UTMS 2003–10 March 3, 2003

Amenable discrete quantum groups

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Abstract. Z.-J. Ruan has shown that several amenability conditions are all equivalent in the case of discrete Kac algebras. In this paper we extend this work to the case of discrete quantum groups with a quite different method. That is, we show that a discrete quantum group, where we do not assume its unimodularity, has an invariant mean if and only if it satisfies a certain condition, which is called strong Voiculescu amenability in the case of Kac algebras.

1. Introduction

For a long time, a great deal of mathematicians have been charmed in beautiful duality theorems. One of them, *Pontryagin's duality theorem*, has been a birthplace of one approach to *quantum groups*. A great deal of efforts had been made to obtain the category which contains all locally compact groups and their duals. The appearance of *Kac algebras* is a major break-through in this attempt. After the work by G. I. Kac [Ka] and M. Takesaki [T1], Kac algebras have been reigning on a vast area of operator algebras. A new interesting example has been given by S. L. Woronowicz [W1], which is the *quantum* $SU(2)$*-group*. This example is out of the category of Kac algebras and suggested a possibility of appearance of a new category wider than that. It was T. Masuda and Y. Nakagami who gave axioms of the *Woronowicz algebra* ([M-N]). They introduced the new idea of a *scaling group* and succeeded in including all the known *quantum groups*. Their axioms were rather complicated and a further simplification of axioms had been longed for. Recently J. Kustermans and S. Vaes have proposed a much easier definition so as to include all the known examples of *quantum groups* ([K-V]). In this paper, we adopt their definition of *quantum groups*.

Amenability of locally compact groups is one of the main themes for operator algebraists and their actions on operator algebras have been studied well from the era of A. Connes' magnificent work [C2]. Let us consider the following basic result for amenable discrete groups (see, for example, [P], [Ky]):

Theorem 1.1. *Let G be a discrete group. Then the following statements are equivalent.*

(1) *It is amenable.*
(2) *The reduced group algebra $C_r^*(G)$ is nuclear.*

(3) *The group von Neumann algebra $L(G)$ is injective.*

Z.-J. Ruan showed the Kac algebra version of the above theorem by using the operator space method [R]. We would like to generalize the above theorem in the case of discrete quantum groups. As we pointed out, the category of discrete quantum groups is much wider than that of Kac algebras. In an attempt of such a generalization, several technical difficulties arise from the existence of the scaling group. In fact we show the main Theorem 3.9 by a quite different way from Z.-J. Ruan's. It asserts that the existence of an invariant mean is equivalent to several conditions. For example, it is equivalent to the nuclearity of the dual C^*-algebra having a character. As a corollary, we obtain the result that the dual discrete quantum group of the quantum $SU(2)$-group has an invariant mean. Moreover we get another equivalent condition, which was called *strong Voiculescu amenability* in [R] in the case of discrete Kac algebras. If the discrete quantum group (M, Δ) is a Kac algebra, we recover most of Z.-J. Ruan's characterization theorem:

Corollary 1.2. *If (M, Δ) is a discrete Kac algebra, the following statements are equivalent.*

(1) *It has an invariant mean.*
(2) *It is strongly Voiculescu amenable.*
(3) *The C^*-algebra $\hat{A}$ is nuclear and has a finite dimensional representation.*
(4) *The C^*-algebra $\hat{A}$ is nuclear.*
(5) *The von Neumann algebra $\hat{M}$ is injective.*

Amenability of quantum groups are studied in [B-M-T1], [B-M-T2], [B-M-T3], [B-C-T], [D-Q-V], for example. We emphasize that our main theorem is more general than their versions of Z.-J. Ruan's result, because we do not assume the tracial property of the invariant state on a compact quantum group. After this work was done, we learned from S. Vaes that E. Blanchard and S. Vaes also have a proof of the implication $1 \Rightarrow 2$ of Theorem 3.9. In Section 2, we prepare several definitions and notations, and we summarize important equalities. In Section 3, we prove our main Theorem 3.9.

Acknowledgements. The author is highly grateful to his supervisor Yasuyuki Kawahigashi, Yoshiomi Nakagami and Stefaan Vaes for all the discussion and encouragement.

2. Definitions and notations

Let A be a C^*-algebra and ω be a state on A. Its GNS-representation is denoted by $\{H_\omega, \xi_\omega, \pi_\omega\}$, where H_ω is the Hilbert space, ξ_ω is the cyclic vector and π_ω is the nondegenerate representation of A on H_ω, satisfying the equality: $\omega(x) = \langle \pi_\omega(x)\xi_\omega | \xi_\omega \rangle$. Let θ be a functional in the dual space A^* and a be in A. We define the functionals θa and $a\theta$ by $(\theta a)(x) = \theta(ax)$ and $(a\theta)(x) = \theta(xa)$. With these operations, the Banach space A^* becomes an A-bimodule. The *multiplier* C^*-algebra of A is denoted by $\mathcal{M}(A)$. The C^*-algebra generated by compact operators on the Hilbert space H is denoted by $\mathcal{K}(H)$. We refer to [S] and [T2] for the weight theory. For a weight φ on a von Neumann algebra M, the notations n_φ and m_φ denote the subsets $n_\varphi = \{x \in M; \varphi(x^*x) < \infty\}$ and $m_\varphi = n_\varphi^* n_\varphi = \text{span}\{x^*y; x, y \in n_\varphi\}$, respectively. The algebra,

$$M_\varphi = \{x \in M; xm_\varphi \subset m_\varphi,\ xm_\varphi \subset m_\varphi,\ \varphi(xy) = \varphi(yx),\ \text{for all } y \in m_\varphi\}$$

is called the *centralizer* of the weight φ. Let φ be a normal semifinite faithful, abbreviated as n.s.f., weight. The corresponding *modular automorphism group* is denoted by $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$. Note that we have $M_\varphi = \{x \in M; \sigma_t^\varphi(x) = x \text{ for all } t \in \mathbb{R}\}$. An operator $x \in M$ is called *analytic* if the continuous function $t \in \mathbb{R} \to \omega(\sigma_t^\varphi(x))$ is extended to an analytic function on the complex plane $\mathbb{C}$ for any normal functional $\omega \in M_*$. The representation associated to the weight φ is denoted by the triple $\{H_\varphi, \Lambda_\varphi, \pi_\varphi\}$, where they denote the Hilbert space, the canonical map $n_\varphi \to H_\varphi$ and the representation, respectively. The autopolar $\mathcal{P}_\varphi^\natural$ is defined to be the closure of the set $\{xJ_\varphi\Lambda_\varphi(x); x \in n_\varphi \cap n_\varphi^*\}$ and the quadruple $\{H_\varphi, \pi_\varphi, J_\varphi, \mathcal{P}_\varphi^\natural\}$ is called the standard representation of M (see [H] for details). One of its remarkable property is that a normal state is a vector state whose corresponding vector is in the autopolar $\mathcal{P}_\varphi^\natural$ and such a vector is uniquely determined. For a nonsingular positive selfadjoint operator h affiliated with M_φ, we define another n.s.f. weight φ_h by $\varphi_h(x) = \lim_{\varepsilon\downarrow 0} \varphi(h_\varepsilon^{\frac{1}{2}} x h_\varepsilon^{\frac{1}{2}})$ for x in M_+, where h_ε denotes $h(1+\varepsilon h)^{-1}$. Let α be an action of a locally compact group G on M. The fixed point algebra is denoted by M^α. Of course, we have $M_\varphi = M^{\sigma^\varphi}$.

In this paper, we use the minimal tensor product for C^*-algebras and von Neumann algebraic tensor product, and the both tensor products are simply denoted by $\otimes$.

We refer to [K-V] for the definition of locally compact quantum group in the following way.

Definition 2.1. A pair (M, Δ) is called a (von Neumann algebraic) locally compact quantum group when it satisfies the following conditions:

(1) M is a von Neumann algebra and $\Delta : M \to M \otimes M$ is a normal and unital $*$-homomorphism satisfying the coassociativity relation : $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.
(2) There exist an n.s.f. left invariant weight φ and an n.s.f. right invariant weight ψ on M. That is, we have $\varphi((\omega \otimes \iota)\Delta(x)) = \omega(1)\varphi(x)$ for all $x \in m_\varphi^+$, $\omega \in M_*^+$, and $\psi((\iota \otimes \omega)\Delta(x)) = \omega(1)\psi(x)$ for all $x \in m_\psi^+$, $\omega \in M_*^+$.

We denote the representation associated to φ by $\{H, \mathit{\Lambda}, \pi\}$ and regard M as a subalgebra of $B(H)$ via π. The symbols J, $\mathit{\Delta}_\varphi$, σ_t^φ are the modular conjugation, the modular operator and the modular automorphism group of φ, respectively.

The Radon-Nikodym derivative $[D\psi : D\varphi]_t$ can be written as $\nu^{\frac{1}{2}it^2}\delta^{it}$, where ν is a positive constant called the *scaling constant* and δ is a positive nonsingular selfadjoint operator called the *modular element* on H affiliated with M.

We define a unitary operator W on $H \otimes H$ by

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Delta(y)(x \otimes 1)) \text{ for } x, y \in n_\varphi.$$

Here, $\Lambda \otimes \Lambda$ denotes the canonical GNS-map for the tensor product weight $\varphi \otimes \varphi$. The operator W satisfies a so-called *pentagonal equation*, i.e., $W_{12}W_{13}W_{23} = W_{23}W_{12}$, where we use the well-known leg-notation. The operator W is called a *multiplicative unitary*.

There exists an *antipode* S on M, which is a densely defined, σ-strongly $*$ closed, linear map from M to M satisfying $(\iota \otimes \omega)(W) \in D(S)$ for all $\omega \in B(H)_*$,

$$S(\iota \otimes \omega)(W) = (\iota \otimes \omega)(W^*)$$

and having a property that the elements $(\iota \otimes \omega)(W)$ form a σ-strong $*$ core for S. This antipode S has a polar decomposition $S = R\tau_{-\frac{i}{2}}$ where R is an anti-automorphism of M and $\{\tau_t\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of automorphisms of M. We call R the *unitary antipode* and $\{\tau_t\}_{t\in\mathbb{R}}$ the *scaling group* of M. We define the nonsingular positive selfadjoint operator P by $P^{it}\Lambda(x) = \nu^{\frac{t}{2}}\Lambda(\tau_t(x))$.

The dual Banach space M^* has a convolution product $*$, i.e., for ω, $\theta \in M^*$ and $x \in M$, we define $\omega * \theta(x) = (\omega \otimes \theta)(\Delta(x))$. With this product, M^* is a Banach algebra. It may not have a $*$-structure. In fact, its involution is defined by $\omega^\sharp = \bar{\omega} \circ S$, therefore, $\sharp$ is a densely defined operation. If the scaling automorphism τ is trivial, M^* becomes a Banach*-algebra.

The *dual locally compact quantum group* $(\hat{M}, \hat{\Delta})$ is defined as follows. Denote $(\omega \otimes \iota)(W)$ by $\lambda(\omega)$ for $\omega \in B(H)^*$. We set the von Neumann algebra $\hat{M}$ to be the σ-weak closure of $\{\lambda(\omega); \omega \in B(H)_*\}$. Its coproduct is defined as $\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W}$, where $\hat{W}$ is a unitary operator $\Sigma W^* \Sigma$ and Σ denotes the flip map on $H \otimes H$. To construct the invariant weights on $(\hat{M}, \hat{\Delta})$, we define

$$\mathcal{I} = \{\omega \in M_*; \text{ there exists } \xi \in H \text{ such that } \omega(x^*) = \langle \xi, \Lambda(x) \rangle \text{ for all } x \in n_\varphi\}.$$

Such a vector ξ is uniquely determined by ω and denoted by $\xi(\omega)$. Then there exists a unique n.s.f. weight $\hat{\varphi}$ on $\hat{M}$ with the GNS-triple $(H, \hat{\Lambda}, \iota)$ such that $\lambda(\mathcal{I}) \subset n_{\hat{\varphi}}$, $\lambda(\mathcal{I})$ is a σ-strong $*$ - norm core for $\hat{\Lambda}$ and $\hat{\Lambda}(\lambda(\omega)) = \xi(\omega)$ for all $\omega \in \mathcal{I}$. The right invariant weight is defined as $\hat{\psi} = \hat{\varphi}\hat{R}$, where $\hat{R}(x) = Jx^*J$. We obtain the dual objects such as $\hat{J}$, $\Delta_{\hat{\varphi}}$, $\sigma_t^{\hat{\varphi}}$, $\hat{S} = \hat{R}\hat{\tau}_{-\frac{i}{2}}$, corresponding to $(\hat{M}, \hat{\Delta})$ in a similar way to the case of (M, Δ).

We often refer to the important results in [K-V], [V] and [VV]. For convenience, we summarize them.

1. $\psi = \varphi R, \quad \sigma_t^{\psi} = R\sigma_{-t}^{\varphi} R, \quad R(x) = \hat{J} x^* \hat{J}, \quad \hat{R}(x) = J x^* J.$

2. The automorphism groups σ^{φ}, σ^{ψ}, τ commute pairwise.

3. We have the following commutation relations, for all $t \in \mathbb{R}$:

$$\Delta\sigma_t^{\varphi} = (\tau_t \otimes \sigma_t^{\varphi})\Delta, \quad \Delta\sigma_t^{\psi} = (\sigma_t^{\psi} \otimes \tau_{-t})\Delta, \quad \Delta\tau_t = (\tau_t \otimes \tau_t)\Delta = (\sigma_t^{\varphi} \otimes \sigma_{-t}^{\psi})\Delta$$

4. There exists a scaling constant $\nu > 0$ satisfying the following identities for all $t \in \mathbb{R}$:

$$\varphi\sigma_t^{\psi} = \nu^t \varphi, \quad \psi\sigma_t^{\varphi} = \nu^{-t}\psi, \quad \varphi\tau_t = \nu^{-t}\varphi, \quad \psi\tau_t = \nu^{-t}\psi.$$

5. $P = \hat{P}, \quad \Delta_{\varphi} = \hat{J}\hat{\delta}^{-1}\hat{J}P, \quad \Delta_{\hat{\varphi}} = J\delta^{-1}JP.$

6. $W \in M \otimes \hat{M}, \quad (\hat{J} \otimes J)W(\hat{J} \otimes J) = W^*, \quad (R \otimes \hat{R})(W) = W.$
   $(P^{it} \otimes P^{it})W(P^{-it} \otimes P^{-it}) = W, \quad (\Delta_{\hat{\varphi}}^{it} \otimes \Delta_{\varphi}^{it})W(\Delta_{\hat{\varphi}}^{-it} \otimes \Delta_{\varphi}^{-it}) = W.$

7. $\tau_t(x) = \Delta_{\hat{\varphi}}^{it} x \Delta_{\hat{\varphi}}^{-it}, \quad \hat{\tau}_t(y) = \Delta_{\varphi}^{it} y \Delta_{\varphi}^{-it}.$

8. $\Delta_{\varphi}^{2it} = \hat{\delta}^{-it}\delta^{-it}J\delta^{-it}J\hat{J}\hat{\delta}^{it}\hat{J}, \quad \Delta_{\hat{\varphi}}^{2it} = \nu^{2it^2}\hat{\delta}^{-it}\delta^{-it}J\delta^{it}J\hat{J}\hat{\delta}^{-it}\hat{J},$
   $P^{2it} = \hat{\delta}^{-it}\delta^{-it}J\delta^{-it}J\hat{J}\hat{\delta}^{-it}\hat{J}.$

9. $\hat{J}J = \nu^{\frac{i}{4}} J\hat{J}.$

10. $\delta^{is}\hat{\delta}^{it} = \nu^{ist}\hat{\delta}^{it}\delta^{is}, \quad R(\delta) = \hat{J}\delta\hat{J} = \delta^{-1}, \quad \sigma_t^{\varphi}(\delta^{is}) = \nu^{ist}\delta^{is},$
    $\sigma_t^{\psi}(\delta^{is}) = \nu^{ist}\delta^{is}, \quad \tau_t(\delta^{is}) = \delta^{is}, \quad \Delta(\delta) = \delta \otimes \delta.$

From the last equalities: $R(\delta) = \hat{J}\delta\hat{J} = \delta^{-1}$, $\Delta(\delta) = \delta \otimes \delta$, we see the spectrum of δ is a closed subgroup of $\mathbb{R}_+ = [0, \infty)$. Therefore, δ is a bounded operator if and only if $\delta = 1$. If the scaling constant ν does not equal to 1, we have the spectrum $\mathrm{Sp}(\delta) = \mathbb{R}_+$. For a locally compact quantum group (M, Δ), we set A to be the norm closure of the linear space $\{(\iota \otimes \omega)(W); \omega \in B(H)_*\}$. Then A is a C^*-algebra and has a *C^*-algebraic locally compact quantum group* structure (see [K-V, Definition 4.1, p. 37] for its definition) by restricting Δ, φ and ψ on A. We regard A as a subalgebra of $B(H)$.

Let $\{X_i\}_{i \in I}$ be a family of normed spaces. We define three normed spaces as follows.

The normed space c_0-$\sum_{i \in I} X_i$ is defined to be the set of elements $(x_i)_{i \in I}$ of $\prod_{i \in I} X_i$ satisfying the condition that for any $\varepsilon > 0$ there exists a finite subset F of I such that $\sup_{i \notin F} \|x_i\| < \varepsilon$. The norm is defined to be $\|(x_i)_{i \in I}\| = \sup_{i \in I} \|x_i\|$.

The normed space l_∞-$\sum_{i \in I} X_i$ is defined to be the set of elements $(x_i)_{i \in I}$ of $\prod_{i \in I} X_i$ satisfying the condition that $\sup_{i \in I} \|x_i\|$ is finite. The norm is defined by $\|(x_i)_{i \in I}\| = \sup_{i \in I} \|x_i\|$.

The normed space l_1-$\sum_{i\in I} X_i$ is defined by the set of elements $(x_i)_{i\in I}$ of $\prod_{i\in I} X_i$ satisfying the condition that $\sum_{i\in I} \|x_i\|$ is finite. The norm is defined by $\|(x_i)_{i\in I}\| = \sum_{i\in I} \|x_i\|$.

They are complete normed space if and only if all normed spaces X_i are complete.

Note that the dual space of c_0-$\sum_{i\in I} X_i$ is identified with l_1-$\sum_{i\in I} X_i^*$ and the dual space of l_1-$\sum_{i\in I} X_i$ is identified with l_∞-$\sum_{i\in I} X_i^*$.

3. Amenability of discrete quantum groups

This section is devoted to a proof of our main result Theorem 3.9. We define an invariant mean on locally compact quantum groups in a similar way to the case of locally compact groups.

Definition 3.1. Let (M, Δ) be a locally compact quantum group.

(1) A state m of M is called a left invariant mean if $\omega * m = \omega(1)m$ for any $\omega \in M_*$.
(2) A state m of M is called a right invariant mean if $m * \omega = \omega(1)m$ for any $\omega \in M_*$.
(3) A state m of M is called an invariant mean if m is left and right invariant.

Remark 3.2. If (M, Δ) has a left invariant mean, it also has an invariant mean. In fact, let m be a left invariant mean, then $m * mR$ is an invariant mean.

Definition 3.3. Let (M, Δ) be a locally compact quantum group.

(1) We say that it satisfies the condition (W_1), if there exists a net of unit vectors $\{\xi_j\}_{j\in\mathcal{J}}$ in H such that for any vector η in H, $\|W^*(\eta\otimes\xi_j) - \eta\otimes\xi_j\| \to 0$.
(2) We say that it satisfies the condition (W_2), if there exists a net of unit vectors $\{\xi_j\}_{j\in\mathcal{J}}$ in H such that for any representation $\{\pi, H_\pi\}$ of A, and for any vector η in H_π, $\|(\pi\otimes\iota)(W)^*(\eta\otimes\xi_j) - \eta\otimes\xi_j\| \to 0$.

Lemma 3.4. *Let (M, Δ) be a locally compact quantum group. Then the following conditions are equivalent.*

(1) *It satisfies the condition* (W_1).
(2) *There exists a net of unit vectors $\{\xi_j\}_{j\in\mathcal{J}}$ in H such that for any functional $\omega \in M_*$, $\lambda(\omega)\xi_j - \omega(1)\xi_j \to 0$ in the norm topology.*
(3) *There exists a net of normal states $\{\omega_j\}_{j\in\mathcal{J}}$ on $\hat{M}$ such that $\{(\iota\otimes\omega_j)(W)\}_{j\in\mathcal{J}}$ is a σ-weakly approximate unit of A.*
(4) *There exists a net of normal states $\{\omega_j\}_{j\in\mathcal{J}}$ on $\hat{M}$ such that* $\mathrm{id} : \hat{A} \to \hat{A}$ *is pointwise-weakly approximated by the net of unital completely positive map $\{(\mathrm{id}\otimes\omega_j)\circ\hat{\Delta}\}_{j\in\mathcal{J}}$ and also by $\{(\omega_j\otimes\mathrm{id})\circ\hat{\Delta}\}_{j\in\mathcal{J}}$.*

(5) *There exists a net of normal states* $\{\omega_j\}_{j\in\mathcal{J}}$ *on* $\hat{M}$ *such that* $\mathrm{id}:\hat{A}\to\hat{A}$ *is pointwise-norm approximated by the net of unital completely positive map* $\{(\mathrm{id}\otimes\omega_j)\circ\hat{\Delta}\}_{j\in\mathcal{J}}$ *and also by* $\{(\omega_j\otimes\mathrm{id})\circ\hat{\Delta}\}_{j\in\mathcal{J}}$.

Proof. $1\Rightarrow 2$. It suffices to prove the statement in the case that ω is a normal state on M by considering linear combinations. Since M is standardly represented, ω is written as $\omega=\omega_\eta$ with a unit vector $\eta\in H$. For any vector $\zeta\in H$,

$$|\langle\lambda(\omega_\eta)\xi_j-\omega_\eta(1)\xi_j|\zeta\rangle|\leq\|\zeta\|\|\eta\|\|W(\eta\otimes\xi_j)-\eta\otimes\xi_j\|,$$

so we get $\|\lambda(\omega_\eta)\xi_j-\omega_\eta(1)\xi_j\|\leq\|\eta\|\|W(\eta\otimes\xi_j)-\eta\otimes\xi_j\|$. Hence $\|\lambda(\omega_\eta)\xi_j-\omega_\eta(1)\xi_j\|$ converges to 0.

$2\Rightarrow 3$. Put $\omega_j=\omega_{\xi_j}$ for any j in $\mathcal{J}$. Then for any operator $a\in A$ and any normal functional $\theta\in M_*$, we have

$$\begin{aligned}|\theta(a(\iota\otimes\omega_j)(W)-a)|&=|\langle\lambda(\theta a)\xi_j-\theta(a)\xi_j|\xi_j\rangle|\\&\leq\|\lambda(\theta a)\xi_j-(\theta a)(1)\xi_j\|.\end{aligned}$$

Therefore, $a(\iota\otimes\omega_j)(W)-a$ converges to 0 σ-weakly. Similarly we see $(\iota\otimes\omega_j)(W)a-a$ converges to 0 σ-weakly.

$3\Rightarrow 4$. Take a net of normal states $\{\omega_j\}_{j\in\mathcal{J}}$ of $\hat{M}_*$ which satisfies the third condition. Let ω be a normal functional on M. By applying Cohen's factorization theorem ([B-D, Theorem 10, p. 61]) to the A-bimodule M_*, there exist a in A and ω' in M_* such that $\omega=a\omega'$. Then for any functional $\theta\in\hat{A}^*$, we have

$$\begin{aligned}\theta((\omega_j\otimes\mathrm{id})\circ\hat{\Delta}(\lambda(\omega)))&=\omega((\iota\otimes\theta)(W)(\iota\otimes\omega_j)(W))\\&=(\omega'(\iota\otimes\theta)(W))((\iota\otimes\omega_j)(W))a),\end{aligned}$$

which converges to $\theta(\lambda(\omega))$. Since the linear subspace $\{\lambda(\omega);\omega\in M_*\}$ is norm dense in $\hat{A}$ and $\{\theta\circ(\omega_j\otimes\mathrm{id})\circ\hat{\Delta}\}_{j\in\mathcal{J}}$ is a norm bounded family, $\theta((\omega_j\otimes\mathrm{id})\circ\hat{\Delta}(x))$ converges to $\theta(x)$ for any operator $x\in\hat{A}$. Similarly we can see that $\theta((\mathrm{id}\otimes\omega_j)\circ\hat{\Delta}(x))$ converges to $\theta(x)$ for $x\in\hat{A}$.

$4\Rightarrow 5$. Take a net of normal states $\{\omega_j\}_{j\in\mathcal{J}}$ on $\hat{M}$ which satisfies the fourth condition. Let $\mathcal{F}$ be the set of finite subsets of $\hat{A}$. Take $F=\{a_1,a_2,\dots,a_k\}$ in $\mathcal{F}$ and n in $\mathbb{N}$. Consider the product Banach space $\hat{A}_F=l_\infty\text{-}\sum_{x\in\mathcal{F}}\hat{A}\times\hat{A}$ and its dual Banach space $\hat{A}_F^*=l_1\text{-}\sum_{x\in\mathcal{F}}(\hat{A}\times\hat{A})^*$. Let $x_F(\omega)$ denote

$$\begin{gathered}\big((\omega\otimes\mathrm{id})\circ\hat{\Delta}(a_1)-a_1,(\mathrm{id}\otimes\omega)\circ\hat{\Delta}(a_1)-a_1,\\(\omega\otimes\mathrm{id})\circ\hat{\Delta}(a_2)-a_2,(\mathrm{id}\otimes\omega)\circ\hat{\Delta}(a_2)-a_2,\\\vdots\\(\omega\otimes\mathrm{id})\circ\hat{\Delta}(a_k)-a_k,(\mathrm{id}\otimes\omega)\circ\hat{\Delta}(a_k)-a_k\big).\end{gathered}$$

Then $x_F(\omega_j)$ converges to 0 weakly. Hence the norm closure of the convex hull of $\{x_F(\omega_j); j \in \mathcal{J}\}$ contains 0. So there exists a normal state $\omega_{(F,n)}$ on $\hat{M}$ such that $\|(\omega_{(F,n)} \otimes \mathrm{id}) \circ \hat{\Delta}(a) - a\| < \frac{1}{n}$, and $\|(\mathrm{id} \otimes \omega_{(F,n)}) \circ \hat{\Delta}(a) - a\| < \frac{1}{n}$ for any element $a \in F$. This new net $\{\omega_{(F,n)}\}_{(F,n) \in \mathcal{F} \times \mathbb{N}}$ is a desired one.

$5 \Rightarrow 4$. It is trivial.

$4 \Rightarrow 3$. Easy to prove by reversing the proof of $3 \Rightarrow 4$.

$3 \Rightarrow 1$. Take such a net of normal states $\{\omega_j\}_{j \in \mathcal{J}}$ on M. Since $\hat{M}$ is standardly represented, there exists a net of unit vectors $\{\xi_j\}_{j \in \mathcal{J}}$ in H with $\omega_j = \omega_{\xi_j}$. Take a vector η in H. Now by Cohen's factorization theorem, there exist an operator $a \in A$ and a vector $\zeta \in H$ with $\eta = a\zeta$. Then we have

$$\begin{aligned}\|W(\eta \otimes \xi_j) - \eta \otimes \xi_j\|^2 &= 2\|a\zeta\|^2 - 2\,\mathrm{Re}(\langle W(a\zeta \otimes \xi_j) | a\zeta \otimes \xi_j\rangle) \\ &= 2\|a\zeta\|^2 - 2\,\mathrm{Re}(\langle (\iota \otimes \omega_j)(W) a\zeta | a\zeta\rangle).\end{aligned}$$

This converges to 0.

□

Lemma 3.5. *Let (M, Δ) be a locally compact quantum group. The following conditions are equivalent.*

1. *It satisfies the condition* (W_2).
2. *There exists a net of unit vectors $\{\xi_j\}_{j \in \mathcal{J}}$ in H such that for any cyclic representation $\{\pi, H_\pi\}$ of A, $\|(\pi \otimes \iota)(W)^*(\eta \otimes \xi_j) - (\eta \otimes \xi_j)\|$ converges to 0, for any vector $\eta \in H_\pi$.*
3. *There exists a net of unit vectors $\{\xi_j\}_{j \in \mathcal{J}}$ in H such that for any functional $\omega \in A^*$, $\lambda(\omega)\xi_j - \omega(1)\xi_j \to 0$ in the norm topology.*
4. *There exists a net of normal states $\{\omega_j\}_{j \in \mathcal{J}}$ in $\hat{M}_*$ such that $\{(\iota \otimes \omega_j)(W)\}_{j \in \mathcal{J}}$ is a weakly approximate unit of A.*
5. *There exists a net of normal states $\{\omega_j\}_{j \in \mathcal{J}}$ in $\hat{M}_*$ such that $\{(\iota \otimes \omega_j)(W)\}_{j \in \mathcal{J}}$ is a norm approximate unit of A.*
6. *There exists a net of normal states $\{\omega_j\}_{j \in \mathcal{J}}$ in $\hat{M}_*$ such that* $\mathrm{id} : \hat{A} \to \hat{A}$ *is approximated in the pointwise norm topology by the net of unital completely positive maps $\{(\mathrm{id} \otimes \omega_j) \circ \hat{\Delta}\}_{j \in \mathcal{J}}$ and also by $\{(\omega_j \otimes \mathrm{id}) \circ \hat{\Delta}\}_{j \in \mathcal{J}}$.*
7. *There exists a character ϱ on $\hat{A}$ such that $(\iota \otimes \varrho)(W) = 1$.*
8. *The C^*-algebra $\hat{A}$ has a character.*
9. *There exists a state ϱ on $\hat{M}$ such that ϱ is an $\hat{A}$-linear map and satisfies $(\iota \otimes \varrho)(W) = 1$.*

Proof. $1 \Rightarrow 2$. It is trivial.

$2 \Rightarrow 3$. Take such a net of unit vectors $\{\xi_j\}_{j \in \mathcal{J}}$ in H. Let ω be a state on A and $\{H_\omega, \pi_\omega, \xi_\omega\}$ be its GNS representation. Then for any ζ in H, we have

$$|\langle \lambda(\omega)\xi_j - \omega(1)\xi_j | \zeta\rangle| \leq \|\zeta\| \|(\pi_\omega \otimes \iota)(W)(\xi_\omega \otimes \xi_j) - \xi_\omega \otimes \xi_j\|.$$

So we get $\|\lambda(\omega)\xi_j - \omega(1)\xi_j\| \leq \|(\pi_\omega \otimes \iota)(W)(\xi_\omega \otimes \xi_j) - \xi_\omega \otimes \xi_j\|$. Hence for any $\omega \in A^*$, $\|\lambda(\omega)\xi_j - \omega(1)\xi_j\|$ converges to 0.

$3 \Rightarrow 4$. Put $\omega_j = \omega_{\xi_j}$ for any j in $\mathcal{J}$. Then for any operator $a \in A$ and any functional $\theta \in A^*$,

$$|\theta(a(\iota \otimes \omega_j)(W) - a)| = |\langle \lambda(\theta a)\xi_j - \theta(a)\xi_j | \xi_j \rangle| \leq \|\lambda(\theta a)\xi_j - (\theta a)(1)\xi_j\|.$$

And therefore, $a(\iota \otimes \omega_j)(W) - a$ converges to 0 weakly. Similarly, we can see $(\iota \otimes \omega_j)(W)a - a$ converges to 0 weakly.

$4 \Rightarrow 5$. Take such a net of normal states $\{\omega_j\}_{j \in \mathcal{J}}$ of $\hat{M}_*$. Let $\mathcal{F}$ be the set of finite subsets of A. Take $F = \{a_1, a_1, \ldots, a_k\}$ in $\mathcal{F}$ and n in $\mathbb{N}$. Consider the product Banach space $A_F = l_\infty\text{-}\sum_{x \in \mathcal{F}} A$ and its dual Banach space $A_F^* = l_1\text{-}\sum_{x \in \mathcal{F}} A^*$. Let $x_F(\omega)$ denote

$$((\iota \otimes \omega)(W)a_1 - a_1, (\iota \otimes \omega)(W)a_2 - a_2, \ldots, (\iota \otimes \omega)(W)a_k - a_k).$$

Now the net of the elements in A_F, $x_F(\omega_j)$ converges to 0 weakly. Hence the norm closure of the convex hull of $\{x_F(\omega_j); j \in \mathcal{J}\}$ contains 0. So there exists a normal state of $\hat{M}_*$, $\omega_{(F,n)}$ such that $\|(\iota \otimes \omega_{(F,n)})(W)a - a\| < \frac{1}{n}$ for any a in F. Then we get a new net of normal states of $\hat{M}_*$, $\{\omega_{(F,n)}\}_{(F,n) \in \mathcal{F} \times \mathbb{N}}$ and it is easy to see this net is a desired one.

$5 \Rightarrow 6$. This is shown in a similar way to the proof of $3 \Rightarrow 4 \Rightarrow 5$ in Lemma 3.4.

$6 \Rightarrow 7$. Take such a net of normal states $\{\omega_j\}_{j \in \mathcal{J}}$ of $\hat{M}_*$. Let ϱ be a weak$*$-accumulating state in $\hat{A}^*$ of the net. Then for any normal functional ω on $\hat{M}$, we have

$$\begin{aligned}(\omega_j \otimes \mathrm{id})(\hat{\Delta}(\lambda(\omega))) &= (\omega_j \otimes \iota \otimes \omega)(\hat{W}_{23}^* \hat{W}_{13}^*) \\ &= (\iota \otimes \omega)(\hat{W}^*(\omega_j \otimes \iota)(\hat{W}^*)).\end{aligned}$$

This equality converges to $\lambda(\omega) = (\iota \otimes \omega)(\hat{W}^*(\varrho \otimes \iota)(\hat{W}^*))$. Therefore we obtain $(\iota \otimes \varrho)(W) = 1$ and we easily see ϱ is a character of $\hat{A}$.

$7 \Rightarrow 8$. It is trivial.

$8 \Rightarrow 7$. Take a character ϱ on $\hat{A}$. Let u be a unitary $(\iota \otimes \varrho)(W)$ in M. Then we have

$$\begin{aligned}\Delta(u) &= (\iota \otimes \iota \otimes \varrho)((\Delta \otimes \iota)(W)) \\ &= (\iota \otimes \iota \otimes \varrho)(W_{13} W_{23}) \\ &= u \otimes u.\end{aligned}$$

Therefore, for any normal functional $\omega \in M_*$ we get

$$\begin{aligned}\lambda(u^*\omega) &= (\omega \otimes \iota)(W(u^* \otimes 1)) \\ &= (\omega \otimes \iota)((1 \otimes u^*)W(1 \otimes u)) \\ &= u^*\lambda(\omega)u.\end{aligned}$$

Then we obtain

$$\begin{aligned}|\omega(1)| &= |(u^*\omega)(u)| \\ &= |\varrho(\lambda(u^*\omega))| \\ &\leq \|\lambda(u^*\omega)\| \\ &= \|u^*\lambda(\omega)u\| \\ &= \|\lambda(\omega)\|.\end{aligned}$$

Hence we can define the character χ on $\hat{A}$ with $\chi(\lambda(\omega)) = \omega(1)$ for $\omega \in M_*$.

$7 \Rightarrow 9$. Take such a character ϱ on $\hat{A}$ and extend it to the state on $\hat{M}$. Denote the extended state by $\bar{\varrho}$. The $\hat{A}$-linearity of $\bar{\varrho}$ easily follows from Stinespring's theorem.

$9 \Rightarrow 3$. Take such a state ϱ on $\hat{M}$ and let $\{\omega_j\}_{j \in \mathcal{J}}$ be a net of normal states on $\hat{M}$ which converges to ϱ weakly$*$. Then for any functional θ on A and for any normal functional ω on $\hat{M}$, we have $\theta((\iota \otimes \omega_j)(W)) = \omega_j((\theta \otimes \iota)(W))$. This converges to $\varrho((\theta \otimes \iota)(W)) = \theta(1)$.

$3 \Rightarrow 2$. Take a state ω of A and let $\{H_\omega, \pi_\omega, \xi_\omega\}$ be the GNS representation of ω. Take a vector η in H_ω. By Cohen's factorization theorem, there exists an operator $a \in A$ and a vector $\zeta \in H$ with $\eta = \pi_\omega(a)\zeta$. Then, we have

$$\begin{aligned}\|(\pi_\omega \otimes \iota)(W)\eta \otimes \xi_j - \eta \otimes \xi_j\|^2 &= 2\|\pi_\omega(a)\zeta\|^2 \\ &\quad - \mathrm{Re}(\langle(\pi_\omega \otimes \iota)(W)\pi_\omega(a)\zeta \otimes \xi_j | \pi_\omega(a)\zeta \otimes \xi_j\rangle) \\ &= 2\|\pi_\omega(a)\zeta\|^2 - \mathrm{Re}(\langle\pi_\omega(a^*(\iota \otimes \omega_j)(W)a)\zeta|\zeta\rangle).\end{aligned}$$

This converges to 0.

$2 \Rightarrow 1$. Let $\{\pi, H_\pi\}$ be a representation of A. We may assume that this representation is nondegenerate. We decompose this representation to the direct sum of cyclic representation. As easily seen, it derives the statement of 1.

□

Therefore, we obtain the following result by the previous two Lemmas.

Corollary 3.6. *Let (M, Δ) be a locally compact quantum group. Then the following statements are equivalent.*

1. *It satisfies the condition* (W_1).
2. *It satisfies the condition* (W_2).

Proof. 2 $\Rightarrow$ 1. It is trivial.

1 $\Rightarrow$ 2. The condition (W_2) is equivalent to the condition 6 in Lemma 3.5 and it is the same as the condition 5 in Lemma 3.4 which is equivalent to the condition (W_1). □

We begin to study the amenability of discrete quantum groups from now. Let us adopt well-established definition of compactness and discreteness as follows (see [M-V]).

Definition 3.7. Let (M, Δ) be a locally compact quantum group.

(1) We call it *unimodular* if its modular element δ equals to 1.
(2) We call it *compact* if the associated C^*-algebra A is unital.
(3) We call it *discrete* if the dual locally compact quantum group $(\hat{M}, \hat{\Delta})$ is compact.

We recall some special properties of compact or discrete quantum groups (*cf.* [W2]).

Remark 3.8. (1) A locally compact quantum group (M, Δ) is compact if and only if its invariant weight φ is finite. So, when we consider a compact quantum group, we assume the invariant weight is normalized. Note that compact quantum groups are unimodular, so $\varphi = \psi$. Of course its scaling constant equals to 1.

(2) If (M, Δ) is discrete, $z = \hat{\lambda}(\hat{\varphi})$ is a one dimensional central projection of M and $Mz = \mathbb{C}z$. Let ε be the normal $*$-homomorphism from M to $\mathbb{C}$ such that $xz = \varepsilon(x)z$. Then one can see easily $(\varepsilon \otimes \iota)\Delta = (\iota \otimes \varepsilon)\Delta = \mathrm{id}$. This ε is called the counit of (M, Δ).

(3) If (M, Δ) is a discrete quantum group, then the C^*-algebra A is a C^*-subalgebra of $\mathcal{K}(H)$.

(4) If (M, Δ) is a discrete quantum group, then there exist central projections $\{z_\alpha\}_{\alpha \in I}$ of M which satisfy the following conditions.
   (a) $1 = \sum_{\alpha \in I} z_\alpha$.
   (b) For any $\alpha \in I$, Mz_α is isomorphic to a matrix algebra $\mathbb{M}_{n_\alpha}(\mathbb{C})$ for some positive integer n_α.
   (c) For any $\alpha \in I$, the multiplicity of the inclusion $Mz_\alpha \subset z_\alpha B(H) z_\alpha$ equals to n_α.
   (d) The associated C^*-algebra A is naturally isomorphic to the C^*-algebra c_0-$\sum_{\alpha \in I} Mz_\alpha$.

(5) If (M, Δ) has a normal invariant mean, then it is compact. In fact, let m be a normal invariant mean and e be the support projection of m. Then for any normal state ω, $(\omega \otimes m)(\Delta(e^\perp)) = (\omega * m)(e^\perp) = m(e^\perp) = 0$. So we get $(\omega \otimes \iota)(\Delta(e^\perp))e = 0$ and this implies $1 \otimes e \leq \Delta(e)$, then $e = 1$ by [V, Lemma 1.7.2, p. 68].

We will show the following main theorem of this paper.

Theorem 3.9. *If (M, Δ) is a discrete quantum group, the following statements are equivalent.*

(1) *It has an invariant mean.*
(2) *It satisfies the condition* (W_1)*.*
(3) *It satisfies the condition* (W_2)*.*
(4) *The C^*-algebra $\hat{A}$ has a character ϱ with $(\iota \otimes \varrho)(W) = 1$.*
(5) *The C^*-algebra $\hat{A}$ has a character.*
(6) *The C^*-algebra $\hat{A}$ is nuclear and has a character.*
(7) *The von Neumann algebra $\hat{M}$ is injective and has an $\hat{A}$-linear state.*

Let (M, Δ) be a discrete quantum group and $\{Mz_\alpha\}_{\alpha \in I}$ be a matrix decomposition as above remark. For a finite subset F in I, let us denote $z_F = \sum_{\alpha \in F} z_\alpha$. Note that $z_F H = \Lambda(M z_F)$ is finite dimensional subspace.

Lemma 3.10. *If F is a finite subset of I, then the linear subspace $K_F = \{\lambda(\omega z_F);\ \omega \in M_*\} \subset \hat{A}$ is finite dimensional.*

Proof. Take a normal functional ω in $\mathcal{I}$ as introduced in Section 2 and an operator x in n_φ. Then we have

$$\begin{aligned}\omega(z_F x^*) &= \omega((z_F x)^*)\\ &= \langle \lambda(\omega) | z_F \Lambda(x) \rangle\\ &= \langle z_F \lambda(\omega) \xi_{\hat{\varphi}} | \Lambda(x) \rangle.\end{aligned}$$

So ωz_F is in $\mathcal{I}$ and we obtain $\lambda(\omega z_F)\xi_{\hat{\varphi}} = z_F \lambda(\omega) \xi_{\hat{\varphi}} \in z_F H$. Since $\xi_{\hat{\varphi}}$ is a separating vector for $\hat{M}$ and $\mathcal{I}$ is norm dense in M_*, the statement follows. □

We give a proof of Theorem 3.9 partly first.

Proof of Theorem 3.9 $(2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 1)$.

Equivalence of conditions 2, 3, 4 and 5 has been already proved in Lemma 3.5. Fix a complete orthonormal system $\{e_p\}_{p \in \mathcal{P}}$ of H.

$3 \Rightarrow 6$. We show that $\hat{A}$ has completely positive approximation property. Set $T_\omega = (\iota \otimes \omega) \circ \hat{\Delta}$ for $\omega \in \hat{M}_*$. By assumption, there exists a net of normal states $\{\omega_j\}_{j \in \mathcal{J}}$ on $\hat{M}$ satisfying the third condition, that is, T_{ω_j} converges to the identity map of $\hat{A}$ in the pointwise norm topology. Since $\hat{M}$ is standardly represented, each ω_j is a vector state defined by a unit vector $\xi_j \in H$. Take a finite subset F

of I. Then for any operator $x \in \hat{A}$, we have

$$\begin{aligned} T_{\omega_{z_F\xi_j}}(x) &= (\iota \otimes \omega_{z_F\xi_j})(\hat{\Delta}(x)) \\ &= \sum_{p\in\mathcal{P}} (\iota \otimes \omega_{z_F\xi_j, e_p})(\hat{W})^*(\iota \otimes \omega_{z_F\xi_j, e_p})((1\otimes x)\hat{W}) \\ &= \sum_{p\in\mathcal{P}} \lambda((\omega_{e_p,\xi_j})z_F)\lambda((\omega_{x^*e_p,\xi_j})z_F)^*. \end{aligned}$$

Since $\lambda((\omega_{e_p,\xi_j})z_F)\lambda((\omega_{x^*e_p,\xi_j})z_F)^*$ is in the finite dimensional linear subspace $K_F K_F^* = \mathrm{span}\{ab^*; a, b \in K_F\} \subset \hat{A}$, so is $T_{\omega_{z_F\xi_j}}(x)$. Hence the completely positive map $T_{\omega_{z_F\xi_j}}$ is finite rank. For a natural number n and $j \in \mathcal{J}$ take an finite subset $F(n, j)$ of I with $\|T_{\omega_{z_{F(n,j)}\xi_j}} - T_{\omega_{\xi_j}}\| < \frac{1}{n}$. Then we get a new net of completely positive maps $\{T_{\omega_{z_{F(n,j)}\xi_j}}\}_{(n,j)\in\mathbb{N}\times\mathcal{J}}$ of finite rank. This net gives a completely positive approximation of the identity map of $\hat{A}$, hence $\hat{A}$ is a nuclear C^*-algebra.

$6 \Rightarrow 7$. The nuclearity $\hat{A}$ implies the injectivity $\hat{M}$. Extend the character on $\hat{A}$ to the state on $\hat{M}$. We can easily see the $\hat{A}$-linearity of this state by Stinespring's theorem.

$7 \Rightarrow 1$. Let ϱ be an $\hat{A}$-linear state on M. There exists a conditional expectation E from $B(H)$ onto $\hat{M}$. We set a state m on M by $m = \varrho \circ E|_M$. Then for any vector $\xi \in H$ and for any operator $x \in M$, we have

$$\begin{aligned} \omega_\xi * m(x) &= m((\omega_\xi \otimes \iota)(\Delta(x))) \\ &= m\Big(\sum_{p\in\mathcal{P}} (\omega_{\xi, e_p} \otimes \iota)(W)^* x (\omega_{\xi, e_p} \otimes \iota)(W)\Big) \\ &= \sum_{p\in\mathcal{P}} m\big((\omega_{\xi, e_p} \otimes \iota)(W)^* x (\omega_{\xi, e_p} \otimes \iota)(W)\big) \\ &= \sum_{p\in\mathcal{P}} \varrho\big(((\omega_{\xi, e_p} \otimes \iota)(W)^* E(x) (\omega_{\xi, e_p} \otimes \iota)(W)\big) \\ &= \sum_{p\in\mathcal{P}} \varrho((\omega_{\xi, e_p} \otimes \iota)(W)^*)\varrho(E(x))\varrho((\omega_{\xi, e_p} \otimes \iota)(W)) \\ &= \sum_{p\in\mathcal{P}} \varrho((\omega_{\xi, e_p} \otimes \iota)(W)^*)\varrho((\omega_{\xi, e_p} \otimes \iota)(W))\varrho(E(x)) \\ &= \sum_{p\in\mathcal{P}} \varrho((\omega_{\xi, e_p} \otimes \iota)(W)^*(\omega_{\xi, e_p} \otimes \iota)(W))\varrho(E(x)) \\ &= \varrho(\omega_\xi(1))\varrho(E(x)) \\ &= \omega_\xi(1)m(x), \end{aligned}$$

where we use the norm convergence of $\sum_{p\in\mathcal{P}} (\omega_{\xi, e_p} \otimes \iota)(W)^* x (\omega_{\xi, e_p} \otimes \iota)(W)$ in the third equality. Therefore, m is a left invariant mean on M. □

Now we are going to prove the implication 1 $\Rightarrow$ 2 of Theorem 3.9. We have to prove several lemmas. We assume that (M, Δ) is discrete throughout the following. Let us denote $L^\infty(\mathbb{R})$ for the von Neumann algebra which consists of essentially bounded measurable functions with respect to Lebesgue measure.

Lemma 3.11. *Let $m_\mathbb{R}$ be an invariant mean of $L^\infty(\mathbb{R})$. For any ω in M_* define the τ-invariant functional ω' by $\omega'(x) = m_\mathbb{R}(\{t \mapsto \omega(\tau_t(x))\})$ for x in M. Then ω' is a normal functional with $\|\omega'\| \leq \|\omega\|$.*

Proof. For ω in M_* set $f_{\omega,x}(t) = \{t \to \omega(\tau_t(x))\}$ in $C^b(\mathbb{R})$. For any finite subset F in I, $|f_{\omega,x}(t) - f_{\omega z_F, x}(t)| \leq \|\omega - \omega z_F\|\|x\|$, hence $\|\omega' - (\omega z_F)'\| \leq \|\omega - \omega z_F\|$. By the normality of ω, $\lim_F (\omega z_F)' = \omega'$. Notice that $M z_F$ is finite dimensional, so $(\omega z_F)' = (\omega z_F)' z_F$ is a normal functional. □

Since $\sigma_t^\varphi = \tau_t = \mathrm{Ad}\delta^{-\frac{i}{2}t}$ and $\delta^{-\frac{i}{2}t}$ is in M, we fix a matrix unit $\{e(\alpha)_{kl}\}_{1 \leq k, l \leq n_\alpha}$ of $M z_\alpha$ for each α in I, such that they are diagonalizing δz_α as $\delta^{-\frac{1}{2}} z_\alpha = \sum_{k=1}^{n_\alpha} \nu(\alpha)_k$ $e(\alpha)_{kk}$, where $\nu(\alpha)_k$ denotes a positive real number.

Lemma 3.12. *If (M, Δ) has an invariant mean m, there exists a net of normal states $\{\omega_j\}_{j \in \mathcal{J}}$ on M, which satisfies the following two conditions:*

(1) *$\|\omega * \omega_j - \omega(1)\omega_j\|$ converges to 0 for any ω in M_*.*
(2) *$\omega_j \circ \tau_t = \omega_j$ for any t in $\mathbb{R}$.*

Proof. Firstly we show the existence of a net satisfying the first condition. Since the convex hull of the vector states is weak$*$-dense in the state space of M, there exists a net of normal states $\{\chi_j\}_{j \in \mathcal{J}}$ in M_* such that $m = \text{w}^*\text{-}\lim_j \chi_j$. Let $\mathcal{F}$ be the set of finite subsets of M_*. For $F = \{\omega_1, \omega_2, \ldots, \omega_k\} \in \mathcal{F}$, consider the Banach space $(M_*)_F = l_1\text{-}\sum_{\omega \in F} M_*$ and its dual Banach space $M_F = l_\infty\text{-}\sum_{\omega \in F} M$. Set

$$x_F(\chi) = (\omega_1 * \chi - \omega_1(1)\chi, \omega_k * \chi - \omega_k(1)\chi, \ldots, \omega_k * \chi - \omega_k(1)\chi),$$

for χ in M_*. Then $x_F(\chi_j)$ converges to 0 weakly. So the norm closure of the convex hull of $\{x_F(\chi_j); j \in \mathcal{J}\}$ contains 0. Hence for any n in $\mathbb{N}$, there exists $\chi_{(F,n)}$ such that $\|\omega * \chi_{(F,n)} - \omega(1)\chi_{(F,n)}\| < \frac{1}{n}$ for an ω in F. The new net $\{\chi_{(F,n)}\}_{(F,n) \in \mathcal{F} \times \mathbb{N}}$ is a desired one.

Next we show existence of a net satisfying the both conditions. Let $\{\omega_j\}_{j \in \mathcal{J}}$ be a net satisfying the first condition. Take an invariant mean $m_\mathbb{R}$ of $L^\infty(\mathbb{R})$ as usual abelian group. For ω in M_*, define the functional ω' by $\omega'(x) = m_\mathbb{R}(\{t \mapsto \omega(\tau_t(x))\})$ for x in M. By the previous Lemma, ω' is also normal. We show that the net $\{\omega_j'\}_{j \in \mathcal{J}}$ satisfies the first condition. For the normal functional $\omega = \omega_{\Lambda(e(\alpha)_{kl}), \Lambda(e(\alpha)_{mn})}$, we have $\omega \circ \tau_{-t} = \nu_k(\alpha)^{it} \nu_l(\alpha)^{-it} \nu_m(\alpha)^{-it} \nu_n(\alpha)^{it} \omega$. Then we obtain

$$\begin{aligned}
&|\omega * \omega_j'(x) - \omega(1)\omega_j'(x)| \\
=&|m_{\mathbb{R}}(\{t \to (\omega \otimes \omega_j \circ \tau_t)(\Delta(x)) - \omega(1)\omega_j(\tau_t(x))\})| \\
=&|m_{\mathbb{R}}(\{t \to (\omega \circ \tau_{-t} \otimes \omega_j)(\Delta(\tau_t(x))) - \omega(\tau_{-t}(1))\omega_j(\tau_t(x))\})| \\
\leq &\sup_{t\in\mathbb{R}}\{|(\omega \circ \tau_{-t}) * \omega_j(\tau_t(x)) - \omega(\tau_{-t}(1))\omega_j(\tau_t(x))|\} \\
\leq &\sup_{t\in\mathbb{R}}\{\|(\omega \circ \tau_{-t}) * \omega_j - \omega \circ \tau_{-t}(1)\omega_j\|\}\|x\| \\
= &\sup_{t\in\mathbb{R}}\{\|\nu_k(\alpha)^{it}\nu_l(\alpha)^{-it}\nu_m(\alpha)^{-it}\nu_n(\alpha)^{it}(\omega * \omega_j - \omega(1)\omega_j)\|\}\|x\| \\
\leq &\|\omega * \omega_j - \omega(1)\omega_j\|\|x\|.
\end{aligned}$$

So for this ω, we have $\|\omega * \omega_j' - \omega(1)\omega_j'\| \leq \|\omega * \omega_j - \omega(1)\omega_j\|$ and therefore $\|\omega * \omega_j' - \omega(1)\omega_j'\|$ converges to 0. By taking the linear combination for ω, we see that $\|\omega * \omega_j' - \omega(1)\omega_j'\|$ converges to 0 for any normal functional with $\omega = \omega z_\alpha$. Take a normal functional ω and a positive ε. Then there exists a finite subset F of I and j_0 in $\mathcal{J}$ such that $\|\omega z_F - \omega\| < \varepsilon$ and $\|\omega z_F * \omega_j' - \omega z_F(1)\omega_j'\| < \varepsilon$ for $j \geq j_0$. Then we have

$$\begin{aligned}
&\|\omega * \omega_j' - \omega(1)\omega_j'\| \\
\leq &\|(\omega - \omega z_F) * \omega_j'\| + \|\omega z_F * \omega_j' - \omega z_F(1)\omega_j'\| + \|(\omega z_F(1) - \omega(1))\omega_j'\| \\
\leq &\varepsilon + \varepsilon + \varepsilon = 3\varepsilon,
\end{aligned}$$

for $j \geq j_0$. Therefore, $\|\omega * \omega_j' - \omega(1)\omega_j'\|$ converges to 0. □

Lemma 3.13. *If (M, Δ) has an invariant mean, there exists a net of unit vectors $\{\xi_j\}_{j\in\mathcal{J}}$ in H which satisfies the following four conditions:*

(1) $\|\omega * \omega_{\xi_j} - \omega(1)\omega_{\xi_j}\|$ *converges to 0 for any ω in M_*.*
(2) $P^{it}\xi_j = \xi_j$ *for any t in $\mathbb{R}$.*
(3) *For any j in $\mathcal{J}$, there exists a finite subset F_j of I such that $z_{F_j}\xi_j = \xi_j$.*
(4) *For any j in $\mathcal{J}$, the vector ξ_j is in the convex cone $\mathcal{P}^\natural_\varphi$.*

Proof. There exists a net of normal state $\{\omega_j\}_{j\in\mathcal{J}}$ which satisfies the two conditions of the previous lemma. If necessary, by cutting and normalizing, we may assume that there exists a finite subset F_j of I such that $\omega_j z_{F_j} = \omega_j$ for any j in $\mathcal{J}$. The von Neumann algebra M is standardly represented, so there exists a unique net of unit vectors $\{\xi_j\}_{j\in\mathcal{J}}$ in $\mathcal{P}^\natural_\varphi$ such that $\omega_j = \omega_{\xi_j}$ and $z_{F_j}\xi_j = \xi_j$. Because of the commutativity of J and P^{it}, we have $P^{it}\mathcal{P}^\natural_\varphi = \mathcal{P}^\natural_\varphi$. From the assumption: $\omega_j = \omega_j \circ \tau_{-t} = \omega_{P^{it}\xi_j}$, the uniqueness of ξ_j implies $P^{it}\xi_j = \xi_j$. □

Let $\{\xi_j\}_{j\in\mathcal{J}}$ be a net of unit vectors in H in the previous Lemma. Since $z_{F_j}H = \Lambda(Mz_{F_j})$, there exists x_j in M such that $\xi_j = \Lambda(x_j)$. The operator $x_j = x_j z_{F_j}$ is in $M^\tau = M_\varphi$ and satisfies $\varphi(x_j^* x_j) = 1$ and $x_j = x_j^*$. We prepare some notations. For an operator X in $M \otimes M$, $X(\alpha)$ denotes the operator $X(z_\alpha \otimes 1)$ in $Mz_\alpha \otimes M$.

For an operator Y in $Mz_\alpha \otimes M$, we obtain the equality $Y = \sum_{k,l=1}^{n_\alpha} e_{kl} \otimes Y_{kl}$, where $\{Y_{kl}\}_{kl}$ are operators in M.

Lemma 3.14. *Let $x = x^*$ be in $M_\varphi \cap M_{z_F}$, where F is a finite subset of I, and α be in I. Then the following inequality holds:*

$$\begin{aligned}&\|\omega_{\hat{J}\Lambda(e(\alpha)_{k1}),\,\hat{J}\Lambda(e(\alpha)_{l1})} * \omega_{\Lambda(x)} - \omega_{\hat{J}\Lambda(e(\alpha)_{k1}),\,\hat{J}\Lambda(e(\alpha)_{l1})}(1)\omega_{\Lambda(x)}\| \\ &\quad \geq \nu(\alpha)_k^{-\frac{1}{2}}\nu(\alpha)_l^{\frac{1}{2}}\varphi(e(\alpha)_{11})\varphi(|X(\alpha)_{kl}|),\end{aligned}$$

where $X = \Delta(x^2) - (1 \otimes x^2)$.

Proof. We simply write e_{kl}, ν_k and $X(\alpha)_{kl}$ for $e(\alpha)_{kl}$, $\nu(\alpha)_k$ and X_{kl} respectively. Since $X(\alpha) = \sum_{1\leq k,\,l\leq n_\alpha}(e_{kl} \otimes X_{kl})$ is in $(M \otimes M)_{\varphi\otimes\varphi} = (M \otimes M)^{\tau\otimes\tau}$ and $\sigma_t^\varphi(e_{kl}) = \nu_k^{it}\nu_l^{-it}e_{kl}$, we have $\sigma_t^\varphi(X_{kl}) = \nu_k^{-it}\nu_l^{it}X_{kl}$. Let $X_{kl} = v_{kl}|X_{kl}|$ be the polar decomposition of X_{kl}. Put $a_{kl} = v_{kl}^*$. Then $\sigma_t^\varphi(a_{kl}) = \nu_k^{it}\nu_l^{-it}a_{kl}$. Then we have

$$\begin{aligned}&(\omega_{\hat{J}\Lambda(e_{k1}),\,\hat{J}\Lambda(e_{l1})} * \omega_{\Lambda(x)} - \omega_{\hat{J}\Lambda(e_{k1}),\,\hat{J}\Lambda(e_{l1})}(1)\omega_{\Lambda(x)})(a_{kl}) \\
&= \langle\Delta(a_{kl})(\hat{J}\Lambda(e_{k1}) \otimes \Lambda(x))|\hat{J}\Lambda(e_{k1}) \otimes \Lambda(x)\rangle - \langle\Lambda(e_{l1})|\Lambda(e_{k1})\rangle\langle a_{kl}\Lambda(x)|\Lambda(x)\rangle \\
&= \langle(1 \otimes a_{kl})W(\hat{J} \otimes J)(\Lambda(e_{k1}) \otimes \Lambda(x))|W(\hat{J} \otimes J)(\Lambda(e_{k1}) \otimes \Lambda(x))\rangle \\
&\qquad - \langle\Lambda(e_{l1})|\Lambda(e_{k1})\rangle\langle a_{kl}\Lambda(x)|\Lambda(x)\rangle \\
&= \langle(1 \otimes Ja_{kl}^*J)W^*(\Lambda(e_{l1}) \otimes \Lambda(x))|W^*\Lambda(e_{k1}) \otimes \Lambda(x)\rangle \\
&\qquad - \langle\Lambda(e_{l1})|\Lambda(e_{k1})\rangle\langle Ja_{kl}^*J\Lambda(x)|\Lambda(x)\rangle \\
&= \langle(\Lambda \otimes \Lambda)(\Delta(x)(e_{l1} \otimes 1)(1 \otimes (\nu_k^{\frac{1}{2}}\nu_l^{-\frac{1}{2}}a_{kl})))|(\Lambda \otimes \Lambda)(\Delta(x)(e_{k1} \otimes 1))\rangle \\
&\qquad - \langle\Lambda(e_{l1})|\Lambda(e_{k1})\rangle\langle\Lambda(x\nu_k^{\frac{1}{2}}\nu_l^{-\frac{1}{2}}a_{kl})|\Lambda(x)\rangle \\
&= \nu_k^{\frac{1}{2}}\nu_l^{-\frac{1}{2}}\{(\varphi \otimes \varphi)((e_{1k} \otimes 1)\Delta(x^2)(e_{l1} \otimes 1)(1 \otimes a_{kl})) \\
&\qquad - (\varphi \otimes \varphi)((e_{1k} \otimes 1)(1 \otimes x^2)(e_{l1} \otimes 1)(1 \otimes a_{kl}))\} \\
&= \nu_k^{\frac{1}{2}}\nu_l^{-\frac{1}{2}}(\varphi \otimes \varphi)(e_{11} \otimes X_{kl}a_{kl}) \\
&= \nu_k^{\frac{1}{2}}\nu_l^{-\frac{1}{2}}\varphi(e_{11})\varphi(X_{kl}a_{kl}) \\
&= \nu_k^{\frac{1}{2}}\nu_l^{-\frac{1}{2}}\nu_k^{-1}\nu_l^{1}\varphi(e_{11})\varphi(a_{kl}X_{kl}) \\
&= \nu_k^{-\frac{1}{2}}\nu_l^{\frac{1}{2}}\varphi(e_{11})\varphi(|X_{kl}|).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\|\omega_{\hat{J}\Lambda(e_{k1}),\,\hat{J}\Lambda(e_{l1})} * \omega_{\Lambda(x)} - \omega_{\hat{J}\Lambda(e_{k1}),\,\hat{J}\Lambda(e_{l1})}(1)\omega_{\Lambda(x)}\| \\ &\quad \geq \nu_k^{-\frac{1}{2}}\nu_l^{\frac{1}{2}}\varphi(e_{11})\varphi(|X_{kl}|).\end{aligned}$$

□

Lemma 3.15. *Let N be a von Neumann algebra and θ be an n.s.f. weight on N. Let $\mathbb{M}_n(\mathbb{C})$ be a matrix algebra with the matrix unit $\{e_{ij}\}_{1\leq i,\,j\leq n}$ and $\chi = \mathrm{Tr}_h$*

be an n.s.f. weight on $\mathbb{M}_n(\mathbb{C})$ *with* $h = \sum_{i=1}^n \lambda_i e_{ii}$, $\lambda_i > 0$. *If* $\{A_j\}_{j\in\mathcal{J}}$ *is a net in* $(\mathbb{M}_n(\mathbb{C}) \otimes N_\theta)_{\chi\otimes\theta}$ *such that* $\theta(|(A_j)_{kl}|) < \infty$ *and* $\lim_j \theta(|(A_j)_{kl}|) = 0$ *for any* $k, l = 1, 2, \ldots, n$, *then* $\lim_j \theta(|A_j|_{kl}) = 0$ *for any* $k, l = 1, 2, \ldots, n$.

Proof. Let $A_j = V_j|A_j|$ be the polar decomposition in $(\mathbb{M}_n(\mathbb{C}) \otimes N_\theta)_{\chi\otimes\theta}$. Then for each k, l, we obtain $\sigma_t^\theta((V_j)_{kl}) = \lambda_k^{-it}\lambda_l^{it}(V_j)_{kl}$ and $\sigma_t^\theta((A_j)_{kl}) = \lambda_k^{-it}\lambda_l^{it}(A_j)_{kl}$. Since $|A_j|_{kl} = \sum_{m=1}^n (V_j^*)_{km}(A_j)_{ml}$ and each $(V_j)_{km}$ is analytic, $|A_j|_{kl}$ is an element of m_θ. Let $(A_j)_{ml} = v_{j,m,l}|(A_j)_{ml}|$ be the polar decomposition. Then we have

$$\begin{aligned}
|\theta(|A_j|_{kl})| &\leq \sum_{m=1}^n |\theta((V_j^*)_{km}(A_j)_{ml})| \\
&= \sum_{m=1}^n \left|\left\langle \Lambda_\theta\big(|(A_j)_{ml}|^{\frac{1}{2}}\big) \,\middle|\, \Lambda_\theta\big(|(A_j)_{ml}|^{\frac{1}{2}} v_{j,m,l}^*(V_j)_{mk}\big)\right\rangle\right| \\
&= \sum_{m=1}^n \left|\left\langle \Lambda_\theta\big(|(A_j)_{ml}|^{\frac{1}{2}}\big) \,\middle|\, J_\theta \sigma_{\frac{i}{2}}^\theta(v_{j,m,l}^*(V_j)_{mk})^* J_\theta \Lambda_\theta\big(|(A_j)_{ml}|^{\frac{1}{2}}\big)\right\rangle\right| \\
&\leq \sum_{m=1}^n \|\sigma_{\frac{i}{2}}^\theta(v_{j,m,l}^*(V_j)_{mk})\| \varphi(|(A_j)_{ml}|) \\
&= \lambda_k^{-\frac{1}{2}}\lambda_l^{\frac{1}{2}} \sum_{m=1}^n \varphi(|(A_j)_{ml}|).
\end{aligned}$$

Hence $\lim_j \theta(|A_j|_{kl}) = 0$. □

Lemma 3.16. *Let* $x = x^*$ *be in* $M_\varphi \cap M_{z_F}$, *where* F *is a finite subset of* I *such that* $\varphi(x^2) = 1$ *and* $\Lambda(x)$ *is in* $\mathcal{P}_\varphi^\natural$. *Let* α *be in* I. *Then the following inequality holds:*

$$\begin{aligned}
&\big\|W^*\big(\Lambda(e(\alpha)_{k1}) \otimes \Lambda(x)\big) - \Lambda(e(\alpha)_{k1}) \otimes \Lambda(x)\big\|^2 \\
&\qquad \leq 2 \max_{1\leq k\leq n_\alpha} \{\nu(\alpha)_k^{-1}\nu(\alpha)_1\} \cdot \varphi(z_\alpha)^{\frac{1}{2}} (\varphi\otimes\varphi)(|X(\alpha)|)^{\frac{1}{2}},
\end{aligned}$$

where $X = \Delta(x^2) - 1 \otimes x^2$.

Proof. We use the notations in Lemma 3.14. We have

$$\begin{aligned}
&\left\|W^*\big(\Lambda(e_{k1})\otimes\Lambda(x)\big)-\Lambda(e_{k1})\otimes\Lambda(x)\right\|^2\\
=&\,2\varphi(e_{11})-2\operatorname{Re}\left\langle W^*\big(\Lambda(e_{k1})\otimes\Lambda(x)\big)|\Lambda(e_{k1})\otimes\Lambda(x)\right\rangle\\
=&\,2\varphi(e_{11})-2\operatorname{Re}(\varphi\otimes\varphi)((e_{1k}\otimes 1)\Delta(x)(e_{k1}\otimes x^*))\\
=&\,2\varphi(e_{11})-2\operatorname{Re}\nu_k^{-1}\nu_1(\varphi\otimes\varphi)((e_{kk}\otimes 1)(1\otimes x^*)\Delta(x))\\
=&\,2\nu_k^{-1}\nu_1\operatorname{Re}\{\varphi(e_{kk})-(\varphi\otimes\varphi)((e_{kk}\otimes 1)(1\otimes x^*)\Delta(x))\}\\
=&\,2\nu_k^{-1}\nu_1\operatorname{Re}\{(\varphi\otimes\varphi)\big((e_{kk}\otimes 1)(1\otimes x^*)(1\otimes x-\Delta(x))\big)\}\\
\leq&\,2\max_{1\leq k\leq n_\alpha}\{\nu_k^{-1}\nu_1\}\cdot\sum_{k=1}^{n_\alpha}\operatorname{Re}\{(\varphi\otimes\varphi)\big((e_{kk}\otimes 1)(1\otimes x^*)(1\otimes x-\Delta(x))\big)\}\\
=&\,2\max_{1\leq k\leq n_\alpha}\{\nu_k^{-1}\nu_1\}\cdot\operatorname{Re}\{(\varphi\otimes\varphi)\big((z_\alpha\otimes 1)(1\otimes x^*)(1\otimes x-\Delta(x))\big)\}\\
\leq&\,2\max_{1\leq k\leq n_\alpha}\{\nu_k^{-1}\nu_1\}\cdot|(\varphi\otimes\varphi)\big((z_\alpha\otimes 1)(1\otimes x^*)(1\otimes x-\Delta(x))\big)|\\
\leq&\,2\max_{1\leq k\leq n_\alpha}\{\nu_k^{-1}\nu_1\}\cdot(\varphi\otimes\varphi)(z_\alpha\otimes x^*x)^{\frac{1}{2}}\\
&\qquad\cdot(\varphi\otimes\varphi)\big((z_\alpha\otimes 1)(1\otimes x-\Delta(x))^*(1\otimes x-\Delta(x))\big)\\
=&\,2\max_{1\leq k\leq n_\alpha}\{\nu_k^{-1}\nu_1\}\cdot\varphi(z_\alpha)^{\frac{1}{2}}\left\|(\Lambda\otimes\Lambda)(z_\alpha\otimes x)-(\Lambda\otimes\Lambda)((z_\alpha\otimes 1)\Delta(x))\right\|.
\end{aligned}$$

From the assumption: $\Lambda(x)=z_F\Lambda(x)\in\mathcal{P}_\varphi^\natural$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}\in Mz_F$ with $\lim_n y_nJ\Lambda(y_n)=\Lambda(x)$. Then we obtain

$$\begin{aligned}
(\Lambda\otimes\Lambda)((z_\alpha\otimes 1)\Delta(x))&=W^*(\Lambda\otimes\Lambda)(z_\alpha\otimes x)\\
&=\lim_n W^*(\Lambda\otimes\Lambda)(z_\alpha\otimes y_n\sigma^\varphi_{\frac{i}{2}}(y_n)^*)\\
&=\lim_n(\Lambda\otimes\Lambda)\big(\Delta(y_n)(\sigma^\varphi_{\frac{i}{2}}\otimes\sigma^\varphi_{\frac{i}{2}})(\Delta(y_n))^*(z_\alpha\otimes 1)\big)\\
&=\lim_n\Delta(y_n)(z_\alpha\otimes 1)(J\otimes J)(\Lambda\otimes\Lambda)(\Delta(y_n)(z_\alpha\otimes 1)).
\end{aligned}$$

Therefore, we see $(\Lambda\otimes\Lambda)((z_\alpha\otimes 1)\Delta(x))\in\mathcal{P}^\natural_{\varphi\otimes\varphi}$. By using the Powers-Størmer inequality (see, for example, [H], [P-S]) we obtain

$$\begin{aligned}
&\|(\Lambda\otimes\Lambda)(z_\alpha\otimes x)-(\Lambda\otimes\Lambda)((z_\alpha\otimes 1)\Delta(x))\|\\
\leq&\,\|\omega_{(\Lambda\otimes\Lambda)(z_\alpha\otimes x)}-\omega_{(\Lambda\otimes\Lambda)((z_\alpha\otimes 1)\Delta(x))}\|^{\frac{1}{2}}\\
=&\,(\varphi\otimes\varphi)(|z_\alpha\otimes x^2-(z_\alpha\otimes 1)\Delta(x^2)|)^{\frac{1}{2}}\\
=&\,(\varphi\otimes\varphi)(|X(\alpha)|)^{\frac{1}{2}}.
\end{aligned}$$

□

Proof of 1 ⇒ 2 *of Theorem 3.9* . By the assumption, we can pick up a net $\{x_j\}_{j\in\mathcal{J}}$ in $M^\tau=M_\varphi$ which satisfy the conditions of Lemma 3.14. Now we apply the Lemma to this net for fixed $\alpha\in I$, then $\varphi(|(X_j(\alpha))_{kl}|)$ converges to 0 for any

$k, l = 1, 2, \ldots, n_\alpha$, where $X_j = \Delta(x_j^2) - 1 \otimes x_j^2$. By Lemma 3.15, it implies that $\varphi((|X_j(\alpha)|)_{kl})$ converges to 0 for any $k, l = 1, 2, \ldots, n_\alpha$. Since we have

$$(\varphi \otimes \varphi)(|X_j(\alpha)|) = \sum_{1 \leq k, l \leq n_\alpha} \varphi(e(\alpha)_{kl}) \varphi((|X_j(\alpha)|)_{kl}),$$

we see $(\varphi \otimes \varphi)(|X_j(\alpha)|)$ converges to 0. By Lemma 3.16, we see $\|W^*(\Lambda(e(\alpha)_{k1}) \otimes \Lambda(x_j)) - \Lambda(e(\alpha)_{k1}) \otimes \Lambda(x_j)\|$ converges to 0 for any $k = 1, 2, \ldots, n_\alpha$. Then we have

$$\begin{aligned}
&\left\| W^*(\Lambda(e(\alpha)_{kl}) \otimes \Lambda(x_j)) - \Lambda(e(\alpha)_{kl}) \otimes \Lambda(x_j) \right\| \\
=& \left\| (J\sigma^\varphi_{\frac{i}{2}}(e(\alpha)_{1l})^* J \otimes 1)(W^*(\Lambda(e(\alpha)_{k1}) \otimes \Lambda(x_j)) - \Lambda(e(\alpha)_{k1}) \otimes \Lambda(x_j)) \right\| \\
\leq& \left\| (J\sigma^\varphi_{\frac{i}{2}}(e(\alpha)_{1l})^* J \otimes 1) \right\| \left\| W^*(\Lambda(e(\alpha)_{k1}) \otimes \Lambda(x_j)) - \Lambda(e(\alpha)_{k1}) \otimes \Lambda(x_j) \right\|.
\end{aligned}$$

This implies that $\|W^*(\Lambda(e(\alpha)_{kl}) \otimes \Lambda(x_j)) - \Lambda(e(\alpha)_{kl}) \otimes \Lambda(x_j)\|$ converges to 0 for any $k, l = 1, 2, \ldots, n_\alpha$. By taking a linear combination, $\|W^*(z_\alpha \eta \otimes \Lambda(x_j)) - z_\alpha \eta \otimes \Lambda(x_j)\|$ converges to 0 for any vector $\eta \in H$. Take a vector $\eta \in H$. For any $\varepsilon > 0$, there exists a finite subset F of I such that $\| \sum_{\alpha \in F} z_\alpha \eta - \eta \| < \varepsilon$. By the above arguments, we can take j_0 in $\mathcal{J}$ such that $\sum_{\alpha \in F} \|W^*(z_\alpha \eta \otimes \Lambda(x_j)) - z_\alpha \eta \otimes \Lambda(x_j)\| < \varepsilon$ for $j \geq j_0$. Then we have

$$\begin{aligned}
&\|W^*(\eta \otimes \Lambda(x_j)) - \eta \otimes \Lambda(x_j)\| \\
=& \Big\| W^*\Big((\eta - \sum_{\alpha \in F} z_\alpha \eta) \otimes \Lambda(x_j)\Big) - \Big(\eta - \sum_{\alpha \in F} z_\alpha \eta\Big) \otimes \Lambda(x_j) \\
&+ W^*\Big(\sum_{\alpha \in F} z_\alpha \eta \otimes \Lambda(x_j)\Big) - \sum_{\alpha \in F} z_\alpha \eta \otimes \Lambda(x_j) \Big\| \\
\leq& 2 \Big\| \sum_{\alpha \in F} z_\alpha \eta - \eta \Big\| \\
&+ \sum_{\alpha \in F} \|W^*(z_\alpha \eta \otimes \Lambda(x_j)) - z_\alpha \eta \otimes \Lambda(x_j)\| \\
\leq& 2\varepsilon + \varepsilon = 3\varepsilon,
\end{aligned}$$

for $j \geq j_0$. Therefore, $\|W^* \eta \otimes \Lambda(x_j) - \eta \otimes \Lambda(x_j)\|$ converges to 0 for any vector $\eta \in H$. This completes the proof of Theorem 4.4. □

We give one example of a discrete quantum group which satisfies the conditions in Theorem 3.9. In [W1], S. L. Woronowicz gave an example of a compact quantum group, so called, $SU_q(2)$-group. It is defined as follows. Let q be a positive number belonging to the set $(0, 1)$, and $\hat{A}_q$ be the unital universal C^*-algebra which is generated by two operators α and γ, which satisfies the following commutation relations:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1,$$

$$\gamma^*\gamma = \gamma\gamma^*, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha.$$

It has a coproduct defined by $\hat{\Delta}(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma$ and $\hat{\Delta}(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$. The universality implies the existence of a surjective $*$-homomorphism π from $\hat{A}_q$ to the C^*-algebra $C(S^1)$ of continuous functions on the circle S^1 in the complex plane. In fact if we put $\alpha = z$, $\gamma = 0$, where z denotes the identity map of S^1, they satisfy the above relations. Therefore, $\hat{A}_q$ has a character. It has also been proved in [W1] that $\hat{A}$ is of type I, in particular, a nuclear C^*-algebra. So we can get the following result by Theorem 3.9.

Corollary 3.17. *The discrete quantum group* $\widehat{SU_q(2)}$ $(q \in (0,1))$ *has an invariant mean.*

In the end of this paper, we state Theorem 3.9 in the case that the von Neumann algebra $\hat{M}$ is a Kac algebra. Recall that the modular operator of $\hat{\varphi}$ equals to $\delta^{-\frac{1}{2}} J \delta^{-\frac{1}{2}} J$. So the discrete quantum group (M, Δ) is unimodular if and only if the invariant weight $\hat{\varphi}$ of $(\hat{M}, \hat{\Delta})$ is a normal tracial state. It is also equivalent to the condition that the discrete quantum group (M, Δ) becomes a Kac algebra. Note that the condition (W_1) is called *strong Voiculescu amenability* in the case of Kac algebras (see [R]).

Corollary 3.18. *Let* (M, Δ) *be a discrete Kac algebra. Then the following statements are equivalent.*

(1) *It has an invariant mean.*
(2) *It is strongly Voiculescu amenable.*
(3) *The* C^**-algebra* $\hat{A}$ *is nuclear and has a finite dimensional representation.*
(4) *The* C^**-algebra* $\hat{A}$ *is nuclear.*
(5) *The von Neumann algebra* $\hat{M}$ *is injective.*

Proof. $1 \Rightarrow 2 \Rightarrow 3$. This has been already proved in Theorem 3.9.

$3 \Rightarrow 4 \Rightarrow 5$. It is trivial.

$5 \Rightarrow 1$. Let E be a conditional expectation from $B(H)$ onto $\hat{M}$. Note that $\hat{\varphi}$ is a normal trace on $\hat{M}$. Take a complete orthonormal system $\{e_p\}_{p \in \mathcal{P}}$. Then for any operator $x \in M$ and for any vector $\xi \in H$, we have

$$\begin{aligned}\omega_\xi * m(x) &= \hat{\varphi}\Big(E\Big(\sum_{p\in\mathcal{P}}(\omega_{\xi,e_p}\otimes\iota)(W)^* x(\omega_{\xi,e_p}\otimes\iota)(W)\Big)\Big)\\ &= \sum_{p\in\mathcal{P}}\hat{\varphi}\big((\omega_{\xi,e_p}\otimes\iota)(W)^*E(x)(\omega_{\xi,e_p}\otimes\iota)(W)\big)\\ &= \sum_{p\in\mathcal{P}}\hat{\varphi}\big((\omega_{\xi,e_p}\otimes\iota)(W)(\omega_{\xi,e_p}\otimes\iota)(W)^*E(x)\big)\\ &= \sum_{p\in\mathcal{P}}\hat{\varphi}\big((\omega_{\hat{J}\xi,\hat{J}e_p}\otimes\iota)(W)^*(\omega_{\hat{J}\xi,\hat{J}e_p}\otimes\iota)(W)E(x)\big)\\ &= \omega_{\hat{J}\xi}(1)\hat{\varphi}(E(x))\\ &= \omega_\xi(1)m(x).\end{aligned}$$

Therefore, m is a left invariant mean on M. □

As we have seen, the nuclearity of a compact Kac algebra leads the amenability of the dual discrete Kac algebra, however, it is now open whether it holds in the case of a compact quantum group or not.

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