

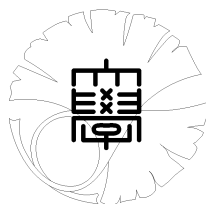
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**Modified elastic wave equations
on Riemannian and Kähler manifolds**

by

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Modified Elastic Wave Equations on Riemannian and Kähler Manifolds

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Abstract

We introduce some geometrically invariant systems of differential equations on any Riemannian manifolds and also on any Kähler manifolds, which are natural extensions of the elastic wave equations on \mathbb{R}^3 . Further we prove the local decomposition theorems of distribution solutions for those systems. In particular, the solutions of our systems on Kähler manifolds are decomposed into 4 solutions with different propagation speeds.

0 Introduction and Results

Introduction

The elastic wave equation on \mathbb{R}^3 is written as follows:

$$\begin{aligned} Pu &:= \rho \frac{\partial^2}{\partial t^2} u - (\lambda + \mu) \text{grad div } u - \mu \Delta u \\ &= \rho \frac{\partial^2}{\partial t^2} u - (\lambda + 2\mu) \text{grad div } u + \mu \text{rot rot } u = f, \end{aligned}$$

where u is a 3-dimensional vector field of the displacement of an elastic body, ρ is the density constant and λ, μ are the Lamé constants. It is well-known that any distribution solution u of $Pu = 0$ is decomposed into a sum $u = u_1 + u_2$ of solutions u_1, u_2 satisfying the following additional equations:

$$\operatorname{rot} u_1 = 0, \quad \operatorname{div} u_2 = 0.$$

We call u_1, u_2 a longitudinal wave solution and a transverse wave solution, respectively. In a physical generalization of this system of equations to Riemannian manifolds we replace $\operatorname{div}, \operatorname{rot}$ with some covariant differentiations. However covariant differentiations do not commute with each other in general.

To begin with, in Chapter 2, we consider the physical generalization $P_{\text{org}} u = 0$ of the elastic wave equation on a Riemannian manifold. In Chapter 3, we introduce a new differential equation $P_{\text{R}} u = 0$ which is a modification on the lower order term of the original equation $P_{\text{org}} u = 0$. Then we show that; the new differential equation admits a decomposition of any solutions into longitudinal wave solutions and transverse wave solutions. However, the original equation does not admit any similar decompositions in general. On the other hand, by its duality, P_{R} and P_{org} operate on 1-differential forms. Through this duality, we can generalize P_{R} and P_{org} to operators on p -differential forms. In Chapter 4, we deal with the differential equations $P_{\text{C}} u = 0, P_{\text{C}}^* u = 0$ on complex manifolds and $P_{\text{K}} u = 0$ on Kähler manifolds. We show any distribution solutions of the differential equations $P_{\text{C}} u = 0$ and $P_{\text{C}}^* u = 0$ admit some decompositions into 2 solutions with different propagation speeds. In the same way, we also show that any distribution solution of the equation $P_{\text{K}} u = 0$ admits a decomposition into 4 solutions with 4 different propagation speeds.

Results

Definition 0.1. Let M be an n -dimensional Riemannian manifold and $\widetilde{M} = \mathbb{R}_t \times M$. Let $u = \sum u^i \partial_i$ be a contravariant vector field on M with parameter t ; precisely, a contravariant vector field on \widetilde{M} with $\langle dt, u \rangle = 0$. We assume the density constant ρ and the Lamé

constants λ, μ are positive. Then we define the original elastic wave equation as follows:

$$\begin{aligned} P_{\text{org}} u^i &:= \rho \frac{\partial^2}{\partial t^2} u^i - \lambda g^{ij} \nabla_j \nabla_k u^k - \mu g^{ik} \nabla_j \nabla_k u^j - \mu g^{jk} \nabla_j \nabla_k u^i \\ &= \rho \frac{\partial^2}{\partial t^2} u^i - \lambda \nabla^i \nabla_k u^k - \mu \nabla_k \nabla^i u^k - \mu \nabla_k \nabla^k u^i = f^i. \end{aligned}$$

where we denote by g_{ij} , all ∇_j the Riemannian metric tensor and the covariant differentiation, respectively.

In this paper, we often omit \sum by Einstein's convention.

For this equation, we define another differential equation on \widetilde{M} which is a modification on the part of order 0 of P_{org} :

$$\begin{aligned} P_{\text{R}} u^i &:= \rho \frac{\partial^2}{\partial t^2} u^i - (\lambda + 2\mu) g^{ij} \nabla_j \nabla_k u^k + \mu g^{ij} \nabla_k \nabla_j u^k - \mu g^{jk} \nabla_j \nabla_k u^i \\ &= \rho \frac{\partial^2}{\partial t^2} u^i - (\lambda + 2\mu) \nabla^i \nabla_k u^k + \mu \nabla_k \nabla^i u^k - \mu \nabla_k \nabla^k u^i \\ &= P_{\text{org}} u^i + 2\mu R_l^i u^l = f^i, \end{aligned}$$

where R_{ij} is the Ricci tensor.

The modified elastic equation above has a simple form when we write it in a covariant form; that is, an equation for 1-forms. At the same time, we extend the equations naturally to equations for p -differential forms for all p .

Let $\bigwedge^{(p)} T^*M$ be a vector bundle of p -differential forms on M . Let $\mathcal{E}_M^{(p)}$ be a sheaf of p -forms on M with C^∞ coefficients, and $\mathcal{D}b_M^{(p)}$ a sheaf of p -currents on M ; that is, p -forms with distribution coefficients. In this article, we do not mean distributions the dual space of $C_0^\infty(M)$. Our distributions behave as “functions” for coordinate transformations. Further we define $\widetilde{\mathcal{E}}_M^{(p)}$ and $\widetilde{\mathcal{D}b}_M^{(p)}$.

Definition 0.2. We denote by $\widetilde{\mathcal{E}}_M^{(p)}$, $\widetilde{\mathcal{D}b}_M^{(p)}$ the sheaves of sections of $\mathcal{E}_M^{(p)}$, $\mathcal{D}b_M^{(p)}$ which do not include the covariant vector dt . That is, setting the projection $\pi : \mathbb{R}_t \times M \rightarrow M$, we define

$$\widetilde{\mathcal{E}}_M^{(p)} := \mathcal{E}_M^{(0)} \otimes_{\pi^{-1}\mathcal{E}_M^{(0)}} \pi^{-1}\mathcal{E}_M^{(p)}, \quad \widetilde{\mathcal{D}b}_M^{(p)} := \mathcal{D}b_M^{(0)} \otimes_{\pi^{-1}\mathcal{E}_M^{(0)}} \pi^{-1}\mathcal{E}_M^{(p)}.$$

For

$$u = \sum_{1 \leq i_1 < \dots < i_p \leq n} u_{i_1 \dots i_p}(t, x) dx^{i_1} \wedge \dots \wedge dx^{i_p} \in \widetilde{\mathcal{D}b}_M^{(p)},$$

we define an operator P_R for $\widetilde{\mathcal{D}b}_M^{(p)}$ on M ($1 \leq p \leq n-1$), where the coefficients $\{u_{i_1 \dots i_p}\}$ are supposed to be alternating with respect to $(i_1 \dots i_p)$.

Definition 0.3. We define sheaf-morphisms $P_R : \widetilde{\mathcal{D}b}_M^{(p)} \rightarrow \widetilde{\mathcal{D}b}_M^{(p)}$ by

$$P_R u := \rho \frac{\partial^2}{\partial t^2} u + (\lambda + 2\mu) d\delta u + \mu \delta du.$$

For $p = 1$, this equation is the covariant form of $P_R u^i$.

When $p = 0$ or n , $P_R u = 0$ reduces to a wave equation. Therefore we suppose $1 \leq p \leq n-1$. Here, d, δ are the exterior differential operator and the associated exterior differential operator on M , respectively.

For $u \in \widetilde{\mathcal{D}b}_M^{(p)}$, we define equations $\mathfrak{M}^R, \mathfrak{M}_1^R, \mathfrak{M}_2^R, \mathfrak{M}_0^R$ below:

$$\begin{aligned} \mathfrak{M}^R &: P_R u = 0, \\ \mathfrak{M}_1^R &: \begin{cases} P_R u = 0, \\ du = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha \Delta)u = 0, \\ du = 0, \end{cases} \\ \mathfrak{M}_2^R &: \begin{cases} P_R u = 0, \\ \delta u = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \beta \Delta)u = 0, \\ \delta u = 0, \end{cases} \\ \mathfrak{M}_0^R &: \begin{cases} P_R u = 0, \\ du = 0, \\ \delta u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ du = 0, \\ \delta u = 0. \end{cases} \end{aligned}$$

Here, $\alpha = (\lambda + 2\mu)/\rho$, $\beta = \mu/\rho$ and $\Delta = d\delta + \delta d : \widetilde{\mathcal{D}b}_M^{(p)} \rightarrow \widetilde{\mathcal{D}b}_M^{(p)}$ is the Laplacian on M .

Further we define subsheaves $Sol(\mathfrak{M}^R; p)$, $Sol(\mathfrak{M}_j^R; p)$, ($j = 0, 1, 2$) of $\widetilde{\mathcal{D}b}_M^{(p)}$ as follows: For $\mathfrak{N}^R = \mathfrak{M}^R, \mathfrak{M}_j^R$,

$$Sol(\mathfrak{N}^R; p) := \left\{ u \in \widetilde{\mathcal{D}b}_M^{(p)} \mid u \text{ satisfies } \mathfrak{N}^R \right\}.$$

Then, we have the theorem below.

Theorem A (Theorem 3.3). *For any distribution solution $u \in Sol(\mathfrak{M}^R; p) \Big|_{\left(\overset{\circ}{t}, \overset{\circ}{x}\right)}$, there exist some germs $u_j \in Sol(\mathfrak{M}_j^R; p) \Big|_{\left(\overset{\circ}{t}, \overset{\circ}{x}\right)}$ ($j = 1, 2$) such that $u = u_1 + u_2$.*

Furthermore, the equation $u = u_1 + u_2 = 0$ implies $u_1, u_2 \in Sol(\mathfrak{M}_0^R; p) \Big|_{\left(\overset{\circ}{t}, \overset{\circ}{x}\right)}$. Equivalently, we have the following exact sequence:

$$0 \longrightarrow Sol(\mathfrak{M}_0^R; p) \xrightarrow{F} Sol(\mathfrak{M}_1^R; p) \oplus Sol(\mathfrak{M}_2^R; p) \xrightarrow{G} Sol(\mathfrak{M}^R; p) \longrightarrow 0,$$

where $F(U) = U \oplus (-U)$, $G(U_1 \oplus U_2) = U_1 + U_2$.

Let X be an n -dimensional complex manifold with a Hermitian metric, and $\bigwedge^{(q,r)} T^*X$ a vector bundle of (q, r) -type differential forms on X . Let $\mathcal{E}_X^{(q,r)}$ be a sheaf of (q, r) -forms on X with C^∞ coefficients, and $\mathcal{D}b_X^{(q,r)}$ a sheaf of (q, r) -currents on X . Setting $\widetilde{X} = \mathbb{R}_t \times X$, we also define $\widetilde{\mathcal{E}}_X^{(q,r)}$, $\widetilde{\mathcal{D}b}_X^{(q,r)}$ similarly to $\widetilde{\mathcal{E}}_M^{(p)}$, $\widetilde{\mathcal{D}b}_M^{(p)}$.

Definition 0.4. We define sheaf-morphisms $P_C, P_C^* : \widetilde{\mathcal{D}b}_X^{(q,r)} \longrightarrow \widetilde{\mathcal{D}b}_X^{(q,r)}$ on \widetilde{X} which are similar to P_R :

$$P_C = \frac{\partial^2}{\partial t^2} + \alpha_1 \partial \bar{\partial} + \alpha_2 \bar{\partial} \partial,$$

$$P_C^* = \frac{\partial^2}{\partial t^2} + \alpha_3 \bar{\partial} \partial + \alpha_4 \partial \bar{\partial},$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are positive constants. Here, $\partial, \bar{\partial}$ are the exterior differential operator, the conjugate exterior differential operator on X , and $\bar{\partial}, \partial$ are the associated operators of $\partial, \bar{\partial}$, respectively.

For $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}$, we define equations $\mathfrak{M}^c, \mathfrak{M}_1^c, \mathfrak{M}_2^c, \mathfrak{M}^{c*}, \mathfrak{M}_3^{c*}, \mathfrak{M}_4^{c*}$ below:

$$\begin{aligned} \mathfrak{M}^c &: P_C u = 0, \\ \mathfrak{M}_1^c &: \begin{cases} P_C u = 0, \\ \partial u = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha_1 \square)u = 0, \\ \partial u = 0, \end{cases} \\ \mathfrak{M}_2^c &: \begin{cases} P_C u = 0, \\ \bar{\partial} u = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha_2 \square)u = 0, \\ \bar{\partial} u = 0, \end{cases} \\ \mathfrak{M}^{c*} &: P_C^* u = 0, \\ \mathfrak{M}_3^{c*} &: \begin{cases} P_C^* u = 0, \\ \bar{\partial} u = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha_3 \bar{\square})u = 0, \\ \bar{\partial} u = 0, \end{cases} \\ \mathfrak{M}_4^{c*} &: \begin{cases} P_C^* u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha_4 \bar{\square})u = 0, \\ \vartheta u = 0. \end{cases} \end{aligned}$$

Here, $\square = \partial\bar{\partial} + \bar{\partial}\partial$ and $\bar{\square} = \bar{\partial}\vartheta + \vartheta\bar{\partial}$ are the complex Laplace-Beltrami operators.

Further we define subsheaves $\mathcal{S}ol(\mathfrak{M}^c; q, r), \mathcal{S}ol(\mathfrak{M}_j^c; q, r)$ ($j = 1, 2$), $\mathcal{S}ol(\mathfrak{M}^{c*}; q, r), \mathcal{S}ol(\mathfrak{M}_k^{c*}; q, r)$ ($k = 3, 4$) of $\widetilde{\mathcal{D}b}_X^{(q,r)}$ as follows: For $\mathfrak{N}^c = \mathfrak{M}^c, \mathfrak{M}_j^c, \mathfrak{M}^{c*}, \mathfrak{M}_k^{c*}$,

$$\mathcal{S}ol(\mathfrak{N}^c; q, r) := \left\{ u \in \widetilde{\mathcal{D}b}_M^{(q,r)} \mid u \text{ satisfies } \mathfrak{N}^c \right\}.$$

Then, we get the following theorems.

Theorem B (Theorem 4.6). *For any distribution solution $u \in \mathcal{S}ol(\mathfrak{M}^c; q, r) \Big|_{\overset{\circ}{(t, \bar{z})}}$, there exist some germs $u_j \in \mathcal{S}ol(\mathfrak{M}_j^c; q, r) \Big|_{\overset{\circ}{(t, \bar{z})}}$ ($j = 1, 2$) such that $u = u_1 + u_2$.*

Theorem B' (Theorem 4.9). *For any distribution solution $u \in \mathcal{S}ol(\mathfrak{M}^{c*}; q, r) \Big|_{\overset{\circ}{(t, \bar{z})}}$, there exist some germs $u_k \in \mathcal{S}ol(\mathfrak{M}_k^{c*}; q, r) \Big|_{\overset{\circ}{(t, \bar{z})}}$ ($k = 3, 4$) such that $u = u_3 + u_4$.*

Now we assume that X is a Kähler manifold; that is, for the Hermitian metric h , we have the equation $d(\sum h_{j\bar{k}}(z)dz^j \wedge d\bar{z}^k) = 0$, and we know that $h_{j\bar{k}}$ can be described as $h_{j\bar{k}} = \partial_j \bar{\partial}_k \phi$ with a smooth real function ϕ locally. Then the following equations for operators on $\widetilde{\mathcal{D}b}_X^{(q,r)}$ are well-known:

$$\begin{cases} \square = \bar{\square} = \frac{1}{2}\Delta, \\ \partial\vartheta + \vartheta\partial = 0, & \bar{\partial}\bar{\vartheta} + \bar{\vartheta}\bar{\partial} = 0, \\ \partial\bar{\partial} + \bar{\partial}\partial = 0, & \vartheta\bar{\vartheta} + \bar{\vartheta}\vartheta = 0. \end{cases} \quad (0.1)$$

As for the relationship between the conditions (0.1) and the Kähler condition, we give a brief introduction and a proof of the equivalency in Appendix.

Definition 0.5. We define sheaf-morphisms $P_K : \widetilde{\mathcal{D}b}_X^{(q,r)} \longrightarrow \widetilde{\mathcal{D}b}_X^{(q,r)}$ on \tilde{X} by

$$P_K = \frac{\partial^2}{\partial t^2} + \alpha_1 \partial \bar{\vartheta} + \alpha_2 \bar{\vartheta} \partial + \alpha_3 \bar{\partial} \vartheta + \alpha_4 \vartheta \bar{\partial}.$$

Here, $\alpha_1, \alpha_2, \alpha_3$ and α_4 are positive constants.

When $q, r = 0$ or n , $P_K u = 0$ reduces to a wave equation. When $q = 0, n$ or $r = 0, n$, P_K stands for P_C^* or P_C , respectively. Therefore, we suppose $1 \leq q, r \leq n - 1$.

For $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}$, we define equations $\mathfrak{M}^K, \mathfrak{M}_i^K$ ($i = 1, 2, 3, 4$), $\mathfrak{M}_{jk}^K, \mathfrak{M}_{jk0}^K$ ($(jk) = (13), (14), (23), (24)$) below:

$$\begin{aligned} \mathfrak{M}^K & : P_K u = 0, \\ \mathfrak{M}_1^K & : \begin{cases} P_K u = 0, \\ \partial u = 0, \end{cases} & \mathfrak{M}_2^K & : \begin{cases} P_K u = 0, \\ \bar{\vartheta} u = 0, \end{cases} \\ \mathfrak{M}_3^K & : \begin{cases} P_K u = 0, \\ \bar{\partial} u = 0, \end{cases} & \mathfrak{M}_4^K & : \begin{cases} P_K u = 0, \\ \vartheta u = 0, \end{cases} \\ \mathfrak{M}_{13}^K & : \begin{cases} P_K u = 0, \\ \partial u = 0, \\ \bar{\partial} u = 0, \end{cases} & \iff & \begin{cases} \left(\partial_t^2 + \frac{\alpha_1 + \alpha_3}{2} \Delta \right) u = 0, \\ \partial u = 0, \\ \bar{\partial} u = 0, \end{cases} \end{aligned}$$

$$\begin{aligned}
\mathfrak{M}_{23}^{\mathbb{K}} &: \begin{cases} P_{\mathbb{K}} u = 0, \\ \bar{\vartheta}u = 0, \\ \bar{\partial}u = 0, \end{cases} \iff \begin{cases} \left(\partial_t^2 + \frac{\alpha_2 + \alpha_3}{2} \Delta \right) u = 0, \\ \bar{\vartheta}u = 0, \\ \bar{\partial}u = 0, \end{cases} \\
\mathfrak{M}_{14}^{\mathbb{K}} &: \begin{cases} P_{\mathbb{K}} u = 0, \\ \partial u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} \left(\partial_t^2 + \frac{\alpha_1 + \alpha_4}{2} \Delta \right) u = 0, \\ \partial u = 0, \\ \vartheta u = 0, \end{cases} \\
\mathfrak{M}_{24}^{\mathbb{K}} &: \begin{cases} P_{\mathbb{K}} u = 0, \\ \bar{\vartheta}u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} \left(\partial_t^2 + \frac{\alpha_2 + \alpha_4}{2} \Delta \right) u = 0, \\ \bar{\vartheta}u = 0, \\ \vartheta u = 0, \end{cases} \\
\mathfrak{M}_{130}^{\mathbb{K}} &: \begin{cases} P_{\mathbb{K}} u = 0, \\ \partial u = 0, \\ \bar{\partial}u = 0, \\ \Delta^2 u = 0, \end{cases} \iff \begin{cases} \partial_t^4 u = \Delta^2 u = 0, \\ \left(\partial_t^2 + \frac{\alpha_1 + \alpha_3}{2} \Delta \right) u = 0, \\ \partial u = 0, \\ \bar{\partial}u = 0, \end{cases} \\
\mathfrak{M}_{230}^{\mathbb{K}} &: \begin{cases} P_{\mathbb{K}} u = 0, \\ \bar{\vartheta}u = 0, \\ \bar{\partial}u = 0, \\ \Delta^2 u = 0, \end{cases} \iff \begin{cases} \partial_t^4 u = \Delta^2 u = 0, \\ \left(\partial_t^2 + \frac{\alpha_2 + \alpha_3}{2} \Delta \right) u = 0, \\ \bar{\vartheta}u = 0, \\ \bar{\partial}u = 0, \end{cases} \\
\mathfrak{M}_{140}^{\mathbb{K}} &: \begin{cases} P_{\mathbb{K}} u = 0, \\ \partial u = 0, \\ \vartheta u = 0, \\ \Delta^2 u = 0, \end{cases} \iff \begin{cases} \partial_t^4 u = \Delta^2 u = 0, \\ \left(\partial_t^2 + \frac{\alpha_1 + \alpha_4}{2} \Delta \right) u = 0, \\ \partial u = 0, \\ \vartheta u = 0, \end{cases} \\
\mathfrak{M}_{240}^{\mathbb{K}} &: \begin{cases} P_{\mathbb{K}} u = 0, \\ \bar{\vartheta}u = 0, \\ \vartheta u = 0, \\ \Delta^2 u = 0, \end{cases} \iff \begin{cases} \partial_t^4 u = \Delta^2 u = 0, \\ \left(\partial_t^2 + \frac{\alpha_2 + \alpha_4}{2} \Delta \right) u = 0, \\ \bar{\vartheta}u = 0, \\ \vartheta u = 0, \end{cases}
\end{aligned}$$

Further we define subsheaves $Sol(\mathfrak{M}^{\mathbb{K}}; q, r)$, $Sol(\mathfrak{M}_i^{\mathbb{K}}; q, r)$ ($i = 1, 2, 3, 4$), $Sol(\mathfrak{M}_{jk}^{\mathbb{K}}; q, r)$, $Sol(\mathfrak{M}_{jk0}^{\mathbb{K}}; q, r)$ ($(jk) = (13), (23), (14), (24)$)

of $\widetilde{\mathcal{D}b}_X^{(q,r)}$ as the sheaves of $\widetilde{\mathcal{D}b}_X^{(q,r)}$ -solutions, respectively.

Then, we have the theorem below.

Theorem C (Theorem 4.15). *For any distribution solution $u \in \text{Sol}(\mathfrak{M}^K; q, r) \Big|_{\left(\overset{\circ}{t}, \overset{\circ}{z}\right)}$, there exist some germs $u_{ij} \in \text{Sol}(\mathfrak{M}_{ij}^K; q, r) \Big|_{\left(\overset{\circ}{t}, \overset{\circ}{z}\right)}$ ($(ij) = (13), (23), (14), (24)$) such that $u = u_{13} + u_{23} + u_{14} + u_{24}$.*

Further, we find that $u = u_{13} + u_{23} + u_{14} + u_{24} = 0$ implies

$$u_{jk} \in \text{Sol}(\mathfrak{M}_{jk0}^K; q, r) \quad ((jk) = (13), (23), (14), (24)).$$

Equivalently, we have the following exact sequence:

$$0 \longrightarrow \bigoplus_{(ij)}' \text{Sol}(\mathfrak{M}_{ij0}^K; q, r) \xrightarrow{G} \bigoplus_{(ij)} \text{Sol}(\mathfrak{M}_{ij}^K; q, r) \xrightarrow{H} \text{Sol}(\mathfrak{M}^K; q, r) \longrightarrow 0.$$

Here,

$$\bigoplus_{(ij)}' \text{Sol}(\mathfrak{M}_{ij0}^K; q, r) := \left\{ (u_{ij}) \in \bigoplus_{(ij)} \text{Sol}(\mathfrak{M}_{ij0}^K; q, r) \mid \sum_{(ij)} u_{ij} = 0 \right\},$$

$$G(U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}) = U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}, \quad H(U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}) = U_{13} + U_{23} + U_{14} + U_{24}.$$

1 Preparation from Riemannian geometry

In this section, we recall some notations and terminologies in Riemannian geometry used in this paper.

We assume that M is oriented. Then, there is a global section Ω of $\mathcal{E}_M^{(n)}$ on M , which never vanishes on M .

Definition 1.1. The inner products $\langle \cdot, \cdot \rangle : \bigwedge^{(1)} T_x^* M \times \bigwedge^{(1)} T_x M \rightarrow \mathbb{R}$, $\langle \cdot, \cdot \rangle^* : \bigwedge^{(p)} T_x^* M \times \bigwedge^{(p)} T_x^* M \rightarrow \mathbb{R}$, are defined as follows. We

choose a local positive orthonormal system $(\omega^1, \dots, \omega^n)$ of C^∞ sections of T^*M concerning the Riemannian metric; that is, there is a positive valued C^∞ function α such that $\omega^1 \wedge \dots \wedge \omega^n = \alpha \Omega > 0$. Then for

$$\sigma = \sum_{1 \leq i \leq n} \sigma_i dx^i, \quad \tau = \sum_{1 \leq i \leq n} \tau^i \partial_i,$$

we define

$$\langle \sigma, \tau \rangle := \sum_{1 \leq i \leq n} \sigma_i \tau^i,$$

and for

$$\begin{aligned} \phi &= \sum_{1 \leq i_1 < \dots < i_p \leq n} \phi_{i_1 \dots i_p} \omega^{i_1} \wedge \dots \wedge \omega^{i_p}, \\ \psi &= \sum_{1 \leq i_1 < \dots < i_p \leq n} \psi_{i_1 \dots i_p} \omega^{i_1} \wedge \dots \wedge \omega^{i_p}, \end{aligned}$$

we define

$$\begin{aligned} \langle \phi, \psi \rangle^* &:= \sum_{1 \leq i_1 < \dots < i_p \leq n} \phi_{i_1 \dots i_p} \psi^{i_1 \dots i_p} \\ &:= \sum_{\substack{1 \leq i_1 < \dots < i_p \leq n \\ 1 \leq j_1 < \dots < j_p \leq n}} \phi_{i_1 \dots i_p} g^{i_1 j_1} \dots g^{i_p j_p} \psi_{j_1 \dots j_p}. \end{aligned}$$

Definition 1.2. We denote by $d : \mathcal{D}b_M^{(p)} \rightarrow \mathcal{D}b_M^{(p+1)}$ the exterior differential operator which acts on $\mathcal{D}b_M^{(p)}$ as a sheaf morphism. Then the following formulas are well-known:

$$\begin{cases} d(\phi \pm \psi) = d\phi \pm d\psi & (\phi, \psi \in \mathcal{D}b_M^{(p)}), \\ d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi & (\phi \in \mathcal{D}b_M^{(p)}, \psi \in \mathcal{D}b_M^{(q)}), \\ d(d\phi) = 0 & (\phi \in \mathcal{D}b_M^{(p)}), \\ \text{For } f \in \mathcal{D}b_M^{(0)}, df := \sum \frac{\partial f}{\partial x_j} dx^j \in \mathcal{D}b_M^{(1)}. \end{cases}$$

Here $0 \leq p \leq n$. If $p = n$, $d\phi = 0$ holds.

Definition 1.3. The isomorphism $*$: $\wedge T^*M \rightarrow \wedge T^*M$ of vector bundle is defined below:

$$\begin{cases} * : \wedge^{(p)} T_x^*M \mapsto \wedge^{(n-p)} T_x^*M \text{ is a linear map,} \\ *(\omega^{i_1} \wedge \dots \wedge \omega^{i_p}) = (-1)^{(i_1-1)+\dots+(i_p-p)} \omega^{j_1} \wedge \dots \wedge \omega^{j_{n-p}}, \\ \text{for any permutation } (i_1, \dots, i_p, j_1, \dots, j_{n-p}) \text{ of } (1, \dots, n). \end{cases}$$

Here $(i_1 \cdots i_p)$ and $(j_1 \cdots j_{n-p})$ are indices satisfying

$$\begin{cases} (i_1 \cdots i_p j_1 \cdots j_{n-p}) \text{ is a permutation of } (1 \cdots n), \\ 1 \leq i_1 < \cdots < i_p \leq n, \quad 1 \leq j_1 < \cdots < j_{n-p} \leq n. \end{cases}$$

Remark 1.4. The definition above does not depend on the choice of the positive orthonormal system $\{\omega^1, \dots, \omega^n\}$.

Proposition 1.5. We set $\phi, \psi \in \bigwedge^{(p)} T_x^* M$. Then we obtain

$$\begin{cases} \phi \wedge * \psi = (*\phi) \wedge \psi = \langle \phi, \psi \rangle^* \omega^1 \wedge \cdots \wedge \omega^n, \\ *1 = \omega^1 \wedge \cdots \wedge \omega^n = \sqrt{g} \, dx^1 \wedge \cdots \wedge dx^n, \\ * \phi = (-1)^{(i_1-1)+\cdots+(i_p-p)} \sqrt{g} \, g^{i_1 j_1} \cdots g^{i_p j_p} \phi_{i_1 \cdots i_p} \, dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}} \\ \quad \in \bigwedge^{(n-p)} T_x^* M. \end{cases}$$

Here $g = \det(g_{kl})$.

Let $U \subset M$ be an open subset. Let $\alpha^{(p)} \in \mathcal{D}b_M^{(p)}(U)$, $\beta^{(p)} \in \mathcal{E}_M^{(p)}(U)$ be sections. We suppose that $\beta^{(p)}$ has a compact support in U . Then the following integral is well-defined.

$$(\alpha^{(p)}, \beta^{(p)}) := \int_M \langle \alpha^{(p)}, \beta^{(p)} \rangle^* \omega^1 \wedge \cdots \wedge \omega^n.$$

Definition 1.6. Let $\alpha^{(p)} \in \mathcal{D}b_M^{(p)}$, $\beta^{(p-1)} \in \mathcal{E}_M^{(p-1)}$ be sections. We suppose $\beta^{(p-1)}$ has a compact support. Then the sheaf morphism $\delta : \mathcal{D}b_M^{(p)} \rightarrow \mathcal{D}b_M^{(p-1)}$ is defined as

$$(\delta \alpha^{(p)}, \beta^{(p-1)}) = (\alpha^{(p)}, d\beta^{(p-1)}).$$

Hence we have

$$\delta = (-1)^{n(p-1)+1} * d *.$$

Definition 1.7. Let \mathfrak{X}_s^r be the sheaf of $\bigotimes^r T_x M \otimes \bigotimes^s T_x^* M$ -valued C^∞ functions, and $\mathcal{D}b_s^r$ the sheaf of $\bigotimes^r T_x M \otimes \bigotimes^s T_x^* M$ -valued distributions. Then, the sheaf morphisms $\nabla : \mathfrak{X}_s^r \rightarrow \mathfrak{X}_{s+1}^r$, $\mathcal{D}b_s^r \rightarrow \mathcal{D}b_{s+1}^r$

are defined as follows:

$$\left\{ \begin{array}{ll} \text{For } a(x) \in \mathfrak{X}_0^0, & \text{we have } \nabla a(x) = \frac{\partial a}{\partial x^j} dx^j. \\ \text{For } \frac{\partial}{\partial x^j} \in \mathfrak{X}_0^1, & \text{we have } \nabla \left(\frac{\partial}{\partial x^j} \right) = \Gamma_{j \ k}^i \frac{\partial}{\partial x^i} \otimes dx^k. \\ \text{For } dx^j \in \mathfrak{X}_1^0, & \text{we have } \nabla (dx^j) = -\Gamma_{i \ k}^j dx^i \otimes dx^k. \\ \text{For } e \in \mathfrak{X}_s^r, f \in \mathfrak{X}_{s'}^{r'}, & \text{we have } \nabla(e \otimes f) = (\nabla e) \otimes f + e \otimes \nabla f. \end{array} \right.$$

Here,

$$\left\{ \Gamma_{i \ k}^j = g^{jl} \Gamma_{ilk} = g^{jl} \cdot \frac{1}{2} \left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^l} \right) \right\}$$

are the Riemann-Christoffel symbols.

Proposition 1.8. *We set*

$$e = e_{i_1 \dots i_s}^{j_1 \dots j_r} dx^{i_1} \otimes \dots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_r}} \in \mathfrak{X}_s^r.$$

Then we have

$$\begin{aligned} \nabla e &= \left(\partial_k e_{i_1 \dots i_s}^{j_1 \dots j_r} + e_{i_1 \dots i_s}^{j_1 \dots j_{p-1} q j_{p+1} \dots j_r} \Gamma_{q \ k}^{j_p} - e_{i_1 \dots i_{p-1} q i_{p+1} \dots i_s}^{j_1 \dots j_r} \Gamma_{i_p \ k}^q \right) \\ &\quad \times dx^{i_1} \otimes \dots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_r}} \otimes dx^k. \end{aligned}$$

Hence we call the following *the covariant differentiation* :

$$\begin{aligned} \nabla_k e &= \left(\partial_k e_{i_1 \dots i_s}^{j_1 \dots j_r} + e_{i_1 \dots i_s}^{j_1 \dots j_{p-1} q j_{p+1} \dots j_r} \Gamma_{q \ k}^{j_p} - e_{i_1 \dots i_{p-1} q i_{p+1} \dots i_s}^{j_1 \dots j_r} \Gamma_{i_p \ k}^q \right) \\ &\quad \times dx^{i_1} \otimes \dots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_r}}. \end{aligned}$$

2 Elastic mechanics on Riemannian manifolds

Let M be an n -dimensional Riemannian manifold with metric g . We consider an elastic body G in M . A motion of G is identified with

an open subset \tilde{G} of \tilde{M} with one parameter family of diffeomorphisms $h_t(\cdot)$ ($t \in \mathbb{R}$):

$$h_s : \tilde{G} \cap \{t = 0\} \xrightarrow{\sim} \tilde{G} \cap \{t = s\}.$$

Then the elastic wave equation for G is formulated as a time development equation for small motions of G ; that is, h_t is close to the identity map and it is expressed as

$$h_t^i(x) = x^i + u^i(x, t)$$

in a local coordinate system (x^1, \dots, x^n) , where $u = \sum u^i(x, t)\partial_i$ is the small displacement vector field.

Then, the differential dh of the map h and its dual map dh^* are given as follows:

$$dh : T_x M \ni \xi^i \longmapsto \eta^i = \xi^j \left(\frac{\partial h^i}{\partial x^j} \right) = \xi^i + \frac{\partial u^i}{\partial x^j} \xi^j \in T_{h(x)} M,$$

$$dh^* : T_{h(x)}^* M \ni \eta_i \longmapsto \xi_j = \eta_i \left(\frac{\partial h^i}{\partial x^j} \right) = \eta_j + \frac{\partial u^i}{\partial x^j} \eta_i \in T_x^* M.$$

Let us calculate the difference between the line element of M

$$ds(x)^2 = g_{ij}(x) dx^i \otimes dx^j$$

and its pull-back ${}^*ds(x)^2$ by h :

$$\begin{aligned} \varepsilon_{ij}(x) dx^i \otimes dx^j &:= \frac{1}{2} \{ {}^*ds(x)^2 - ds(x)^2 \} \\ &= \frac{1}{2} \left\{ g_{kl}(h(x)) \frac{\partial h^k}{\partial x^i} \frac{\partial h^l}{\partial x^j} dx^i \otimes dx^j - g_{ij}(x) dx^i \otimes dx^j \right\}. \end{aligned}$$

Here ε_{ij} is called the strain tensor. By ignoring the non-linear terms of u^m , $\partial_k u^m$, $\partial_l u^m$, we have

$$\varepsilon_{kl} = \frac{1}{2} (g_{km} \partial_l u^m + g_{ml} \partial_k u^m + u^m \partial_m g_{kl}).$$

On Riemannian spaces, equations

$$\nabla_m g_{kl} = \partial_m g_{kl} - \Gamma_{ml}^n g_{kn} - \Gamma_{mk}^n g_{nl} = 0$$

hold, so we have

$$\begin{aligned}
\varepsilon_{kl} &= \frac{1}{2} \{g_{km} \partial_l u^m + g_{ml} \partial_k u^m + u^m (\Gamma_{ml}^n g_{kn} + \Gamma_{mk}^n g_{nl})\} \\
&= \frac{1}{2} \{g_{km} (\partial_l u^m + u^n \Gamma_{nl}^m) + g_{ml} (\partial_k u^m + u^n \Gamma_{nk}^m)\} \\
&= \frac{1}{2} (g_{km} \nabla_l u^m + g_{ml} \nabla_k u^m). \tag{2.1}
\end{aligned}$$

In physics, we assume that the stress tensor σ^{ji} has a linear relationship with the strain tensor at each point. Hence there exists the elastic coefficient tensor E^{jilk} such that

$$\frac{\sigma^{ji}}{\sqrt{g}} = E^{jilk} \varepsilon_{kl}. \tag{2.2}$$

As a physical assumption for E^{ijkl} , we have the equations:

$$E^{ijkl} = E^{jikl} = E^{ijlk}.$$

In particular, it is well-known that the elastic coefficient tensor of an isotropic elastic body has the following form:

$$E^{ijkl} = \lambda g^{ij} g^{kl} + \mu g^{ik} g^{jl} + \mu g^{il} g^{jk}, \tag{2.3}$$

where λ and μ are the two Lámé constants. Therefore, from (2.1), (2.2) and (2.3), we get

$$\begin{aligned}
\frac{\sigma^{ji}}{\sqrt{g}} &= E^{ijkl} \varepsilon_{kl} \\
&= (\lambda g^{ij} g^{kl} + \mu g^{ik} g^{jl} + \mu g^{il} g^{jk}) \left(\frac{1}{2} g_{mk} \nabla_l u^m + \frac{1}{2} g_{ml} \nabla_k u^m \right) \\
&= \lambda g^{ij} \nabla_l u^l + \mu g^{jl} \nabla_l u^i + \mu g^{il} \nabla_l u^j. \tag{2.4}
\end{aligned}$$

Hence the equation of power-balance between the stress of the elastic body and the external force is written as follows:

$$df^i = \frac{\sigma^{ji}}{\sqrt{g}} dS_j,$$

where df^i is the external force vector for the surface element dS_j .

In order to introduce the elastic wave equation, we consider a small neighborhood V of a point x in M , whose boundary is given by a smooth closed surface S . Then the equation of motion for this small part V of the elastic body is written as below:

$$\begin{aligned} \int_V \rho \frac{\partial^2 u^i}{\partial t^2} dV &= - \int_S df^i = - \int_S \frac{\sigma^{ji}}{\sqrt{g}} dS_j \\ &= \int_V \nabla_j \left(\frac{\sigma^{ji}}{\sqrt{g}} \right) dV. \end{aligned}$$

Here $dV = \sqrt{g} dx^1 \cdots dx^k$ with $\sqrt{g} = \sqrt{\det(g_{kl})}$. We divide both sides by the volume $|V|$ and shrink V to $\{x\}$. Then from (2.4), we have the elastic equation on Riemannian manifolds:

$$\begin{aligned} \rho \frac{\partial^2 u^i}{\partial t^2} &= \nabla_j \left(\frac{\sigma^{ji}}{\sqrt{g}} \right) \\ &= \lambda g^{ij} \nabla_j \nabla_k u^k + \mu g^{jk} \nabla_j \nabla_k u^i + \mu g^{ik} \nabla_j \nabla_k u^j. \end{aligned} \quad (2.5)$$

3 Decomposition of modified elastic wave equations on Riemannian manifolds

We set $\alpha = (\lambda + 2\mu)/\rho$, $\beta = \mu/\rho$. Then, we have the next lemmas.

Lemma 3.1. *For any $u \in \text{Sol}(\mathfrak{M}^{\mathbb{R}}; p)$ and the variety*

$$\begin{aligned} V_{\mathbb{R}} &:= \bigcup_{k=1,2} \left\{ (t, x; \tau, \xi) \mid \tau^2 - c_k |\xi|^2 = 0 \right\} \\ &= \bigcup_{k=1,2} \left\{ (t, x; \tau, \xi) \mid \tau^2 - c_k g^{ij}(x) \xi_i \xi_j = 0 \right\}, \end{aligned}$$

we have $\text{WF}(u) \subset V_{\mathbb{R}}$, where $c_1 = \alpha$, $c_2 = \beta$.

Proof. The symbol of the second-order operator $\rho^{-1}P_{\mathbb{R}}(t, x, \partial_t, \partial_x)$ at $\overset{\circ}{p}t = (\overset{\circ}{x}; \overset{\circ}{\xi}) \in T^*M$ is defined as a linear operator

$$\sigma_2(\rho^{-1}P_{\mathbb{R}})(\overset{\circ}{p}t) : \bigwedge^{(p)} T_x^* M \rightarrow \bigwedge^{(p)} T_x^* M$$

given by the following:

$$\begin{aligned} & \sigma_2(\rho^{-1}P_R)(\overset{\circ}{pt})U \\ & := \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda^2} e^{-\lambda i([x, \overset{\circ}{\xi}] + t\overset{\circ}{\tau})} \rho^{-1}P_R(\overset{\circ}{t}, \overset{\circ}{x}, \overset{\circ}{\partial}_t, \overset{\circ}{\partial}_x) \left(e^{\lambda i([x, \overset{\circ}{\xi}] + t\overset{\circ}{\tau})} U \right), \end{aligned}$$

where we consider the coefficients of $U \in \bigwedge^{(p)} T_x^* M$ as constants in a local coordinate system. Here $[x, \overset{\circ}{\xi}] = x_l \overset{\circ}{\xi}_l$. We suppose $(\overset{\circ}{\tau}, \overset{\circ}{\xi}) \neq (0, 0)$. Therefore we get

$$\begin{aligned} \sigma_2(\rho^{-1}P_R)(\overset{\circ}{pt})U &= \sigma_2(\overset{\circ}{\partial}_t^2 + \alpha d\delta + \beta \delta d)U \\ &= -\overset{\circ}{\tau}^2 U + (-1)^{n(p-1)} \alpha (\tilde{\xi} \wedge (*(\tilde{\xi} \wedge *U))) \\ &\quad + (-1)^{np} \beta (*(\tilde{\xi} \wedge *(\tilde{\xi} \wedge U))). \end{aligned}$$

Here $\tilde{\xi} := \sum \overset{\circ}{\xi}_j dx^j$. Let $\omega = (\omega_1, \dots, \omega_n)$ be a positive orthonormal system at $T_x^* M$ satisfying $\omega_1 = \tilde{\xi}/|\tilde{\xi}|$. We set $U = U_1 + U_2$ where $U_1 = \sum_{1 \in I} U_{1I} \omega^I$, $U_2 = \sum_{1 \notin I} U_{2I} \omega^I$, then we get $\tilde{\xi} \wedge U_1 = 0$, $\tilde{\xi} \wedge *U_2 = 0$. Therefore we have

$$\begin{aligned} *(\tilde{\xi} \wedge *(\tilde{\xi} \wedge U)) &= *(\tilde{\xi} \wedge *(|\tilde{\xi}| \sum_{1 \notin I} U_{2I} \omega_1 \wedge \omega^I)) \\ &= (-1)^{np} |\tilde{\xi}|^2 U_2, \\ \tilde{\xi} \wedge (*(\tilde{\xi} \wedge *U)) &= (-1)^{n(p-1)} |\tilde{\xi}|^2 U_1, \\ \sigma_2(\rho^{-1}P_R)(\overset{\circ}{pt})U &= (-\overset{\circ}{\tau}^2 + \alpha |\tilde{\xi}|^2) U_1 + (-\overset{\circ}{\tau}^2 + \beta |\tilde{\xi}|^2) U_2. \end{aligned}$$

Thus we find that $\sigma_2(\rho^{-1}P_R)(\overset{\circ}{pt})$ has 2 eigenvalues $(-\overset{\circ}{\tau}^2 + \alpha |\tilde{\xi}|^2)$, $(-\overset{\circ}{\tau}^2 + \beta |\tilde{\xi}|^2)$, and their multiplicities are ${}_{n-1}C_{p-1}$, ${}_{n-1}C_p$, respectively. Hence we get

$$\det(\sigma_2(\rho^{-1}P_R)(\overset{\circ}{pt})) = (-\overset{\circ}{\tau}^2 + \alpha |\tilde{\xi}|^2)^{{}_{n-1}C_{p-1}} (-\overset{\circ}{\tau}^2 + \beta |\tilde{\xi}|^2)^{{}_{n-1}C_p}.$$

If $\sigma_2(\rho^{-1}P_R)(\overset{\circ}{pt})$ is an isomorphism, $\overset{\circ}{pt}$ does not belong to the characteristic variety of $P_R u = 0$. Therefore, for a characteristic point

$\overset{\circ}{p}t$, we have $\det(\sigma_2(\rho^{-1}P_R)(\overset{\circ}{p}t)) = 0$. Hence,

$$\begin{aligned} \text{WF}(u) \subset V_R &= \bigcup_{k=1,2} \left\{ (t, x; \tau, \xi) \mid \tau^2 - c_k |\xi|^2 = 0 \right\} \\ &= \bigcup_{k=1,2} \left\{ (t, x; \tau, \xi) \mid \tau^2 - c_k g^{ij}(x) \xi_i \xi_j = 0 \right\}. \end{aligned}$$

■

Lemma 3.2. *For a germ $u \in \widetilde{\mathcal{D}b}_M^{(p)}$ at $(\overset{\circ}{t}, \overset{\circ}{x})$, we assume $\text{WF}(u) \not\subset (\overset{\circ}{t}, \overset{\circ}{x}; \pm dt)$. Then, there exists a germ $w \in \widetilde{\mathcal{D}b}_M^{(p)}$ at $(\overset{\circ}{t}, \overset{\circ}{x})$, which satisfies $\Delta w = u$ and $\text{WF}(w) \not\subset (\overset{\circ}{t}, \overset{\circ}{x}; \pm dt)$.*

Proof. For a differential form $u = \sum u_J(t, x) dx^J$, we write $\Delta u = \sum \{P_{JK}(x, \partial_x) u_K(t, x)\} dx^J$. Δ is an elliptic operator on $\mathcal{D}b_M^{(p)}$. Therefore, in a neighborhood of $(\overset{\circ}{x}, \overset{\circ}{x}) \in M \times M$, there exist integral kernels $\{G_{KL}(x, y)\}_{KL}$, which satisfy

$$\begin{cases} \sum P_{JK}(x, \partial_x) G_{KL}(x, y) = \delta_{J,L} \cdot \delta(x - y), \\ \text{WF}(G_{KL}) \subset \{(x, y; \xi, \eta) \mid x = y, \xi = -\eta\}. \end{cases}$$

Here, $\delta_{J,L}$ is a Kronecker's delta. Hence,

$$w \equiv \sum \left(\int G_{KL}(x, y) u_L(t, y) \psi(y) dy \right) dx^K$$

satisfies $\Delta w = u$ at $(\overset{\circ}{t}, \overset{\circ}{x})$ and $\text{WF}(w) \not\subset (\overset{\circ}{t}, \overset{\circ}{x}; \pm dt)$. Here $\psi \in C_0^\infty(M)$ has a compact support in a small neighborhood of $\overset{\circ}{x}$ and $\psi(y) \equiv 1$ near $\overset{\circ}{x}$. ■

Then, we get the following theorem.

Theorem 3.3. *For any distribution solution $u \in \text{Sol}(\mathfrak{M}^R; p) \Big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$, there exist some germs $u_j \in \text{Sol}(\mathfrak{M}_j^R; p) \Big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$ ($j = 1, 2$) such that $u = u_1 + u_2$.*

Furthermore, the equation $u = u_1 + u_2 = 0$ implies $u_1, u_2 \in \mathcal{Sol}(\mathfrak{M}_0^{\mathbb{R}}; p) \Big|_{(t, \overset{\circ}{x})}$. Equivalent saying, we have the following exact sequence:

$$0 \longrightarrow \mathcal{Sol}(\mathfrak{M}_0^{\mathbb{R}}; p) \xrightarrow{F} \mathcal{Sol}(\mathfrak{M}_1^{\mathbb{R}}; p) \oplus \mathcal{Sol}(\mathfrak{M}_2^{\mathbb{R}}; p) \xrightarrow{G} \mathcal{Sol}(\mathfrak{M}^{\mathbb{R}}; p) \longrightarrow 0,$$

where $F(U) = U \oplus (-U)$, $G(U_1 \oplus U_2) = U_1 + U_2$.

Proof. For $u \in \widetilde{\mathcal{D}b}_M^{(p)} \Big|_{(t, \overset{\circ}{x})}$, we suppose u_1, u_2 are of the form;

$$u_2 = \delta v, \quad u_1 = u - \delta v.$$

Here $v \in \widetilde{\mathcal{D}b}_M^{(p+1)} \Big|_{(t, \overset{\circ}{x})}$. Hence we have only to impose the following conditions on v :

$$\begin{cases} P_{\mathbb{R}}(\delta v) = 0, \\ d(u - \delta v) = 0, \\ dv = 0. \end{cases}$$

Since the equation $P_{\mathbb{R}}(\delta v) = (\partial_t^2 + \beta \delta d) \delta v = \delta (\partial_t^2 + \beta \Delta) v$ holds, it is sufficient to impose the conditions below:

$$\begin{cases} \partial_t^2 v + \beta \Delta v = 0, \\ d\delta v = du, \\ dv = 0, \end{cases} \iff \begin{cases} \partial_t^2 v = -\beta du, \\ \Delta v = du, \\ dv = 0. \end{cases}$$

From Lemma 3.1 and Lemma 3.2, we have a w satisfying $\Delta w = u$ and $\text{WF}(w) \not\supset (t, \overset{\circ}{x}; \pm dt)$. Then it is sufficient to impose the following equations on v :

$$\begin{cases} \partial_t^2(v - dw) = -d(\beta u + \partial_t^2 w), \\ \Delta(v - dw) = 0, \\ d(v - dw) = 0. \end{cases}$$

Therefore we can take v as follows:

$$v - dw = -d \int_t^t ds \int_t^s (\beta u(\tau, x) + \partial_t^2 w(\tau, x)) d\tau.$$

Then u_2 satisfies the following wave equation:

$$\rho \frac{\partial^2}{\partial t^2} u_2 + \beta \Delta u_2 = 0.$$

Hence $u \in \mathcal{Sol}(\mathfrak{M}^{\mathbb{R}}; p) \Big|_{(t, \hat{x})}^{\circ}$ can be decomposed as a sum $u = u_1 + u_2$, where $u_j \in \mathcal{Sol}(\mathfrak{M}_j^{\mathbb{R}}; p) \Big|_{(t, \hat{x})}^{\circ}$ ($j=1,2$).

When $u = u_1 + u_2 = 0$ holds, the equations $\delta u_1 = 0$ and $du_2 = 0$ imply $u_1, u_2 \in \mathcal{Sol}(\mathfrak{M}_0^{\mathbb{R}}; p) \Big|_{(t, \hat{x})}^{\circ}$.

Therefore we have an exact sequence

$$0 \longrightarrow \mathcal{Sol}(\mathfrak{M}_0^{\mathbb{R}}; p) \longrightarrow \mathcal{Sol}(\mathfrak{M}_1^{\mathbb{R}}; p) \oplus \mathcal{Sol}(\mathfrak{M}_2^{\mathbb{R}}; p) \longrightarrow \mathcal{Sol}(\mathfrak{M}^{\mathbb{R}}; p) \longrightarrow 0.$$

■

Remark 3.4. For the case $p = 1$, the contravariant form of this decomposition means the decomposition $u^i = u_1^i + u_2^i \in \widetilde{\mathcal{D}b}_0^1$ satisfying the conditions as follows:

$$\nabla_i u_1^i = 0, \quad \nabla^i u_2^j - \nabla^j u_2^i = 0.$$

Remark 3.5. In Einstein spaces satisfying $R_{ij} = \lambda g_{ij}$, a distribution solution u of $P_{\text{org}} u = 0$ has a similar decomposition $u = u_1 + u_2$. However, if M is not an Einstein space, a distribution solution u of $P_{\text{org}} u = 0$ does not necessarily admit any decomposition of solutions above.

4 Decomposition of modified elastic wave equations on complex manifolds

We extend the results on a Riemannian manifold to ones in a complex manifold X with a Hermitian metric h . Firstly, we recall operators on complex manifolds.

Definition 4.1. We denote by $\partial : \mathcal{D}b_X^{(q,r)} \rightarrow \mathcal{D}b_X^{(q+1,r)}$ the exterior differential operator which acts on $\mathcal{D}b_X^{(q,r)}$ as a sheaf morphism and $\bar{\partial} : \mathcal{D}b_X^{(q,r)} \rightarrow \mathcal{D}b_X^{(q,r+1)}$ the conjugate exterior differential operator. For a section

$$\phi = \phi_{i_1 \dots i_q \bar{j}_1 \dots \bar{j}_r} dz^{i_1} \wedge \dots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_r} \quad \text{of } \mathcal{D}b_X^{(q,r)},$$

the following formulas are well-known:

$$\left\{ \begin{array}{l} d\phi = (\partial + \bar{\partial})\phi, \\ \partial\phi = \frac{\partial\phi}{\partial z^k} dz^k \wedge dz^{i_1} \wedge \dots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_r} \in \mathcal{D}b_X^{(q+1,r)}, \\ \bar{\partial}\phi = \frac{\partial\phi}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^{i_1} \wedge \dots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_r} \in \mathcal{D}b_X^{(q,r+1)}. \end{array} \right.$$

Definition 4.2. The linear operator $*$ on X induces isomorphisms $\bigwedge^{(q,r)} T^*X \rightarrow \bigwedge^{(n-r,n-q)} T^*X$ of vector bundle. Hence we have sheaf-morphisms $*$: $\mathcal{D}b_X^{(q,r)} \rightarrow \mathcal{D}b_X^{(n-r,n-q)}$ on X as follows: For

$$\psi = \psi_{I\bar{J}} \omega^I \wedge \bar{\omega}^{\bar{J}} \in \mathcal{D}b_X^{(q,r)},$$

we have

$$*\psi = \delta \left(\begin{array}{ccc} 1 \cdots n & \bar{1} \cdots \bar{n} \\ I & \bar{J} & \bar{J}^C \\ & & I^C \end{array} \right) \psi_{I\bar{J}} \omega^{J^C} \wedge \bar{\omega}^{I^C} \in \mathcal{D}b_X^{(n-r,n-q)},$$

where $\{\omega^1, \dots, \omega^n\}$ is a local orthonormal system of C^∞ sections of T^*X concerning the Hermitian metric and $I^C := \{1, \dots, n\} \setminus I$. Here $\delta(\cdot) = \pm 1$ is the signature of the permutation $(I\bar{J}\bar{J}^C I^C)$ of $(1 \cdots n \bar{1} \cdots \bar{n})$.

Let $U \subset X$ be an open subset. Let $\alpha^{(q,r)} = \alpha_{I\bar{J}} \omega^I \wedge \bar{\omega}^{\bar{J}} \in \mathcal{D}b_X^{(q,r)}(U)$, $\beta^{(q,r)} = \beta_{I\bar{J}} \omega^I \wedge \bar{\omega}^{\bar{J}} \in \mathcal{E}_X^{(q,r)}(U)$ be sections. We suppose that $\beta^{(q,r)}$ has a compact support in U . Then the following integral is well-defined.

$$\langle \alpha^{(q,r)}, \beta^{(q,r)} \rangle := \int_X \langle \alpha^{(q,r)}, \beta^{(q,r)} \rangle^* \omega^1 \wedge \dots \wedge \omega^n \wedge \bar{\omega}^1 \wedge \dots \wedge \bar{\omega}^n,$$

where, $\langle \alpha^{(q,r)}, \beta^{(q,r)} \rangle^* = \sum_{I,\bar{J}} \alpha_{I\bar{J}} \bar{\beta}^{I\bar{J}}$.

Definition 4.3. Let $\alpha^{(q,r)} \in \mathcal{D}b_X^{(q,r)}$, $\beta^{(q-1,r)} \in \mathcal{E}_X^{(q-1,r)}$, and $\gamma^{(q,r-1)} \in \mathcal{E}_X^{(q,r-1)}$ be sections. We suppose $\beta^{(q-1,r)}$ and $\gamma^{(q,r-1)}$ have compact supports. Then sheaf morphisms $\bar{\vartheta} : \mathcal{D}b_X^{(q,r)} \rightarrow \mathcal{D}b_X^{(q-1,r)}$ and $\vartheta : \mathcal{D}b_X^{(q,r)} \rightarrow \mathcal{D}b_X^{(q,r-1)}$ are defined as

$$\begin{aligned}(\bar{\vartheta}\alpha^{(q,r)}, \beta^{(q-1,r)}) &= (\alpha^{(q,r)}, \partial\beta^{(q-1,r)}), \\(\vartheta\alpha^{(q,r)}, \gamma^{(q,r-1)}) &= (\alpha^{(q,r)}, \bar{\partial}\gamma^{(q,r-1)}).\end{aligned}$$

Further they satisfy the following equations:

$$\begin{cases} \delta &= \bar{\vartheta} + \vartheta, \\ \bar{\vartheta} &= - * \bar{\partial} *, \\ \vartheta &= - * \partial *. \end{cases}$$

Then, we get the lemmas below.

Lemma 4.4. For any $u \in \text{Sol}(\mathfrak{M}^c; q, r)$ and the variety

$$\begin{aligned}V_C &:= \bigcup_{k=1,2} \left\{ (t, z; \tau, \zeta) \mid \tau^2 - \alpha_k |\zeta|^2 = 0 \right\} \\ &= \bigcup_{k=1,2} \left\{ (t, z; \tau, \zeta) \mid \tau^2 - \alpha_k h^{ij}(z) \zeta_i \bar{\zeta}_j = 0 \right\},\end{aligned}$$

we have $\text{WF}(u) \subset V_C$, where (τ, ζ) is identified with the real cotangent vector $\tau dt + 2\text{Re}(\zeta dz)$.

Proof. The symbol of the second-order operator $P_C(t, z, \partial_t, \partial_z, \bar{\partial}_z)$ at $\overset{\circ}{p}t = (\overset{\circ}{z}; \overset{\circ}{\zeta}) \in T^*X$ is defined as a linear operator

$$\sigma_2(P_C)(\overset{\circ}{p}t) : \bigwedge^{(q,r)} T_{\overset{\circ}{z}}^* X \rightarrow \bigwedge^{(q,r)} T_{\overset{\circ}{z}}^* X$$

given by the following:

$$\begin{aligned}&\sigma_2(P_C)(\overset{\circ}{p}t)U \\ &:= \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda^2} e^{-\lambda i(2\text{Re}[z, \overset{\circ}{\zeta}] + t\bar{t})} P_C(\overset{\circ}{t}, \overset{\circ}{z}, \overset{\circ}{\partial}_t, \overset{\circ}{\partial}_z, \bar{\overset{\circ}{\partial}}_z) \left(e^{\lambda i(2\text{Re}[z, \overset{\circ}{\zeta}] + t\bar{t})} U \right),\end{aligned}$$

where we consider the coefficients of $U \in \bigwedge^{(q,r)} T_z^* X$ as constants in a local coordinate system. Here, $[z, \overset{\circ}{\zeta}] = z_l \overset{\circ}{\zeta}_l$. We suppose $(\overset{\circ}{\tau}, \overset{\circ}{\zeta}) \neq (0, 0)$. Therefore we get

$$\begin{aligned} \sigma_2(P_C)(\overset{\circ}{pt})U &= \sigma_2(\partial_t^2 + \alpha_1 \partial \bar{\vartheta} + \alpha_2 \bar{\vartheta} \partial)U \\ &= -\overset{\circ}{\tau}^2 U - \alpha_1 (\Xi \wedge (*(\bar{\Xi} \wedge *U))) \\ &\quad - \alpha_2 (*(\bar{\Xi} \wedge *(\Xi \wedge U))). \end{aligned}$$

Here $\Xi := \sum \overset{\circ}{\zeta}_j dz^j$. Let $\omega = (\omega_1, \dots, \omega_n)$ be an orthonormal system of $T_z^* X$ satisfying $\omega_1 = \Xi/|\Xi|$. When we set $U = U_1 + U_2$ where $U_1 = \sum_{1 \in I} U_{1I\bar{J}} \omega^I \wedge \bar{\omega}^{\bar{J}}$, $U_2 = \sum_{1 \notin I} U_{2I\bar{J}} \omega^I \wedge \bar{\omega}^{\bar{J}}$, we get $\Xi \wedge U_1 = 0$, $\bar{\Xi} \wedge *U_2 = 0$. Therefore we have

$$\begin{aligned} *(\bar{\Xi} \wedge *(\Xi \wedge U)) &= *(\bar{\Xi} \wedge *(|\Xi| \sum_{1 \notin I} U_{2I\bar{J}} \omega_1 \wedge \omega^I \wedge \bar{\omega}^{\bar{J}})) \\ &= \delta \left(\begin{matrix} 1 & \cdots & n\bar{1} & \cdots & \bar{n} \\ I & \bar{J} & \bar{J}^C & I^C \end{matrix} \right) |\Xi| \\ &\quad \times \left\{ *(\bar{\Xi} \wedge \sum_{1 \notin I} U_{2I\bar{J}} \omega^{J^C} \wedge \bar{\omega}^{\{1, I\}^C}) \right\} \\ &= -|\Xi|^2 U_2, \\ \Xi \wedge (*(\bar{\Xi} \wedge *U)) &= -|\Xi|^2 U_1, \\ \sigma_2(P_C)(\overset{\circ}{pt})U &= (-\overset{\circ}{\tau}^2 + \alpha_1 |\Xi|^2) U_1 + (-\overset{\circ}{\tau}^2 + \alpha_2 |\Xi|^2) U_2. \end{aligned}$$

Thus we find that $\sigma_2(P_C)(\overset{\circ}{pt})$ has 2 eigenvalues $(-\overset{\circ}{\tau}^2 + \alpha_1 |\Xi|^2)$, $(-\overset{\circ}{\tau}^2 + \alpha_2 |\Xi|^2)$, and their multiplicities are ${}_{n-1}C_{q-1} \cdot {}_n C_r$, ${}_{n-1}C_q \cdot {}_n C_r$, respectively. Hence we get

$$\det(\sigma_2(P_C)(\overset{\circ}{pt})) = (-\overset{\circ}{\tau}^2 + \alpha_1 |\Xi|^2)^{{}_{n-1}C_{q-1} \cdot {}_n C_r} (-\overset{\circ}{\tau}^2 + \alpha_2 |\Xi|^2)^{{}_{n-1}C_q \cdot {}_n C_r}.$$

If $\sigma_2(P_C)(\overset{\circ}{pt})$ is an isomorphism, $\overset{\circ}{pt}$ does not belong to the characteristic variety of $P_C u = 0$. Therefore, for a characteristic point $\overset{\circ}{pt}$,

we have $\det(\sigma_2(P_C)(\overset{\circ}{p}t)) = 0$. Hence,

$$\begin{aligned} \text{WF}(u) \subset V_C &= \bigcup_{k=1,2} \left\{ (t, z; \tau, \zeta) \mid \tau^2 - \alpha_k |\zeta|^2 = 0 \right\} \\ &= \bigcup_{k=1,2} \left\{ (t, z; \tau, \zeta) \mid \tau^2 - \alpha_k h^{ij}(z) \zeta_i \bar{\zeta}_j = 0 \right\}. \end{aligned}$$

■

Lemma 4.5. *For a germ $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}$ at $(\overset{\circ}{t}, \overset{\circ}{z})$, we assume $\text{WF}(u) \not\subset (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$. Then, there exists a germ $w \in \widetilde{\mathcal{D}b}_X^{(q,r)}$ at $(\overset{\circ}{t}, \overset{\circ}{z})$, which satisfies $\square w = u$ and $\text{WF}(w) \not\subset (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$.*

Proof. For $u = \sum u_{J\bar{L}}(t, z) dz^J \wedge d\bar{z}^L$, we write

$$\square u = \sum \{ P_{JK}(z, \partial_z, \bar{\partial}_z) u_{K\bar{L}}(t, z) \} dz^J \wedge d\bar{z}^L.$$

\square is an elliptic operator on $\mathcal{D}b_X^{(q,r)}$. Therefore, in a neighborhood of $(\overset{\circ}{z}, \overset{\circ}{z}) \in X \times X$, there exist integral kernels $\{G_{KM}(z, z')\}_{KM}$, which satisfy

$$\begin{aligned} &\sum P_{JK}(z, \partial_z, \bar{\partial}_z) G_{KM}(z, z') \\ &= \delta_{J,M} \cdot \delta(\text{Re } z - \text{Re } z') \cdot \delta(\text{Im } z - \text{Im } z'), \\ \text{WF}(G_{KM}) &\subset \{ (z, z'; \zeta dz + \bar{\zeta} d\bar{z} + \zeta' dz' + \bar{\zeta}' d\bar{z}') \mid z = z', \zeta = -\zeta' \}. \end{aligned}$$

Here, $\delta_{J,M}$ is a Kronecker's delta. Hence,

$$w \equiv \sum \left(\int G_{KM}(z, z') u_{M\bar{L}}(t, z') \psi(z') d(\text{Re } z') d(\text{Im } z') \right) dz^K \wedge d\bar{z}^L$$

satisfies $\square w = u$ at $(\overset{\circ}{t}, \overset{\circ}{z})$ and $\text{WF}(w) \not\subset (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$. Here $\psi \in C_0^\infty(X)$ has a compact support in a small neighborhood of $\overset{\circ}{z}$ and $\psi(z') \equiv 1$ near $\overset{\circ}{z}$. ■

Then, we get the following theorem.

Theorem 4.6. For any distribution solution $u \in \text{Sol}(\mathfrak{M}^c; q, r) \Big|_{(t, \dot{z})}$, there exist some germs $u_j \in \text{Sol}(\mathfrak{M}_j^c; q, r) \Big|_{(t, \dot{z})}$ ($j = 1, 2$) such that $u = u_1 + u_2$.

Proof. For $u \in \widetilde{\mathcal{D}b}_X^{(q,r)} \Big|_{(t, \dot{z})}$, we suppose u_1, u_2 are of the form;

$$u_1 = \partial v, \quad u_2 = u - \partial v.$$

Here $v \in \widetilde{\mathcal{D}b}_X^{(q-1,r)} \Big|_{(t, \dot{z})}$. Hence we have only to impose the following conditions on v :

$$\begin{cases} P_C(\partial v) = 0, \\ \bar{\partial}(u - \partial v) = 0, \\ \bar{\partial}v = 0. \end{cases}$$

Since the equation $P_C(\partial v) = (\partial_t^2 + \alpha_1 \bar{\partial}) \partial v = \partial(\partial_t^2 + \alpha_1 \square)v = 0$ holds, it is sufficient to impose the conditions below:

$$\begin{cases} \partial_t^2 v + \alpha_1 \square v = 0, \\ \bar{\partial} \partial v = \bar{\partial} u, \\ \bar{\partial} v = 0, \end{cases} \iff \begin{cases} \partial_t^2 v = -\alpha_1 \bar{\partial} u, \\ \square v = \bar{\partial} u, \\ \bar{\partial} v = 0. \end{cases}$$

From Lemma 4.4 and Lemma 4.5, we have a w satisfying $\square w = u$ and $\text{WF}(w) \not\supset (t, \dot{z}; \pm dt)$. Then it is sufficient to impose the following equations on v :

$$\begin{cases} \partial_t^2(v - \bar{\partial} w) = -\bar{\partial}(\alpha_1 u + \partial_t^2 w), \\ \square(v - \bar{\partial} w) = 0, \\ \bar{\partial}(v - \bar{\partial} w) = 0. \end{cases}$$

Therefore we can take v as follows:

$$v - \bar{\partial} w = -\bar{\partial} \int_t^t ds \int_t^s (\alpha_1 u(\tau, z) + \partial_t^2 w(\tau, z)) d\tau.$$

Then u_1 satisfies the following wave equation:

$$\frac{\partial^2}{\partial t^2} u_1 + \alpha_1 \square u_1 = 0.$$

Hence $u \in \mathcal{S}ol(\mathfrak{M}^c; q, r) \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$ can be decomposed as a sum $u = u_1 + u_2$, where $u_j \in \mathcal{S}ol(\mathfrak{M}_j^c; q, r) \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$ ($j=1,2$). \blacksquare

In the same way, we obtain the following similar lemmas and theorems as adjoint versions of Lemma 4.4, Lemma 4.5, and Theorem 4.6. We omit the proofs because the arguments go in a similar way.

Lemma 4.7. *For any $u \in \mathcal{S}ol(\mathfrak{M}^{c*}; q, r)$ and the variety*

$$\begin{aligned} V_C^* &:= \bigcup_{k=3,4} \left\{ (t, z; \tau, \zeta) \mid \tau^2 - \alpha_k |\zeta|^2 = 0 \right\} \\ &= \bigcup_{k=3,4} \left\{ (t, z; \tau, \zeta) \mid \tau^2 - \alpha_k h^{ij}(z) \zeta_i \bar{\zeta}_j = 0 \right\}, \end{aligned}$$

we have $\text{WF}(u) \subset V_C^*$, where (τ, ζ) is identified with the real cotangent vector $\tau dt + 2\text{Re}(\zeta dz)$.

Lemma 4.8. *For a germ $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}$ at $(\overset{\circ}{t}, \overset{\circ}{z})$, we assume $\text{WF}(u) \not\subset (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$. Then, there exists a germ $w \in \widetilde{\mathcal{D}b}_X^{(q,r)}$ at $(\overset{\circ}{t}, \overset{\circ}{z})$, which satisfies $\bar{\square}w = u$ and $\text{WF}(w) \not\subset (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$.*

Theorem 4.9. *For any distribution solution $u \in \mathcal{S}ol(\mathfrak{M}^{c*}; q, r) \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$, there exist some germs $u_k \in \mathcal{S}ol(\mathfrak{M}_k^{c*}; q, r) \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$ ($k = 3, 4$) such that $u = u_3 + u_4$.*

Now we assume that X is a Kähler manifold. For the sheaf-morphisms P_K , we have the following lemmas.

Lemma 4.10. *For any $u \in \mathcal{S}ol(\mathfrak{M}^k; q, r)$ and the variety*

$$\begin{aligned} V_K &:= \bigcup_{k=1,2,3,4} \left\{ (t, z; \tau, \zeta) \mid (\tau^2 - \beta_k |\zeta|^2 = 0) \right\} \\ &= \bigcup_{k=1,2,3,4} \left\{ (t, z; \tau, \zeta) \mid (\tau^2 - \beta_k h^{ij}(z) \zeta_i \bar{\zeta}_j = 0) \right\}, \end{aligned}$$

we have $\text{WF}(u) \subset V_K$, where $\beta_1 = \alpha_1 + \alpha_3$, $\beta_2 = \alpha_2 + \alpha_3$, $\beta_3 = \alpha_1 + \alpha_4$, $\beta_4 = \alpha_2 + \alpha_4$ and (τ, ζ) is identified with the real cotangent vector $\tau dt + 2\text{Re}(\zeta dz)$.

Proof. Similarly to Lemma 4.4, we get the symbol of the second-order operator

$$\begin{aligned} \sigma_2(P_K)(\overset{\circ}{p}t)U &= \sigma_2(\partial_t^2 + \alpha_1 \partial \bar{\vartheta} + \alpha_2 \bar{\vartheta} \partial + \alpha_3 \bar{\partial} \vartheta + \alpha_4 \vartheta \bar{\partial})U \\ &= -\overset{\circ}{\tau}^2 U - \alpha_1 \left(\Xi \wedge \left(* \left(\bar{\Xi} \wedge *U \right) \right) \right) \\ &\quad - \alpha_2 \left(* \left(\bar{\Xi} \wedge * \left(\Xi \wedge U \right) \right) \right) \\ &\quad - \alpha_3 \left(\bar{\Xi} \wedge \left(* \left(\Xi \wedge *U \right) \right) \right) \\ &\quad - \alpha_4 \left(* \left(\Xi \wedge * \left(\bar{\Xi} \wedge U \right) \right) \right). \end{aligned}$$

Here $\Xi := \sum \overset{\circ}{\zeta}_j dz^j$. Let $\omega = (\omega_1, \dots, \omega_n)$ be an orthonormal system of T_z^*X satisfying $\omega_1 = \Xi/|\Xi|$. We set $U = U_1 + U_2 + U_3 + U_4$ where

$$\begin{aligned} U_1 &= \sum_{1 \in I, 1 \in J} U_{1I\bar{J}} \omega^I \wedge \bar{\omega}^J, & U_2 &= \sum_{1 \notin I, 1 \in J} U_{2I\bar{J}} \omega^I \wedge \bar{\omega}^J, \\ U_3 &= \sum_{1 \in I, 1 \notin J} U_{3I\bar{J}} \omega^I \wedge \bar{\omega}^J, & U_4 &= \sum_{1 \notin I, 1 \notin J} U_{4I\bar{J}} \omega^I \wedge \bar{\omega}^J. \end{aligned}$$

Then we get

$$\begin{cases} \Xi \wedge U_1 = 0, \\ \bar{\Xi} \wedge U_1 = 0, \end{cases} \quad \begin{cases} \bar{\Xi} \wedge U_2 = 0, \\ \bar{\Xi} \wedge *U_2 = 0. \end{cases} \\ \begin{cases} \Xi \wedge U_3 = 0, \\ \Xi \wedge *U_3 = 0, \end{cases} \quad \begin{cases} \bar{\Xi} \wedge *U_4 = 0, \\ \Xi \wedge *U_4 = 0. \end{cases}$$

Therefore we have

$$\begin{aligned}
& * \left(\bar{\Xi} \wedge * \left(\Xi \wedge U \right) \right) \\
& \quad = * \left(\bar{\Xi} \wedge * \left(|\Xi| \left(\sum_{1 \neq I} (U_{2I\bar{J}} + U_{4I\bar{J}}) \omega_1 \wedge \omega^I \wedge \bar{\omega}^J \right) \right) \right) \\
& \quad = -|\Xi|^2 (U_2 + U_4), \\
\Xi \wedge \left(* \left(\bar{\Xi} \wedge * U \right) \right) & = -|\Xi|^2 (U_1 + U_3), \\
* \left(\Xi \wedge * \left(\bar{\Xi} \wedge U \right) \right) & = -|\Xi|^2 (U_3 + U_4), \\
\bar{\Xi} \wedge \left(* \left(\Xi \wedge * U \right) \right) & = -|\Xi|^2 (U_1 + U_2).
\end{aligned}$$

Hence,

$$\sigma_2(P_K)(\overset{\circ}{pt})U = \sum_{k=1,2,3,4} \left(-\overset{\circ}{\tau}^2 + \beta_k |\Xi|^2 \right) U_k.$$

Thus we find that $\sigma_2(P_K)(\overset{\circ}{pt})$ has 4 eigenvalues $-\overset{\circ}{\tau}^2 + \beta_k |\Xi|^2$ ($k = 1, 2, 3, 4$), and their multiplicities are γ_k , respectively. Here $\gamma_1 = {}_{n-1}C_{q-1} \cdot {}_{n-1}C_{r-1}$, $\gamma_2 = {}_{n-1}C_q \cdot {}_{n-1}C_{r-1}$, $\gamma_3 = {}_{n-1}C_{q-1} \cdot {}_{n-1}C_r$, and $\gamma_4 = {}_{n-1}C_q \cdot {}_{n-1}C_r$. Hence we get

$$\det(\sigma_2(P_K)(\overset{\circ}{pt})) = \prod_{k=1}^4 \left(-\overset{\circ}{\tau}^2 + \beta_k |\Xi|^2 \right)^{\gamma_k}.$$

If $\sigma_2(P_K)(\overset{\circ}{pt})$ is an isomorphism, $\overset{\circ}{pt}$ does not belong to the characteristic variety of $P_K u = 0$. Therefore, for a characteristic point $\overset{\circ}{pt}$, we have $\det(\sigma_2(P_K)(\overset{\circ}{pt})) = 0$. Hence,

$$\begin{aligned}
\text{WF}(u) \subset V_K & = \bigcup_{k=1,2,3,4} \left\{ (t, z; \tau, \zeta) \mid (\tau^2 - \beta_k |\zeta|^2 = 0) \right\} \\
& = \bigcup_{k=1,2,3,4} \left\{ (t, z; \tau, \zeta) \mid (\tau^2 - \beta_k h^{ij}(z) \zeta_i \bar{\zeta}_j = 0) \right\}.
\end{aligned}$$

■

Lemma 4.11. *For any germ $u \in \mathcal{S}ol(\mathfrak{M}^K; q, r) \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$, there exist some germs $u_j \in \mathcal{S}ol(\mathfrak{M}_j^K; q, r) \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$ ($j = 1, 2$) such that $u = u_1 + u_2$.*

Proof. For $u \in \widetilde{\mathcal{D}b}_X^{(q,r)} \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$, we suppose u_1, u_2 are of the form;

$$u_1 = \partial v_1, \quad u_2 = u - \partial v_1.$$

Here $v_1 \in \widetilde{\mathcal{D}b}_X^{(q-1,r)} \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$. Hence we have only to impose the following conditions on v_1 :

$$\begin{cases} P_K (\partial v_1) = 0, \\ \bar{\vartheta}(u - \partial v_1) = 0, \\ \bar{\vartheta}v_1 = 0. \end{cases}$$

Since the equation

$$\begin{aligned} P_K (\partial v_1) &= (\partial_t^2 + \alpha_1 \partial \bar{\vartheta} + \alpha_3 \bar{\partial} \vartheta + \alpha_4 \vartheta \bar{\partial}) \partial v_1 \\ &= \partial (\partial_t^2 + \alpha_1 \square + \alpha_3 \bar{\partial} \vartheta + \alpha_4 \vartheta \bar{\partial}) v_1 = 0 \end{aligned}$$

holds, it is sufficient to impose the conditions below:

$$\begin{cases} \left(\partial_t^2 + \alpha_1 \square + \alpha_3 \bar{\partial} \vartheta + \alpha_4 \vartheta \bar{\partial} \right) v_1 = 0, \\ \bar{\vartheta}(u - \partial v_1) = 0, \\ \bar{\vartheta}v_1 = 0, \end{cases} \iff \begin{cases} P_C^* v_1 = -\alpha_1 \bar{\vartheta}u, \\ \square v_1 = \bar{\vartheta}u, \\ \bar{\vartheta}v_1 = 0. \end{cases}$$

From Lemma 4.10 and Lemma 4.5, we have a w satisfying $\square w = u$ and $\text{WF}(w) \not\cong (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$. Then, when we put

$$v_1 - \bar{\vartheta}w =: \varphi, \quad -\alpha_1 \bar{\vartheta}u - \bar{\vartheta} P_C^* w =: \psi,$$

and use $\square = \bar{\square}$, it is sufficient to impose the following equations on φ :

$$\begin{cases} P_C^* \varphi = \left(\partial_t^2 + \alpha_3 \bar{\partial} \vartheta + \alpha_4 \vartheta \bar{\partial} \right) \varphi = \psi, \\ \square \varphi = 0, \\ \bar{\vartheta} \varphi = 0, \end{cases} \iff \begin{cases} \left(\partial_t^2 + (\alpha_4 - \alpha_3) \vartheta \bar{\partial} \right) \varphi = \psi, \\ \square \varphi = 0, \\ \bar{\vartheta} \varphi = 0. \end{cases}$$

Setting $\mathcal{F} := \left\{ \Psi \in \widetilde{\mathcal{D}b}_X^{(q,r)} \Big|_{\left(\overset{\circ}{t}, \overset{\circ}{z}\right)} ; \square\Psi = 0, \bar{\vartheta}\Psi = 0 \right\}$, we note that

$$\vartheta\bar{\partial}(\mathcal{F}) \subset \mathcal{F}, \quad (\vartheta\bar{\partial})^2\mathcal{F} = 0.$$

Since $\psi \in \mathcal{F}$ and $\text{WF}(\psi) \not\supset \left(\overset{\circ}{t}, \overset{\circ}{z}; \pm dt\right)$, then we can take φ as follows:

$$\begin{aligned} \varphi &= \int_{\overset{\circ}{t}}^t ds \int_{\overset{\circ}{t}}^s \psi(s', z) ds' \\ &\quad + (\alpha_3 - \alpha_4) \int_{\overset{\circ}{t}}^t ds \int_{\overset{\circ}{t}}^s ds' \int_{\overset{\circ}{t}}^{s'} d\tau \int_{\overset{\circ}{t}}^{\tau} \vartheta\bar{\partial}\psi(\tau', z) d\tau'. \end{aligned}$$

Hence $u \in \text{Sol}(\mathfrak{M}^k; q, r) \Big|_{\left(\overset{\circ}{t}, \overset{\circ}{z}\right)}$ can be decomposed into a sum $u = u_1 + u_2$, where $u_j \in \text{Sol}(\mathfrak{M}_j^k; q, r) \Big|_{\left(\overset{\circ}{t}, \overset{\circ}{z}\right)}$ ($j=1,2$). \blacksquare

By the proof above, we get the following more precise form of u_1, u_2 .

Lemma 4.12. *Let $u = u_1 + u_2$ be a decomposition. We can write u_1, u_2 in the preceding lemma as follows:*

$$u_1 = \partial v_1, \quad u_2 = \bar{\vartheta}v_2,$$

where $v_1 \in \widetilde{\mathcal{D}b}_X^{(q-1,r)} \Big|_{\left(\overset{\circ}{t}, \overset{\circ}{z}\right)}$, $v_2 \in \widetilde{\mathcal{D}b}_X^{(q+1,r)} \Big|_{\left(\overset{\circ}{t}, \overset{\circ}{z}\right)}$ satisfying $\text{WF}(v_j) \not\supset \left(\overset{\circ}{t}, \overset{\circ}{z}; \pm dt\right)$ ($j = 1, 2$) and the equations

$$\begin{cases} \partial_t^2 v_1 + \alpha_1 \square v_1 + \alpha_3 \bar{\partial}\vartheta v_1 + \alpha_4 \vartheta\bar{\partial}v_1 = 0, \\ \partial_t^2 v_2 + \alpha_2 \square v_2 + \alpha_3 \bar{\partial}\vartheta v_2 + \alpha_4 \vartheta\bar{\partial}v_2 = 0. \end{cases} \quad (4.1)$$

Proof. We define $D_t^{-l} := \int_{\overset{\circ}{t}}^t dt_1 \int_{\overset{\circ}{t}}^{t_1} dt_2 \cdots \int_{\overset{\circ}{t}}^{t_{l-1}} dt_l$. Putting $v_1 = \bar{\vartheta}w +$

$D_t^{-2}\psi + (\alpha_3 - \alpha_4)\vartheta\bar{\partial}D_t^{-4}\psi$, we calculate the difference:

$$\begin{aligned}
u_2 &\equiv u - u_1 = u - \partial v_1 \\
&= \bar{\vartheta}\partial w - D_t^{-2}\partial\psi + (\alpha_4 - \alpha_3)\vartheta\bar{\partial}D_t^{-4}\partial\psi \\
&= \bar{\vartheta}\partial w - (D_t^{-2} - (\alpha_4 - \alpha_3)\vartheta\bar{\partial}D_t^{-4})\partial\bar{\vartheta}(-\alpha_1 u - P_C^* w) \\
&= \bar{\vartheta}\partial w \\
&\quad - (D_t^{-2} - (\alpha_4 - \alpha_3)\vartheta\bar{\partial}D_t^{-4})(\bar{\vartheta}\partial(\alpha_1 u + P_C^* w) + (\alpha_2 - \alpha_1)\bar{\vartheta}\partial u) \\
&= \bar{\vartheta}\partial\left(w - (D_t^{-2} - (\alpha_4 - \alpha_3)\vartheta\bar{\partial}D_t^{-4})(\alpha_2 u + P_C^* w)\right).
\end{aligned}$$

Hence, we can get

$$\begin{aligned}
v_1 &= \bar{\vartheta}\left(w - (D_t^{-2} - (\alpha_4 - \alpha_3)\vartheta\bar{\partial}D_t^{-4})(\alpha_1 u + P_C^* w)\right), \\
v_2 &= \partial\left(w - (D_t^{-2} - (\alpha_4 - \alpha_3)\vartheta\bar{\partial}D_t^{-4})(\alpha_2 u + P_C^* w)\right).
\end{aligned}$$

In particular, $\text{WF}(v_j) = \not\equiv (t, z; \pm dt)$ ($j = 1, 2$). The first equation of (4.1) is already obtained in the proof of Lemma 4.11. The second equation of (4.1) is also obtained in a similar way as follows: Set $\Phi = v_2 - \partial w$. Then Φ satisfies

$$\begin{cases} \left(\partial_t^2 + (\alpha_4 - \alpha_3)\vartheta\bar{\partial}\right)\Phi = \Psi := -\partial(\alpha_2 u + P_C^* w), \\ \square\Phi = 0, \\ \partial\Phi = 0. \end{cases}$$

Hence,

$$\begin{cases} P_C^*\Phi = \Psi, \\ \square\Phi = 0, \\ \partial\Phi = 0, \end{cases} \iff \begin{cases} P_C^* v_2 = -\alpha_2 \partial u, \\ \square v_2 = \partial u, \\ \partial v_2 = 0. \end{cases}$$

Thus we obtain

$$(P_C^* + \alpha_2 \square)v_2 = 0.$$

This is just the second equation of (4.1). ■

Then we have the following lemma.

Lemma 4.13. For any germ $u_1 = \partial v_1 \in \mathcal{S}ol(\mathfrak{M}_1^k; q, r) \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$ with a $v_1 \in \widetilde{\mathcal{D}b}_X^{(q-1, r)} \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$ satisfying (4.1) and $\text{WF}(v_1) \not\supseteq (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$, there exist some germs $u_{1j} \in \mathcal{S}ol(\mathfrak{M}_{1j}^k; q, r) \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$ ($j = 3, 4$) such that $u_1 = u_{13} + u_{14}$.

Proof. For $u_1 = \partial v_1 \in \widetilde{\mathcal{D}b}_X^{(q, r)} \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$, we suppose u_{13}, u_{14} are of the form;

$$u_{13} = \partial \bar{\partial} V, \quad u_{14} = u_1 - \partial \bar{\partial} V = \partial(v_1 - \bar{\partial} V).$$

Here $V \in \widetilde{\mathcal{D}b}_X^{(q-1, r-1)} \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$. Hence we have only to impose the following conditions on V :

$$\begin{cases} P_k(\partial \bar{\partial} V) = 0, \\ \vartheta(v_1 - \bar{\partial} V) = 0, \\ \vartheta V = 0. \end{cases}$$

Since the equation

$$\begin{aligned} P_k(\partial \bar{\partial} V) &= (\partial_t^2 + \alpha_1 \partial \bar{\partial} + \alpha_2 \bar{\partial} \partial + \alpha_3 \bar{\partial} \vartheta + \alpha_4 \vartheta \bar{\partial}) \partial \bar{\partial} V \\ &= \partial \bar{\partial} (\partial_t^2 + \alpha_1 \square + \alpha_3 \bar{\square}) V = \partial \bar{\partial} (\partial_t^2 + (\alpha_1 + \alpha_3) \bar{\square}) V \end{aligned}$$

holds, it is sufficient to impose the conditions below:

$$\begin{cases} (\partial_t^2 + (\alpha_1 + \alpha_3) \bar{\square}) V = 0, \\ \vartheta(v_1 - \bar{\partial} V) = 0, \\ \vartheta V = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + (\alpha_1 + \alpha_3) \bar{\square}) V = 0, \\ \bar{\square} V = \vartheta v_1, \\ \vartheta V = 0. \end{cases}$$

From Lemma 4.10 and Lemma 4.8, we have a w satisfying $\bar{\square} w' = v_1$ and $\text{WF}(w') \not\supseteq (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$. Then it is sufficient to impose the following equations on V :

$$\begin{cases} \partial_t^2(V - \vartheta w') = -\vartheta((\alpha_1 + \alpha_3)v_1 + \partial_t^2 w'), \\ \bar{\square}(V - \vartheta w') = 0, \\ \vartheta(V - \vartheta w') = 0. \end{cases}$$

Therefore we get

$$V - \vartheta w' = -\vartheta D_t^{-2} \left((\alpha_1 + \alpha_3) v_1(t, z) + \partial_t^2 w'(t, z) \right). \quad (4.2)$$

Indeed the $\overline{\square}$ -closedness of right side of (4.2) follows from (4.1). Hence $u_1 \in \mathcal{S}ol(\mathfrak{M}_1^K; q, r) \Big|_{\overset{\circ}{(t, \dot{z})}}$ can be decomposed as a sum $u_1 = u_{13} + u_{14}$, where $u_{1j} \in \mathcal{S}ol(\mathfrak{M}_{1j}^K; q, r) \Big|_{\overset{\circ}{(t, \dot{z})}}$ ($j=3,4$). \blacksquare

In the same way, we obtain the following similar lemma as adjoint version of Lemma 4.13. We omit the proof because the arguments go in a similar way.

Lemma 4.14. *For any germ $u_2 = \overline{\vartheta} v_2 \in \mathcal{S}ol(\mathfrak{M}_2^K; q, r) \Big|_{\overset{\circ}{(t, \dot{z})}}$ with a $v_2 \in \widetilde{\mathcal{D}b}_X^{(q+1, r)} \Big|_{\overset{\circ}{(t, \dot{z})}}$ satisfying (4.1) and $\text{WF}(v_2) \not\ni \overset{\circ}{(t, \dot{z})} \pm dt$, there exist some germs $u_{2k} \in \mathcal{S}ol(\mathfrak{M}_{2k}^K; q, r) \Big|_{\overset{\circ}{(t, \dot{z})}}$ ($k = 3, 4$) such that $u_2 = u_{23} + u_{24}$.*

Then we have the following theorem.

Theorem 4.15. *For any distribution solution $u \in \mathcal{S}ol(\mathfrak{M}^K; q, r) \Big|_{\overset{\circ}{(t, \dot{z})}}$, there exist some germs $u_{ij} \in \mathcal{S}ol(\mathfrak{M}_{ij}^K; q, r) \Big|_{\overset{\circ}{(t, \dot{z})}}$ ($(ij) = (13), (23), (14), (24)$) such that $u = u_{13} + u_{23} + u_{14} + u_{24}$.*

Further, we find that $u = u_{13} + u_{23} + u_{14} + u_{24} = 0$ implies

$$u_{jk} \in \mathcal{S}ol(\mathfrak{M}_{jk0}^K; q, r) \quad ((jk) = (13), (23), (14), (24)).$$

Equivalently, we have the following exact sequence:

$$0 \longrightarrow \bigoplus_{(ij)}' \mathcal{S}ol(\mathfrak{M}_{ij0}^K; q, r) \xrightarrow{G} \bigoplus_{(ij)} \mathcal{S}ol(\mathfrak{M}_{ij}^K; q, r) \xrightarrow{H} \mathcal{S}ol(\mathfrak{M}^K; q, r) \longrightarrow 0.$$

Here,

$$\bigoplus'_{(ij)} \mathcal{S}ol(\mathfrak{M}_{ij0}^K; q, r) := \left\{ (u_{ij}) \in \bigoplus_{(ij)} \mathcal{S}ol(\mathfrak{M}_{ij0}^K; q, r) \mid \sum_{(ij)} u_{ij} = 0 \right\},$$

$$G(U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}) = U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}, \quad H(U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}) = U_{13} + U_{23} + U_{14} + U_{24}.$$

Proof. By virtue of Lemma 4.11, 4.12, 4.13 and 4.14, we find that $u \in \mathcal{S}ol(\mathfrak{M}^K; q, r) \Big|_{\left(\begin{smallmatrix} \circ \\ (t, \bar{z}) \end{smallmatrix}\right)}$ is decomposed as a sum $u = u_{13} + u_{23} + u_{14} + u_{24}$ by using $u_{ij} \in \mathcal{S}ol(\mathfrak{M}_{ij}^K; q, r) \Big|_{\left(\begin{smallmatrix} \circ \\ (t, \bar{z}) \end{smallmatrix}\right)}$ $((i, j) = (1, 3), (2, 3), (1, 4), (2, 4))$.

When $u = u_{13} + u_{23} + u_{14} + u_{24} = 0$ is holds, we set $w = u_{13} + u_{23} = -u_{14} - u_{24}$. Then w satisfies $\bar{\partial}w = 0$, $\partial w = 0$. Hence we have $\Delta w = 2\bar{\square}w = 0$, and so $\Delta u_{13} = -\Delta u_{23}$. By a similar argument we obtain $\Delta^2 u_{13} = 0$. Therefore we have $u_{13} \in \mathcal{S}ol(\mathfrak{M}_{130}^K; q, r) \Big|_{\left(\begin{smallmatrix} \circ \\ (t, \bar{z}) \end{smallmatrix}\right)}$. In a similar way we conclude that

$$u_{jk} \in \mathcal{S}ol(\mathfrak{M}_{jk0}^K; q, r) \quad ((jk) = (23), (14), (24)).$$

This completes the proof of Theorem 4.15. ■

Example 4.16 We assume $X = \mathbb{C}^2$. Then, X is a Kähler manifold with the complex Euclidean metric. We find a solution $u \in \widetilde{\mathcal{D}b}_X^{(1,1)}$ of the form with $\zeta \equiv \zeta_1 dz^1 + \zeta_2 dz^2$ where $(\zeta_1, \zeta_2) \in \mathbb{C}^2 \setminus \{0\}$;

$$u(t, z) = U(t)e^{i(z \cdot \zeta + \bar{z} \cdot \bar{\zeta})}.$$

Then,

$$\begin{aligned} P_K u &= U'' + (\alpha_1 - \alpha_2) \zeta \wedge \left(* (\bar{\zeta} \wedge *U) \right) + \alpha_2 |\zeta|^2 U \\ &\quad + (\alpha_3 - \alpha_4) \bar{\zeta} \wedge \left(* (\zeta \wedge *U) \right) + \alpha_4 |\zeta|^2 U = 0. \end{aligned}$$

We put

$$U(t) = c_1(t) \zeta \wedge \bar{\zeta} + c_2(t) \zeta \wedge \bar{\zeta}^\perp + c_3(t) \zeta^\perp \wedge \bar{\zeta} + c_4(t) \zeta^\perp \wedge \bar{\zeta}^\perp,$$

where $\zeta^\perp = \bar{\zeta}_2 dz^1 - \bar{\zeta}_1 dz^2$, $|\zeta| = |\zeta^\perp|$ hold. Then, we get

$$\begin{aligned} & \left(c_1'' + (\alpha_1 + \alpha_3)|\zeta|^2 c_1 \right) \zeta \wedge \bar{\zeta} + \left(c_2'' + (\alpha_1 + \alpha_4)|\zeta|^2 c_2 \right) \zeta \wedge \bar{\zeta}^\perp \\ & \left(c_3'' + (\alpha_2 + \alpha_3)|\zeta|^2 c_3 \right) \zeta^\perp \wedge \bar{\zeta} + \left(c_4'' + (\alpha_2 + \alpha_4)|\zeta|^2 c_4 \right) \zeta^\perp \wedge \bar{\zeta}^\perp = 0. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} c_1(t) &= A_{13}^+ \exp(i\sqrt{\alpha_1 + \alpha_3}|\zeta|t) + A_{13}^- \exp(-i\sqrt{\alpha_1 + \alpha_3}|\zeta|t), \\ c_2(t) &= A_{14}^+ \exp(i\sqrt{\alpha_1 + \alpha_4}|\zeta|t) + A_{14}^- \exp(-i\sqrt{\alpha_1 + \alpha_4}|\zeta|t), \\ c_3(t) &= A_{23}^+ \exp(i\sqrt{\alpha_2 + \alpha_3}|\zeta|t) + A_{23}^- \exp(-i\sqrt{\alpha_2 + \alpha_3}|\zeta|t), \\ c_4(t) &= A_{24}^+ \exp(i\sqrt{\alpha_2 + \alpha_4}|\zeta|t) + A_{24}^- \exp(-i\sqrt{\alpha_2 + \alpha_4}|\zeta|t). \end{aligned}$$

Since

$$\begin{aligned} U(0) &= (A_{13}^+ + A_{13}^-) \zeta \wedge \bar{\zeta} + (A_{14}^+ + A_{14}^-) \zeta \wedge \bar{\zeta}^\perp \\ & \quad + (A_{23}^+ + A_{23}^-) \zeta^\perp \wedge \bar{\zeta} + (A_{24}^+ + A_{24}^-) \zeta^\perp \wedge \bar{\zeta}^\perp, \\ \frac{\partial}{\partial t} U(0) &= i\sqrt{\alpha_1 + \alpha_3}|\zeta|(A_{13}^+ - A_{13}^-) \zeta \wedge \bar{\zeta} \\ & \quad + i\sqrt{\alpha_1 + \alpha_4}|\zeta|(A_{14}^+ - A_{14}^-) \zeta \wedge \bar{\zeta}^\perp \\ & \quad + i\sqrt{\alpha_2 + \alpha_3}|\zeta|(A_{23}^+ - A_{23}^-) \zeta^\perp \wedge \bar{\zeta} \\ & \quad + i\sqrt{\alpha_2 + \alpha_4}|\zeta|(A_{24}^+ - A_{24}^-) \zeta^\perp \wedge \bar{\zeta}^\perp, \end{aligned}$$

we get

$$\begin{aligned} A_{13}^+ &= \frac{\langle U(0), \zeta \wedge \bar{\zeta} \rangle^*}{2|\zeta|^4} - i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta \wedge \bar{\zeta} \rangle^*}{2\sqrt{\alpha_1 + \alpha_3}|\zeta|^5}, \\ A_{13}^- &= \frac{\langle U(0), \zeta \wedge \bar{\zeta} \rangle^*}{2|\zeta|^4} + i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta \wedge \bar{\zeta} \rangle^*}{2\sqrt{\alpha_1 + \alpha_3}|\zeta|^5}, \\ A_{14}^+ &= \frac{\langle U(0), \zeta \wedge \bar{\zeta}^\perp \rangle^*}{2|\zeta|^4} - i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta \wedge \bar{\zeta}^\perp \rangle^*}{2\sqrt{\alpha_1 + \alpha_4}|\zeta|^5}, \\ A_{14}^- &= \frac{\langle U(0), \zeta \wedge \bar{\zeta}^\perp \rangle^*}{2|\zeta|^4} + i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta \wedge \bar{\zeta}^\perp \rangle^*}{2\sqrt{\alpha_1 + \alpha_4}|\zeta|^5}, \end{aligned}$$

$$\begin{aligned}
A_{23}^+ &= \frac{\langle U(0), \zeta^\perp \wedge \bar{\zeta} \rangle^*}{2|\zeta|^4} - i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^\perp \wedge \bar{\zeta} \rangle^*}{2\sqrt{\alpha_2 + \alpha_3} |\zeta|^5}, \\
A_{23}^- &= \frac{\langle U(0), \zeta^\perp \wedge \bar{\zeta} \rangle^*}{2|\zeta|^4} + i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^\perp \wedge \bar{\zeta} \rangle^*}{2\sqrt{\alpha_2 + \alpha_3} |\zeta|^5}, \\
A_{24}^+ &= \frac{\langle U(0), \zeta^\perp \wedge \bar{\zeta}^\perp \rangle^*}{2|\zeta|^4} - i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^\perp \wedge \bar{\zeta}^\perp \rangle^*}{2\sqrt{\alpha_2 + \alpha_4} |\zeta|^5}, \\
A_{24}^- &= \frac{\langle U(0), \zeta^\perp \wedge \bar{\zeta}^\perp \rangle^*}{2|\zeta|^4} + i \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^\perp \wedge \bar{\zeta}^\perp \rangle^*}{2\sqrt{\alpha_2 + \alpha_4} |\zeta|^5}.
\end{aligned}$$

A Appendix

As an appendix, we give a brief introduction to Kähler manifolds. Further we give a proof of the equivalency of the conditions (0.1) and the Kähler condition in A2 and A3.

A1. The Kähler condition on Hermitian manifolds

Let X be an n -dimensional Hermitian manifold. Taking a holomorphic local coordinates (z^1, \dots, z^n) , we define a positive definite Hermitian matrix $h_{j\bar{k}}(z) = \langle \partial_{z^j}, \partial_{z^k} \rangle_z$, when $\langle \cdot, \cdot \rangle$ is the Hermitian metric. Then the $(1, 1)$ -form

$$\Phi = h_{j\bar{k}} dz^j \wedge d\bar{z}^k$$

does not depend on the choice of the coordinates.

Definition A.1. We call a Hermitian manifold (X, h) a Kähler manifold when the condition

$$d\Phi = 0 \tag{A.1}$$

is satisfied. We call (A.1) a Kähler condition.

Proposition A.2. *In order that (A.1) holds, it is necessary and sufficient that there exists a real-valued C^∞ function φ satisfying $\Phi = \partial\bar{\partial}\varphi$ locally.*

Proof. The Kähler condition (A.1) is equivalent to $\partial\Phi = 0$ and $\bar{\partial}\Phi = 0$. Hence we obtain

$$\frac{\partial h_{i\bar{j}}}{\partial z^k} - \frac{\partial h_{k\bar{j}}}{\partial z^i} = 0, \quad \frac{\partial h_{i\bar{j}}}{\partial \bar{z}^k} - \frac{\partial h_{i\bar{k}}}{\partial \bar{z}^j} = 0.$$

Because of the equation $\bar{\partial}(h_{i\bar{j}} d\bar{z}^j) = 0$ for any i , there exist C^∞ functions f_i satisfying

$$h_{i\bar{j}} = \frac{\partial f_i}{\partial \bar{z}^j}$$

from the Dolbeault lemma.

On the other hand, by the equation

$$0 = \frac{\partial h_{i\bar{j}}}{\partial z^k} - \frac{\partial h_{k\bar{j}}}{\partial z^i} = \frac{\partial}{\partial \bar{z}^j} \left(\frac{\partial f_i}{\partial z^k} - \frac{\partial f_k}{\partial z^i} \right),$$

we find that

$$\frac{\partial f_i}{\partial z^k} - \frac{\partial f_k}{\partial z^i} =: F_{ik}$$

are holomorphic functions. Because of the equation

$$d(F_{ik} dz^i \wedge dz^k) = \partial(F_{ik} dz^i \wedge dz^k) = 0,$$

there exist holomorphic functions F_i satisfying

$$F_{ik} = \frac{\partial F_i}{\partial z^k} - \frac{\partial F_k}{\partial z^i}$$

from the complex version of the de Rham theorem. By the equation

$$\frac{\partial(f_i - F_i)}{\partial z^k} = \frac{\partial(f_k - F_k)}{\partial z^i},$$

there exist a C^∞ function φ satisfying

$$f_i - F_i = \frac{\partial \varphi}{\partial z^i}$$

from the Dolbeault lemma. Therefore we obtain

$$h_{i\bar{j}} = \frac{\partial f_i}{\partial \bar{z}^j} = \frac{\partial}{\partial \bar{z}^j} \left(F_i + \frac{\partial \varphi}{\partial z^i} \right) = \frac{\partial^2 \varphi}{\partial \bar{z}^j \partial z^i}.$$

Because $h_{i\bar{j}} = \bar{h}_{j\bar{i}}$, we can replace φ by its real part. ■

To explain the meaning of $d\Phi = 0$, we introduce the canonical connection \mathcal{D} induced by Φ . Let $\mathfrak{X}_{(s,t)}^{(q,r)}$ be a sheaf of $\bigotimes^{(q,r)} T_z X \otimes \bigwedge^{(s,t)} T_z^* X$ -valued C^∞ functions.

Definition A.3. A holomorphic connection $\mathcal{D} : \mathfrak{X}_{(s,t)}^{(1,0)} \rightarrow \mathfrak{X}_{(s+1,t)}^{(1,0)}$ is defined as follows:

$$\text{For } a \in \mathcal{E}^{(s,t)}, f \in \mathfrak{X}_{(s',t')}^{(1,0)}, \quad \text{we have } \mathcal{D}(a \wedge f) = (\partial a) \wedge f \\ + (-1)^{s+t} a \wedge \mathcal{D}f.$$

To find the holomorphic connection induced by the Hermitian metric, we put

$$\mathcal{D} \left(\frac{\partial}{\partial z^j} \right) = \omega_j^i{}_k \frac{\partial}{\partial z^i} \otimes dz^k.$$

Since \mathcal{D} induced by Φ satisfy the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{X}_{(0,0)}^{(1,0)} \ni \frac{\partial}{\partial z^j} & \xrightarrow{\Phi} & h_{j\bar{l}} d\bar{z}^l \in \mathcal{E}_X^{(0,1)} \\ \mathcal{D} \downarrow & & \downarrow \partial \\ \mathfrak{X}_{(1,0)}^{(1,0)} \ni \omega_j^i{}_k \frac{\partial}{\partial z^i} \otimes dz^k & \xrightarrow{\Phi} & \frac{\partial h_{j\bar{l}}}{\partial z^k} dz^k \wedge d\bar{z}^l \in \mathcal{E}_X^{(1,1)}, \end{array}$$

we get

$$\omega_j^i{}_k = h^{i\bar{l}} \partial_k h_{j\bar{l}}.$$

Hence we have

$$\mathcal{D} \left(\alpha^j \frac{\partial}{\partial z^j} \right) = \left(\frac{\partial \alpha^j}{\partial z^k} + \omega_i^j{}_k \alpha^i \right) \frac{\partial}{\partial z^j} \otimes dz^k.$$

By the canonical duality between ∂_{z^j} and dz^j , \mathcal{D} induces a holomorphic connection on the cotangent bundle. Indeed we have

$$\mathcal{D}(dz^j) = -\omega_i^j{}_k dz^i \otimes dz^k.$$

Then, if this \mathcal{D} is compatible with the exterior differential (torsion-free condition), we must impose the condition $\mathcal{D}(dz^j) = 0$ in $\bigwedge^{(2,0)} T^* X$; that is,

$$\omega_i^j{}_k = \omega_k^j{}_i.$$

This condition is just the Kähler condition.

Proposition A.4. *The Kähler condition is equivalent to the condition that the holomorphic connection induced by the Hermitian metric is torsion-free.*

Now we consider the difference between the Riemannian connection ∇ and the holomorphic connection \mathcal{D} .

Let $X^{\mathbb{R}}$ be the underlying real manifold of X . Then a complex vector $\xi^j \partial_{z^j} \in TX$ corresponds to the real vector $\text{Re}(\xi^j \partial_{z^j}) \in TX^{\mathbb{R}}$. We define the induced Riemannian metric as

$$\langle \text{Re}(\xi^j \partial_{z^j}), \text{Re}(\eta^k \partial_{z^k}) \rangle_{\mathbb{R}} := \frac{1}{2} \text{Re}(\xi^j \bar{\eta}^k h_{j\bar{k}}).$$

By expanding the Riemannian metric to the Hermitian inner product on $\mathbb{C} \otimes_{\mathbb{R}} TX^{\mathbb{R}}$, we get

$$\langle \partial_{z^j}, \partial_{z^k} \rangle_{\mathbb{R}} = h_{j\bar{k}}, \quad \langle \partial_{\bar{z}^j}, \partial_{\bar{z}^k} \rangle_{\mathbb{R}} = h_{k\bar{j}}, \quad \langle \partial_{z^j}, \partial_{\bar{z}^k} \rangle_{\mathbb{R}} = 0.$$

Then we have the following well-known fact.

Proposition A.5. *Let ∇ be the induced Riemannian connection. By calculating $\nabla(\partial/\partial x)$ and $\nabla(\partial/\partial y)$ ($z = x + iy$), we obtain the following expression:*

$$\begin{aligned} \nabla \left(\frac{\partial}{\partial z^j} \right) &= h^{i\bar{l}} \frac{\partial h_{j\bar{l}}}{\partial z^k} dz^k \otimes \frac{\partial}{\partial z^i} \\ &+ \frac{1}{2} \left[h^{i\bar{l}} \left(\frac{\partial h_{k\bar{l}}}{\partial z^j} - \frac{\partial h_{j\bar{l}}}{\partial z^k} \right) dz^k \otimes \frac{\partial}{\partial z^i} \right. \\ &+ h^{i\bar{l}} \left(\frac{\partial h_{j\bar{l}}}{\partial \bar{z}^k} - \frac{\partial h_{j\bar{k}}}{\partial \bar{z}^l} \right) d\bar{z}^k \otimes \frac{\partial}{\partial z^i} \\ &\left. + h^{i\bar{l}} \left(\frac{\partial h_{l\bar{k}}}{\partial z^j} - \frac{\partial h_{j\bar{k}}}{\partial z^l} \right) d\bar{z}^k \otimes \frac{\partial}{\partial \bar{z}^i} \right]. \end{aligned}$$

In particular, if $\nabla(\partial/\partial_{z^j})$ does not include $(\partial/\partial_{\bar{z}^i})$ components, X satisfies the Kähler condition. Further the canonical holomorphic connection and the induced Riemannian connection coincide with each other. In other words, the induced Riemannian connection preserves the complex structure if and only if X is a Kähler manifold.

A2. Equations on Kähler manifolds

In this section we prove the commutativity or anti-commutativity of operators on Kähler manifolds. To begin with, we introduce another important operator on a Kähler manifold.

Definition A.6. Let $\Phi = h_{i\bar{j}}(z)dz^i \wedge d\bar{z}^j$ be the Hermitian metric form. Then we can define the sheaf morphism

$$\Phi \wedge \cdot : \mathcal{E}_X^{(q,r)} \ni f(z) \mapsto \Phi(z) \wedge f(z) \in \mathcal{E}_X^{(q+1,r+1)}.$$

Indeed, $\Phi \wedge \cdot$ is a differential operator of order 0. Hence we can also define the adjoint operator $A(\Phi)$ of $\Phi \wedge \cdot$ with respect to the inner products in $\mathcal{E}_X^{(q,r)}$ and $\mathcal{E}_X^{(q-1,r-1)}$. That is, for $f \in \mathcal{E}_X^{(q,r)}$ and $g \in \mathcal{E}_X^{(q-1,r-1)}$ with compact support we have

$$(A(\Phi)f, g) = (f, \Phi \wedge g).$$

$A(\Phi)$ is also a differential operator of order 0.

Proposition A.7. *We have the following equations for differential operators:*

$$A(\Phi)\partial - \partial A(\Phi) = -\vartheta, \tag{A.2}$$

$$A(\Phi)\bar{\partial} - \bar{\partial} A(\Phi) = \bar{\vartheta}. \tag{A.3}$$

We prove this proposition at the end of this section. Once we admit these equations, we can easily prove the commutativity and the anticommutativity for the operators.

Theorem A.8. *The operators $\partial, \bar{\partial}, \bar{\vartheta}, \vartheta, \square, \bar{\square}$ satisfy the following equations:*

$$\begin{cases} \partial\bar{\partial} + \bar{\partial}\partial = 0, & \vartheta\bar{\vartheta} + \bar{\vartheta}\vartheta = 0. \\ \partial\vartheta + \vartheta\partial = 0, & \bar{\partial}\bar{\vartheta} + \bar{\vartheta}\bar{\partial} = 0, \\ \square = \bar{\square} = \frac{1}{2}\Delta, \end{cases}$$

Proof. The first equation follows directly from the definitions of $\partial, \bar{\partial}$. Further, the second equation is just the adjoint of the first equation. The third and the fourth equation follow directly from the previous proposition:

$$\partial\vartheta + \vartheta\partial = \partial(\partial A(\Phi) - A(\Phi)\partial) + (\partial A(\Phi) - A(\Phi)\partial)\partial = 0.$$

Moreover, we also have the following equations:

$$\begin{aligned}\square &= \partial\bar{\vartheta} + \bar{\vartheta}\partial = \partial A(\Phi)\bar{\partial} - \partial\bar{\partial}A(\Phi) + A(\Phi)\bar{\partial}\partial - \bar{\partial}A(\Phi)\partial, \\ \bar{\square} &= \bar{\partial}\vartheta + \vartheta\bar{\partial} = \bar{\partial}\partial A(\Phi) - \bar{\partial}A(\Phi)\partial + \partial A(\Phi)\bar{\partial} - A(\Phi)\partial\bar{\partial}.\end{aligned}$$

Therefore we have

$$\square - \bar{\square} = A(\Phi)(\partial\bar{\partial} + \bar{\partial}\partial) - (\partial\bar{\partial} + \bar{\partial}\partial)A(\Phi) = 0 - 0 = 0,$$

and

$$\begin{aligned}\Delta &= d\delta + \delta d = (\partial + \bar{\partial})(\bar{\vartheta} + \vartheta) + (\bar{\vartheta} + \vartheta)(\partial + \bar{\partial}) \\ &= \square + \bar{\square} + (\partial\vartheta + \vartheta\partial) + (\bar{\partial}\bar{\vartheta} + \bar{\vartheta}\bar{\partial}) = 2\square = 2\bar{\square}.\end{aligned}$$

This completes the proof. ■

The remaining problem is to prove Proposition A.7. The following proposition is very useful for the proof of Proposition A.7.

Proposition A.9. *For any point p on a Kähler manifold X we can take a coordinate system around p (an adapted coordinate system at p) satisfying the following 2 conditions:*

- (1) $z^1(p) = \cdots = z^n(p) = 0$,
- (2) $h_{i\bar{j}}(z) = \delta_{ij} + O(|z|^2)$; that is,

$$h_{i\bar{j}}(0) = \delta_{ij} \quad \text{and} \quad \partial_z h_{i\bar{j}}(0) = \partial_{\bar{z}} h_{i\bar{j}}(0) = 0.$$

Proof. By using a suitable complex linear transformation, we can take a complex coordinate system (z_1, \dots, z_n) around p such that

$$z^1(p) = \cdots = z^n(p) = 0, \quad h_{i\bar{j}} = \delta_{ij}.$$

Consider the Taylor expansion of $h_{i\bar{j}}(z)$ at $z = 0$ as follows:

$$h_{i\bar{j}}(z) = \delta_{ij} + C_{ijk}z^k + \bar{C}_{jik}\bar{z}^k + O(|z|^2),$$

where $\{C_{ijk}; i, j, k = 1, \dots, n\}$ are some complex constants. Then the Kähler condition at p induces the following identities:

$$C_{ijk} = C_{kji} \quad (\forall i, j, k). \quad (\text{A.4})$$

Consider the following coordinate change around $w = 0$:

$$z^j = w^j - \frac{1}{2} \sum_{i,k} C_{ijk} w^i w^k \quad (j = 1, \dots, n).$$

Then we have the following under the equations (A.4):

$$\begin{aligned} h_{i\bar{j}}(z(w)) \frac{\partial z^i}{\partial w^\alpha} \frac{\partial \bar{z}^j}{\partial \bar{w}^\beta} &= (\delta_{ij} + C_{ijk}w^k + \bar{C}_{jik}\bar{w}^k + O(|w|^2)) \\ &\quad \times \left(\delta_{i\alpha} - \frac{C_{\alpha ik'}w^{k'} + C_{k'i\alpha}w^{k'}}{2} + O(|w|^2) \right) \\ &\quad \times \left(\delta_{j\beta} - \frac{\bar{C}_{\beta jk''}\bar{w}^{k''} + \bar{C}_{k''j\beta}\bar{w}^{k''}}{2} + O(|w|^2) \right) \\ &= \delta_{\alpha\beta} + O(|w|^2). \end{aligned}$$

■

Consider the difference

$$B = A(\Phi)\partial - \partial A(\Phi) + \vartheta$$

in the equation (A.2). Then, B is a differential operator of order at most 1. Hence in order to prove $B = 0$, we have only to show that all the coefficients of B vanish at any given point p . Further, it is clear that the value of every coefficient of B at a given point p is determined only by

$$h_{i\bar{j}}(p), \partial h_{i\bar{j}}(p)/\partial z^k, \partial h_{i\bar{j}}(p)/\partial \bar{z}^k.$$

On the other hand, under the adapted coordinate system at p , $h_{i\bar{j}}$ is approximately identified with \mathbb{C}^n up to the first order at p . Hence we

have only to prove Proposition A.7 for $X = \mathbb{C}^n$ with the Euclidean metric. From now on we set $X = \mathbb{C}^n$ and

$$\Phi(z) = \sum dz^j \wedge d\bar{z}^j.$$

Let us prove the equations (A.2) and (A.3). Since (A.3) is the complex conjugate to (A.2), we have only to prove (A.2). Further, taking the operator conjugate, the equation (A.2) is equivalent to the following:

$$\bar{\partial}f = \Phi \wedge (\bar{\partial}f) - \bar{\partial}(\Phi \wedge f) \quad \forall f \in \mathcal{E}_X^{(q,r)}. \quad (\text{A.5})$$

Let $f = \sum f_{I\bar{J}} dz^I \wedge d\bar{z}^{\bar{J}}$ be any (q, r) -form with C^∞ -coefficients. Then we have

$$\begin{aligned} (\bar{\partial}f)_{K\bar{J}} &= - \sum_s \frac{\partial f_{sK\bar{J}}}{\partial \bar{z}^s}, \\ (\Phi \wedge f)_{H\bar{L}} &= \sum_{\alpha, \beta} (-1)^{q+\alpha+\beta} \delta_{h_\alpha, \ell_\beta} f_{h_1 \cdots h_{q+1} \bar{\ell}_1 \cdots \bar{\ell}_{r+1}}. \end{aligned}$$

Here, $|K| = q - 1, |H| = q + 1, |L| = r + 1$ and $h_1 \cdots h_{q+1} = h_1 \cdots h_{\alpha-1} h_{\alpha+1} \cdots h_{r+1}$. Therefore, for $|I| = q, |L| = r + 1$

$$\begin{aligned} &(\Phi \wedge (\bar{\partial}f) - \bar{\partial}(\Phi \wedge f))_{I\bar{L}} \\ &= \sum_{\alpha, \beta, s} (-1)^{1+q-1+\alpha+\beta} \delta_{i_\alpha, \ell_\beta} \frac{\partial f_{s i_1 \cdots i_q \bar{\ell}_1 \cdots \bar{\ell}_{r+1}}}{\partial \bar{z}^s} + \sum_s \frac{\partial (\Phi \wedge f)_{s I \bar{L}}}{\partial \bar{z}^s} \\ &= \sum_{\alpha, \beta, s} (-1)^{q+\alpha+\beta} \delta_{i_\alpha, \ell_\beta} \frac{\partial f_{s i_1 \cdots i_q \bar{\ell}_1 \cdots \bar{\ell}_{r+1}}}{\partial \bar{z}^s} \\ &\quad + \sum_{\beta, s} (-1)^{q+1+\beta} \delta_{s, \ell_\beta} \frac{\partial f_{I \bar{\ell}_1 \cdots \bar{\ell}_{r+1}}}{\partial \bar{z}^s} \\ &\quad + \sum_{\alpha, \beta, s} (-1)^{q+\alpha+1+\beta} \delta_{i_\alpha, \ell_\beta} \frac{\partial f_{s i_1 \cdots i_q \bar{\ell}_1 \cdots \bar{\ell}_{r+1}}}{\partial \bar{z}^s} \end{aligned}$$

$$= \sum_{\beta} (-1)^{q+1+\beta} \frac{\partial f_{I\bar{l}_1 \cdots \bar{l}_{r+1}}}{\partial \bar{z}^{\ell_{\beta}}} = (\bar{\partial} f)_{I\bar{L}}.$$

This completes the proof of Proposition A.7.

A3. Necessity conditions of Kähler manifolds

Conversely, we show that the conditions (0,1) imply the Kähler condition.

Theorem A.10. *If $\square = \bar{\square}$ as operators on $\mathcal{E}_X^{(1,1)}$ on a Hermitian manifold X , then X is a Kähler manifold.*

Proof. We fix any point p on X . Then by using a suitable complex linear transformation we can take a local holomorphic coordinate system (z^1, \dots, z^n) around p satisfying the following:

$$\begin{cases} z^1(p) = \cdots = z^n(p) = 0, \\ h_{i\bar{j}}(z) = \delta_{ij} + \omega_{ijl} z^l + \bar{\omega}_{j\bar{i}l} \bar{z}^l + O(|z|^2). \end{cases}$$

Then we get

$$\begin{cases} h^{j\bar{k}}(z) &= \delta_{jk} - \omega_{kjl} z^l - \bar{\omega}_{jkl} \bar{z}^l + O(|z|^2), \\ \det(h_{jk}) &= 1 + \omega_{jjl} z^l + \bar{\omega}_{jjl} \bar{z}^l + O(|z|^2), \\ |\det(h_{jk})| &= 1 + \omega_{jjl} z^l + \bar{\omega}_{jjl} \bar{z}^l + O(|z|^2). \end{cases}$$

Let $f = f_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and $g = g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ be C^∞ (1,1)-forms with compact supports. Therefore we have

$$\begin{aligned} (\bar{\partial} f, g) &= (\partial f, \partial g) \\ &= \int \frac{\partial f_{i\bar{j}}}{\partial z^k} \overline{\left(\frac{\partial g_{\alpha\bar{\beta}}}{\partial z^l} \right)} (h^{k\bar{l}} h^{i\bar{\alpha}} - h^{i\bar{l}} h^{k\bar{\alpha}}) h^{\beta\bar{j}} |\det(h_{**})| dz^{\{\text{all}\}} \wedge d\bar{z}^{\{\text{all}\}} \\ &= - \int \frac{\partial}{\partial \bar{z}^l} \left(\frac{\partial f_{i\bar{j}}}{\partial z^k} (h^{k\bar{l}} h^{i\bar{\alpha}} - h^{i\bar{l}} h^{k\bar{\alpha}}) h^{\beta\bar{j}} |\det(h_{**})| \right) |\det(h_{**})|^{-1} \overline{g_{\alpha\bar{\beta}}} dV. \end{aligned}$$

Here, $\{\text{all}\} = \{1, 2, \dots, n\}$, $dV = |\det(h_{**})| dz^{\{\text{all}\}} \wedge d\bar{z}^{\{\text{all}\}}$. Hence we have

$$\begin{aligned} & (\bar{\partial}f)_{i\bar{j}} h^{i\alpha} h^{\beta j} \\ &= -\frac{\partial}{\partial \bar{z}^l} \left(\frac{\partial f_{i\bar{j}}}{\partial z^k} (h^{k\bar{l}} h^{i\bar{\alpha}} - h^{i\bar{l}} h^{k\bar{\alpha}}) h^{\beta \bar{j}} |\det(h_{**})| \right) |\det(h_{**})|^{-1}. \end{aligned}$$

Taking the values at $z = 0$, we obtain the following:

$$\begin{aligned} (\bar{\partial}f)_{\alpha\bar{\beta}} &= -\frac{\partial}{\partial \bar{z}^l} \left(\frac{\partial f_{i\bar{j}}}{\partial z^k} (h^{k\bar{l}} h^{i\bar{\alpha}} - h^{i\bar{l}} h^{k\bar{\alpha}}) h^{\beta \bar{j}} |\det(h_{**})| \right) \\ &= -\frac{\partial^2 f_{\alpha\bar{\beta}}}{\partial z^k \partial \bar{z}^k} + \frac{\partial^2 f_{i\bar{\beta}}}{\partial z^\alpha \partial \bar{z}^i} + \overline{\omega_{kll}} \frac{\partial f_{\alpha\bar{\beta}}}{\partial z^k} + \overline{\omega_{i\alpha k}} \frac{\partial f_{i\bar{\beta}}}{\partial z^k} - \overline{\omega_{ill}} \frac{\partial f_{i\bar{\beta}}}{\partial z^\alpha} - \overline{\omega_{k\alpha l}} \frac{\partial f_{l\bar{\beta}}}{\partial z^k} \\ &\quad + \overline{\omega_{\beta j k}} \frac{\partial f_{\alpha\bar{j}}}{\partial z^k} - \overline{\omega_{\beta j l}} \frac{\partial f_{l\bar{j}}}{\partial z^\alpha} - \overline{\omega_{kkl}} \frac{\partial f_{\alpha\bar{\beta}}}{\partial z^l} + \overline{\omega_{kkl}} \frac{\partial f_{l\bar{\beta}}}{\partial z^\alpha} \\ &= -\frac{\partial^2 f_{\alpha\bar{\beta}}}{\partial z^k \partial \bar{z}^k} + \frac{\partial^2 f_{i\bar{\beta}}}{\partial z^\alpha \partial \bar{z}^i} + (\overline{\omega_{llk}} - \overline{\omega_{kll}}) \left(\frac{\partial f_{k\bar{\beta}}}{\partial z^\alpha} - \frac{\partial f_{\alpha\bar{\beta}}}{\partial z^k} \right) \\ &\quad + (\overline{\omega_{i\alpha k}} - \overline{\omega_{k\alpha i}}) \frac{\partial f_{i\bar{\beta}}}{\partial z^k} + \overline{\omega_{\beta j k}} \left(\frac{\partial f_{\alpha\bar{j}}}{\partial z^k} - \frac{\partial f_{k\bar{j}}}{\partial z^\alpha} \right). \end{aligned}$$

On the other hand, let $f = f_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and $g = g_{\bar{j}} d\bar{z}^j$ be $C^\infty(1, 1)$ - and $C^\infty(0, 1)$ -forms with compact supports, respectively. Then we have

$$\begin{aligned} (\bar{\partial}f, g) &= (f, \partial g) = \int f_{i\bar{k}} h^{i\bar{l}} h^{j\bar{k}} \overline{\left(\frac{\partial g_{\bar{j}}}{\partial z^l} \right)} |\det(h_{**})| dz^{\{\text{all}\}} \wedge d\bar{z}^{\{\text{all}\}} \\ &= -\int |\det(h_{**})|^{-1} \frac{\partial}{\partial \bar{z}^l} \left(f_{i\bar{k}} h^{i\bar{l}} h^{j\bar{k}} |\det(h_{**})| \right) \overline{g_{\bar{j}}} dV. \end{aligned}$$

Therefore,

$$(\bar{\partial}f)_{\bar{\beta}} = -|\det(h_{**})|^{-1} h_{j\bar{\beta}} \frac{\partial}{\partial \bar{z}^l} \left(f_{i\bar{k}} h^{i\bar{l}} h^{j\bar{k}} |\det(h_{**})| \right).$$

Consequently we obtain the following:

$$\begin{aligned} (\partial\bar{\partial}f)_{\alpha\bar{\beta}} &= -\frac{\partial}{\partial z^\alpha} \left(|\det(h_{**})|^{-1} h_{j\bar{\beta}} \frac{\partial}{\partial \bar{z}^l} \left(f_{i\bar{k}} h^{i\bar{l}} h^{j\bar{k}} |\det(h_{**})| \right) \right) \\ &= -\frac{\partial}{\partial z^\alpha} \left(h^{i\bar{l}} \frac{\partial f_{i\bar{\beta}}}{\partial \bar{z}^l} + |\det(h_{**})|^{-1} h_{j\bar{\beta}} f_{i\bar{k}} \frac{\partial}{\partial \bar{z}^l} \left(h^{i\bar{l}} h^{j\bar{k}} |\det(h_{**})| \right) \right). \end{aligned}$$

Taking the values at $z = 0$ again, we get

$$\begin{aligned} (\partial\bar{\partial}f)_{\alpha\bar{\beta}} &= -\frac{\partial^2 f_{i\bar{\beta}}}{\partial z^\alpha \partial \bar{z}^i} + \omega_{li\alpha} \frac{\partial f_{i\bar{\beta}}}{\partial \bar{z}^l} + \overline{\omega_{ill}} \frac{\partial f_{i\bar{\beta}}}{\partial z^\alpha} + \overline{\omega_{\beta ki}} \frac{\partial f_{i\bar{k}}}{\partial z^\alpha} - \overline{\omega_{kki}} \frac{\partial f_{i\bar{\beta}}}{\partial z^\alpha} \\ &\quad + C_{\alpha\beta jk} f_{j\bar{k}}. \end{aligned}$$

Here $C_{\alpha\beta jk}$ are some constants which are independent of f and df . Therefore we have the following equalities at $z = 0$:

$$\begin{aligned} (\square f)_{\alpha\bar{\beta}} &= (\partial\bar{\partial}f)_{\alpha\bar{\beta}} + (\bar{\partial}\partial f)_{\alpha\bar{\beta}} \\ &= -\frac{\partial^2 f_{\alpha\bar{\beta}}}{\partial z^k \partial \bar{z}^k} + (\overline{\omega_{ill}} - \overline{\omega_{lli}}) \frac{\partial f_{\alpha\bar{\beta}}}{\partial z^i} + (\overline{\omega_{j\alpha k}} - \overline{\omega_{k\alpha j}}) \frac{\partial f_{j\bar{\beta}}}{\partial z^k} \\ &\quad + \overline{\omega_{\beta jk}} \frac{\partial f_{\alpha\bar{j}}}{\partial z^k} + \omega_{li\alpha} \frac{\partial f_{i\bar{\beta}}}{\partial \bar{z}^l} + C_{\alpha\beta jk} f_{j\bar{k}}. \end{aligned}$$

Since $\bar{\square}$ is the complex conjugate to \square , we also have

$$\begin{aligned} (\bar{\square} f)_{\alpha\bar{\beta}} &= -\frac{\partial^2 f_{\alpha\bar{\beta}}}{\partial z^k \partial \bar{z}^k} + (\omega_{ill} - \omega_{lli}) \frac{\partial f_{\alpha\bar{\beta}}}{\partial \bar{z}^i} + (\omega_{j\beta k} - \omega_{k\beta j}) \frac{\partial f_{\alpha\bar{j}}}{\partial \bar{z}^k} \\ &\quad + \omega_{\alpha jk} \frac{\partial f_{j\bar{\beta}}}{\partial \bar{z}^k} + \overline{\omega_{li\beta}} \frac{\partial f_{\alpha\bar{i}}}{\partial z^l} + \overline{C_{\beta\alpha kj}} f_{j\bar{k}}. \end{aligned}$$

When the equation

$$\begin{aligned} 0 &= (\square f - \bar{\square} f)_{\alpha\bar{\beta}} \\ &= (\overline{\omega_{ill}} - \overline{\omega_{lli}}) \frac{\partial f_{\alpha\bar{\beta}}}{\partial z^i} + (\overline{\omega_{j\alpha k}} - \overline{\omega_{k\alpha j}}) \frac{\partial f_{j\bar{\beta}}}{\partial z^k} + (\overline{\omega_{\beta jk}} - \overline{\omega_{kj\beta}}) \frac{\partial f_{\alpha\bar{j}}}{\partial z^k} \\ &\quad - (\omega_{ill} - \omega_{lli}) \frac{\partial f_{\alpha\bar{\beta}}}{\partial \bar{z}^i} - (\omega_{j\beta k} - \omega_{k\beta j}) \frac{\partial f_{\alpha\bar{j}}}{\partial \bar{z}^k} - (\omega_{\alpha jk} - \omega_{kj\alpha}) \frac{\partial f_{j\bar{\beta}}}{\partial \bar{z}^k} \\ &\quad + (C_{\alpha\beta jk} - \overline{C_{\beta\alpha kj}}) f_{j\bar{k}} \end{aligned}$$

holds, for any given $\mu, \nu, \kappa \in \{1, \dots, n\}$ we can choose f such that

$$\partial f_{j\bar{k}}(0)/\partial z^i = \delta_{i\mu} \delta_{j\nu} \delta_{k\kappa}, \quad \partial f_{j'\bar{k}'}(0)/\partial \bar{z}^{i'} = 0, \quad f_{j''\bar{k}''}(0) = 0,$$

and we have

$$0 = (\overline{\omega_{\mu ll}} - \overline{\omega_{ll\mu}}) \delta_{\alpha\nu} \delta_{\beta\kappa} + (\overline{\omega_{\nu\alpha\mu}} - \overline{\omega_{\mu\alpha\nu}}) \delta_{\beta\kappa} + (\overline{\omega_{\beta\kappa\mu}} - \overline{\omega_{\mu\kappa\beta}}) \delta_{\alpha\nu}.$$

If we choose $\alpha = \nu$ and $\beta \neq \kappa$, then we see that

$$\overline{\omega_{\beta\kappa\mu}} - \overline{\omega_{\mu\kappa\beta}} = 0$$

holds for any β, κ, μ with $\beta \neq \kappa$. Further if we take $\mu = \kappa \neq \beta$, we have $\overline{\omega_{\beta\kappa\kappa}} - \overline{\omega_{\kappa\kappa\beta}} = 0$ (not the summation in κ) for any $\beta \neq \kappa$. Consequently we get

$$\omega_{ijk} = \omega_{kji} \quad \text{for any } i, j, k.$$

This is equivalent to the Kähler condition at p . That is, $d\Phi(p) = 0$. Since p is an arbitrary point of X , so X is a Kähler manifold. ■

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