UTMS 2003–12 March 5, 2003

One unique continuation for a linearized Benjamin-Bona-Mahony equation

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ONE UNIQUE CONTINUATION FOR A LINEARIZED BENJAMIN-BONA-MAHONY EQUATION

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Abstract. For a linearized Benjamin-Bona-Mahony equation:

$$\partial_t u - \partial_x^2 \partial_t u = p(x,t)\partial_x u + q(x,t)u, \quad x \in (0,1),\ 0 < t < T,$$

we prove a unique continuation property by a Carleman estimate. The main result is: if $u(1,t) = \partial_x u(1,t) = 0$ for $t \in (0,T)$ and $u(x,0) = 0$ for $x \in (0,1)$, then $u(x,t) = 0$ for $(x,t) \in (0,1) \times (0,T)$.

§1. Introduction.

We consider a linearized Benjamin-Bona-Mahony equation:

$$\partial_t u(x,t) - \partial_x^2 \partial_t u(x,t) = p(x,t)\partial_x u(x,t) + q(x,t)u(x,t), \quad 0 < x < 1,\ 0 < t < T. \tag{1.1}$$

Here we assume that

$$p \in L^\infty((0,1) \times (0,T)), \quad q \in L^\infty(0,T; L^2(0,1)). \tag{1.2}$$

Here and henceforth, we set

$$\partial_x = \frac{\partial}{\partial x},\ \partial_x^2 = \frac{\partial^2}{\partial x^2},\ \partial_t = \frac{\partial}{\partial t}.$$

1991 *Mathematics Subject Classification.* 35B60, 35Q35, 35R25.

Key words and phrases. unique continuation, Benjamin-Bona-Mahony equation, Carleman estimate.

We discuss a unique continuation property: for some open subset $\omega \subset (0,1)$, can we conclude that

$$u(x,t) = 0 \text{ for } (x,t) \in \omega \times (0,T) \text{ or } u(1,t) = \partial_x u(1,t) = 0 \text{ for } t \in (0,T)$$
$$\text{implies that } u(x,t) = 0 \text{ for } (x,t) \in (0,1) \times (0,T)? \tag{1.3}$$

However, this is not necessarily true, as is pointed out in Zhang and Zuazua [11]: let $p = q \equiv 0$, and let us take $u(x,t) = u_0(x) \not\equiv 0$, a t-independent function such that $u_0 \in C_0^\infty(0,1)$ and $\mathrm{supp}\, u_0 \subset (0,1) \setminus \overline{\omega}$. Then u_0 satisfies (1.1), but u_0 does not vanish identically over $(0,1)$.

This counterexample demonstrates one difficulty in the unique continuation property which is different from other cases such as the KdV equation. We are suggested that other boundary condition may guarantee the unique continuation, as long as we assume that u vanishes in a cylindrical subdomain $\omega \times (0,T)$ or that the lateral Cauchy data (i.e. $u(1,t) = \partial_x u(1,t) = 0$ for $t \in (0,T)$) vanish. In fact, Zhang and Zuazua [11] proves that (1.3) implies $u \equiv 0$ in $(0,1) \times (0,\infty)$ provided that $u(0,t) = u(1,t) = 0$, $0 < t < T$ and p, q are independent of t.

In this paper, assuming that $u(x,0) = 0$, $0 < x < 1$, we prove that $u(1,t) = \partial_x u(1,t) = 0$, $0 < t < T$, implies $u(x,t) = 0$, $0 < x < 1$, $0 < t < T$.

Theorem. *Let $\partial_x^j \partial_t^k u \in C([0,1] \times [0,T])$ with $j = 0,1,2$ and $k = 0,1$, satisfy (1.1). Assume (1.2). If*

$$u(1,t) = \partial_x u(1,t) = 0, \qquad 0 < t < T \tag{1.4}$$

and

$$u(x,0) = 0, \qquad 0 < x < 1, \tag{1.5}$$

then $u(x,t) = 0$, $0 < x < 1$, $0 < t < T$.

Our proof is based on a Carleman estimate, so that we can prove conditional stability in the continuation by a usual method (e.g., Isakov [Chapter 3, 5]) and our argument is valid for a multidimensional analogue

$$\partial_t u - \partial_t \Delta u = \sum_{i=1}^{n} p_i(x,t)\frac{\partial u}{\partial x_i} + q(x,t)u$$

for $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $t \in (0,T)$. Furthermore we can prove the theorem under a weaker regularity assumption on u, but, for simplicity, we mainly consider the classical solution u.

For the Benjamin-Bona-Mahony equation, we can prove the unique continuation not across the characteristics (e.g., Theorems 3.1 - 3.2 in Davila and Menzala [3]). For the unique continuation for other dispersive equations such as the KdV equation, we refer to Isakov [6]. In [3] and [6], the main tool is a Carleman estimate. As for applications of the Carleman estimate to the unique continuation, we further refer to Hörmander [4], Isakov [7], Saut and Scheurer [9], Tataru [10].

This paper is composed of three sections. Section 2 is preliminaries where we establish a simple Carleman estimate and an integral inequality. Section 3 is devoted to the proof of our main theorem.

§2. Preliminaries.

Let $Q \subset (0,1) \times (-T,T) \equiv \{(x,t); 0 < x < 1, -T < t < T\}$ be a subdomain such that the boundary ∂Q is of piecewise C^2. Then we have

Lemma 1. *Let* $\varphi \in C(\mathbb{R}^2)$, *and for any* $T \in (-T,T)$ *let* $\varphi = \varphi(\cdot, t) \in C^\infty(\mathbb{R})$ *satisfy*

$$(\partial_x^2 \varphi)(x,t) > 0, \qquad (x,t) \in \overline{Q} \tag{2.1}$$

and

$$|(\partial_x\varphi)(x,t)| > 0, \qquad (x,t) \in \overline{Q}. \tag{2.2}$$

Then there exist constants $C = C(\varphi, Q) > 0$ and $S = S(\varphi, Q) > 0$ such that

$$\int_Q (s|\partial_x u|^2 + s^3|u|^2)e^{2s\varphi}dxdt \le C\int_Q |\partial_x^2 u|^2 e^{2s\varphi}dxdt \tag{2.3}$$

for all $s > S$ and $u \in C^2(Q)$ such that $u = \partial_x u = 0$ on ∂Q.

Proof. We will apply the argument in Bukhgeim [1]. By a usual density argument, it suffices to prove the lemma for $u \in C_0^\infty(Q)$. We set $v = e^{s\varphi}u$ and $Lv = -e^{s\varphi}\partial_x^2(e^{-s\varphi}v)$. Then [the right-hand side of (2.3)] $= C\int_Q |Lv|^2dxdt$, and for the proof, it is sufficient to prove that

$$\int_Q (s|\partial_x v|^2 + s^3|v|^2)dxdt \le C\int_Q |Lv|^2dxdt.$$

We directly calculate:

$$Lv = -\partial_x^2 v + 2s(\partial_x\varphi)(\partial_x v) + (s(\partial_x^2\varphi) - s^2(\partial_x\varphi)^2)v.$$

Formally we calculate the adjoint operator L^* of L to obtain

$$L^*v = -\partial_x^2 v - 2s(\partial_x\varphi)(\partial_x v) - (s(\partial_x^2\varphi) + s^2(\partial_x\varphi)^2)v.$$

Setting $L_+ = \frac{1}{2}(L + L^*)$ and $L_- = \frac{1}{2}(L - L^*)$, we have $L = L_+ + L_-$, so that

$$\begin{aligned}\|Lv\|^2_{L^2(Q)} &= \|L_+v\|^2_{L^2(Q)} + \|L_-v\|^2_{L^2(Q)} + (L_+v, L_-v)_{L^2(Q)} + (L_-v, L_+v)_{L^2(Q)}\\ &\ge (L_+v, L_-v)_{L^2(Q)} + (L_-v, L_+v)_{L^2(Q)} = ((L_+L_- - L_-L_+)v, v)_{L^2(Q)}\end{aligned}$$

by integration by parts and $v = \partial_x v = 0$ on ∂Q. Here and henceforth we set $(u,v)_{L^2(Q)} = \int_Q uvdxdt$ and $\|u\|_{L^2(Q)} = (u,u)^{\frac{1}{2}}_{L^2(Q)}$.

Direct calculations yield

$$(L_+L_- - L_-L_+)v = -s(\partial_x^4\varphi)v - 4s(\partial_x^3\varphi)(\partial_x v)$$

$$-4s(\partial_x^2\varphi)(\partial_x^2 v) + 2s^3(\partial_x\varphi)\partial_x(|\partial_x\varphi|^2)v.$$

Henceforth $C > 0$ denotes a generic constant which is dependent on φ, Q, but independent of s. By integration by parts and the Schwarz inequality, we have

$$\|Lv\|^2_{L^2(Q)} \geq 4s^3 \int_Q (\partial_x\varphi)^2(\partial_x^2\varphi)v^2 dxdt - Cs\int_Q v^2 dxdt$$

$$+4s\int_Q (\partial_x^2\varphi)|\partial_x v|^2 dxdt - C\int_Q |\partial_x v|^2 dxdt.$$

Therefore under assumptions (2.1) and (2.2), if we take $S > 0$ sufficiently large, then the proof of Lemma 1 is complete.

We arbitrarily choose $\mu \in (0,1)$ and $\delta > 0$, and set

$$\begin{cases} \psi(x,t) = 1 - (x-(1+\delta))^2 - \left(\dfrac{|t|}{T}\right)^{\mu}, \\ \varphi(x,t) = e^{\lambda\psi(x,t)} - 1. \end{cases} \tag{2.4}$$

If we fix $\lambda > 0$ sufficiently large, then φ satisfies (2.1) and (2.2) in $Q \subset (0,1) \times (-T,T)$. We further set

$$Q(\varepsilon) = \{(x,t) \in (0,1)\times\mathbb{R};\ \varphi(x,t) > \varepsilon\} \tag{2.5}$$

for $\varepsilon \geq 0$.

We set

$$h_\varepsilon(t) = 1 + \delta - \left(1 - \frac{1}{\lambda}\log(1+\varepsilon) - \left(\frac{t}{T}\right)^{\mu}\right)^{\frac{1}{2}}, \qquad t > 0.$$

Then, for sufficiently large $\lambda > 0$ and small $\varepsilon \geq 0$, we have

$$Q(\varepsilon) = \{(x,t) \in (0,1)\times\mathbb{R};\ h_\varepsilon(|t|) < x < 1\}. \tag{2.6}$$

We can directly verify that $(x,t) \in Q(\varepsilon)$ implies

$$|t| < T\left(1 - \delta^2 - \frac{1}{\lambda}\log(1+\varepsilon)\right)^{\frac{1}{\mu}} < T. \tag{2.7}$$

Moreover we can prove the following integral inequality:

Lemma 2. *There exists a constant $C = C(\varphi, \varepsilon, T) > 0$ such that*

$$\int_{Q(\varepsilon)} \left| \int_0^t |u(x,\eta)| d\eta \right|^2 e^{2s\varphi} dxdt \le C \int_{Q(\varepsilon)} |u(x,t)|^2 e^{2s\varphi} dxdt$$

for all $s > 0$ and $u \in L^2(Q(\varepsilon))$.

Since for any $x \in (0,1)$, the function $\varphi(x,t)$ is decreasing in $t > 0$ and increasing in $t < 0$, the proof is directly done (e.g., Isakov [p.153, 5], Klibanov [8]). This lemma was used originally by Bukhgeim and Klibanov [2] for proving the uniqueness in some inverse problems by Carleman estimates. The lemma is simple but essential for applications of Carleman estimates to inverse problems. Also see Bukhgeim [1].

§3. Proof of Theorem.

We extend u, p, q to odd functions in $t \in (-T,T)$, which are denoted by the same notations. Then by (1.5), we see

$$\partial_x^2(\partial_t u)(x,t) = \partial_t u(x,t) - p(x,t)\partial_x u(x,t) - q(x,t)u(x,t), \quad 0 < x < 1,\ -T < t < T, \tag{3.1}$$

and

$$u(1,t) = \partial_x u(1,t) = 0, \qquad -T < t < T. \tag{3.2}$$

The proof is done along the argument towards the unique continuation by means of a Carleman estimate (e.g., Hörmander [4], Isakov [5]) except for the application of Lemma 2.

Let $\varepsilon > 0$ be sufficiently small and let $\chi = \chi(x,t)$ such that

$$\begin{cases} \chi \in C_0^\infty(\mathbb{R}^2), \quad 0 \le \chi \le 1, \\ \chi(x,t) = \begin{cases} 1, & (x,t) \in Q(2\varepsilon) \\ 0, & (x,t) \in Q(0) \setminus Q(\varepsilon). \end{cases} \end{cases} \tag{3.3}$$

We set $y = u\chi$. Then $\partial_x^j \partial_t^k y \in C([0,1] \times [-T,T])$ for $j = 0,1,2$ and $k = 0,1$, and

$$y = \partial_x y = 0 \qquad \text{on } \partial Q(0) \tag{3.4}$$

by (1.4), (2.6) and (2.7). Moreover we directly see

$$\partial_x^2(\partial_t y)(x,t) = (\partial_t y) - p(\partial_x y) - qy + I, \tag{3.5}$$

where

$$\begin{aligned} I = &((\partial_x^2 \partial_t \chi) - \partial_t \chi + p(\partial_x \chi))u + 2(\partial_x \partial_t \chi)\partial_x u \\ &+(\partial_x^2 \chi)(\partial_t u) + (\partial_t \chi)\partial_x^2 u + 2(\partial_x \chi)(\partial_x \partial_t u), \quad (x,t) \in Q(0). \end{aligned} \tag{3.6}$$

Moreover by (2.6) and (2.7), we see that $Q(0) \subset (0,1) \times (-T,T)$, and (2.1) and (2.2) are true in $Q(0)$. Consequently, in terms of (3.4), applying Lemma 1 to $\partial_t y$ in (3.5), we have

$$\begin{aligned} &\int_{Q(0)} (s|\partial_x \partial_t y|^2 + s^3 |\partial_t y|^2) e^{2s\varphi} dxdt \\ \le &C \int_{Q(0)} (|\partial_t y|^2 + p^2 |\partial_x y|^2 + q^2 y^2) e^{2s\varphi} dxdt + \int_{Q(0)} |I(x,t)|^2 e^{2s\varphi} dxdt. \end{aligned} \tag{3.7}$$

Here and henceforth $C > 0$ denotes a generic constant which is independent of $s > 0$ and $m \in \mathbb{N}$. By defintion (3.6) of I, we have $I \ne 0$ only in $Q(\varepsilon) \setminus Q(2\varepsilon)$, so that

$$\int_{Q(0)} |I(x,t)|^2 e^{2s\varphi} dxdt \le Ce^{4s\varepsilon} \max_{(x,t) \in [0,1] \times [-T,T]} |I(x,t)|^2 \le Ce^{4s\varepsilon}. \tag{3.8}$$

On the other hand, by (1.5), we have

$$(\partial_x y)(x,t) = \int_0^t (\partial_x\partial_t y)(x,\eta)d\eta, \quad y(x,t) = \int_0^t (\partial_t y)(x,\eta)d\eta.$$

Therefore, by Lemma 2, we obtain

$$\begin{aligned}&\int_{Q(0)} |\partial_x y|^2 e^{2s\varphi} dxdt = \int_{Q(0)} \left|\int_0^t (\partial_x\partial_t y)(x,\eta)d\eta\right|^2 e^{2s\varphi} dxdt\\ \le & C\int_{Q(0)} |(\partial_x\partial_t y)(x,t)|e^{2s\varphi} dxdt \end{aligned}\tag{3.9}$$

and

$$\int_{Q(0)} |y(x,t)|^2 e^{2s\varphi} dxdt \le C\int_{Q(0)} |(\partial_t y)(x,t)|^2 e^{2s\varphi} dxdt. \tag{3.10}$$

Noting (2.6), (2.7), the Sobolev embedding and the Schwarz inequality, we have

$$\begin{aligned}&\int_{Q(0)} q^2y^2e^{2s\varphi}dxdt = \int_{-T(1-\delta^2)^{1/\mu}}^{T(1-\delta^2)^{1/\mu}} \left(\int_{h_0(|t|)}^1 q^2(x,t)y^2(x,t)e^{2s\varphi(x,t)}dx\right) dt\\ \le & \int_{-T(1-\delta^2)^{1/\mu}}^{T(1-\delta^2)^{1/\mu}} \|q^2(\cdot,t)\|_{L^1(0,1)}\|(ye^{s\varphi})(\cdot,t)\|^2_{L^\infty(h_0(|t|),1)}dt\\ \le & C\|q\|^2_{L^\infty(-T,T;L^2(0,1))}\int_{-T(1-\delta^2)^{1/\mu}}^{T(1-\delta^2)^{1/\mu}} \left(\int_{h_0(|t|)}^1 |\partial_x(ye^{s\varphi})(x,t)|^2dx\right) dt\\ \le & C\int_{Q(0)} (s^2y^2 + |\partial_x y|^2)e^{2s\varphi}dxdt. \end{aligned}\tag{3.11}$$

Applying (3.8) - (3.11) in (3.7), we obtain

$$\begin{aligned}&\int_{Q(0)} (s|\partial_x\partial_t y|^2 + s^3|\partial_t y|^2)e^{2s\varphi}dxdt\\ \le & C\int_{Q(0)} ((1+s^2)|\partial_t y|^2 + |\partial_x\partial_t y|^2)e^{2s\varphi}dxdt + Ce^{4s\varepsilon}.\end{aligned}$$

Taking $s > 0$ sufficiently large, we can absorb the first term at the right-hand side into the left-hand side, so that

$$\int_{Q(0)} (s|\partial_x\partial_t y|^2 + s^3|\partial_t y|^2)e^{2s\varphi}dxdt \le Ce^{4s\varepsilon}.$$

Since $Q(3\varepsilon) = \{(x,t) \in (0,1)\times\mathbb{R};\ \varphi(x,t) > 3\varepsilon\} \subset Q(0)$, we have

$$e^{6s\varepsilon}\int_{Q(3\varepsilon)} s^3|\partial_t y|^2 dxdt \le \int_{Q(3\varepsilon)} (s|\partial_x\partial_t y|^2 + s^3|\partial_t y|^2)e^{2s\varphi}dxdt \le Ce^{4s\varepsilon}$$

for all large $s > 0$. That is,

$$\int_{Q(3\varepsilon)} |\partial_t y|^2 dxdt \le Cs^{-3}e^{-2s\varepsilon}$$

for all large $s > 0$. Taking $s \longrightarrow \infty$, we obtain $\partial_t y(x,t) = 0$ for $(x,t) \in Q(3\varepsilon)$. Again, by $y(x,0) = 0$, $0 < x < 1$, we have $y(x,t) = 0$ for $(x,t) \in Q(3\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we see that $y(x,t) = 0$ if

$$(1+\delta) - \left(1 - \left(\frac{t}{T}\right)^{\mu}\right)^{\frac{1}{2}} < x < 1, \quad 0 < t < T(1-\delta^2)^{\frac{1}{2}}.$$

Since $\delta > 0$ is arbitrary, we have $y(x,t) = 0$ if $1 - \left(1 - \left(\frac{t}{T}\right)^{\mu}\right)^{\frac{1}{2}} < x < 1$ and $0 < t < T$. Next, because $\mu \in (0,1)$ is arbitrary, we can let μ tend to 0, so that $y(x,t) = 0$ if $0 < t < T$ and $0 < x < 1$. Thus the proof of the theorem is complete.

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