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by

Tetsuhiro Moriyama

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

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TETSUHIRO MORIYAMA

Abstract. The mapping class group of a surface acts on the homology group of the configuration space of n-points on that surface. The kernels of the actions give a structure of the filtration of the mapping class group parameterized by the number of the points n. In this paper, we will prove that the filtration coincides with the filtration defined by using the lower central series of the fundamental group of the surface.

1. Introduction

Let Σ be a compact oriented surface of genus g with boundary $\partial\Sigma \cong S^1$ and let $p_0 \in \partial\Sigma$ be a base point. Let $\mathrm{Diff}_+(\Sigma, \partial\Sigma)$ be the orientation preserving diffeomorphism group on Σ relative to $\partial\Sigma$, and let $\mathcal{M}_{g,1} = \pi_0(\mathrm{Diff}_+(\Sigma, \partial\Sigma))$ be the mapping class group. $\mathcal{M}_{g,1}$ has the well-known descending filtration $\{\mathcal{M}_{g,1}(n)\}_{n\geq 0}$ defined by using the lower central series of the fundamental group $\pi_1(\Sigma, p_0)$. $\mathcal{M}_{g,1}(n)$ is defined to be the kernel of the natural action on the n-th lower central quotient of $\pi_1(\Sigma, p_0)$. See Section 2 for precise definitions (See Morita [6] [7] [8] for details, or Johnson's earlier results [4] [5]).

Let Δ_n be the big-diagonal subset of the n-th Cartesian product Σ^n, and let A_n be the subset of Σ^n such that $(\Sigma, p_0)^n = (\Sigma^n, A_n)$. Then the diagonal action of $\mathrm{Diff}_+(\Sigma, \partial\Sigma)$ on Σ^n preserves $\Delta_n \cup A_n$. We will consider the induced linear representation of $\mathcal{M}_{g,1}$ on $H_n = H_n(\Sigma^n, \Delta_n \cup A_n; \mathbb{Z})$. Let $F_n(\Sigma) = \Sigma^n - \Delta_n$ be the configuration space of ordered n-points on Σ. Then H_n is isomorphic to $H^n(F_n(\Sigma) \cup A_n, A_n; \mathbb{Z})$ as an $\mathcal{M}_{g,1}$-module. In this paper, we will consider $(\Sigma^n, \Delta_n \cup A_n)$ rather than $(F_n(\Sigma) \cup A_n, A_n)$.

Our Main Theorem is that the kernel of the representation of $\mathcal{M}_{g,1}$ on H_n coincides with $\mathcal{M}_{g,1}(n)$ (Theorem 2.1). In section 5, we will define an $\mathcal{M}_{g,1}$-equivariant homomorphism $\phi_n : \mathbb{Z}\pi_1(\Sigma, p_0) \to H_n$. Roughly speaking, $\phi_n(\gamma)$ is the homology class of the domain of integration for the Chen's iterated integrals ([1]) along a path γ. By comparing the action on H_n with $\pi_1(\Sigma, p_0)$ via ϕ_n, we will prove the Main Theorem. In Section 2, we will introduce notations and state the Main Theorem more precisely.

Similar results are already shown by Beilinson (unpublished, see [3]) for any connected topological manifolds X. Roughly speaking, he considered the n-dimensional homology group of X^n relative to the subset consisting of all the elements $(x_1, x_2, \ldots, x_n) \in X^n$ such that $x_i = x_{i+1}$ for some $0 \leq i \leq n$, where $x_0 = x_{n+1}$ is a base point of X. He proved that there exists an isomorphism from J/J^{n+1} to such a homology group, where J is the augmentation ideal of the group ring $\mathbb{Z}\pi_1(X, x_0)$.

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His idea is based on Chen's iterated integrals. From his result, if $X = \Sigma$ then the kernels of the action of $\mathcal{M}_{g,1}$ on these two groups are equal, which is $\mathcal{M}_{g,1}(n)$ (see Lemma 7.1). Our case is a little complicated because we must consider all the combinations $x_i = p_0$ and $x_j = x_k$ $(1 \leq i, j, k \leq n,\ j \neq k)$.

In Section 3, we will study fundamental properties of H_n. We also introduce an algebra structure of $\hat{H} = \prod_{n=0}^{\infty} H_n$, which has an $\mathcal{M}_{g,1}$-action, filtration and symmetric group action (Lemma 3.1). Often, $\hat{H}$ is easier than H_n.

In Section 4, we will construct a relative cell decomposition of $(\Sigma^n, \Delta_n \cup A_n)$ up to homotopy. $(\Sigma^n, \Delta_n \cup A_n)$ is obtained by attaching n-cells to $D_n \cup A_n$ (Proposition 4.2). Therefore we will obtain a basis of H_n.

In Section 5, we will define the homomorphism ϕ_n, and the formal series homomorphism $\Phi = \sum_{n=0}^{\infty} \phi_n : \mathbb{Z}\pi_1(\Sigma, p_0) \to \hat{H}$. Then Φ is an algebra homomorphism (Proposition 5.2). Moreover, Φ is injective, and so $\mathcal{M}_{g,1}$-action on $\hat{H}$ is faithful (Remark 6.3).

In Section 6, we study the kernels and images of ϕ_n and Φ, and then we will describe the relation between the cell decomposition and the image of ϕ_n. Finally, we will prove that the $\mathfrak{S}_n$-module H_n is generated by all elements of the form $\phi_{n_1}(\gamma_1)\phi_{n_2}(\gamma_2)\cdots\phi_{n_k}(\gamma_k)$, where $n_i \geq 0, \sum_{i=1}^{k} n_i = n$ and $\gamma_i \in \pi_1(\Sigma, p_0)$ (Proposition 6.5). Namely, the action of $\mathcal{M}_{g,1}$ on H_n is determined by the action on $\phi_{n_i}(\gamma_i)$.

In Section 7, we will prove the Main Theorem by using the results of the previous sections.

2. Main Results

Let $\pi_1(\Sigma, p_0) = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ be the lower central series of $\pi_1(\Sigma, p_0)$. Namely, $\Gamma_0 = \pi_1(\Sigma, p_0)$ and $\Gamma_n = [\Gamma_{n-1}, \Gamma_0]$ $(n \geq 1)$. Let

$$\rho_n : \mathcal{M}_{g,1} \to \mathrm{Aut}\,(\Gamma_0/\Gamma_n)$$

be the action induced from the natural action on $\pi_1(\Sigma, p_0)$. We will write $\mathcal{M}_{g,1}(n) = \mathrm{Ker}\,\rho_n$ for the kernel. $\mathcal{M}_{g,1}(1)$ is nothing but the Torelli group, which is the subgroup of $\mathcal{M}_{g,1}$ consisting of all the elements which acts on $H_1(\Sigma;\mathbb{Z})$ trivially. For any integer $n \geq 1$ and any a pair of space (X, Y), define the subspaces $\Delta_n(X)$, $A_n(X, Y)$ of X^n to be

$$\begin{aligned} \Delta_n(X) =& \{ (x_1, \ldots, x_n) \in X^n \mid x_i = x_j \text{ for some } i \neq j \}, \\ A_n(X, Y) =& \{ (x_1, \ldots, x_n) \in X^n \mid x_i \in Y \text{ for some } i \}, \end{aligned}$$

and write $(X, Y)^{\overline{n}} = \big(X^n, \Delta_n(X) \cup A_n(X, Y)\big)$. In the case $n = 0$, we will denote both $(X, Y)^0$ and $(X, Y)^{\overline{0}}$ by a set consisting of one point. Moreover, we will simply write $\Delta_n = \Delta_n(\Sigma)$ and $A_n = A_n(\Sigma, p_0)$.

The diagonal action on Σ^n of $\mathrm{Diff}_+(\Sigma, \partial\Sigma)$ preserves $\Delta_n \cup A_n$. The induced action on the homology group $H_*((\Sigma, p_0)^{\overline{n}}; \mathbb{Z})$ does not depend on the choice of the isotopy classes of a diffeomorphism. By Proposition 3.3, we have only to consider the n-dimensional homology group H_n. Therefore, we have a linear representation

$$\rho'_n : \mathcal{M}_{g,1} \to \mathrm{GL}(H_n),$$

and let $\mathcal{M}_{g,1}(n)' = \mathrm{Ker}\,\rho'_n$. It is easy to see that $\mathcal{M}_{g,1}(n) = \mathcal{M}_{g,1}(n)'$ for $n = 0, 1$ by definition. Our Main Theorem is the following.

Theorem 2.1 (Main Theorem)**.** *For any integer $n \geq 0$, we have*

$$\mathcal{M}_{g,1}(n)' = \mathcal{M}_{g,1}(n).$$

Since $\mathcal{M}_{g,1}(n)$ is not the unit group for any $n \geq 0$, we have the following corollary.

Corollary 2.2. *The representation ρ'_n is not faithful for any $n \geq 0$.*

3. Homology group of $(\Sigma, p_0)^{\overline{n}}$

We introduce a formal series algebra $\hat{H} = \prod_{n=0}^{\infty} H_n$, whose elements are infinite formal sums of the type $\sum_{n\geq 0} v_n$ ($v_n \in H_n$). We will construct some structures on $\hat{H}$ as follows. The representation ρ'_n induces the infinite dimensional linear representation

$$\rho' = \prod_{n\geq 0} \rho'_n : \mathcal{M}_{g,1} \to \mathrm{GL}(\hat{H}).$$

The natural map $(\Sigma, p_0)^{\overline{m}} \times (\Sigma, p_0)^{\overline{n}} \to (\Sigma, p_0)^{\overline{m+n}}$ induces the product $\mu_{m,n} : H_m \otimes H_n \to H_{m+n}$. The unit of $\hat{H}$ is $[(\Sigma, p_0)^{\overline{0}}] \in H_0$. We will simply write $vw = \mu_{m,n}(v, w)$ for any $v \in H_m$, $w \in H_n$. Let $\mathcal{F}$ be the descending filtration of $\hat{H}$ such that $\mathcal{F}_n\hat{H} = \prod_{i\geq n} H_i$, and let $\mathfrak{S}_n$ be the n-th permutation group. Here $\mathfrak{S}_0$ is the unit group. There are natural actions of $\mathfrak{S}_n$ on H_n, and the product group $\mathfrak{S} = \prod_{n\geq 0} \mathfrak{S}_n$ on $\hat{H}$. Therefore, we have the following Lemma.

Lemma 3.1. *H_n is an $(\mathfrak{S}_n \times \mathcal{M}_{g,1})$-module, and hence, $\hat{H}$ is an $(\mathfrak{S} \times \mathcal{M}_{g,1})$-module. Moreover, $\hat{H}$ has the structure of the non-commutative associative filtered $\mathcal{M}_{g,1}$-algebra with action ρ', product μ and filtration $\mathcal{F}$.*

Now, we will study some fundamental properties of $\hat{H}$. Set $Y_n = (\Delta_{n-1} \times \Sigma) \cup A_n$, and then we have $(\Sigma, p_0)^{\overline{n-1}} \times (\Sigma, p_0)^{\overline{1}} = (\Sigma^n, Y_n)$. For $i = 1, 2, \ldots, n-1$, let $f_i : (\Sigma, p_0)^{\overline{n-1}} \to (\Delta_n \cup A_n, Y_n)$ be the map defined by $f_i(x_1, x_2, \ldots, x_{n-1}) = (x_1, x_2, \ldots, x_{n-1}, x_i)$, and set

$$f = \coprod_{i=1}^{n-1} f_i : \coprod^{n-1} (\Sigma, p_0)^{\overline{n-1}} \to (\Delta_n \cup A_{n-1}, Y_n).$$

Lemma 3.2. *The induced homology homomorphism*

$$f_* : \overset{n-1}{\oplus} H_*\big((\Sigma, p_0)^{\overline{n-1}}; \mathbb{Z}\big) \to H_*(\Delta_n \cup A_n, Y_n; \mathbb{Z})$$

is an isomorphism as $\mathcal{M}_{g,1}$-module.

Proof. Let $f' : \coprod^{n-1} (\Delta_{n-1} \cup A_{n-1}) \to Y_n$ be the restriction of f to $\coprod^{n-1} (\Delta_{n-1} \cup A_{n-1})$. and let $Y_n \underset{f'}{\cup} \big(\coprod^{n-1} \Sigma^{n-1}\big)$ be the attaching space. Then we have an isomorphism

$$\overset{n-1}{\oplus} H_*\big((\Sigma, p_0)^{\overline{n-1}}; \mathbb{Z}\big) \cong H_*\big(Y_n \underset{f'}{\cup} \coprod_{i=1}^{n-1} \Sigma^{n-1}, Y_n; \mathbb{Z}\big)$$

by the excision theorem. Now f and the identity on Y_n induce a homeomorphism

$$(Y_n \underset{f'}{\cup} \coprod_{i=1}^{n-1} \Sigma^{n-1}, Y_n) \to (\Delta_n \cup A_n, Y_n),$$

and this induces an isomorphism

$$\overset{n-1}{\oplus} H_*\big((\Sigma,p_0)^{\overline{n-1}};\mathbb{Z}\big) \xrightarrow{\cong} H_*(\Delta_n \cup A_n, Y_n).$$

This isomorphism is f_*, and it is $\mathcal{M}_{g,1}$-equivariant. □

Let us write $\partial_* : H_*((\Sigma,p_0)^{\overline{n}};\mathbb{Z}) \to H_{*-1}(\Delta_n \cup A_n, Y_n;\mathbb{Z})$ for the connecting homomorphism of the homology exact sequence of the triple $(\Sigma^n, \Delta_n \cup A_n, Y_n)$. Let

$$\partial'_* : H_*((\Sigma,p_0)^{\overline{n}};\mathbb{Z}) \to \overset{n-1}{\oplus} H_{*-1}((\Sigma,p_0)^{\overline{n-1}};\mathbb{Z})$$

be $\partial'_* = {f_*}^{-1} \circ \partial_*$, which is $\mathcal{M}_{g,1}$-equivariant.

Proposition 3.3. *Let $n \geq 0$ be an integer.*

1. *If $k \neq n$, then $H_k((\Sigma,p_0)^{\overline{n}};\mathbb{Z}) = 0$.*
2. *If $n \geq 1$, then we have a short exact sequence*

$$0 \to H_{n-1} \otimes H_1 \overset{\mu_{n-1,1}}{\to} H_n \overset{\partial'_*}{\to} \overset{n-1}{\oplus} H_{n-1} \to 0$$

as an $\mathcal{M}_{g,1}$-module. In particular, H_n is a free abelian group of rank

$$2g(2g+1)\cdots(2g+(n-1)).$$

Proof. (1) is obvious if $n \leq 1$, and (2) are obvious if $n = 1$, so we suppose $n \geq 2$. Let us consider the homology exact sequence of the triple $(\Sigma^n, \Delta_n \cup A_n, Y_n)$:

$$\cdots \longrightarrow H_k(\Sigma^n, Y_n;\mathbb{Z}) \longrightarrow H_k((\Sigma,p_0)^{\overline{n}};\mathbb{Z}) \xrightarrow{\partial_*} H_{k-1}(\Delta_n \cup A_n, Y_n;\mathbb{Z}) \longrightarrow \cdots .$$

By Lemma 3.2, we can replace the right group with $\overset{n-1}{\oplus} H_{k-1}\big((\Sigma,p_0)^{\overline{n-1}};\mathbb{Z}\big)$, and ∂_* with ∂'_*. The left group is isomorphic to $H_{k-1}((\Sigma,p_0)^{\overline{n-1}}) \otimes H_1$. By the assumption of induction on n, the groups on both sides of the sequence are zero if $k \neq n$, and hence we have $H_k((\Sigma,p_0)^{\overline{n}};\mathbb{Z}) = 0$. So, we have proved (1). In the case $k = n$, we have (2). We can compute the rank of H_n by induction on n. □

Let $\operatorname{gr}\hat{H} = \oplus_{n=0}^{\infty} \operatorname{gr}_n \hat{H}$, $\operatorname{gr}_n \hat{H} = H_n$ be the associated graded algebra of $\hat{H}$. Let $T[H_1] = \oplus_{n\geq 0} H_1^{\otimes n}$ be the free tensor algebra generated by H_1 over $\mathbb{Z}$, and let $T[[H_1]] = \prod_{n\geq 0} H_1^{\otimes n}$ be its completed algebra. By Proposition 3.3, we obtain some corollaries as follows.

Corollary 3.4. *Let $n \geq 1$ be an integer.*

1. $\mathcal{M}_{g,1}(n-1)' \supset \mathcal{M}_{g,1}(n)'$.
2. *The homomorphism ${H_1}^{\otimes n} \to H_n$ of the products of n-elements in H_1 is injective. Moreover, it induces injective graded ring homomorphisms $T[H_1] \to \operatorname{gr}\hat{H}$ and $T[[H_1]] \to \hat{H}$.*

Proof. (1) is immediate because H_n has the $\mathcal{M}_{g,1}$-submodule $H_{n-1} \otimes H_1$. We will prove (2). The product $H_1^{\otimes n} \to H_n$ is represented as the composition of the homomorphisms as follows:

$$H_1^{\otimes n} \overset{\mu_{1,1}\otimes id_{n-2}}{\longrightarrow} H_2 \otimes H_1^{\otimes n-2} \overset{\mu_{2,1}\otimes id_{n-3}}{\longrightarrow} \cdots \overset{\mu_{n-2,1}\otimes id_1}{\longrightarrow} H_{n-1} \otimes H_1 \overset{\mu_{n-1,1}}{\longrightarrow} H_n.$$

Here, id_i is the identity on $H_1^{\otimes i}$ $(1 \leq i \leq n-2)$. Each homomorphism is injective by Proposition 3.3, and therefore, so is the composition.

The maps $T[H_1] \to \operatorname{gr}\hat{H}$ and $T[[H_1]] \to \hat{H}$ preserve the product because these maps are induced by the natural map $\coprod_{n=0}^{\infty} (\Sigma,p_0)^n \to \coprod_{n=0}^{\infty} (\Sigma,p_0)^{\overline{n}}$ □

4. Cell decomposition of $(\Sigma, p_0)^{\overline{n}}$

Let $\alpha_1, \alpha_2, \ldots, \alpha_{2g}$ be free generators for $\pi_1(\Sigma, p_0)$, and fix an embedded circle $(S_i^1, p_0) \subset (\Sigma, p_0)$ such that S_i^1 represents $\alpha_i\,(1 \le i \le 2g)$. Let $C = \bigvee_{i=1}^{2g} S_i^1$. We can assume that each S_i^1 intersects each other only on p_0 and that the inclusion $C \hookrightarrow \Sigma$ is a homotopy equivalence relative to p_0. Then the induced map $(C, p_0)^{\overline{n}} \to (\Sigma, p_0)^{\overline{n}}$ is also a homotopy equivalence, and hence, we have an isomorphism $H_n((C, p_0)^{\overline{n}}; \mathbb{Z}) \cong H_n$. From now on we will simply denote $\Delta_n(C)$ and $A_n(C, p_0)$ by Δ'_n and A'_n respectively.

It is easy to see that $C^n - (\Delta'_n \cup A'_n)$ consists of $2g(2g+1)\cdots(2g+(n-1))$ domains. We will construct a cell decomposition of C^n relative to $\Delta'_n \cup A'_n$ such that each cell corresponds to some domain of $C^n - (\Delta'_n \cup A'_n)$. Let $x = (x_1, x_2, \ldots, x_n) \in C^n - (\Delta'_n \cup A'_n)$. Suppose that the k_i points $x_{\sigma_j(1)}, x_{\sigma_j(2)}, \ldots, x_{\sigma_j(k_j)}$ are contained

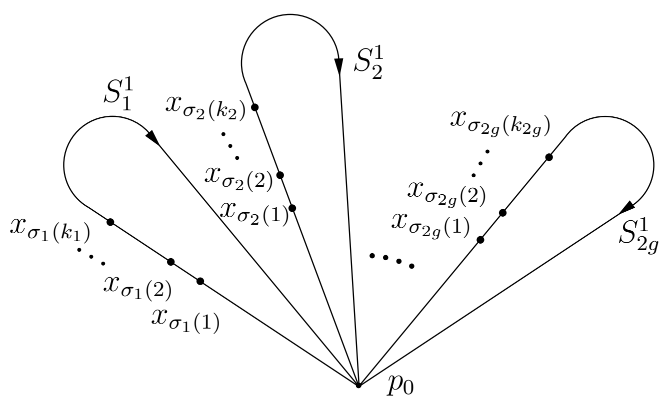

Figure 1. A point x on $C^n - (\Delta'_n \cup A'_n)$

in S_i^1 so that the ordering corresponds with the orientation of α_i (Figure 1), where $i, j, k_j, \sigma_j(i)$ satisfies that

$$\sum_{j=1}^{2g} k_j = n, \quad \{\sigma_j(i) \mid i, j\} = \{1, 2, \ldots, n\}$$
$$k_i \ge 0, \quad 1 \le i \le k_j, \quad 1 \le j \le 2g.$$

Then we have data $\{(k_j, \sigma_j)\}_{j=1}^{2g}$, and we define an element $\sigma \in \mathfrak{S}_n$ by $\sigma = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_{2g}$, namely,

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$$
$$\sigma(k_1 + \cdots + k_{j-1} + i) = \sigma_j(i), \qquad 1 < i \le k_j. \tag{1}$$

If we write $k = (k_1, k_2, \ldots, k_{2g})$, then we have new data (k, σ). K_n will denote the set consisting of all $2g$-tuple of non-negative integers such that the total sum is equal to n, then $k \in K_n$. Since (k, σ) does not depend on the choice of the point on a domain, we obtain a map

$$h : \pi_0(C^n - (\Delta'_n \cup A'_n)) \to K_n \times \mathfrak{S}_n.$$

The map h is bijective because we can define the inverse h^{-1} by tracing the above process in the reverse direction. Therefore we have the following Lemma.

Lemma 4.1. *The map h defined as above is a bijection.*

Now let Δ^n be the n-simplex with coordinates

$$\Delta^n = \{(t_1, \dots, t_n) \mid 0 \le t_1 \le t_2 \le \cdots \le t_n \le 1\}.$$

Proposition 4.2. *Let $e_{(k,\sigma)}$ be the n-cell corresponding to the domain of $C^n - (\Delta'_n \cup A'_n)$ by the map h, a more explicit definition is given in the proof. Then we have a cell decomposition*

$$C^n \cong (\Delta'_n \cup A'_n) \cup \left(\bigcup_{(k,\sigma)\in K_n \times \mathfrak{S}_n} e_{(k,\sigma)} \right)$$

of C^n relative to $\Delta'_n \cup A'_n$. If we write $[e_{(k,\sigma)}] \in H_n$ for the homology class of $e_{(k,\sigma)}$, then the set $\{[e_{(k,\sigma)}] \mid (k,\sigma) \in K_n \times \mathfrak{S}_n\}$ is a basis of H_n over $\mathbb{Z}$. Therefore, H_n is isomorphic to $\mathbb{Z}K_n \otimes \mathbb{Z}\mathfrak{S}_n$.

Proof. Fix a data $(k,\sigma) \in K_n \times \mathfrak{S}_n$, and let $\{(k_j, \sigma_j)\}_{j=1}^{2g}$ be the associated data which is determined from (k,σ) by using the formula (1). For $i = 1, 2, \dots, 2g$, fix a path $\tilde{\alpha}_j : [0,1] \to S^1_j$ which represents α_j. We express the coordinates of points on $\Delta^{k_1} \times \Delta^{k_2} \times \cdots \times \Delta^{k_{2g}}$ as follows:

$$(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{2g}) \in \Delta^{k_1} \times \Delta^{k_2} \times \cdots \Delta^{k_{2g}},$$
$$\mathbf{t}_j = (t_{j,1}, t_{j,2}, \dots, t_{j,k_j}) \in \Delta^{k_j} \quad (j = 1, 2, \dots, 2g).$$

Then we define a map $e_{(k,\sigma)}$ by

$$e_{(k,\sigma)} : \Delta^{k_1} \times \Delta^{k_2} \times \cdots \times \Delta^{k_{2g}} \to C$$

$$e_{(k,\sigma)}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{2g}) = (x_1, x_2, \dots, x_n)$$
$$x_{\sigma_j(i)} = \tilde{\alpha}_j(t_{j,i}) \quad (1 \le j \le 2g,\ 1 \le i \le k_j).$$

$(\Delta'_n \cup A'_n) \cup (\cup_{(k,\sigma)} e_{(k,\sigma)})$ denotes the attaching space obtained by attaching $\Delta^{k_1} \times \Delta^{k_2} \times \cdots \times \Delta^{k_{2g}}$ by using the restricted map $e_{(k,\sigma)}|_{\partial(\Delta^{k_1}\times\cdots\times\Delta^{k_{2g}})}$, then the attaching space is homeomorphic to C^n. Therefore, we can consider $e_{(k,\sigma)}$ as an n-cell of C^n relative to $\Delta'_n \cup A'_n$. Each domain $\mathrm{Int}\, e_{(k,\sigma)} \subset C^n - (\Delta'_n \cup A'_n)$ corresponds to $h^{-1}(k,\sigma)$. Then since all cells have dimension n, it follows that H_n is a free abelian group, and $[e_{(k,\sigma)}]\ (k \in K_n,\ \sigma \in \mathfrak{S}_n)$ is a basis. □

Let $1_n \in \mathfrak{S}_n$ be the unit. The following Corollary is immediately from Proposition 4.2.

Corollary 4.3. *H_n is a free $\mathfrak{S}_n$-module with a basis $\{[e_{(k,1_n)}] \mid k \in K_n\}$, and so H_n has rank $2g(2g+1)\cdots(2g+n-1)/n!$.*

Proof. We have $\sigma_*([e_{(k,\tau)}]) = [e_{(k,\sigma\tau)}]$ for any $k \in K_n$ and $\sigma, \tau \in \mathfrak{S}_n$, where σ_* is the action of σ on H_n. Hence, $[e_{(k,1_n)}]$'s form a basis of the $\mathfrak{S}_n$-module H_n. □

Remark 4.4. $H_n(\Sigma^n/\mathfrak{S}_n, (\Delta_n \cup A_n)/\mathfrak{S}_n; \mathbb{Z})$ is isomorphic to the n-th symmetric tensor power $S^n H_1$ of H_1. The rank is $2g(2g+1)\cdots(2g+(n-1))/n!$, and the kernel of the representation $\mathcal{M}_{g,1} \to \mathrm{GL}(S^n H_1)$ is the Torelli group for any $n \ge 1$.

5. Definition of the map Φ

Let $\gamma \in \pi_1(\Sigma, p_0)$ be an element, and fix a path $\tilde{\gamma}$ such that the homotopy class is γ. For an integer $n \geq 1$, we define an n-chain $c^n_{\tilde{\gamma}} : \Delta^n \to \Sigma^n$ by

$$c^n_{\tilde{\gamma}}(t_1, t_2, \dots, t_n) = (\tilde{\gamma}(t_1), \tilde{\gamma}(t_2), \dots, \tilde{\gamma}(t_n)),$$

for $(t_1, t_2, \dots, t_n) \in \Delta^n$.Then the homology class $[c^n_{\tilde{\gamma}}] \in H_n$ does not depend on the choice of $\tilde{\gamma}$.

Definition 5.1. Define the additive homomorphism $\phi_n : \mathbb{Z}\pi_1(\Sigma, p_0) \to H_n$ such that

$$\phi_n(\gamma) = \begin{cases} [c^n_{\tilde{\gamma}}], & \text{if } n \geq 1 \\ 1, & \text{if } n = 0 \end{cases}$$

for any $\gamma \in \pi_1(\Sigma, p_0)$, and define the map $\Phi : \mathbb{Z}\pi_1(\Sigma, p_0) \to \hat{H}$ to be the formal series $\Phi = \sum_{n=0}^{\infty} \phi_n$.

Clearly, Φ is $\mathcal{M}_{g,1}$-equivariant. We will write $I = \operatorname{Ker} \phi_0$ to denote the augmentation ideal of $\mathbb{Z}\pi(\Sigma, p_0)$. Then $\mathbb{Z}\pi_1(\Sigma, p_0)$ is a filtered $\mathcal{M}_{g,1}$-algebra with the filtration $\{I^n\}_{n \geq 0}$.

Proposition 5.2. *Φ is a filtered $\mathcal{M}_{g,1}$-algebra homomorphism.*

Namely, Φ satisfies $\Phi(I^n) \subset \mathcal{F}_n \hat{H}$ and preserves the product structure.

Proof. We have only to prove that Φ preserves the product and the filtration.

Φ preserves the product if and only if

$$\phi_n(\gamma\delta) = \sum_{k=0}^{n} \phi_k(\gamma)\, \phi_{n-k}(\delta) \tag{2}$$

for any $\gamma, \delta \in \pi_1(\Sigma, p_0)$ and $n \geq 0$. To prove this, we consider the partition of Δ^n as follows:

$$\Delta^n = D_0 \cup D_1 \cup \cdots \cup D_n,$$
$$D_k = \left\{ (x_1, \dots, x_n) \;\middle|\; x_k \leq \tfrac{1}{2} \leq x_{k+1} \right\}, \quad (1 \leq k \leq n).$$

Here $x_0 = 0,\ x_{n+1} = 1$. Let $\tilde{\gamma}, \tilde{\delta} : ([0,1], \{0,1\}) \to (\Sigma, p_0)$ be paths which represent $\gamma,\ \delta$. Let $\tilde{\gamma}\tilde{\delta}$ be the path such that

$$\tilde{\gamma}\tilde{\delta}(t) = \begin{cases} \tilde{\gamma}(2t) & 0 \leq t \leq \frac{1}{2}, \\ \tilde{\delta}(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

which represents $\gamma\delta$. Then, the homology class $[c^n_{\tilde{\gamma}\tilde{\delta}}|_{D_k}] \in H_n$ of the restriction $c^n_{\tilde{\gamma}\tilde{\delta}}|_{D_k}$ to D_k is well-defined, and hence, we have

$$\phi_n(\gamma\delta) = [c^n_{\tilde{\gamma}}|_{D_0}] + [c^n_{\tilde{\gamma}}|_{D_1}] + \cdots + [c^n_{\tilde{\gamma}}|_{D_n}].$$

The equation $[c^n_{\tilde{\gamma}\tilde{\delta}}|_{D_k}] = [c^k_{\tilde{\gamma}}][c^{n-k}_{\tilde{\delta}}]$ is shown by the natural direct product decomposition $D_k \cong \Delta^k \times \Delta^{n-k}$. Therefore, we obtain equation (2) as required.

By the Lemma 5.3 (1) which follows this proof, the restriction $(\phi_0 + \phi_1 + \cdots + \phi_{n-1})|_{I^n}$ is zero. Hence Φ preserves the filtration. □

Lemma 5.3. *Let $n \geq 1$ be an integer. For any element of the form $(\gamma_1 - 1)(\gamma_2 - 1)\cdots(\gamma_n - 1) \in I^n$, ($\gamma_i \in \pi_1(\Sigma, p_0)$), we have*

$$\Phi\big((\gamma_1 - 1)(\gamma_2 - 1)\cdots(\gamma_n - 1)\big) \equiv \phi_1(\gamma_1)\phi_1(\gamma_2)\cdots\phi_1(\gamma_n) \pmod{\mathcal{F}_{n+1}\hat{H}}.$$

In particular, we have

1. $\operatorname{Ker}\phi_{n-1} \supset I^n$,
2. $\phi_n\big((\gamma_1 - 1)(\gamma_2 - 1)\cdots(\gamma_n - 1)\big) = \phi_1(\gamma_1)\phi_1(\gamma_2)\cdots\phi_1(\gamma_n)$.

Proof. It is immediately because of the facts $\Phi(\gamma_i - 1) \equiv \phi_1(\gamma_i) \pmod{\mathcal{F}_2}$ and that Φ is a ring-homomorphism. □

6. Properties of Φ

Let $q_n : \mathbb{Z}\pi_1(\Sigma, p_0) \to \mathbb{Z}\pi_1(\Sigma, p_0)/I^{n+1}$ be the quotient map. Since $\operatorname{Ker}\phi_n \supset I^{n+1}$ (Proposition 5.3 (1)), ϕ_n induces the homomorphism

$$\phi'_n : \mathbb{Z}\pi_1(\Sigma, p_0)/I^{n+1} \to H_n$$

which satisfies $\phi'_n \circ q_n = \phi_n$. The associated graded homomorphism

$$\operatorname{gr}\Phi : \operatorname{gr}\mathbb{Z}\pi_1(\Sigma, p_0) \to \operatorname{gr}\hat{H}$$

is given by $\operatorname{gr}_n \mathbb{Z}\pi_1(\Sigma, p_0) = I^n/I^{n+1}$, $\operatorname{gr}_n \hat{H} = H_n$ and $\operatorname{gr}_n \Phi = \phi'_n|_{I^n/I^{n+1}}$ on each n.

Lemma 6.1. $\operatorname{gr}\Phi$ *is an isomorphism onto the subalgebra $T[H_1] \subset \operatorname{gr}\hat{H}$.*

Proof. Clearly, $\operatorname{gr}_0 \Phi$ is an isomorphism, and suppose $n \geq 1$. By Lemma 5.3,

$$\operatorname{gr}_n \Phi\left((\gamma_1 - 1)(\gamma_2 - 1)\cdots(\gamma_n - 1)\right) = \phi_1(\gamma_1)\phi_1(\gamma_2)\cdots\phi_1(\gamma_n)$$

for $\gamma_i \in \mathbb{Z}\pi_1(\Sigma, p_0)$ $(i = 1, 2, \ldots, n)$. Therefore $\operatorname{Im}(\operatorname{gr}_n \Phi) = H_1^{\otimes n} \subset H_n$, and it is easy to see that $\operatorname{gr}_n \Phi$ is injective. □

Let $\Phi_n : \mathbb{Z}\pi_1(\Sigma, p_0)/I^{n+1} \to \hat{H}/\mathcal{F}_{n+1}\hat{H}$ be the homomorphism induced by Φ which can be written $\Phi_n = \phi'_0 + \phi'_1 + \cdots + \phi'_n$.

Proposition 6.2. Φ_n *is injective.*

Proof. By Lemma 6.1, $\operatorname{gr}_n \Phi$ is injective for any $n \geq 0$. Since Φ preserves the filtrations, there exists a commutative diagram as follows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^n/I^{n+1} & \longrightarrow & \mathbb{Z}\pi_1(\Sigma, p_0)/I^{n+1} & \longrightarrow & \mathbb{Z}\pi_1(\Sigma, p_0)/I^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \operatorname{gr}_n\Phi} & & \downarrow{\scriptstyle \Phi_n} & & \downarrow{\scriptstyle \Phi_{n-1}} & & \\
0 & \longrightarrow & H_n & \longrightarrow & \hat{H}/\mathcal{F}_{n+1}\hat{H} & \longrightarrow & \hat{H}/\mathcal{F}_n\hat{H} & \longrightarrow & 0
\end{array}$$

Now we can prove Proposition by induction on n. □

Remark 6.3. By Proposition 6.2, we have $\operatorname{Ker}\Phi \subset \cap_{n\geq 0} I^n$. Since $\pi_1(\Sigma, p_0)$ is a free group, we have $\cap_{n\geq 0} I^n = 0$ (Fox [2]). Therefore, Φ is injective. The action of $\mathcal{M}_{g,1}$ on $\pi_1(\Sigma, p_0)$ is faithful due originally to Nielsen. Consequently, the representation $\rho' : \mathcal{M}_{g,1} \to \operatorname{GL}(\hat{H})$ is faithful.

Lemma 6.4. *If $(k, \sigma) \in K_n \times \mathfrak{S}_n$, $k = (k_1, \ldots, k_{2g})$, then we have*

$$[e_{(k,\sigma)}] = \sigma_*\left(\phi_{k_1}(\alpha_1)\,\phi_{k_2}(\alpha_2)\,\cdots\,\phi_{k_{2g}}(\alpha_g)\right).$$

Proof. Since $[e_{(k,\sigma)}] = \sigma_*[e_{(k,1_n)}]$, we have only to prove Lemma in case $\sigma = 1_n$. Let $l_i = (0, \cdots, 0, k_i, 0, \dots, 0) \in K_{k_i}$ be the $2g$-tuple of integers such that the i-th component is k_i and the other components are equal to zero. Referring to the construction of the cells in the proof of Proposition 4.2, we can then verify that

$$[e_{(k,1_n)}] = [e_{(l_1,1_{k_1})}][e_{(l_2,1_{k_2})}]\cdots[e_{(l_{2g},1_{k_{2g}})}],$$
$$[e_{(l_i,1_{k_i})}] = \phi_{k_i}(\alpha_i).$$

□

Let R be the subalgebra of $\operatorname{gr}\hat{H}$ generated by all the elements in $\cup_{n\geq 0}\operatorname{Im}\phi_n$ over $\mathbb{Z}$, and let $R_n = R\cap H_n$.

Proposition 6.5. *R_n generates H_n as an $\mathfrak{S}_n$-module.*

Proof. By Corollary 4.3, $\{[e_{(k,1_n)}] \mid k \in K_n\}$ generates H_n as an $\mathfrak{S}_n$-module. $[e_{(k,1_n)}]$ is contained in R_n by Lemma 6.4. Therefore, R_n generates H_n as an $\mathfrak{S}_n$-module. □

7. Proof of The Main Theorem

Lemma 7.1. *For any integer $n \geq 0$, the kernel of the representation of $\mathcal{M}_{g,1}$ on $\mathbb{Z}\pi_1(\Sigma, p_0)/I^{n+1}$ is $\mathcal{M}_{g,1}(n)$.*

This Lemma is proved easily by using the fact that $\gamma \in \pi_1(\Sigma, p_0)$ is contained in Γ_{n+1} if and only if $\gamma - 1 \in I^{n+1}$ ([2]).

We now have everything ready to prove the Main Theorem.

Proof of Main Theorem. First we will prove that $\mathcal{M}_{g,1}(n)' \subset \mathcal{M}_{g,1}(n)$. Let K be the kernel of the representation of $\mathcal{M}_{g,1}$ on $\hat{H}/\mathcal{F}_{n+1}\hat{H}$. By Proposition 6.2, we can consider $\mathbb{Z}\pi_1(\Sigma, p_0)/I^{n+1}$ as an $\mathcal{M}_{g,1}$-submodule of $\hat{H}/\mathcal{F}_{n+1}\hat{H}$, and therefore $K \subset \mathcal{M}_{g,1}(n)$ by Lemma 7.1. Since the representation on $\hat{H}/\mathcal{F}_{n+1}\hat{H}$ is $\oplus_{i=1}^n \rho'_i$, we have that $K = \cap_{i=1}^n \mathcal{M}_{g,1}(n)' = \mathcal{M}_{g,1}(n)'$ by Corollary 3.4 (1). Hence, we have $\mathcal{M}_{g,1}(n)' \subset \mathcal{M}_{g,1}(n)$.

Next we will prove the converse $\mathcal{M}_{g,1}(n)' \supset \mathcal{M}_{g,1}(n)$. H_n is generated by R_n as an $\mathfrak{S}_n$-module (Proposition 6.5), so we have only to prove that $\mathcal{M}_{g,1}(n)$ acts on $\operatorname{Im}\phi_m$ trivially for $m = 1, 2, \dots, n$. Since ϕ'_m is $\mathcal{M}_{g,1}$-equivariant, we have $\varphi_*(\phi_m(\gamma)) = \phi'_m(\varphi_*(q_n(\gamma)))$ for any $\varphi \in \mathcal{M}_{g,1}(n)$ and $\gamma \in \pi_1(\Sigma, p_0)$. By Lemma 7.1, $\varphi_*(q_n(\gamma)) = q_n(\gamma)$. Hence, we have $\varphi_* \circ \phi_m = \phi_m$ if $\varphi \in \mathcal{M}_{g,1}(n)$. □

This completes the prove of the Main Theorem.

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Department of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153-8914, Japan

E-mail address: `tetsuhir@ms.u-tokyo.ac.jp`

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