell)(\lambda_\ell-\lambda_{\ell-1}-s_{\ell-1})=f(\ell-2,\ell)(\lambda_\ell-\lambda_{\ell-1})-s_{\ell-1}f(\ell-2,\ell-1).$$

It follows from (3.18) that $f(k,\ell)$ is a polynomial of degree at most 1 with respect to $s_{\ell-1}$ because $f(k,\nu)$ does not contain $s_{\ell-1}$ for $\nu<\ell$. Hence $h(\ell)=f(\ell-2,\ell)|_{s_{\ell-1}=0}$. Moreover by putting $s_{\ell-1}=0$ in (3.18), it is clear that $f(\ell-2,\ell)|_{s_{\ell-1}=0}$ does not contain $\lambda_{\ell-1}$. Hence $h(\ell)=f(\ell-2,\ell-1)|_{\lambda_{\ell-1}\mapsto\lambda_\ell}$ and we get

$$f(\ell-1,\ell)=\prod_{\nu=1}^{\ell-1}(\lambda_\ell-\lambda_\nu-s_\nu) \tag{3.19}$$

by the induction on ℓ. Thus we have ii).

Now we put

$$f(\ell-1,\ell)=\sum_{\nu=0}^{\ell-1}c(\nu,\ell)(\lambda_\ell-\lambda_{\nu+1})(\lambda_\ell-\lambda_{\nu+2})\cdots(\lambda_\ell-\lambda_{\ell-1}) \tag{3.20}$$

with homogeneous polynomials $c(\nu,\ell)$ of $(\lambda_1,\ldots,\lambda_{\ell-1},s_1,\ldots,s_{\ell-1})$ with degree ν. Here $c(\nu,\ell)$ does not contain λ_ℓ. Then by the induction on $k=\ell-1,\ell-2,\ldots,0$, (3.18) shows

$$\begin{aligned}f(k,\ell)&=\sum_{\nu=0}^{k}c(\nu,\ell)(\lambda_\ell-\lambda_{\nu+1})(\lambda_\ell-\lambda_{\nu+2})\cdots(\lambda_\ell-\lambda_k),\\ -\sum_{\nu=1}^{\ell-1}s_\nu f(k-1,\nu)&=c(k,\ell)\end{aligned} \tag{3.21}$$

because $\sum_{\nu=1}^{\ell-1}s_\nu f(k-1,\nu)$ does not contain λ_ℓ. We will show

$$g(\ell,\ell)=\sum_{k=0}^{\ell-1}(t-\lambda_\ell)(t-\lambda_{\ell-1})\cdots(t-\lambda_{k+2})f(k,\ell) \tag{3.22}$$

$$=\prod_{\nu=1}^{\ell-1}(t-\lambda_\nu-s_\nu). \tag{3.23}$$

Note that (3.22) is a direct consequence of (3.18). Denoting

$$g_k(\ell)=\sum_{\nu=0}^{\ell-1}c(\nu,\ell)(\lambda_\ell-\lambda_{\nu+1})\cdots(\lambda_\ell-\lambda_{k-1})(\lambda_\ell-\lambda_k)(t-\lambda_{k+1})\cdots(t-\lambda_{\ell-1})$$

for $k=0,\ldots,\ell-1$, we have

$$\begin{aligned}&g_{k-1}(\ell)-g_k(\ell)\\ &\quad=\sum_{\nu=0}^{k-1}c(\nu,\ell)(\lambda_\ell-\lambda_{\nu+1})\cdots(\lambda_\ell-\lambda_{k-1})(t-\lambda_\ell)(t-\lambda_{k+1})\cdots(t-\lambda_{\ell-1})\\ &\quad=(t-\lambda_\ell)(t-\lambda_{\ell-1})\cdots(t-\lambda_{k+1})f(k-1,\ell)\end{aligned}$$

from (3.21) and therefore (3.22) shows

$$g(\ell,\ell)=g_{\ell-1}(\ell)+\sum_{k=1}^{\ell-1}(g_{k-1}(\ell)-g_k(\ell))=g_0(\ell)=f(\ell-1,\ell)|_{\lambda_\ell\mapsto t},$$

which implies (3.23). Since $f(k,\ell)=0$ for $k\ge\ell$, (3.18) shows

$$g(k,\ell)=\prod_{\nu=1}^{\ell-1}(t-\lambda_\nu-s_\nu)\prod_{\nu=\ell+1}^{k}(t-\lambda_\nu)\quad\text{if}\ \ k\ge\ell, \tag{3.24}$$

from which iii) follows.

In general we have proved the following lemma.

Lemma 3.5. *The functions $f(k,\ell)$ and $g(k,\ell)$ of $\lambda_1,\lambda_2,\ldots,s_1,s_2,\ldots$ and t which are recursively defined by* (3.18) *satisfy* (3.19), (3.24) *and $f(k,\ell)=0$ for $k\ge\ell\ge1$.*

Now suppose $\mathfrak{g}=\mathfrak{sp}_n$ or $\mathfrak{o}_{2n}$. Then

$$\begin{aligned}F^k_{n+1n+1}(\lambda)&\equiv F^{k-1}_{n+1n+1}(\lambda)(F_{n+1}-\lambda_k)+\sum_{\nu=1}^{n}(F^{k-1}_{n+1n+1}(\lambda)-F^{n-1}_{\nu\nu}(\lambda))\\&\qquad+\delta(F^{k-1}_{n+1n+1}(\lambda)-F^{k-1}_{nn}(\lambda))\equiv0\mod U(\mathfrak{g})\mathfrak{n}.\end{aligned}\tag{3.25}$$

Hence

$$\begin{aligned}&F^k_{n+1n+1}(\lambda)-F^k_{nn}(\lambda)\\&\qquad\equiv F^{k-1}_{n+1n+1}(\lambda)(F_{n+1}-\lambda_k+n+\delta)-F^{k-1}_{nn}(\lambda)(F_n-\lambda_k+n+\delta)\\&\qquad\equiv0\mod U(\mathfrak{g})\mathfrak{n}+U(\mathfrak{g})(F_{n+1}-F_n)\end{aligned}\tag{3.26}$$

by the induction on k and

(3.27)
$$F^k_{n+1n+1}(\lambda)\equiv0\mod\sum_{\nu=1}^{n}U(\mathfrak{g})F^{k-1}_{\nu\nu}(\lambda)+U(\mathfrak{g})\mathfrak{n}+U(\mathfrak{g})(F_{n+1}-\lambda_k+n+\delta).$$

Since $F_{n+1}=-F_n$, we have from (3.25)

$$F^L_{n+1n+1}(\lambda_1,\ldots,\lambda_{L-1},n_{L-1})\equiv0\mod\tilde J(\lambda)_{n_{L-1}}+\sum_{\nu=n_{L-1}+1}^{n}U(\mathfrak{g})F_\nu$$

in the case $\lambda_L=n_{L-1}$ and from (3.27) with $-(\lambda_L-n_{L-1})-\lambda_{L+1}+n+\delta=0$

$$F^{L+1}_{n+1n+1}(\lambda_1,\ldots,\lambda_L,-\lambda_L+n_{L-1}+n+\delta)\equiv0\mod\tilde J(\lambda)_n.$$

Suppose $i<n$. Then

$$F^k_{\bar i\bar i}(\lambda)\equiv F^{k-1}_{\bar i\bar i}(\lambda)(F_{\bar i}-\lambda_k)+\sum_{\nu=1}^{\bar i-1}\bigl(F^{k-1}_{\bar i\bar i}(\lambda)-F^{k-1}_{\nu\nu}(\lambda)\bigr)+\delta(F^{k-1}_{\bar i\bar i}(\lambda)-F^{k-1}_{ii}(\lambda))\mod U(\mathfrak{g})\mathfrak{n}$$

and therefore

(3.28)
$$\begin{aligned}F^k_{\bar i+1\bar i+1}(\lambda)-F^k_{\bar i\bar i}(\lambda)&\equiv F^{k-1}_{\bar i+1\bar i+1}(\lambda)(F_{\bar i+1}-\lambda_k+\bar i+\delta)-F^{k-1}_{\bar i\bar i}(\lambda)(F_{\bar i}-\lambda_k+\bar i+\delta)\\&\qquad+\delta(F^{k-1}_{i-1i-1}(\lambda)-F^{k-1}_{ii}(\lambda))\mod U(\mathfrak{g})\\&\equiv0\mod U(\mathfrak{g})\mathfrak{n}+U(\mathfrak{g})(F_i-F_{i-1})+U(\mathfrak{g})(F_{\overline{i-1}}-F_{\bar i}),\end{aligned}$$

$$F^k_{\bar n_p\bar n_p}(\lambda)\equiv0\mod\sum_{\nu=1}^{\bar n_p-1}U(\mathfrak{g})F^{k-1}_{\nu\nu}(\lambda)+U(\mathfrak{g})\mathfrak{n}+U(\mathfrak{g})(F_{\bar n_p}-\lambda_k+\bar n_p-1+\delta).$$

Note that $F_{\bar n_p}-\lambda_k+\bar n_p-1+\delta=(F_{\bar n_p}+\lambda_p-n_{p-1})-\lambda_k-\lambda_p+n_{p-1}-n_p+2n+\delta$. Since $F_{\bar\nu}=-F_\nu$, we have

$$F^k_{\bar i+1\bar i+1}(\lambda)\equiv F^k_{\bar i\bar i}(\lambda)\mod\tilde J(\lambda)_n\quad\text{for}\quad\bar n_p\le\bar i<\bar n_{p-1}\tag{3.29}$$

and hence by the induction on $p=L,L-1,\ldots,1$, we have

$$(3.30)\quad F^{2L+1-p}_{\bar n_p\bar n_p}(\lambda_1,\dots,\lambda_L,-\lambda_L+n_{L-1}-n_L+2n+\delta,\\ \dots,-\lambda_p+n_{p-1}-n_p+2n+\delta)\equiv 0\mod \tilde J(\lambda)_n$$

and if $\lambda_L=n_{L-1}$, then

$$(3.31)\quad F^{2L-p}_{\bar n_p\bar n_p}(\lambda_1,\dots,\lambda_{L-1},n_{L-1},\lambda_{L-1}+n_{L-2}-n_{L-1}+2n+\delta,\\ \dots,-\lambda_p+n_{p-1}-n_p+2n+\delta)\equiv 0\mod \tilde J(\lambda)_n.$$

Suppose $\lambda_L=n_{L-1}$ and $\mathfrak g=\mathfrak{sp}_n$. Then from (3.13) we have
(3.32)

$$F^k_{nn+1}(\lambda)\equiv F^{k-1}_{nn+1}(\lambda)(F_{n+1n+1}-\lambda_k+n+\delta)+F_{nn+1}F^{k-1}_{nn}(\lambda)\mod U(\mathfrak g)\mathfrak n\\ \equiv F^L_{nn+1}(\lambda)\prod_{\nu=L+1}^{k}(-\lambda_\nu+n+\delta)\mod \tilde J(\lambda)_n\quad\text{if}\quad k\ge L.$$

It follows from Lemma 3.5 with $t=n+1$ that

$$\bar H_{nn+1}(\Theta,\lambda)\equiv F_{nn+1}\prod_{\nu=1}^{L-1}(-\lambda_\nu+n_{\nu-1}-n_\nu+n+\delta)\prod_{\nu=1}^{L-1}(\lambda_\nu-n_{\nu-1}+n_\nu-n)\\ \mod \tilde J(\lambda)_n.$$

Thus we have proved iv).

Lastly suppose $\mathfrak g=\mathfrak o_{2n+1}$. Note that $F_{n+1}=0$ and $F_{n+2}=-F_n$. Then

$$F^k_{n+1n+1}(\lambda)\equiv F^{k-1}_{n+1n+1}(F_{n+1}-\lambda_k)+\sum_{\nu=1}^{n}\big(F^{k-1}_{n+1n+1}(\lambda)-F^{k-1}_{\nu\nu}(\lambda)\big)\mod U(\mathfrak g)\mathfrak n\\ \equiv 0\mod \sum_{\nu=1}^{n}U(\mathfrak g)F^{k-1}_{\nu\nu}(\lambda)+U(\mathfrak g)(-\lambda_k+n),$$

$$F^k_{n+2n+2}(\lambda)\equiv F^{k-1}_{n+2n+2}(F_{n+2}-\lambda_k)+\sum_{\nu=1}^{n+1}\big(F^{k-1}_{n+2n+2}(\lambda)-F^{k-1}_{\nu\nu}(\lambda)\big)\\ -\big(F^{k-1}_{n+2n+2}(\lambda)-F^{k-1}_{nn}(\lambda)\big)\mod U(\mathfrak g)\mathfrak n\\ \equiv 0\mod \sum_{\nu=1}^{n+1}U(\mathfrak g)F^{k-1}_{\nu\nu}(\lambda)+U(\mathfrak g)(-F_n-\lambda_k+n)$$

and

$$F^{L+1}_{n+1n+1}(\lambda_1,\dots,\lambda_L,n)\equiv 0\mod \tilde J(\lambda)_n,\\ F^{L+2}_{n+2n+2}(\lambda_1,\dots,\lambda_L,n,-\lambda_L+n_{L-1}+n)\equiv 0\mod \tilde J(\lambda)_n,$$

Since

$$F^k_{n+1n+1}(\lambda)-F^k_{nn}(\lambda)\equiv F^{k-1}_{n+1n+1}(F_{n+1}-\lambda_k+n)-F^{k-1}_{nn}(F_n-\lambda_k+n)\\ \mod U(\mathfrak g)\mathfrak n$$

$$F^k_{n+2n+2}(\lambda)-F^k_{n+1n+1}(\lambda)\equiv F^{k-1}_{n+2n+2}(F_{n+2}-\lambda_k+n)-F^{k-1}_{n+1n+1}(F_{n+1}-\lambda_k+n)\\ -(F^{k-1}_{n+1n+1}(\lambda)-F^{k-1}_{nn}(\lambda))\mod U(\mathfrak g)\mathfrak n,$$

we have

$$F^k_{n+2n+2}(\lambda)\equiv F^k_{n+1n+1}(\lambda)\equiv F^k_{nn}(\lambda)\mod U(\mathfrak g)\mathfrak n+U(\mathfrak g)F_n$$

and

$$F^L_{n+1n+1}(\lambda)\equiv F^L_{n+2n+2}(\lambda)\equiv 0\mod \tilde J(\lambda)_n\quad\text{if}\quad \lambda_L=n_{L-1}.$$

Note that (3.28) is valid if $\bar{i} < n$. But since $F_{\bar{n}_p} - \lambda_k + \bar{n}_p - 1 + \delta = -\lambda_k - (F_{n_p} - \lambda_p + n_{p-1}) - \lambda_p + n_{p-1} - n_p + 2n$, we have

$$F^{2L+2-p}_{\bar{n}_p\bar{n}_p}(\lambda_1, \ldots, \lambda_L, n, -\lambda_L + n_{L-1} - n_L + 2n, \ldots, -\lambda_p + n_{p-1} - n_p + 2n) \equiv 0 \mod \tilde{J}(\lambda)_n$$

for $p = L, L-1, \ldots, 1$. Similarly we have (3.31) with $\delta = 0$ if $\lambda_L = n_{L-1}$. Moreover (3.32) is valid with $\delta = 0$ and we have iv) as in the case of $\mathfrak{g} = \mathfrak{sp}_n$. □

4. Generalized Verma modules

Retain the notation in the previous section. Let $\Theta = \{(0 <)n_1 < n_2 < \cdots < n_L(= n)\}$ be the sequence of strictly increasing positive integers ending at n. Put

$$H_\Theta = \sum_{k=1}^{L}\sum_{i=1}^{n_k} F_i \quad \text{and} \quad H_{\bar{\Theta}} = \sum_{k=1}^{L-1}\sum_{i=1}^{n_k} F_i.$$

Recall that $F_i = F_{ii}$, $F = \big(F_{ij}\big) \in M(N, \mathfrak{g})$, $\mathfrak{n} = \sum_{i>j} \mathbb{C}F_{ij}$, $\mathfrak{a} = \sum_i \mathbb{C}F_i$, $\bar{\mathfrak{n}} = \sum_{i<j} \mathbb{C}F_{ij}$ and $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \bar{\mathfrak{n}}$. Note that $F_{ij} = E_{ij}$ in the case $\mathfrak{g} = \mathfrak{gl}_n$ and $F_{ij} = E_{ij} + \sigma_{\mathfrak{g}}(E_{ij})$ in the case $\mathfrak{g} = \mathfrak{o}_{2n+1}$, $\mathfrak{sp}_n$ or $\mathfrak{o}_{2n}$. Here $\sigma_{\mathfrak{g}}$ is the involution of $\mathfrak{gl}_N$ to define $\mathfrak{g}$ in (3.1) so that $\mathfrak{g}$ is the subalgebra of $\mathfrak{gl}_N$ fixed by $\sigma_{\mathfrak{g}}$. Let G be the analytic subgroup of $GL(N, \mathbb{C})$ with the Lie algebra $\mathfrak{g}$. Namely $G = GL(n, \mathbb{C})$, $O(2n+1, \mathbb{C})$, $Sp(n, \mathbb{C})$ or $O(2n, \mathbb{C})$.

Define

$$(4.1) \qquad \begin{cases} \mathfrak{m}_\Theta = \{X \in \mathfrak{g};\, \mathrm{ad}(H_\Theta)X = 0\}, \\ \mathfrak{n}_\Theta = \{X \in \mathfrak{n};\, \langle X, \mathfrak{m}_\Theta\rangle = 0\},\ \bar{\mathfrak{n}}_\Theta = \{X \in \bar{\mathfrak{n}};\, \langle X, \mathfrak{m}_\Theta\rangle = 0\}, \\ \mathfrak{p}_\Theta = \mathfrak{m}_\Theta + \mathfrak{n}_\Theta. \end{cases}$$

We similarly define $m_{\bar{\Theta}}$, $\mathfrak{n}_{\bar{\Theta}}$, $\bar{\mathfrak{n}}_{\bar{\Theta}}$ and $\mathfrak{p}_{\bar{\Theta}}$ replacing Θ by $\bar{\Theta}$ in the above definition. Then $\mathfrak{n} = \mathfrak{n}_{\{1,2,\ldots,n\}}$, $\bar{\mathfrak{n}} = \bar{\mathfrak{n}}_{\{1,2,\ldots,n\}}$, $\mathfrak{a} = \mathfrak{a}_{\{1,2,\ldots,n\}}$ and $\mathfrak{p}_\Theta$ and $\mathfrak{p}_{\bar{\Theta}}$ are parabolic subalgebras of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b} = \mathfrak{n} + \mathfrak{a}$.

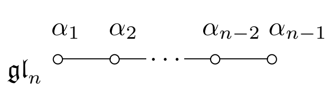
Let $\{e_1, \ldots, e_n\}$ be the dual bases of $\{F_1, \ldots, F_n\}$. Then the fundamental system $\Psi(\mathfrak{a})$ for the pair $(\mathfrak{n}, \mathfrak{a})$ is

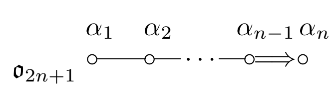
$$(4.2) \qquad \Psi(\mathfrak{a}) = \begin{cases} \{e_2 - e_1,\, e_3 - e_2, \ldots, e_n - e_{n-1}\} & \text{if } \mathfrak{g} = \mathfrak{gl}_n, \\ \{e_2 - e_1,\, e_3 - e_2, \ldots, e_n - e_{n-1}, -e_n\} & \text{if } \mathfrak{g} = \mathfrak{o}_{2n+1}, \\ \{e_2 - e_1,\, e_3 - e_2, \ldots, e_n - e_{n-1}, -2e_n\} & \text{if } \mathfrak{g} = \mathfrak{sp}_n, \\ \{e_2 - e_1,\, e_3 - e_2, \ldots, e_n - e_{n-1}, -e_n - e_{n-1}\} & \text{if } \mathfrak{g} = \mathfrak{o}_{2n}. \end{cases}$$

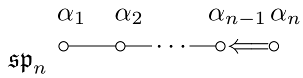
We put $\alpha_j = e_{j+1} - e_j$ for $j = 1, \ldots, n-1$ and $\alpha_n = -e_n$ or $-2e_n$ or $-e_n - e_{n-1}$ if $\mathfrak{g} = \mathfrak{o}_{2n+1}$ or $\mathfrak{sp}_n$ or $\mathfrak{o}_{2n}$, respectively. Then the fundamental system for $(\mathfrak{m}_\Theta \cap \mathfrak{n}, \mathfrak{a})$ is $\Psi(\mathfrak{a}) \setminus \{\alpha_{n_1}, \ldots, \alpha_{n_{L-1}}\}$ and that for $(\mathfrak{m}_{\bar{\Theta}} \cap \mathfrak{n}, \mathfrak{a})$ is

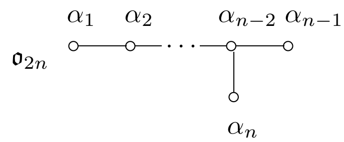
$$\begin{cases} \Psi(\mathfrak{a}) \setminus \{\alpha_{n_1}, \ldots, \alpha_{n_{L-1}}, \alpha_n\} & \text{if } \mathfrak{g} = \mathfrak{o}_{2n+1} \text{ or } \mathfrak{sp}_n, \\ \Psi(\mathfrak{a}) \setminus \{\alpha_{n_1}, \ldots, \alpha_{n_{L-1}}\} & \text{if } \mathfrak{g} = \mathfrak{o}_{2n} \text{ and } n_{L-1} \neq n-1, \\ \Psi(\mathfrak{a}) \setminus \{\alpha_{n_1}, \ldots, \alpha_{n_{L-1}}, \alpha_n\} & \text{if } \mathfrak{g} = \mathfrak{o}_{2n} \text{ and } n_{L-1} = n-1. \end{cases}$$

Then the Dynkin diagram of $\mathfrak{g}$ is as follows:









Fix $\lambda = (\lambda_1, \ldots, \lambda_L) \in \mathbb{C}^L$ and define a character λ_Θ of $\mathfrak{p}_\Theta$

$$\lambda_\Theta(X + \sum_{i=1}^{n} C_i F_i) = \sum_{i=1}^{n} C_i \lambda_{\iota_\Theta(i)} \quad \text{for } X \in \mathfrak{n}_\Theta + [\mathfrak{m}_\Theta, \mathfrak{m}_\Theta]. \tag{4.3}$$

We similarly define a character $\lambda_{\bar\Theta}$ of $\mathfrak{p}_{\bar\Theta}$ if $\lambda_L = 0$.

We introduce the *homogenized universal enveloping algebra*

$$U^\epsilon(\mathfrak{g}) = \left(\sum_{k=0}^{\infty} \otimes^k \mathfrak{g}\right) / \langle X \otimes Y - Y \otimes X - \epsilon[X, Y];\ X,\ Y \in \mathfrak{g}\rangle. \tag{4.4}$$

of $\mathfrak{g}$ as in [O4]. Here ϵ is a central element of $U^\epsilon(\mathfrak{g})$. Let $U^\epsilon(\mathfrak{g})^{(m)}$ be the image of $\sum_{k=0}^{m} \otimes^k \mathfrak{g}$ in $U^\epsilon(\mathfrak{g})$ and let $U^\epsilon(\mathfrak{g})^G$ be the subalgebra of G-invariants of $U^\epsilon(\mathfrak{g})$. Fix generators $\Delta_1, \ldots, \Delta_n$ of $U^\epsilon(\mathfrak{g})^G$ so that

$$\begin{cases} \Delta_j \in U^\epsilon(\mathfrak{g})^{(j)} \quad (1 \le j \le n) & \text{if } \mathfrak{g} = \mathfrak{gl}_n, \\ \Delta_j \in U^\epsilon(\mathfrak{g})^{(2j)} \quad (1 \le j \le n) & \text{if } \mathfrak{g} = \mathfrak{o}_{2n+1} \text{ or } \mathfrak{sp}_n, \\ \Delta_j \in U^\epsilon(\mathfrak{g})^{(2j)} \quad (1 \le j < n), \quad \Delta_n \in U^\epsilon(\mathfrak{g})^{(n)} & \text{if } \mathfrak{g} = \mathfrak{o}_{2n}. \end{cases} \tag{4.5}$$

If $\mathfrak{g} = \mathfrak{o}_{2n}$, we assume that Δ_n changes into $-\Delta_n$ by the outer automorphism of o_{2n} which maps $(F_1, \ldots, F_{n-1}, F_n)$ to $(F_1, \ldots, F_{n-1}, -F_n)$. Moreover put

$$\begin{cases} J^\epsilon_\Theta(\lambda) = \sum_{X \in \mathfrak{p}_\Theta} U^\epsilon(\mathfrak{g})(X - \lambda_\Theta(X)), \quad M^\epsilon_\Theta(\lambda) = U^\epsilon(\mathfrak{g})/J^\epsilon_\Theta(\lambda), \\ J^\epsilon_{\bar\Theta}(\lambda) = \sum_{X \in \mathfrak{p}_{\bar\Theta}} U^\epsilon(\mathfrak{g})(X - \lambda_{\bar\Theta}(X)), \quad M^\epsilon_{\bar\Theta}(\lambda) = U^\epsilon(\mathfrak{g})/J^\epsilon_{\bar\Theta}(\lambda), \\ J^\epsilon(\lambda_\Theta) = \sum_{X \in \mathfrak{b}} U^\epsilon(\mathfrak{g})(X - \lambda_\Theta(X)), \quad M^\epsilon(\lambda_\Theta) = U^\epsilon(\mathfrak{g})/J^\epsilon(\lambda_\Theta). \end{cases} \tag{4.6}$$

For a $U^\epsilon(\mathfrak{g})$-module M the annihilator of M is denoted by $\operatorname{Ann}(M)$ and put $\operatorname{Ann}_G(M) = \bigcap_{g \in G} \operatorname{Ad}(g) \operatorname{Ann}(M)$. Note that $\operatorname{Ann}_G(M) = \operatorname{Ann}(M)$ if $\epsilon \ne 0$. When $\epsilon = 1$, $U^\epsilon(\mathfrak{g})$ is the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ and we will sometimes omit the superfix ϵ for $J^\epsilon_\Theta(\lambda)$ and $M^\epsilon_\Theta(\lambda)$ etc. Then $M_\Theta(\lambda)$ and $M_{\bar\Theta}(\lambda)$ are generalized Verma modules which are quotients of the Verma module $M(\lambda_\Theta)$.

Remark 4.1. i) The parabolic subalgebra $\mathfrak{p}$ containing the Borel subalgebra $\mathfrak{b}$ uniquely corresponds to $\mathfrak{p}_\Theta$ or $\mathfrak{p}_{\Theta'}$ and therefore we will sometimes use the notation $\mathfrak{m}_\mathfrak{p}$, $\mathfrak{n}_\mathfrak{p}$, $\bar{\mathfrak{n}}_\mathfrak{p}$, $\lambda_\mathfrak{p}$, $J^\epsilon_\mathfrak{p}(\lambda)$, $M^\epsilon_\mathfrak{p}(\lambda)$ and $M^\epsilon(\lambda_\mathfrak{p})$ for $\mathfrak{m}_{\Theta'}$, $\mathfrak{n}_{\Theta'}$, $\bar{\mathfrak{n}}_{\Theta'}$, λ_Θ, $L^\epsilon_{\Theta'}(\lambda)$, $M^\epsilon_{\Theta'}(\lambda)$ and $M^\epsilon(\lambda_\Theta)$, respectively, by this correspondence. If $\mathfrak{o}_{2n}$ and $\lambda_L = 0$, we may put $\Theta' = \Theta$ or $\Theta' = \bar\Theta$.

ii) Suppose $\mathfrak{g} = \mathfrak{o}_{2n}$. Then we have not considered the parabolic subalgebra $\mathfrak{p}$ such that the fundamental system for $(\mathfrak{m}_\mathfrak{p}, \mathfrak{a})$ contains α_{n-1} and does not contains α_n. But this is reduced to the case when it contains α_n and does not contains α_{n-1} by the outer automorphism of $\mathfrak{o}_{2n}$. If $\lambda_L = 0$, then $M_\Theta(\lambda) = M_{\bar\Theta}(\lambda)$. Note that the condition $\lambda_L = 0$ corresponds to the fact that $M_\Theta(\lambda)$ is invariant under the outer automorphism.

Let $\rho \in \mathfrak{a}^*$ with $\rho(H) = \frac{1}{2}\,\mathrm{Trace}(\mathrm{ad}(H))|_{\mathfrak{n}}$ for $H \in \mathfrak{a}$. Then with δ in (3.14)

$$\rho = \begin{cases} \sum_{\nu=1}^{n}\big(\nu - \frac{n+1}{2}\big)e_\nu & \text{if } \mathfrak{g} = \mathfrak{gl}_n, \\ \sum_{\nu=1}^{n}\big(\nu - n - \frac{1}{2}\big)e_\nu = \sum_{\nu=1}^{n}\big(\nu - n - \frac{\delta+1}{2}\big)e_\nu & \text{if } \mathfrak{g} = \mathfrak{o}_{2n+1}, \\ \sum_{\nu=1}^{n}\big(\nu - n - 1\big)e_\nu = \sum_{\nu=1}^{n}\big(\nu - n - \frac{\delta+1}{2}\big)e_\nu & \text{if } \mathfrak{g} = \mathfrak{sp}_n, \\ \sum_{\nu=1}^{n}\big(\nu - n\big)e_\nu = \sum_{\nu=1}^{n}\big(\nu - n - \frac{\delta+1}{2}\big)e_\nu & \text{if } \mathfrak{g} = \mathfrak{o}_{2n}. \end{cases} \tag{4.7}$$

We define $\bar\lambda = (\bar\lambda_1, \dots, \bar\lambda_n) \in \mathbb{C}^n$ by

$$\lambda_\Theta|_{\mathfrak{a}} + \epsilon\rho = \bar\lambda_1 e_1 + \bar\lambda_2 e_2 + \cdots + \bar\lambda_n e_n. \tag{4.8}$$

For $P \in U^\epsilon(\mathfrak{g})$ let $\omega(P)$ and $\bar\omega(P)$ denotes the elements of $S(\mathfrak{a}) \simeq U^\epsilon(\mathfrak{a})$ with

$$\begin{aligned} &P - \omega(P) \in \bar{\mathfrak{n}}U^\epsilon(\mathfrak{g}) + U^\epsilon(\mathfrak{g})\mathfrak{n}, \\ &\bar\omega(P)(\mu + \epsilon\rho) = \omega(P)(\mu) \quad \text{for } \forall\mu \in \mathfrak{a}^*. \end{aligned} \tag{4.9}$$

Then $\bar\omega$ induces the Harish-Chandra isomorphism

$$\bar\omega : U^\epsilon(\mathfrak{g})^G \ \xrightarrow{\sim}\ S(\mathfrak{a})^W. \tag{4.10}$$

Here W is the Weyl group for the pair $(\mathfrak{g}, \mathfrak{a})$ and $S(\mathfrak{a})^W$ denotes the totality of W-invariants in $S(\mathfrak{a})$.

Definition 4.2. Retain the above notation and define polynomials

$$\begin{cases} q^\epsilon_\Theta(\mathfrak{gl}_n; x, \lambda) = \prod_{j=1}^{L}(x - \lambda_j - n_{j-1}\epsilon), \\ q^\epsilon_\Theta(\mathfrak{o}_{2n+1}; x, \lambda) = (x - n\epsilon)\prod_{j=1}^{L}(x - \lambda_j - n_{j-1}\epsilon)(x + \lambda_j + (n_j - 2n)\epsilon), \\ q^\epsilon_\Theta(\mathfrak{sp}_n; x, \lambda) = \prod_{j=1}^{L}(x - \lambda_j - n_{j-1}\epsilon)(x + \lambda_j + (n_j - 2n - 1)\epsilon), \\ q^\epsilon_\Theta(\mathfrak{o}_{2n}; x, \lambda) = \prod_{j=1}^{L}(x - \lambda_j - n_{j-1}\epsilon)(x + \lambda_j + (n_j - 2n + 1)\epsilon) \end{cases} \tag{4.11}$$

and if $\mathfrak{g} = \mathfrak{sp}_n$ or $\mathfrak{o}_{2n+1}$ or $\mathfrak{o}_{2n}$,

$$q^\epsilon_{\bar\Theta}(\mathfrak{g}; x, \lambda) = (x - n_{L-1}\epsilon)\prod_{j=1}^{L-1}(x - \lambda_j - n_{j-1}\epsilon)(x + \lambda_j + (n_j - 2n - \delta)\epsilon) \tag{4.12}$$

with the δ given by (3.3). Furthermore define two-sided ideals of $U^\epsilon(\mathfrak{g})$

$$\begin{cases} I^\epsilon_\Theta(\lambda) = \sum_{i=1}^{N}\sum_{j=1}^{N} U^\epsilon(\mathfrak{g})q^\epsilon_\Theta(\mathfrak{g}; F, \lambda)_{ij} + \sum_{j\in J} U^\epsilon(\mathfrak{g})\Big(\Delta_j - \omega(\Delta_j)(\lambda_\Theta)\Big), \\ I^\epsilon_{\bar\Theta}(\lambda) = \sum_{i=1}^{N}\sum_{j=1}^{N} U^\epsilon(\mathfrak{g})q^\epsilon_{\bar\Theta}(\mathfrak{g}; F, \lambda)_{ij} + \sum_{j\in \bar J} U^\epsilon(\mathfrak{g})\Big(\Delta_j - \omega(\Delta_j)(\lambda_\Theta)\Big) \end{cases} \tag{4.13}$$

with

$$\begin{cases} J = \{1, 2, \dots, L-1\} & \text{if } \mathfrak{g} = \mathfrak{gl}_n, \\ J = \{1, 2, \dots, L\},\ \ \bar J = \{1, 2, \dots, L-1\} & \text{if } \mathfrak{g} = \mathfrak{o}_{2n+1}, \\ J = \bar J = \{1, 2, \dots, L-1\} & \text{if } \mathfrak{g} = \mathfrak{sp}_n, \\ J = \bar J = \{1, 2, \dots, L-1\} \cup \{n\} & \text{if } \mathfrak{g} = \mathfrak{o}_{2n}. \end{cases} \tag{4.14}$$

Remark 4.3. i) Let $p(x)$ and $q(x)$ be monic polynomials with $q(x) \in \mathbb{C}[x]p(x)$. Then

$$\begin{cases} \sum_{i=1}^{N} p(F)_{ii} \in U^\epsilon(\mathfrak{g})^G, \\ \sum_{i=1}^{N} p(F)_{ii} - \sum_{i=1}^{N} F_i^{\deg p} \in \bar{\mathfrak{n}}U^\epsilon(\mathfrak{g}) + U^\epsilon(\mathfrak{g})\mathfrak{n} + U^\epsilon(\mathfrak{g})^{(\deg p - 1)}, \\ q(F)_{ij} \in \sum_{\substack{1\le\mu\le N \\ 1\le\nu\le N}} U^\epsilon(\mathfrak{g})p(F)_{\mu\nu}. \end{cases}$$

Hence it is clear

$$I^\epsilon_{\Theta'}(\lambda) \supset \sum_{D\in U^\epsilon(\mathfrak{g})^G} U^\epsilon(\mathfrak{g})(D - \omega(D)(\lambda_\Theta)) \quad \text{for } \Theta' = \Theta \text{ and } \bar{\Theta}. \tag{4.15}$$

Note that it is known that the right hand side of the above equals $\mathrm{Ann}_G(M^\epsilon(\lambda_\Theta))$.

ii) $I^\epsilon_\Theta(\lambda)$ and $I^\epsilon_{\bar\Theta}(\lambda)$ are homogeneous ideals with respect to $(\mathfrak{g}, \lambda, \epsilon)$.

Now we give the main theorem in this paper:

Theorem 4.4. i) *Let* $\mathfrak{g} = \mathfrak{gl}_n$, $\mathfrak{o}_{2n+1}$, $\mathfrak{sp}_n$ *or* $\mathfrak{o}_{2n}$. *Then*

$$\begin{cases} I^\epsilon_\Theta(\lambda) \subset \mathrm{Ann}\big(M^\epsilon_\Theta(\lambda)\big), \\ q^\epsilon_\Theta(\mathfrak{g}; F, \lambda)_{ii+1} \equiv r^\epsilon_i(\mathfrak{g}; \Theta, \lambda)F_{ii+1} \mod J^\epsilon(\lambda_\Theta) & \text{if } n_{k-1} < i < n_k, \\ J^\epsilon_\Theta(\lambda) = I^\epsilon_\Theta(\lambda) + J^\epsilon(\lambda_\Theta) & \text{if } r^\epsilon(\mathfrak{g}; \Theta, \lambda) \neq 0. \end{cases} \tag{4.16}$$

Here

$$\begin{aligned} r^\epsilon_i(\mathfrak{gl}_n; \Theta, \lambda) &= \prod_{\nu=1}^{k-1}\big(\lambda_k - \lambda_\nu - (n_\nu - i)\epsilon\big) \prod_{\nu=k+1}^{L}\big(\lambda_k - \lambda_\nu - (n_{\nu-1} - i)\epsilon\big) \\ &= \prod_{\nu=1}^{k-1}(\bar\lambda_i - \bar\lambda_{n_\nu}) \prod_{\nu=k+1}^{L}(\bar\lambda_{i+1} - \bar\lambda_{n_{\nu-1}+1}), \end{aligned}$$

$$\begin{aligned} r^\epsilon_i(\mathfrak{sp}_n; \Theta, \lambda) = r^\epsilon_i(\mathfrak{o}_{2n}; \Theta, \lambda) \\ &= r^\epsilon_i(\mathfrak{gl}_n; \Theta, \lambda) \prod_{\nu=1}^{L}\big(\lambda_k + \lambda_\nu + (n_\nu - 2n - \delta + i)\epsilon\big) \\ &= \prod_{\nu=1}^{k-1}(\bar\lambda_i - \bar\lambda_{n_\nu}) \prod_{\nu=k+1}^{L}(\bar\lambda_{i+1} - \bar\lambda_{n_{\nu-1}+1}) \prod_{\nu=1}^{L}(\bar\lambda_{i+1} + \bar\lambda_{n_\nu}), \end{aligned}$$

$$\begin{aligned} r^\epsilon_i(\mathfrak{o}_{2n+1}; \Theta, \lambda) &= r^\epsilon_i(\mathfrak{gl}_n; \Theta, \lambda)\big(\lambda_k - (n - i)\epsilon\big) \prod_{\nu=1}^{L}\big(\lambda_k + \lambda_\nu + (n_\nu - 2n + i)\epsilon\big) \\ &= \frac{1}{2}(\bar\lambda_i + \bar\lambda_{i+1}) \prod_{\nu=1}^{k-1}(\bar\lambda_i - \bar\lambda_{n_\nu}) \prod_{\nu=k+1}^{L}(\bar\lambda_{i+1} - \bar\lambda_{n_{\nu-1}+1}) \prod_{\nu=1}^{L}(\bar\lambda_{i+1} + \bar\lambda_{n_\nu}) \end{aligned}$$

with i *and* k *are integers with* $n_{k-1} < i < n_k$ *and*

$$r^\epsilon(\mathfrak{g}; \Theta, \lambda) = \prod_{k=1}^{L} \prod_{n_{k-1}<i<n_k} r^\epsilon_i(\mathfrak{g}; \Theta, \lambda). \tag{4.17}$$

ii) Suppose $\lambda_L = 0$. If $\mathfrak{g} = \mathfrak{sp}_n$ or $\mathfrak{o}_{2n+1}$ or $\mathfrak{o}_{2n}$, then

(4.18)
$$\begin{cases} I^\epsilon_{\bar\Theta}(\lambda) \subset \operatorname{Ann}\bigl(M^\epsilon_{\bar\Theta}(\lambda)\bigr), \\ q^\epsilon_{\bar\Theta}(\mathfrak{g}; F, \lambda)_{ii+1} \equiv r^\epsilon_i(\mathfrak{g}; \bar\Theta, \lambda) F_{ii+1} \mod J^\epsilon(\lambda_\Theta) & \text{if } \iota_\Theta(i) = \iota_\Theta(i+1), \\ q^\epsilon_{\bar\Theta}(\mathfrak{g}; F, \lambda)_{nn+1} \equiv \bar r^\epsilon(\mathfrak{g}; \bar\Theta, \lambda) F_{nn+1} \mod J^\epsilon(\lambda_\Theta) & \text{if } \mathfrak{g} \neq \mathfrak{o}_{2n}, \\ J^\epsilon_{\bar\Theta}(\lambda) = I^\epsilon_{\bar\Theta}(\lambda) + J^\epsilon(\lambda_\Theta) & \text{if } r^\epsilon(\mathfrak{g}; \bar\Theta, \lambda) \neq 0 \end{cases}$$

with denoting

$$\begin{aligned} r^\epsilon_i(\mathfrak{g}; \bar\Theta, \lambda) &= \prod_{\nu=1}^{k-1} \bigl(\lambda_k - \lambda_\nu - (n_\nu - i)\epsilon\bigr) \prod_{\nu=k+1}^{L-1} \bigl(\lambda_k - \lambda_\nu - (n_{\nu-1} - i)\epsilon\bigr), \\ &\quad \cdot \bigl(\lambda_k - (n-i)\epsilon\bigr) \prod_{\nu=1}^{L-1} (\lambda_k + \lambda_\nu + (n_\nu - 2n - \delta + i)\epsilon) \\ &= (\bar\lambda_{i+1} - \bar\lambda_n) \prod_{\nu=1}^{k-1} (\bar\lambda_i - \bar\lambda_{n_\nu}) \prod_{\nu=k+1}^{L-1} (\bar\lambda_{i+1} - \bar\lambda_{n_{\nu-1}+1}) \prod_{\nu=1}^{L} (\bar\lambda_{i+1} + \bar\lambda_{n_\nu}) \\ &\qquad \text{if } n_{k-1} < i < n_k, \end{aligned}$$

$$\begin{aligned} \bar r^\epsilon_o(\mathfrak{g}; \bar\Theta, \lambda) &= (-1)^{L-1} \prod_{\nu=1}^{L-1} \bigl(\lambda_\nu + (n_\nu - n)\epsilon\bigr)\bigl(\lambda_\nu + (n_\nu - n - \delta)\epsilon\bigr) \\ &= \begin{cases} (-1)^{L-1} \prod_{\nu=1}^{L-1} \bar\lambda_{n_\nu}(\bar\lambda_{n_\nu} - \bar\lambda_n) & \text{if } \mathfrak{g} = \mathfrak{sp}_n, \\ (-1)^{L-1} \prod_{\nu=1}^{L-1} (\bar\lambda_{n_\nu} - \bar\lambda_n)^2 & \text{if } \mathfrak{g} = \mathfrak{so}_{2n+1}, \end{cases} \end{aligned}$$

and

$$\begin{cases} r^\epsilon(\mathfrak{o}_{2n}; \bar\Theta, \lambda) = \prod_{k=1}^{L} \prod_{n_{k-1} < i < n_k} r^\epsilon_i(\mathfrak{g}; \bar\Theta, \lambda), \\ r^\epsilon(\mathfrak{g}; \bar\Theta, \lambda) = r^\epsilon(\mathfrak{o}_{2n}; \bar\Theta, \lambda) \bar r^\epsilon_o(\mathfrak{g}; \bar\Theta, \lambda) & \text{if } \mathfrak{g} = \mathfrak{sp}_n \text{ or } \mathfrak{o}_{2n+1}. \end{cases} \tag{4.19}$$

Proof. Note that the parameter λ_ν in Proposition 3.4 changes into $\lambda_\nu - n_{\nu-1}$ in the theorem. Then for $\Theta' = \Theta$ or $\bar\Theta$ Proposition 3.4 shows that

$$q^\epsilon_{\Theta'}(\mathfrak{g}; F, \lambda)_{ij} \equiv 0 \mod \bar{\mathfrak{n}} U^\epsilon(\mathfrak{g}) + J^\epsilon(\lambda_{\Theta'}),$$

which assures $I^\epsilon_{\Theta'}(\lambda) \subset \operatorname{Ann}\bigl(M^\epsilon_{\Theta'}(\lambda)\bigr)$ (cf. [O4, Lemma 2.11 and Remark 2.12]) because $M^\epsilon_{\Theta'}(\lambda)$ is irreducible $\mathfrak{g}$-module for generic (λ, ϵ) and $\sum \mathbb{C} q^\epsilon_{\Theta'}(\mathfrak{g}; F, \lambda)_{ij}$ is $\mathfrak{g}$-invariant.

Other statements of the theorem are direct consequences of Proposition 3.4. □

Remark 4.5. If the infinitesimal character of $M^\epsilon(\lambda_\Theta)$ is regular, $r^\epsilon(\mathfrak{g}; \Theta, \lambda) \neq 0$ and $r^\epsilon(\mathfrak{g}; \bar\Theta, \lambda) \neq 0$.

It is proved by [BG] and [Jo] that for $\mu \in \mathfrak{a}^*$ the map

$$\begin{aligned} &\{I;\ I \text{ is the two sided ideal of } U(\mathfrak{g}) \text{ with } I \supset \operatorname{Ann}\bigl(M(\mu)\bigr)\} \\ &\qquad \ni I \mapsto I + J(\mu) \in \{J;\ J \text{ is the left ideal of } U(\mathfrak{g}) \text{ with } J \supset J(\mu)\} \end{aligned} \tag{4.20}$$

is injective if μ is dominant

$$2\frac{\langle \mu + \rho, \alpha \rangle}{\langle \alpha, \alpha \rangle} \notin \{-1, -2, \ldots\} \quad \text{for any root } \alpha \text{ for the pair } (\mathfrak{n}, \mathfrak{a}). \tag{4.21}$$

Since $J_{\Theta'}(\lambda) = I_{\Theta'}(\lambda) + J(\lambda_{\Theta'}) \subset \operatorname{Ann}\bigl(M_{\Theta'}(\lambda)\bigr) + J(\lambda_{\Theta'}) \subset J_{\Theta'}(\lambda)$ by Theorem 4.4, we have the following corollary by this injectivity.

Corollary 4.6. *If $\lambda_\Theta|_{\mathfrak{a}} + \rho$ is dominant and $r^1(\mathfrak{g};\Theta',\lambda) \neq 0$, then*

$$\operatorname{Ann}\bigl(M_{\Theta'}(\lambda)\bigr) = I^1_{\Theta'}(\lambda)$$

for $\Theta' = \Theta$ or $\bar{\Theta}$.

Remark 4.7. Suppose $\mathfrak{g} = \mathfrak{gl}_n$. Then another generator system of $\operatorname{Ann}_G(M^\epsilon_\Theta(\lambda))$ is given for every $(\Theta,\epsilon,\lambda)$. It is interesting to express them by the generators constructed in this note, which is done by [Sa] when $\mathfrak{p}_\Theta$ is a maximal parabolic subalgebra. In the case of the minimal parabolic subalgebra, that is, in the case of the central elements of $U(\mathfrak{g})$, it is studied by [I1], [I2] and [Um]. In general, it may be considered as a generalization of Newton's formula for symmetric polynomials.

Remark 4.8. Considering the m-th exterior product of the natural representation of $\mathfrak{gl}_n$, we may put $p(E) = \bigl(E_{IJ}\bigr)_{\#I=\#J=m} \in M\bigl(\binom{n}{m}, U(\mathfrak{g})\bigr)$ in Lemma 2.1, where $I = \{i_1,\dots,i_m\}$, $J = \{j_1,\dots,j_m\}$ with $1 \le i_1 < \cdots < i_m \le n$ and $1 \le j_1 < \cdots < j_m \le n$ and $E_{IJ} = \det\bigl(E_{i_\mu j_\nu} + (\mu - m)\epsilon\delta_{i_\mu j_\nu}\bigr)_{\substack{1\le\mu\le m\\ 1\le\nu\le m}}$. Here $\det\bigl(A_{ij}\bigr) = \sum_{\sigma\in\mathfrak{S}_n} A_{\sigma(1)1}\cdots A_{\sigma(n)n}$. The study of $f(p(E))$ for polynomials $f(x)$ may be interesting because it may be a *quantization* of the ideals of the *rank varieties* (cf. [ES]) defined by the condition $\operatorname{rank} f(A) = m$ for $A \in M(n,\mathbb{C})$.

Remark 4.9. For $\mathfrak{g} = \mathfrak{o}_n$ or $\mathfrak{sp}_n$ we may expect an explicit generator system for $\operatorname{Ann}_G(M^\epsilon_{\mathfrak{p}}(\lambda))$ which are of the same type given by [O4] for $\mathfrak{gl}_n$. It should be a *quantization* of determinants and Pfaffians (and elementary divisors for the singular case). The quantization of Pfaffians for $\mathfrak{o}_n$ is studied by [I2], [IU] and [Od] etc. It is shown by [Od] that it gives $\operatorname{Ann}(M_{\mathfrak{p}}(\lambda))$ for the expected $\mathfrak{p}$.

Remark 4.10. We have considered $\sum_{i,j}\mathbb{C}f(p(E))_{ij}$ for the construction of a two-sided ideal of $U(\mathfrak{g})$ with a required property. We may pick up a $\mathfrak{g}$-invariant subspace V of $\sum_{i,j}\mathbb{C}f(p(E))_{ij}$ to get a refined result. Moreover for a certain problem (cf. [O1]) related to a symmetric pair $(\mathfrak{g},\mathfrak{k})$ it is useful to study $\mathfrak{k}$-invariant subspaces of $\sum_{i,j}\mathbb{C}f(p(E))_{ij}$ which should have required zeros under the map of Harish-Chandra homomorphism for the pair. This will be discussed in another paper [OSh].

In the case when $\epsilon = 0$ we have the following.

Theorem 4.11. *Let $\lambda \in \mathfrak{a}$ and suppose that the centralizer of λ in $\mathfrak{g}$ equals $\mathfrak{m}_{\Theta'}$ with $\Theta' = \Theta$ or $\bar{\Theta}$. Then*

$$I^0_{\Theta'}(\lambda) = \{f \in S(\mathfrak{g});\ f|_{\operatorname{Ad}(G)\lambda} = 0\}.$$

Proof. It is clear from Theorem 4.4 that the element of $I^0_{\Theta'}(\lambda)$ vanishes on λ and therefore $I^0_{\Theta'}(\lambda)$ vanishes on $\operatorname{Ad}(G)\lambda$ because $I^0_{\Theta'}(\lambda)$ is G-stable.

We will prove that the dimension of the space $\sum_{i=1}^N\sum_{j=1}^N \mathbb{C}dq^0_{\Theta'}(\mathfrak{g};F,\lambda)_{ij}|_\lambda$ is not smaller than $\dim\mathfrak{m}_{\Theta'}$. This is shown by the direct calculation and it is almost the same in any case and therefore we give it in the case when $\mathfrak{g} = \mathfrak{sp}_n$ and $\Theta' = \bar{\Theta}$.

Put $\Theta = \{n_1,\dots,n_L\}$ and $\lambda = (\lambda_1,\dots,\lambda_L)$. Note that $\lambda_L = 0$ and $q^0_{\bar{\Theta}}(\mathfrak{sp}_\mathfrak{n};x,\lambda) = x\prod_{1\le\nu<L}(x-\lambda_\nu)(x+\lambda_\nu)$. If $n_{k-1} \le i < n_k$ and $n_{k-1} \le j < n_k$ and $k < L$, we have

$$dq^0_{\bar{\Theta}}(\mathfrak{sp}_\mathfrak{n};F,\lambda)_{ij}|_{\lambda_\Theta} = 2\lambda_k^2 \prod_{1\le\nu<L,\ \nu\ne k}(\lambda_k-\lambda_\nu)(\lambda_k+\lambda_\nu)dF_{ij}.$$

If $n_{L-1} \le i < 2n - n_{L-1}$ and $n_{L-1} \le 2n - n_{L-1}$, then

$$dq^0_{\bar{\Theta}}(\mathfrak{sp}_\mathfrak{n};F,\lambda)_{ij}|_{\lambda_\Theta} = \prod_{1\le\nu<L}(-\lambda_\nu)(\lambda_\nu)dF_{ij}.$$

The assumption of the proposition implies $\lambda_k \neq 0$, $\lambda_\nu \neq 0$ and $\lambda_k^2 \neq \lambda_\nu^2$ in the above and therefore we get the required result.

Put $V = \{X \in \mathfrak{g};\, f(X) = 0\ (\forall f \in I_{\Theta'}(\lambda))\}$. Since $[\lambda, \mathfrak{g}] = \mathfrak{n}_\Theta + \bar{\mathfrak{n}}_\Theta$, the tangent space of $\mathrm{Ad}(G)\lambda$ at λ is isomorphic $\mathfrak{n}_\Theta + \bar{\mathfrak{n}}_\Theta$. Since $\mathrm{Ad}(G)\lambda \subset V$, it follows from the above calculation of the dimension that $\mathrm{Ad}(G)\lambda$ and V are equal in a neighborhood of λ. In particular, V is non-singular at λ.

Let $X \in \mathfrak{g}$ with $f(X) = 0$ for all $f \in I^0_{\Theta'}(\lambda)$. We will show $X \in \mathrm{Ad}(G)\lambda$, which completes the proof of the theorem. Let $X = X_s + X_n$ be the Jordan decomposition of X. Here X_s is semisimple and X_n is nilpotent. By the action of the element of $Ad(G)$, we may assume $X_s \in \mathfrak{a}$ and $X_n \in \mathfrak{n}$. Then it is clear that $f(X_s + tX_n) = 0$ for all $f \in I^0_{\Theta'}(\lambda)$ and $t \in \mathbb{C}$. Moreover it is also clear that X_s is a transformation of λ_Θ under a suitable element of the Weyl group of the root system for the pair $(\mathfrak{g}, \mathfrak{a})$ and therefore we may assume $X_s = \lambda$. Since the tangent space of V and λ is isomorphic to $\mathfrak{n}_\Theta + \bar{\mathfrak{n}}_\Theta$, we have $X_n \in \mathfrak{n}_\Theta$. Hence $X_n = 0$ because $[X_s, X_n] = 0$. and therefore $X \in \mathrm{Ad}(G)\lambda$. □

Remark 4.12. Theorem 4.11 shows that we have constructed a generator system of the defining ideal of the adjoint orbit of any semisimple element of any classical Lie algebra. In fact, for any $\lambda \in \mathfrak{a}$ in the orbit the centralizer of λ in $\mathfrak{g}$ is $\mathfrak{m}_\Theta$ or $\mathfrak{m}_{\bar{\Theta}}$ or $\mathfrak{g}$ with a suitable Θ.

On the other hand [O4] constructed a generator system of the ideal corresponding to the closure of an arbitrary conjugacy class of $\mathfrak{gl}_n$, which is of a different type from the one given here.

We will generalize the Cayley-Hamilton theorem in the linear algebra. Put

$$\bar{d}_{\mathfrak{g}}(x) = \begin{cases} \prod_{i=1}^{n} \bigl(x - F_i - \frac{n-1}{2}\epsilon\bigr) & \text{if } \mathfrak{g} = \mathfrak{gl}_n, \\ \prod_{i=1}^{n} \bigl(x - F_i - n\epsilon\bigr)\bigl(x + F_i - n\epsilon\bigr) & \text{if } \mathfrak{g} = \mathfrak{sp}_n, \\ \prod_{i=1}^{n} \bigl(x - F_i - (n-1)\epsilon\bigr)\bigl(x + F_i - (n-1)\epsilon\bigr) & \text{if } \mathfrak{g} = \mathfrak{o}_{2n}, \\ (x - n\epsilon) \prod_{i=1}^{n} \bigl(x - F_i - (n - \frac{1}{2})\epsilon\bigr)\bigl(x + F_i - (n - \frac{1}{2})\epsilon\bigr) & \text{if } \mathfrak{g} = \mathfrak{o}_{2n+1}. \end{cases}$$

Here we note that if $\Theta = \{1, 2, \ldots, n\}$, then $n_j = j$, $\lambda_i + n_{i-1}\epsilon = \bar{\lambda}_i + (n + \frac{\delta-1}{2})\epsilon$ and $\lambda_i + (n_i - 2n - \delta)\epsilon = \bar{\lambda}_i - (n + \frac{\delta-1}{2})\epsilon$. Since $\bar{d}_{\mathfrak{g}}(x) \in S(\mathfrak{a})^W[x]$, there exists $d_{\mathfrak{g}}(x) \in U^\epsilon(\mathfrak{g})^G[x]$ with

$$\bar{\omega}(d_{\mathfrak{g}}(x)) = \bar{d}_{\mathfrak{g}}(x), \tag{4.22}$$

which is equivalent to $d_{\mathfrak{g}}(x) \equiv \bar{d}_{\mathfrak{g}}(x)(\mu) \mod J^\epsilon(\mu - \epsilon\rho)$. Then Theorem 4.4 assures $d_{\mathfrak{g}}(F) \equiv \bar{d}_{\mathfrak{g}}(F)(\mu) \equiv 0 \mod J^\epsilon(\mu - \epsilon\rho)$. Hence $\bar{\omega}(d_{\mathfrak{g}}(F))(\mu) = 0$ for any $\mu \in \mathfrak{a}^*$ and therefore $\bar{\omega}(d_{\mathfrak{g}}(F)) = 0$, which assures $d_{\mathfrak{g}}(F) = 0$ because $\sum_{i,j} \mathbb{C} d_{\mathfrak{g}}(F)_{ij}$ is $\mathfrak{g}$-invariant (cf. [O4, Lemma 2.12]). Thus we have the following corollary.

Corollary 4.13 (The Cayley-Hamilton theorem for the natural representation of the classical Lie algebra $\mathfrak{g}$).

$$d_{\mathfrak{g}}(F) = 0.$$

Remark 4.14. This result for $\mathfrak{gl}_n$ and $\mathfrak{o}_n$ is given by [Um] and [I2], respectively. A much general result is given by [Go2] (cf. [OO]).

Remark 4.15. Suppose $\mathfrak{g} = \mathfrak{gl}_n$. Then it follows from [O3] that

$$d_{\mathfrak{g}}(x) = \det\Bigl(x - F_{ij} - (i - n)\epsilon\delta_{ij}\Bigr)_{\substack{1 \le i \le n \\ 1 \le j \le n}}.$$

In [O4] we define another generator system of $\mathrm{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr)$ for any $(\Theta,\lambda,\epsilon)$ by using "elementary divisors" in place of the "minimal polynomial" $q^\epsilon_\Theta(\mathfrak{gl}_n;x,\lambda)$.

Proposition 4.16. *Suppose $\mathfrak{g}=\mathfrak{gl}_n$ and let π be its natural representation. Then the characteristic polynomial of $F=\bigl(E_{ij}\bigr)$ in $U^\epsilon(\mathfrak{g})$ equals $d^\epsilon_{\mathfrak{g}}(x)$ and the minimal polynomial of $(F,M^\epsilon_\Theta(\lambda))$ equals $q^\epsilon_\Theta(\mathfrak{g};x,\lambda)$.*

Proof. Suppose $\epsilon=0$ and identify the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ with $\mathfrak{g}$ by the bilinear form (2.1). Put $J^0_\Theta(\lambda)^\perp=\{X\in\mathfrak{g}^*;\langle X,Y\rangle=0\ (\forall Y\in J^0_\Theta(\lambda))\}$. Then the condition $q(F)M^0_\Theta(\lambda)=0$ for a polynomial $q(x)$ is equivalent to $q(F)(J^0_\Theta(\lambda)^\perp)=0$, which also equivalent to $q(A_\Theta(\lambda))=0$ for a generic element $A_{\Theta,\lambda}$ of $J^0_\Theta(\lambda)^\perp$ because the closure of $\bigcup_{g\in GL(n,\mathbb{C})}gA_{\Theta,\lambda}g^{-1}$ equals $\bigcup_{g\in GL(n,\mathbb{C})}g(J^0_\Theta(\lambda)^\perp)g^{-1}$ (cf. [O4, §2]). In fact

$$A_{\Theta,\lambda}=\begin{pmatrix}\lambda_1 I_{n'_1} & & & & \text{\Large 0}\\ A_{21} & \lambda_2 I_{n'_2} & & & \\ A_{31} & A_{32} & \lambda_3 I_{n'_3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ A_{L1} & A_{L2} & A_{L3} & \cdots & \lambda_L I_{n'_L}\end{pmatrix}$$

with generic $A_{ij}\in M(n'_i,n'_j,\mathbb{C})$. Hence our minimal polynomial is the same as that of $A_{\Theta,\lambda}$ in the linear algebra and the claim in the lemma for the minimal polynomial is clear.

We may assume $\epsilon=1$. Let $p(x)$ be the minimal polynomial of $(F,M_\Theta(\lambda))$ with a fixed λ. Define a homogeneous and monic polynomial $p(x,\epsilon)$ of (x,ϵ) with $p(x)=p(x,1)$. Then $p(x,\epsilon)M^\epsilon_\Theta(\epsilon\lambda)=0$ for $\epsilon\in\mathbb{C}$. If follows from the result in the case $\epsilon=0$ that the degree of $p(x)$ should not be smaller than that of $q_\Theta(\mathfrak{g};x,\lambda)$. Hence $q_\Theta(\mathfrak{g};x,\lambda)$ is the minimal polynomial for $M^\epsilon_\Theta(\lambda)$.

Since the degree of the minimal polynomial $q_\Theta(\mathfrak{g};x,\lambda)$ for $\Theta=\{1,2,\dots,n\}$ is n, the degree of the characteristic polynomial is not smaller than n. Hence $d_{\mathfrak{g}}(x)$ is the characteristic polynomial. □

Remark 4.17. Let $\mathfrak{g}$ is $\mathfrak{o}_{2n+1}$, $\mathfrak{sp}_n$ or $\mathfrak{o}_{2n}$ and let F be the matrix defined through the natural representation of $\mathfrak{g}$. Then it may be expected that $d_{\mathfrak{g}}(x)$ is the characteristic polynomial of F and $q^\epsilon_{\Theta'}(\mathfrak{g};x,\lambda)$ is the minimal polynomial of $(F,M^\epsilon_{\Theta'}(\lambda))$ with $\Theta=\Theta$ or Θ'. We will not discuss this problem in this note but it should be remarked that these concept is the motivation of the construction of our two-sided ideal $I^\epsilon_{\Theta'}(\lambda)$.

Definition 4.18. The non-zero element $q(x,\lambda,\epsilon)\in\mathbb{C}[x,\lambda,\epsilon]$ is called the *global minimal polynomial* of $(F,M_{\mathfrak{p}}(\lambda))$ if $q(x,\lambda,\epsilon)$ satisfies $q(F,\lambda,\epsilon)M_{\mathfrak{p}}(\lambda)=0$ for any (λ,ϵ) in the parameter space and any other non-zero polynomial whose degree with respect to x is smaller than that of $q(x,\lambda,\epsilon)$ does not satisfies this.

Theorem 4.19. *The polynomials $q^\epsilon_{\Theta'}(\mathfrak{g};x,\lambda)$ in Definition 4.2 are the global minimal polynomials of $(F,M_{\Theta'}(\lambda))$ for $\Theta'=\Theta$ and $\bar\Theta$.*

Proof. Put $\epsilon=0$ and consider the generic λ. Then the minimality of the degree of the polynomial is clear by evaluating $q_{\Theta'}(\mathfrak{g};F,\lambda)$ at generic λ. □

5. Integral transforms on generalized flag manifolds

Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\mathfrak{p}$ be a parabolic subalgebra containing a Borel subalgebra $\mathfrak{b}$. For a holomorphic character λ of $\mathfrak{p}$ we define left ideals

$$\begin{cases}J_{\mathfrak{p}}(\lambda)=\sum_{X\in\mathfrak{p}}(X-\lambda(X)),\\ J_{\mathfrak{b}}(\lambda)=\sum_{X\in\mathfrak{b}}(X-\lambda(X))\end{cases}\tag{5.1}$$

of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Let $I_{\mathfrak{p}}(\lambda)$ be the two-sided ideal of $U(\mathfrak{g})$ which satisfies

$$I_{\mathfrak{p}}(\lambda) \subset J_{\mathfrak{p}}(\lambda). \tag{5.2}$$

Let G be a connected real semisimple Lie group and let P be a parabolic subgroup of P such that the complexifications of Lie(G) and Lie(P) equal $\mathfrak{g}$ and $\mathfrak{p}$, respectively. Let L_λ be a line bundle over G/P such that the local section of L_λ is killed by $J_{\mathfrak{p}}(\lambda)$. Then the image of any $\mathfrak{g}$-equivalent map of the space of sections of L_λ over an open subset of G/P is killed by $I_{\mathfrak{p}}(\lambda)$. Here the element of $I_{\mathfrak{p}}(\lambda)$ is identifies with a left invariant differential operator but it may be identified with a right invariant differential operator through the anti-automorphism of $U(\mathfrak{g})$ ($X \mapsto -X$, $XY \mapsto (-Y)(-X)$ for $X,\ Y \in \mathfrak{g}$) because $I_{\mathfrak{p}}(\lambda)$ is a two-sided ideal. If the $\mathfrak{g}$-equivariant map is an integral transform to the space of functions on a homogeneous space X of G or sections of a vector bundle over X, it is a natural question how the system of differential equations induced from $I_{\mathfrak{p}}(\lambda)$ characterizes the image.

The same problem may be considered when L_λ is the holomorphic line bundle over the complexification of G/P.

5.1. **Penrose transformations.** Let $G_{\mathbb{C}}$ be a reductive complex Lie group with the Lie algebra $\mathfrak{g}$. Let G be a real from of $G_{\mathbb{C}}$ and let $P_{\mathbb{C}}$ be a parabolic subalgebra of $G_{\mathbb{C}}$ with the Lie algebra $\mathfrak{p}$ and let V be a G-orbit in $G_{\mathbb{C}}$. Suppose $\mathcal{O}_\lambda$ is a holomorphic line bundle over $G_{\mathbb{C}}/P_{\mathbb{C}}$ which is killed by $J_{\mathfrak{p}}(\lambda)$. Then the image of any G-equivariant map

$$\mathcal{T} : H^*_V(G_{\mathbb{C}}/P_{\mathbb{C}}, \mathcal{O}_\lambda) \to E, \tag{5.3}$$

is killed by $I_{\mathfrak{p}}(\lambda)$.

This is obvious because $I_{\mathfrak{p}}(\lambda)$ is a two-sided ideal. Here E is usually a space of sections of a certain line (or vector) bundle over a homogeneous space of G. In this case $I_{\mathfrak{p}}(\lambda)$ is identified with a system of differential equations and we may identify the element of $I_{\mathfrak{p}}(\lambda)$ as a right invariant differential operator on G through the anti-automorphism of the universal enveloping algebra or a left invariant differential operator on G.

5.2. **Poisson transformations.** Let G be a connected semisimple Lie group with finite center, let K be a maximal compact subgroup of G and let P be a parabolic subalgebra of G with the Langlands decomposition $P = MAN$ and let P_o be a minimal parabolic subgroup with the Langlands decomposition $P_o = M_oA_oN_o$ satisfying $M_o \subset M$, $A_o \supset A$, $N_o \supset N$ and $P_o \subset P$. Let λ be an element of the complexification $\mathfrak{a}^*$ of the dual of the Lie algebra of A and put

$$\mathcal{B}(G/P, L_\lambda) = \{f \in \mathcal{B}(G); f(xman) = a^\lambda f(x) \quad (\forall m \in M,\ \forall a \in A,\ \forall n \in N)\}$$

which is the space of hyperfunction sections of spherical degenerate principal series. Let $\mathfrak{p}$ be a complexification of the Lie algebra of P. The Poisson transformation of the space $\mathcal{B}(G/P, L_\lambda)$ is defined by

$$\mathcal{P}^\lambda : \mathcal{B}(G/P, L_\lambda) \to \mathcal{B}(G/K),\ f \mapsto (\mathcal{P}^\lambda f)(x) = \int_K f(xk)dk \tag{5.4}$$

with the normalized Haar measure dk on K. Let $\mathbb{D}(G/K)$ be the ring of invariant differential operator of G and let χ_λ be the algebra homomorphism of $\mathbb{D}(G/K)$ to $\mathbb{C}$ so that the image of $\mathcal{P}_\lambda$ is in the solution space $\mathcal{A}(G/K, \mathcal{M}_\lambda)$ of the system

$$\mathcal{M}_\lambda : Du = \chi_\lambda(D)u \quad (\forall D \in \mathbb{D}(G/K)) \tag{5.5}$$

for $u \in \mathcal{A}(G/K)$. Here $\mathcal{A}(G/K)$ denotes the space of real analytic functions on G/K.

Note that $\mathcal{B}(G/P, L_\lambda)$ is the subspace of the space of hyperfunction sections of spherical principal series

$$\mathcal{B}(G/P_o, L_\lambda) = \{f \in \mathcal{B}(G); f(xman) = a^\lambda f(x) \quad (\forall m \in M_o,\ \forall a \in A_o,\ \forall n \in N_o)\}.$$

Here λ is extended to the complexification $\mathfrak{a}_o^*$ of the dual of the Lie algebra $\mathfrak{a}_0$ of A_o so that it takes the value 0 on $\mathrm{Lie}(M) \cap \mathrm{Lie}(A_o)$.

Theorem 5.1. *Suppose that the Poisson transform*

$$\mathcal{P}_o^\lambda : \mathcal{B}(G/P_o, L_\lambda) \to \mathcal{A}(G/K, \mathcal{M}_\lambda),\ f \mapsto (\mathcal{P}_o^\lambda f)(x) = \int_K f(xk)dk \tag{5.6}$$

for the boundary G/P_o of G/K is bijective. Assume the condition

$$J_\mathfrak{p}(\lambda) = I_\mathfrak{p}(\lambda) + J_\mathfrak{b}(\lambda) \tag{5.7}$$

for a two-sided ideal $I_\mathfrak{p}(\lambda)$ of $U(\mathfrak{g})$. Then the Poisson transform $\mathcal{P}^\lambda$ for the boundary G/P is a G-isomorphism onto the simultaneous solution space of the system $\mathcal{M}_\lambda$ and the system defined by $I_\mathfrak{p}(\lambda)$.

Proof. Since $\mathcal{B}(G/P, L_\lambda)$ is a subspace of $\mathcal{B}(G/P_o, L_\lambda)$ and $\mathcal{P}^\lambda$ is a G-equivariant map, the image of $\mathcal{P}_o^\lambda$ satisfies the systems $\mathcal{M}_\lambda$ and $I_\mathfrak{p}(\lambda)$.

Suppose the function $u \in \mathcal{A}(G/K, \mathcal{M}_\lambda)$ satisfies $I_\mathfrak{p}(\lambda)$. Since the function $(\mathcal{P}_o^\lambda)^{-1}u \in \mathcal{B}(G/P_o, L_\lambda)$ also satisfies $I_\mathfrak{p}(\lambda)$, the condition (5.7) assures $(\mathcal{P}_o^\lambda)^{-1}u \in \mathcal{B}(G/P, L_\lambda)$ because we may assume $\mathbb{C} \otimes_\mathbb{R} \mathrm{Lie}(P_o) \supset \mathfrak{b}$. □

Remark 5.2. i) The above theorem with its proof is based on the idea given by [O3] which explains it in the case when $G = GL(n, \mathbb{R})$.

ii) The bijectivity of $\mathcal{P}_o^\lambda$ is equivalent to the condition $e(\lambda+\rho) \neq 0$ by [K–]. This condition is introduced by [He] for the injectivity of $\mathcal{P}_o^\lambda$. Here

$$e(\lambda) = \prod_{\alpha \in \Sigma_o^+} \left\{ \Gamma\left(\frac{\langle \lambda, \alpha\rangle}{2\langle \alpha, \alpha\rangle} + \frac{m_\alpha}{4} + \frac{1}{2}\right) \Gamma\left(\frac{\langle \lambda, \alpha\rangle}{2\langle \alpha, \alpha\rangle} + \frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2}\right)\right\}, \tag{5.8}$$

Σ^+ is the set of the positive system for the pair $(\mathfrak{g}, \mathfrak{a}_0)$ so that $\mathrm{Lie}(N)$ corresponds to the positive root spaces. Moreover $\Sigma_o^+ = \{\alpha \in \Sigma^+;\ \frac{1}{2}\alpha \notin \Sigma^+\}$, m_α is the multiplicity of the root $\alpha \in \Sigma^+$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$.

iii) Suppose G is simple and of the classical type and suppose the condition $e(\lambda + \rho) \neq 0$. Let $I_\mathfrak{p}(\lambda)$ be the system given by (4.13). Then if moreover the infinitesimal character of $\mathcal{B}(G/P, L_\lambda)$ is regular, $\mathcal{P}^\lambda$ is G-isomorphic to the solution space of the system of differential equations $I_\mathfrak{p}(\lambda)$ on G/K since Theorem 4.4 assures (5.7). This is because the natural map of $U(\mathfrak{g})^G$ to $\mathbb{D}(G/K)$ is surjective and therefore it follows from Remark 4.3 i) that $\mathcal{M}_\lambda$ is contained in $I_\mathfrak{p}(\lambda)$. Here the function on G/K is identified with the right K-invariant function on G. Note that all the assumption are valid when $\lambda = 0$.

iv) Owing to [K–] the abstract existence of the system of differential equations characterizing the image of $\mathcal{P}^\lambda$ is clear (cf. [OSh]) but a certain existence theorem of the system in the case $\lambda = 0$ is given by [Jn]. More precise study for this problem including the relation to the *Hua operators* will be discussed in [OSh].

5.3. **Radon transformations.** Let G be a semisimple Lie group of G and let P_1 and P_2 be maximal parabolic subgroups of G. For characters λ_j of P_j we put

$$\mathcal{B}(G/P_j, L_{\lambda_j}) = \{f \in \mathcal{B}(G);\ f(xp) = \lambda_j(p)f(x)\ (\forall p \in P_j)\}$$

for $j = 1$ and 2. If there exists a G-equivariant map $\mathcal{R} : \mathcal{B}(G/P_1, L_{\lambda_1}) \to \mathcal{B}(G/P_2, L_{\lambda_2})$, then the image of $\mathcal{R}$ satisfies the system $I_\mathfrak{p}(\lambda)$. Here $\mathfrak{p}$ and λ correspond to P_1 and λ_1, respectively.

Some special cases of these transformations and their relations to Aomoto-Gelfand hypergeometric functions are discussed in [O3], [Se] and [Ta].

6. Closure of ideals

Now we will consider the non-regular λ which are excluded in Theorem 4.4. We begin with a general consideration.

Definition 6.1. Let M be a C^∞-manifold and let U be an open subset of $\mathbb{C}^\ell$. We denote by $\mathcal{D}'(M)$ the space of distributions on M. Suppose that meromorphic functions $f_1(\lambda),\ldots,f_n(\lambda)$ of U with values in $\mathcal{D}'(M)$ are given. Moreover suppose there exists a non-zero holomorphic function $r(\lambda)$ on U such that $f_1,\ldots,f_n$ are holomorphic on $U_r=\{\lambda\in U;\, r(\lambda)\neq 0\}$ and $\dim V_\lambda = m$ for any $\lambda\in U_r$. For $\lambda\in U$ we define

$$\begin{aligned}\bar V_\mu = \{f(0);\ & f \text{ is a holomorphic function on } \{t\in\mathbb{C};\,|t|<1\} \text{ valued in } \mathcal{D}'(M)\\ & \text{and there exists a holomorphic curve } c:\{t\in\mathbb{C};\,|t|<1\}\to U \text{ such that}\\ & c(t)\in U_r \text{ and } f(t)\in V_{c(t)} \text{ for } 0<|t|\ll 1 \text{ and } c(0)=\mu\}.\end{aligned}$$

We call $\bar V_\mu$ the *closure* of the holomorphic family of the spaces V_λ ($\lambda\in U_r$) at μ. It follows from [OS, Proposition 2.21] that $\dim \bar V_\mu \geq m$. We define that a point $\mu\in U\setminus U_r$ is a *removable* (resp. *un-removable*) *singular point* if $\dim V_\mu = m$ (resp. $\dim V_\mu > m$). Note that $\bar V_\lambda = V_\lambda$ if $\lambda\in U_r$, which follows from the last statement in Lemma 6.3 by replacing μ and U_r by λ ($\in U_r$) and $U_r\setminus\{\lambda\}$.

Example 6.2. The origin $\lambda=(\lambda_1,\lambda_2)=0$ is a removable singular point of $V_\lambda = \mathbb{C}(x+\lambda_1)+\mathbb{C}(\lambda_2 x+\lambda_1 y_2+\lambda_1^2)$ and an un-removable singular point of $V_\lambda=\mathbb{C}(\lambda_1 x+\lambda_2 y)$.

Lemma 6.3. i) *If μ is a removable singular point of the spaces V_λ, then there exist a neighborhood U_μ of μ and holomorphic functions $h_1(\lambda),\ldots,h_m(\lambda)$ on U_μ valued in $\mathcal{D}'(M)$ such that they are linearly independent for any $\lambda\in U_\mu$ and they span V_λ for any $\lambda\in U_\mu\cap U_r$. On the other hand, the existence of $h_j(\lambda)$ $(j=1,\ldots,m)$ with these property implies that μ is a removable singular point.*

ii) *If U is convex and there is no un-removable singular point in U, we may choose $U_\mu=U$ in* i).

Proof. i) Suppose $\dim\bar V_\mu = m$. We may assume $f_1(\lambda),\ldots,f_m(\lambda)$ are linearly independent for a generic point λ in U_r. Fix a curve c to U with $c(0)=\mu$ and $c(t)\in U_r$ for $0<|t|\ll 1$. Then [OS, Proposition 2.21] assures the existence of holomorphic functions $v_i(t)$ $(1\le i\le m)$ on $\{t\in\mathbb{C};\,|t|<1\}$ valued in $\mathcal{D}'(M)$ and a holomorphic curve $c:\{t\in\mathbb{C};\,|t|<1\}\to U$ such that $c(0)=\mu$, $c(t)\in U_r$ and $v_i(t)\in V_{c(t)}$ for $0<|t|\ll 1$ and $v_1(t),\ldots,v_m(t)$ are linearly independent for any t. Then the set $\{v_1(0),\ldots,v_m(0)\}$ is a basis of V_μ. Fix test functions $\phi_1,\ldots,\phi_m$ so that $\langle v_i(0),\phi_j\rangle=\delta_{ij}$ and put $c_{ij}(\lambda)=\langle f_i(\lambda),\phi_j\rangle$. If $0<|t|\ll 1$, then $v_i(t),\ldots,v_m(t)$ span $V_{c(t)}=\sum_{i=1}^m \mathbb{C}f_i(c(t))$ and therefore $f_i(c(t))=\sum_{j=1}^m c_{ij}(c(t))v_j(t)$, which means $\det\Bigl(c_{ij}(\lambda)\Bigr)$ is not identically zero. Let $\Bigl(d_{ij}(\lambda)\Bigr)$ be the inverse of $\Bigl(c_{ij}(\lambda)\Bigr)$ and define $h_i(\lambda)=\sum_{j=1}^m d_{ij}(\lambda)f_j(\lambda)$ so that $\langle h_i(c(t)),\phi_j\rangle=\delta_{ij}$ for $0<|t|\ll 1$.

Suppose $h_k(\lambda)$ has a pole at $\lambda=\mu$. Then there exists a test function ϕ such that $\langle h_k(\lambda),\phi\rangle$ has a pole at μ. Then it follows from Weierstrass' preparation theorem that there exists a curve $c(t)$ as above and moreover $\langle h_k(c(t)),\phi\rangle$ has a pole at the origin. Choose a positive integer ℓ so that the function $\tilde h(t)=t^\ell h_k(c(t))$ is holomorphically extends to $t=0$ and $\tilde h(0)\neq 0$. Since $\langle\tilde h(t),\phi_j\rangle = t^\ell\delta_{kj}$, $\langle\tilde h(0),\phi_j\rangle = 0$ for $j=1,\ldots,m$, which contradicts to the facts $\langle v_i(0),\phi_j\rangle=\delta_{ij}$ because $0\neq\tilde h(0)\in V_\mu=\sum_{i=1}^m\mathbb{C}v_i(0)$ by definition.

Thus we have proved that $h_i(\lambda)$ are holomorphic functions on λ in a neighborhood of U_μ of μ. Since $\langle h_i(\lambda),\phi_j\rangle=\delta_{ij}$, they are the required functions. In fact,

$f_i(\lambda) = \sum_{j=1}^m \langle f_i(\lambda), \phi_j \rangle h_j(\lambda)$ for generic λ and therefore $V_\lambda \subset \sum_{j=1}^m \mathbb{C}h_j(\lambda)$ for $\lambda \in U_\mu \cap U_r$.

Now suppose the existence of $h_1, \ldots, h_m$ and consider the function f to define $\bar{V}_\mu$ in Definition 6.1. Then under the above notation, $f(t) = \sum_{j=1}^m \langle f(t), \phi_j \rangle h_j(c(t))$ for $0 < |t| \ll 1$ and therefore $f(0) = \sum_{j=1}^m \langle f(0), \phi_j \rangle h_j(c(0))$, which means $\dim V_\mu = m$.

ii) The claim in i) reduces the global existence of h_i to the second problem of Cousin and it is solved for the convex open domain by Oka's principle. □

Remark 6.4. i) Replacing "meromorphic" and "holomorphic" by "rational" and "regular", respectively, we have also Lemma 6.3 in the algebraic sense.

ii) When M is a finite set in Lemma 6.3, $\mathcal{D}'(M)$ is a finite dimensional vector space V over $\mathbb{C}$ and $f_i(\lambda)$ are the elements of V with a meromorphic parameter λ.

Definition 6.5. Fix a base $\{X_1, \ldots, X_m\}$ of $\mathfrak{g}$. Let

$$q_\nu(\lambda, \epsilon) = \sum_{\alpha_1 \geq 0, \ldots, \alpha_m \geq 0} q_{\nu,\alpha}(\lambda, \epsilon) X_1^{\alpha_1} \cdots X_m^{\alpha_m} \tag{6.1}$$

be elements of $U^\epsilon(\mathfrak{g})$ for $(\lambda, \epsilon) \in \mathbb{C}^{r+1}$ and $\nu = 1, \ldots, k$. Here $q_{\nu,\alpha}$ are polynomial functions of (λ, ϵ) and $q_{\nu,\alpha} = 0$ if $\alpha_1 + \cdots + \alpha_m$ is sufficiently large. Let $I(\lambda, \epsilon)$ is the left ideal of $U^\epsilon(\mathfrak{g})$ generated by q_ν for $\nu = 1, \ldots, k$. Put $d_j = \max_{\lambda,\epsilon} \dim I(\lambda, \epsilon) \cap U^\epsilon(\mathfrak{g})^{(j)}$ for $j = 1, 2, \ldots$. Then we can find

$$p_{j,\mu}(\lambda, \epsilon) = \sum_{\alpha_1 \geq 0, \ldots, \alpha_m \geq 0} p_{j,\mu,\alpha}(\lambda, \epsilon) X_1^{\alpha_1} \cdots X_m^{\alpha_m}$$

such that $p_{j,\mu}(\lambda, \epsilon) \in I(\lambda, \epsilon) \cap U^\epsilon(\mathfrak{g})^{(j)}$ for any (λ, ϵ), $p_{j,\mu,\alpha}(\lambda, \epsilon)$ are polynomial functions and $p_{j,1}(\lambda, \epsilon), \ldots, p_{j,d_j}(\lambda, \epsilon)$ are linearly independent for generic (λ, ϵ). Then we denote by $\bar{I}(\lambda, \epsilon)^{(j)}$ the closure of the holomorphic family $\sum_{\mu=1}^{d_j} \mathbb{C}p_{j,\mu}$ at (λ, ϵ) and put $\bar{I}(\lambda, \epsilon) = \bigcup_{j=1}^\infty \bar{I}(\lambda, \epsilon)^{(j)}$. We call $\bar{I}(\lambda, \epsilon)$ the closure of the ideal $I(\lambda, \epsilon)$ with respect to the parameter (λ, ϵ). We call a point $(\lambda, \epsilon) \in \mathbb{C}^{r+1}$ is an un-removable singular point if (λ, ϵ) is an un-removable singular point of $\sum_{\mu=1}^{d_j} \mathbb{C}p_{j,\mu}$ for a certain j. Note that $\bar{I}(\lambda, \epsilon)$ does not depend on the choice of $\{X_1, \ldots, X_m\}$ or $p_{j,\mu}$.

Let $\bar{I}^\epsilon_{\Theta'}(\lambda)$ be the closure of the two-sided ideal $I^\epsilon_{\Theta'}(\lambda)$ given by (4.13) for $\Theta' = \Theta$ or $\bar{\Theta}$. Then we give some conjectures.

Conjecture 1. There exists no un-removable singular point in the parameter (λ, ϵ) of the holomorphic family $I^\epsilon_{\Theta'}(\lambda)$.

Conjecture 2. $\bar{I}^1_{\Theta'}(\lambda) = \operatorname{Ann}(M_{\Theta'}(\lambda))$.

Conjecture 3. $\bar{I}_{\Theta'}(\lambda) = I_{\Theta'}(\lambda)$ if λ is regular.

Conjecture 4. Let $I_\mathfrak{p}(\lambda)$ be a two-sided ideal of $U(\mathfrak{g})$ satisfying (5.7) under the notation in §5. Then

$$\operatorname{Ann}(M_{\Theta'}(\lambda)) = I_\mathfrak{p}(\lambda) + \operatorname{Ann}(M(\lambda_\Theta)). \tag{6.2}$$

Conjecture 5. The condition (5.7) is valid if $I_\mathfrak{p}(\lambda) = \operatorname{Ann}(U(\mathfrak{g})/J_\mathfrak{p}(\lambda))$ and the infinitesimal character of $U(\mathfrak{g})/J_\mathfrak{p}(\lambda)$ is regular.

Remark 6.6. i) It is clear that $\bar{I}^1_{\Theta'}(\lambda) \subset \operatorname{Ann}(M_{\Theta'}(\lambda))$.

ii) Conjecture 1 is equivalent to the existence of a generator system $\{q_\nu(\lambda, \epsilon);\ \nu = 1, \ldots, k\}$ of the form (6.1) such that $\bar{I}^\epsilon_{\Theta'}(\lambda) = \sum_{\nu=1}^k U^\epsilon(\mathfrak{g}) q_\nu(\lambda, \epsilon)$ for any fixed (λ, ϵ). It is also equivalent to the fact that the graded ring

$$\operatorname{gr}\bigl(\bar{I}^\epsilon_{\Theta'}(\lambda)\bigr) = \bigoplus_{j=1}^\infty \Bigl(\bar{I}^\epsilon_{\Theta'}(\lambda) \cap U^\epsilon(\mathfrak{g})^{(j)} / \bar{I}^\epsilon_{\Theta'}(\lambda) \cap U^\epsilon(\mathfrak{g})^{(j-1)}\Bigr)$$

does not depend on (λ, ϵ) because they are also equivalent to the fact that the dimension of the vector space $\bar{I}^{\epsilon}_{\Theta'}(\lambda) \cap U^{\epsilon}(\mathfrak{g})^{(j)}$ does not depend on (λ, ϵ) and the space is spanned by homogeneous elements with respect to $(\mathfrak{g}, \lambda, \epsilon)$.

iii) Conjecture 1 and Conjecture 2 are true if $\mathfrak{g} = \mathfrak{gl}_n$ because there exist $q_\nu(\lambda, \epsilon)$ $(\nu = 1, \dots, k)$ of the form (6.1) such that $\mathrm{Ann}_G(M^{\epsilon}_{\Theta}(\lambda))$ is generated by $q_\nu(\lambda, \epsilon)$ $(\nu = 1, \dots, k)$ for any (λ, ϵ) (cf. [O4]). In this case $\mathrm{gr}\big(I^{\epsilon}_{\Theta}(\lambda)\big)$ is a prime ideal of $S(\mathfrak{g})$ but this is not true in general.

iv) If $\mathrm{gr}\big(I^0_{\Theta'}(\lambda)\big)$ is a prime ideal for generic λ, then Conjecture 1 and 2 are true, which is proved by the same argument as in [O4]. Note that $I^0_{\Theta'}(\lambda)$ is the defining ideal of $\mathrm{Ad}(G)\lambda$ for generic $\lambda \in \mathfrak{a}_{\Theta'}$ by Theorem 4.11.

v) The condition (5.7) with $I_{\mathfrak{p}}(\lambda) = I_{\Theta'}(\lambda)$ are true if $\mathfrak{g}$ is of the classical type and moreover the infinitesimal character of $M_{\Theta'}(\lambda)$ is regular (cf. the proof of Corollary 4.6).

vi) Conjecture 4 implies Conjecture 2 and 3 if $\mathfrak{g}$ is of the classical type and the infinitesimal character is regular. Note that they are true if moreover the infinitesimal character is dominant.

vii) Conjecture 5 is true if $\mathfrak{g}$ is of the classical type. If $\mathfrak{g}$ is of the exceptional type, Conjecture 5 is also true except for the λ belonging to a finite number of complex hypersurfaces which are explicitly given (cf. [OO]).

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