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On the determination of wave speed and potential in a hyperbolic equation by two measurements

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On the determination of wave speed and potential in a hyperbolic equation by two measurements

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Abstract. We discuss a problem of finding a speed of sound $c(x)$ and a potential $q(x)$ in a second-order hyperbolic equation from two boundary observations. The coefficients are assumed to be unknown inside a disc in $\mathbb{R}^2$. On a suitable bounded part of the cylindrical surface, we are given Cauchy data for solutions to a hyperbolic equation with zero initial data and sources located on the lines $\{(x,t) \in \mathbb{R}^3 | \, x \cdot \nu = 0, t = 0\}$ for two distinct unit vectors $\nu = \nu^{(k)}$, $k = 1, 2$. We obtain a conditional stability estimate under a priori assumptions on smallness of $c(x) - 1$ and $q(x)$.

§1. Statement of the inverse problem and main results

In the papers [2], [6] - [9], a new method for obtaining conditional stability estimates for problems related to determination of coefficients for linear hyperbolic equations has been proposed. This method uses a single observation for finding one unknown coefficient.

By our method, we can prove the stability in determining coefficients by means of a finite number of measurements where initial data are zero and impulsive inputs are added. As other methodology for inverse problems with a finite number of measurements, we refer to [1], [4], [5] and the references therein. However in those papers, we have to assume some positivity or non-degeneracy of initial values, which is not practical. For our method, we need not such restrictions on initial data, which is very practical. On the other hand, we have to assume that unknown coefficients should be close to fixed reference coefficients which are constant.

An analysis shows that the problem with several unknown coefficients under the derivatives of the first order can also be successfully studied by this method (see [7], [8]). However its application to determination of coefficients under derivatives of different orders meets some

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difficulties. Recently the problems of finding a damping coefficient and a potential from two measurements, and the speed of sound and damping, were considered in papers [3] and [10], respectively. In this paper, by two measurements, we consider the inverse problem where coefficients of the leading term and the lowest term are unknown. The technique of this paper differs from [3] and [10], but keeps some common features with them.

Let $u = u(x, t)$, $x \in \mathbb{R}^2$, satisfy the equation

$$u_{tt} - c^2(\Delta u + qu) = 2\delta(t)\,\delta(x \cdot \nu), \quad (x, t) \in \mathbb{R}^3, \tag{1.1}$$

and the zero initial condition

$$u|_{t<0} = 0. \tag{1.2}$$

Here ν is a unit vector and the symbol $x \cdot \nu$ means the scalar product of the vectors x and ν. The solution to problem (1.1) - (1.2) depends on the parameter ν, i.e., $u = u(x, t, \nu)$.

Assume that supports of the coefficients $q(x)$ and $c(x) - 1$ are located strictly inside the disc $B := \{x \in \mathbb{R}^2 |\, |x - x^0| < r\}$ and B belongs to the half-plane $x \cdot \nu > 0$. Suppose also that $q(x)$ and $c(x) > 0$ are smooth functions in $\mathbb{R}^2$ (see below).

Introduce the function $\tau(x, \nu)$ as the solution to the following problem for the eikonal equation:

$$|\nabla \tau|^2 = c^{-2}(x), \quad \tau|_{x \cdot \nu = 0} = 0. \tag{1.3}$$

Let $G(\nu)$ be the cylindrical domain $G(\nu) := \{(x, t) |\, x \in B, \tau(x, \nu) < t < T + \tau(x, \nu)\}$ where T is a positive number. Denote by $S(\nu)$ the lateral boundary of this domain and by $\Sigma_0(\nu)$ and $\Sigma_T(\nu)$ the lower and upper basements, respectively. That is, $S(\nu) := \{(x, t) |\, x \in \partial B, \tau(x, \nu) \leq t \leq T + \tau(x, \nu)\}$, $\Sigma_0(\nu) := \{(x, t) |\, x \in B, t = \tau(x, \nu)\}$, $\Sigma_T(\nu) := \{(x, t) |\, x \in B, t = T + \tau(x, \nu)\}$, $\partial B := \{x \in \mathbb{R}^2 |\, |x - x^0| = r\}$.

Consider the problem of determination of $q(x)$ and $c(x)$. Let the following information be known. We take distinct unit vectors $\nu^{(1)}$ and $\nu^{(2)}$ such that B belongs to the half plane $x \cdot \nu^{(k)} > 0$ for $k = 1, 2$. Then we are given the traces of the functions $\tau(x, \nu^{(k)})$ on ∂B, and the traces on $S(\nu^{(k)}) := S_k$ of solutions and its normal derivatives to problem (1.1) - (1.2) with $\nu = \nu^{(k)}$, that is,

$$u(x,t,\nu^{(k)}) = f^{(k)}(x,t), \quad \frac{\partial}{\partial n} u(x,t,\nu^{(k)}) = g^{(k)}(x,t), \quad (x,t) \in S_k;$$
$$\tau(x,\nu^{(k)}) = \tau^{(k)}(x), \quad x \in \partial B; \quad k = 1,2. \tag{1.4}$$

The problem is: find $q(x)$ and $c(x)$ from given data, i.e., from $f^{(k)}$, $g^{(k)}$, $\tau^{(k)}$, $k = 1,2$.

For fixed constants $q_0 > 0$ and $d > 0$, let $\Lambda(q_0, d)$ be the set of functions (q, c) satisfying the following two conditions:

1) supp $q(x)$, supp $(c(x) - 1) \subset \Omega \subset B$, $\text{dist}(\partial B, \Omega) \geq d$,

2) $\|q\|_{\mathbf{C}^{17}(\mathbb{R}^n)} \leq q_0$, $\|c - 1\|_{\mathbf{C}^{19}(\mathbb{R}^n)} \leq q_0$.

In particular, we note that $\nu^{(1)}$ and $\nu^{(2)}$ are linearly independent.

We prove here the following stability and uniqueness theorems.

Theorem 1.1. *Let $(q_j, c_j) \in \Lambda(q_0, d)$, and let $\{f_j^{(k)}, g_j^{(k)}, \tau_j^{(k)}\}$ be the data corresponding to the solution to (1.1) - (1.2) with $q = q_j(x)$, $c = c_j(x)$ and $\nu = \nu^{(k)}$, $k, j = 1,2$. Moreover let the condition $4r/T < 1$ be satisfied. Then there exist positive numbers q^* and C depending on T, r, d and $|\nu^{(1)} - \nu^{(2)}|$ such that for all $q_0 \leq q^*$ the following inequality holds:*

$$\begin{aligned} &\|q_1 - q_2\|^2_{\mathbf{L}^2(B)} + \|c_1 - c_2\|^2_{\mathbf{H}^2(B)} \\ &\leq C \sum_{k=1}^{2} \Big(\|\widehat{f}_1^{(k)} - \widehat{f}_2^{(k)}\|^2_{\mathbf{H}^3(\partial B \times \{0\})} + \|(\widehat{f}_1^{(k)} - \widehat{f}_2^{(k)})_t\|^2_{\mathbf{H}^2(\partial B \times (0,T))} \\ &+ \|(\widehat{g}_1^{(k)} - \widehat{g}_2^{(k)})_t\|^2_{\mathbf{H}^1(\partial B \times (0,T))} + \|\tau_1^{(k)} - \tau_2^{(k)}\|^2_{\mathbf{H}^5(\partial B)} \Big), \end{aligned} \tag{1.5}$$

where $\widehat{f}_j^{(k)}(x,t) = f_j^{(k)}(x, t - \tau_j^{(k)}(x))$ and $\widehat{g}_j^{(k)}(x,t) = g_j^{(k)}(x, t - \tau_j^{(k)}(x))$.

Theorem 1.2. *Let the conditions the Theorem 1.1 be fulfilled. Then one can find a number $q^* > 0$ such that if $(q_j, c_j) \in \Lambda(q^*, d)$, $j = 1,2$, and the corresponding data partly coincide, namely,*

$$f_1^{(k)}(x,t) = f_2^{(k)}(x,t), \ (x,t) \in S_k; \quad \tau_1^{(k)}(x) = \tau_2^{(k)}(x), \ x \in \partial B; \ k = 1,2, \tag{1.6}$$

then $q_1(x) = q_2(x)$ and $c_1(x) = c_2(x)$.

Theorem 1.1 is proven in §2. To prove Theorem 1.2 we use the following assertion proven in [10] (see §4). If $u_1(x,t,\nu) = u_2(x,t,\nu)$ on $S(\nu)$ and $\tau_1(x,\nu) = \tau_2(x,\nu)$ on ∂B, then $(\nabla u_1 \cdot \mathbf{n}) = (\nabla u_2 \cdot \mathbf{n})$ on $S(\nu)$, where $\mathbf{n}$ is the outward unit normal to $S(\nu)$. Then Theorem 1.2 is a simple corollary of Theorem 1.1.

In §2 we also use the following lemma, whose proof is similar to Lemma 1.1 in [10] and we omit it here.

Lemma 1.1. *For each fixed $T_0 > 0$ there exists positive number $q^* = q^*(T_0)$ such that for $(q,c) \in \Lambda(q_0,d)$ and $q_0 \le q^*$ the solution to problem (1.1) - (1.2) in the domain $K(T_0,\nu) := \{(x,t)|\, t \le T_0 - \tau(x,\nu)\}$ can be represented in the form*

$$u(x,t,\nu) = \sum_{k=0}^{5} \alpha_k(x,\nu)\,\theta_k(t-\tau(x,\nu)) + u_5(x,t,\nu), \tag{1.7}$$

where $\theta_0(t)$ is the Heaviside function: $\theta_0(t) = 1$ for $t \ge 0$ and $\theta_0(t) = 0$ for $t < 0$, $\theta_k(t) = \frac{t^k\theta_0(t)}{k!}$, the coefficients $\alpha_k(x,\nu)$ are given in the form

$$\alpha_0(x,\nu) = \exp(\varphi(x,\nu)),\ \varphi(x,\nu) = -\frac{1}{2}\int\limits_{\Gamma(x,\nu)} c^2(\xi)\Delta\tau(\xi,\nu)\,ds,$$
$$\alpha_k(x,\nu) = \frac{\alpha_0(x,\nu)}{2}\int\limits_{\Gamma(x,\nu)} \frac{c^2(\xi)(\Delta\alpha_{k-1}(\xi,\nu) + q(\xi)\alpha_{k-1}(\xi,\nu))}{\alpha_0(\xi,\nu)}\,ds, \tag{1.8}$$
$$k = 1,\dots,m,$$

where $\Gamma(x,\nu)$ is the geodesic line joining the line $\{\xi \in \mathbb{R}^2 | \xi\cdot\nu = 0\}$ and x with respect to ds, and ds is the element of the Riemannian length: $ds = c^{-1}(x)(\sum_{k=1}^2 dx_k^2)^{1/2}$. Then $\tau(x,\nu) \in \mathbf{C}^{19}(\Omega(T_0,\nu))$, $\alpha_k(x,\nu) \in \mathbf{C}^{17-2k}(\Omega(T_0,\nu))$ for $\Omega(T_0,\nu) := \{x \in \mathbb{R}^2 |\, \tau(x,\nu) \le T_0/2\}$, and the function $u_5(x,t,\nu)$ vanishes for $t \le \tau(x,\nu)$ and belongs to $\mathbf{H}^6(K(T_0,\nu))$ for fixed ν. Moreover there exists a positive number C depending on T, r and q_0 such that C does not increase as q_0 decreases and that the following inequalities hold

$$\|u-1\|_{\mathbf{H}^6(G(\nu))} \le Cq_0, \quad \|\tau(x,\nu) - x\cdot\nu\|_{\mathbf{C}^{18}(B)} \le Cq_0. \tag{1.9}$$

Corollary. *If* $(q, c) \in \Lambda(q_0, d)$ *and* q_0 *is sufficiently small, then the function* $u(x, t, \nu)$ *is continuous on the closure of domain* $G(\nu)$ *together with all derivatives up to the fourth-order.*

§2. Proof of Theorem 1.1

Introduce the function $\widehat{u}(x, t, \nu) := u(x, t + \tau(x, \nu), \nu)$. Then, by (1.1) and (1.7), the function $\widehat{u}(x, t, \nu)$ for $(x, t) \in B \times (0, T)$, satisfies

$$
\begin{aligned}
2\nabla \widehat{u}_t \cdot \nabla \tau - \Delta \widehat{u} - q\widehat{u} + (\Delta\tau)\, \widehat{u}_t = 0, \quad (x, t) \in B \times (0, T);\\
\widehat{u}(x, +0, \nu) = \alpha_0(x, \nu), \quad \widehat{u}_t(x, +0, \nu) = \alpha_1(x, \nu),
\end{aligned}
\tag{2.1}
$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$.

Substituting (1.7) into (1.1) and equating the terms of $\delta(t - \tau(x, \nu))$, $\theta_0(t - \tau(x, \nu))$, we see that the functions $\varphi(x, \nu) \equiv \ln \alpha_0(x, \nu)$ and $\alpha_1(x, \nu)$ satisfy the first-order differential equations:

$$
\begin{aligned}
&2\nabla\varphi \cdot \nabla\tau + \Delta\tau = 0,\\
&2\nabla\alpha_1 \cdot \nabla\tau + \alpha_1 \Delta\tau - \Delta\alpha_0 - q\alpha_0 = 0.
\end{aligned}
\tag{2.2}
$$

The latter of these equations and equation (2.1) can be rewritten respectively in the forms

$$
2(\nabla\alpha_1 \cdot \nabla\tau - \alpha_1 \nabla\varphi \cdot \nabla\tau) - \alpha_0(\Delta\varphi + |\nabla\varphi|^2 + q) = 0 \tag{2.3}
$$

and

$$
\begin{aligned}
2\nabla \widehat{u}_t \cdot \nabla\tau - \Delta\widehat{u} - 2(\nabla\varphi \cdot \nabla\tau)\widehat{u}_t - q\widehat{u} = 0, \quad (x, t) \in B \times (0, T);\\
\widehat{u}(x, +0, \nu) = \alpha_0(x, \nu), \quad \widehat{u}_t(x, +0, \nu) = \alpha_1(x, \nu).
\end{aligned}
\tag{2.4}
$$

Introduce $v(x, t, \nu) = \ln \widehat{u}(x, t, \nu)$ and assume that q_0 is small enough in order that the function $\widehat{u}(x, t, \nu)$ is positive in $B \times (0, T)$. The function $v(x, t, \nu)$ satisfies the relations

$$
\begin{aligned}
&2\nabla v_t \cdot \nabla\tau - \Delta v - |\nabla v|^2\\
&\quad + 2(\nabla v \cdot \nabla\tau - \nabla\varphi \cdot \nabla\tau)v_t - q = 0, \quad (x, t) \in B \times (0, T);\\
&\quad v(x, +0, \nu) = \varphi(x, \nu), \quad v_t(x, +0, \nu) = \beta(x, \nu),
\end{aligned}
\tag{2.5}
$$

where $\beta(x,\nu) = \alpha_1(x,\nu)/\alpha_0(x,\nu)$ solves the equation

$$2\nabla\beta\cdot\nabla\tau - \Delta\varphi - |\nabla\varphi|^2 - q = 0. \tag{2.6}$$

Let $(q_j, c_j) \in \Lambda(q_0, d)$ for $j = 1, 2$. Denote the functions u, $\widehat{u}$, v, φ, α_0, β, τ corresponding to the coefficients (q_j, c_j) by u_j, $\widehat{u}_j$, v_j, φ_j, α_{0j}, β_j, τ_j and introduce the differences

$$\widetilde{u} = \widehat{u}_1 - \widehat{u}_2,\ \widetilde{v} = v_1 - v_2,\ \widetilde{\varphi} = \varphi_1 - \varphi_2,\ \widetilde{\alpha}_0 = \alpha_{01} - \alpha_{02},$$
$$\widetilde{\beta} = \beta_1 - \beta_2,\ \widetilde{\tau} = \tau_1 - \tau_2,\ \widetilde{c} = c_1 - c_2,\ \widetilde{q} = q_1 - q_2.$$

Then we can obtain the relations

$$\begin{gathered} 2\nabla\widetilde{v}_t\cdot\nabla\tau_1 - \Delta\widetilde{v} + a_1\cdot\nabla\widetilde{v} + a_2\,\widetilde{v}_t + a_3\cdot\nabla\widetilde{\tau} \\ +a_4\cdot\nabla\widetilde{\varphi} - \widetilde{q} = 0, \quad (x,t)\in B\times(0,T); \\ \widetilde{v}(x,+0,\nu) = \widetilde{\varphi}(x,\nu), \quad \widetilde{v}_t(x,+0,\nu) = \widetilde{\beta}(x,\nu), \end{gathered} \tag{2.7}$$

where $a_1 = -\nabla(v_1+v_2)+2(v_2)_t\nabla\tau_2$, $a_2 = 2(\nabla v_1\cdot\nabla\tau_1 - \nabla\varphi_1\cdot\nabla\tau_1)$, $a_3 = 2\nabla(v_2)_t + 2(v_2)_t(\nabla v_1 - \nabla\varphi_1)$, $a_4 = -2(v_2)_t\nabla\tau_2$. From equations (2.2) and (2.6), it follows that the functions $\widetilde{\alpha}_0(x,\nu)$, $\widetilde{\beta}(x,\nu)$, $\widetilde{\varphi}(x,\nu)$ satisfy the relations

$$\begin{gathered} \widetilde{\alpha}_0 = b_1\widetilde{\varphi}, \quad \nabla\widetilde{\varphi}\cdot b_2 + \nabla\widetilde{\tau}\cdot b_3 + \Delta\widetilde{\tau} = 0, \\ \Delta\widetilde{\varphi} + \nabla\widetilde{\varphi}\cdot h_1 + \nabla\widetilde{\beta}\cdot h_2 + \nabla\widetilde{\tau}\cdot h_3 + \widetilde{q} = 0, \end{gathered} \tag{2.8}$$

where

$$b_1 = \int_0^1 \exp[\varphi_2(1-\eta) + \varphi_1\eta]\,d\eta, \quad b_2 = \nabla(\tau_1+\tau_2), \quad b_3 = \nabla(\varphi_1+\varphi_2),$$
$$h_1 = \nabla(\varphi_1+\varphi_2), \quad h_2 = -\nabla(\tau_1+\tau_2), \quad h_3 = -\nabla(\beta_1+\beta_2).$$

Introduce the function $w(x,t,\nu) := \widetilde{v}_t(x,t,\nu)$. Then

$$\begin{gathered} 2\nabla w_t\cdot\nabla\tau_1 - \Delta w + a_1\cdot\nabla w + a_2\,w_t + (a_2)_t w + (a_1)_t\cdot\nabla\widetilde{v} \\ +(a_3)_t\cdot\nabla\widetilde{\tau} + (a_4)_t\cdot\nabla\widetilde{\varphi} = 0, \quad (x,t)\in B\times(0,T); \\ w(x,+0,\nu) = \widetilde{\beta}(x,\nu). \end{gathered} \tag{2.9}$$

Note that the function $\widetilde{v}$ can be represented in the form

$$\widetilde{v}(x,t,\nu) \;=\; \widetilde{\varphi}(x,\nu) + \int\limits_0^t w(x,\eta,\nu)\,d\eta, \quad (x,t) \in B \times (0,T). \tag{2.10}$$

¿From Lemma 1.1 and the embedding theorems, by the definition we have

$$\begin{aligned} \max_{1\le k\le 4} \|a_k\|_{\mathbf{C}^2(B\times(0,T))} \le Cq_0, \\ \max_{k=1,2} \|b_k\|_{\mathbf{C^1}(B\times(0,T))} \le C, \quad \|b_3\|_{\mathbf{C^1}(B\times(0,T))} \le Cq_0, \\ \|h_2\|_{\mathbf{C}(B\times(0,T))} \le C, \quad \max_{k=1,3} \|h_k\|_{\mathbf{C}(B\times(0,T))} \le Cq_0 \end{aligned} \tag{2.11}$$

Here and henceforth $C > 0$ denotes a generic constant which depends on T, r, q_0 and does not increase as q_0 decreases. Therefore relations (2.8) – (2.11) lead to the following inequalities

$$\begin{aligned} &\|\widetilde{q}\|^2_{\mathbf{L}^2(B)} \le C\left(\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)} + \|\widetilde{\tau}\|^2_{\mathbf{H}^1(B)} + \|\widetilde{\beta}\|^2_{\mathbf{H}^1(B)}\right), \\ &\|\Delta\widetilde{\tau}\|^2_{\mathbf{H}^1(B)} \le C\left(\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)} + q_0^2\|\widetilde{\tau}\|^2_{\mathbf{H}^2(B)}\right), \\ &\|2\nabla w_t\cdot\nabla\tau_1 - \Delta w\|^2_{\mathbf{H}^1(B\times(0,T))} \le Cq_0^2\left(\|w\|^2_{\mathbf{H}^2(B\times(0,T))} + \|\widetilde{\tau}\|^2_{\mathbf{H}^2(B)} + \|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)}\right). \end{aligned} \tag{2.12}$$

We will use the obvious inequality:

$$\|\widetilde{\tau}\,\|^2_{\mathbf{H}^3(B)} \le C\left(\|\Delta\widetilde{\tau}\,\|^2_{\mathbf{H}^1(B)} + \sum_{|\gamma|\le 3}\|D^\gamma\widetilde{\tau}\,\|^2_{\mathbf{L}^2(\partial B)}\right), \tag{2.13}$$

where

$$D^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1}\partial x_2^{\gamma_n}}, \quad \gamma = (\gamma_1,\gamma_2), \quad |\gamma| = \gamma_1 + \gamma_2.$$

Since $\operatorname{supp}(c(x)-1) \subset \Omega \subset B$ and $\operatorname{dist}(\partial B,\Omega) \ge d$, the function $\widetilde{\tau}(x,\nu)$ vanishes together with all its derivatives on ∂B anywhere except the set $\partial B_+(\nu) := \{x \in \partial B \,|\, \nu\cdot(x-x^0) > \sqrt{r^2-(r-d)^2}\}$. Moreover, since outside of B, the function $\widetilde{\tau}$ satisfies the equation $\nabla\widetilde{\tau}\cdot\nabla(\tau_1+\tau_2) = 0$, all its derivatives of $\widetilde{\tau}$ on $\partial B_+(\nu)$ can be expressed via the derivatives along $\partial B_+(\nu)$. Therefore we have

$$\sum_{|\gamma|\le 3}\|D^\gamma\widetilde{\tau}\,\|^2_{\mathbf{L}^2(\partial B)} \le C\|\widetilde{\tau}\,\|^2_{\mathbf{H}^3(\partial B)}. \tag{2.14}$$

Using the second inequality in (2.12), from (2.13) and (2.14) we find

$$\|\widetilde{\tau}\|^2_{\mathbf{H}^3(B)} \le C\left(\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)} + q_0^2\|\widetilde{\tau}\|^2_{\mathbf{H}^2(B)} + \|\widetilde{\tau}\|^2_{\mathbf{H}^3(\partial B)}\right). \tag{2.15}$$

Consequently, for small q_0, we obtain the inequality

$$\|\widetilde{\tau}\|^2_{\mathbf{H}^3(B)} \le C\left(\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)} + \|\widetilde{\tau}\|^2_{\mathbf{H}^3(\partial B)}\right). \tag{2.16}$$

The following lemma is one key, which can be proved by the multiplier method similarly to Lemma 4.3.6 from [8] (see also [6]).

Lemma 2.1. *Let $c \in \Lambda(q_0, d)$, $4r/T < 1$ and $z(x,t) \in \mathbf{H}^2(B \times (0,T))$. Then for sufficiently small q_0, there exists a positive constant C such that the following inequality holds:*

$$\begin{aligned}&\|z\|^2_{\mathbf{H}^1(B\times(0,T))} + \|z\|^2_{\mathbf{H}^1(B\times\{0\})}\\ &\le C\Big(\|2\nabla z_t\cdot\nabla\tau - \Delta z\|^2_{\mathbf{L}^2(B\times(0,T))} + \|z\|^2_{\mathbf{H}^1(\partial B\times(0,T))} + \|\nabla z\cdot\mathbf{n}\|^2_{\mathbf{L}^2(\partial B\times(0,T))}\Big).\end{aligned} \tag{2.17}$$

Applying (2.17) with $\tau = \tau_1$ to the function $w(x,t,\nu)$ and its first derivatives and using the third inequality in (2.12), we obtain

$$\begin{aligned}&\|w\|^2_{\mathbf{H}^2(B\times(0,T))} + \|w\|^2_{\mathbf{H}^2(B\times\{0\})}\\ &\le C[q_0^2(\|w\|^2_{\mathbf{H}^2(B\times(0,T))} + \|\widetilde{\tau}\|^2_{\mathbf{H}^2(B)} + \|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)}) + \varepsilon^2(\nu)],\end{aligned} \tag{2.18}$$

where

$$\varepsilon^2(\nu) = \|(\widehat{f}_1 - \widehat{f}_2)_t\|^2_{\mathbf{H}^2(\partial B\times(0,T))} + \|(\widehat{g}_1 - \widehat{g}_2)_t\|^2_{\mathbf{H}^1(\partial B\times(0,T))}. \tag{2.19}$$

and $\widehat{f}_j(x,t) = f_j(x, t-\tau_j(x,\nu))$, $\widehat{g}_j(x,t) = g_j(x,t-\tau_j(x,\nu))$, $j = 1,2$.

From relation (2.18) for sufficiently small q_0, we derive the inequality

$$\|\widetilde{\beta}\|^2_{\mathbf{H}^2(B)} = \|w\|^2_{\mathbf{H}^2(B\times\{0\})} \le C[q_0^2(\|\widetilde{\tau}\|^2_{\mathbf{H}^2(B)} + \|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)}) + \varepsilon^2(\nu)]. \tag{2.20}$$

Then from the first inequality in (2.12), we see that

$$\|\widetilde{q}\|^2_{\mathbf{L}^2(B)} \le C\left(\|\widetilde{\varphi}\|^2_{\mathbf{H}^2(B)} + \|\widetilde{\tau}\|^2_{\mathbf{H}^2(B)} + \varepsilon^2(\nu)\right). \tag{2.21}$$

Consider inequalities (2.16), (2.20), (2.21), the second and third relations in (2.8) for $\nu = \nu^{(k)}$, $k = 1, 2$. We set $\widetilde{\alpha}_0(x, \nu^{(k)}) = \widetilde{\alpha}_{0k}(x)$, $\widetilde{\varphi}(x, \nu^{(k)}) = \widetilde{\varphi}_k(x)$, $\widetilde{\beta}(x, \nu^{(k)}) = \widetilde{\beta}_k(x)$, $\tau(x, \nu^{(k)}) = \tau_k(x)$, $b_j(x, \nu^{(k)}) = b_{jk}(x)$, $j = 1, 3$, $b_2(x, \nu^{(k)}) = \rho_k(x)$, $\varepsilon^2(\nu^{(k)}) = \varepsilon_k^2$.

Inequalities (2.16), (2.20) and (2.21) lead to the relations:

$$\|\widetilde{\tau}_k\|^2_{\mathbf{H}^3(B)} \le C \left(\|\widetilde{\varphi}_k\|^2_{\mathbf{H}^2(B)} + \|\widetilde{\tau}_k\|^2_{\mathbf{H}^3(\partial B)} \right), \quad k = 1, 2, \tag{2.22}$$

$$\|\widetilde{\beta}_k\|^2_{\mathbf{H}^2(B)} \le C \Big(q_0^2 (\|\widetilde{\tau}_k\|^2_{\mathbf{H}^2(B)} + \|\widetilde{\varphi}_k\|^2_{\mathbf{H}^2(B)}) + \varepsilon_k^2 \Big), \quad k = 1, 2, \tag{2.23}$$

$$\|\widetilde{q}\|^2_{\mathbf{L}^2(B)} \le C \left(\|\widetilde{\varphi}_k\|^2_{\mathbf{H}^2(B)} + \|\widetilde{\tau}_k\|^2_{\mathbf{H}^2(B)} + \varepsilon_k^2 \right), \quad k = 1, 2. \tag{2.24}$$

By (2.8), we have

$$\nabla \widetilde{\varphi}_k \cdot \rho_k + \nabla \widetilde{\tau}_k \cdot b_{3k} + \Delta \widetilde{\tau}_k = 0, \quad k = 1, 2, \tag{2.25}$$

$$\Delta \widetilde{\varphi}_k + \nabla \widetilde{\varphi}_k \cdot h_{1k} + \nabla \widetilde{\beta}_k \cdot h_{2k} + \nabla \widetilde{\tau}_k \cdot h_{3k} + \widetilde{q} = 0, \quad k = 1, 2. \tag{2.26}$$

Moreover the eikonal equation implies

$$\nabla \widetilde{\tau}_k \cdot \rho_k + \widetilde{c}\,(c_1 + c_2) c_1^{-2} c_2^{-2} = 0, \quad k = 1, 2. \tag{2.27}$$

Setting $k = 1, 2$ in (2.26) and subtracting, we find that

$$\Delta \widehat{\varphi} = \sum_{k=1}^{2} (-1)^k (\nabla \widetilde{\varphi}_k \cdot h_{1k} + \nabla \widetilde{\beta}_k \cdot h_{2k} + \nabla \widetilde{\tau}_k \cdot h_{3k}), \tag{2.28}$$

where $\widehat{\varphi} = \widetilde{\varphi}_1 - \widetilde{\varphi}_2$. Using inequalities (2.22) and (2.23), we obtain

$$\|\Delta \widehat{\varphi}\|^2_{\mathbf{H}^1(B)} \le C \sum_{k=1}^{2} \left(q_0^2 \|\widetilde{\varphi}_k\|^2_{\mathbf{H}^2(B)} + \widetilde{\varepsilon}_k^2 + \|\widetilde{\tau}_k\|^2_{\mathbf{H}^3(\partial B)} \right). \tag{2.29}$$

Similarly to $\widetilde{\tau}(x, \nu)$, we can prove

$$\|\widehat{\varphi}\|^2_{\mathbf{H}^3(B)} \le C \left(\|\Delta \widehat{\varphi}\|^2_{\mathbf{H}^1(B)} + \sum_{|\gamma| \le 3} \|D^\gamma \widehat{\varphi}\|^2_{\mathbf{L}^2(\partial B)} \right). \tag{2.30}$$

The function $\widehat{\varphi} = \widetilde{\varphi}_1 - \widetilde{\varphi}_2$ and each function $\widetilde{\varphi}_k(x)$ vanish together with its derivatives on ∂B anywhere except the set $\partial B_+(\nu)$. Since outside of B, the function $\widetilde{\varphi}_k$ satisfies the equation

$\nabla\widetilde{\varphi}_k\cdot\rho_k+\nabla\widetilde{\tau}_k\cdot b_{3k}+\Delta\widetilde{\tau}_k=0$ and $\widetilde{\tau}_k$ satisfies the relation $\nabla\widetilde{\tau}_k\cdot\rho_k=0$, all their derivatives on $\partial B_+(\nu)$ can be expressed via the derivatives along $\partial B_+(\nu)$. Therefore we have

$$\begin{aligned}&\sum_{|\gamma|\le 5}\|D^\gamma\widetilde{\tau}_k\|^2_{\mathbf{L}^2(\partial B)}\le C\|\widetilde{\tau}_k\|^2_{\mathbf{H}^5(\partial B)},\quad k=1,2\\ &\sum_{|\gamma|\le 3}\|D^\gamma\widehat{\varphi}\|^2_{\mathbf{L}^2(\partial B)}\le C\sum_{k=1}^{2}\left(\|\widetilde{\varphi}_k\|^2_{\mathbf{H}^3(\partial B)}+\|\widetilde{\tau}_k\|^2_{\mathbf{H}^5(\partial B)}\right).\end{aligned}\tag{2.31}$$

Here $\widetilde{\varphi}_k=\widetilde{\alpha}_{0k}/b_{1k}$,

$$\widetilde{\alpha}_{0k}=\widetilde{v}_k(x,+0)=\ln\widehat{u}_1(x,+0,\nu^{(k)})-\ln\widehat{u}_2(x,+0,\nu^{(k)})=\widetilde{u}(x,+0,\nu^{(k)})\,B_k(x)$$

and the function $B_k(x)$ is defined by the formula

$$B_k(x)=\int_0^1\frac{d\eta}{\eta\,\widehat{u}_1(x,+0,\nu^{(k)})+(1-\eta)\widehat{u}_2(x,+0,\nu^{(k)})}$$

and bounded in B together with all the derivatives up to the third-order. On the other hand, $\widetilde{u}(x,+0,\nu^{(k)})=\widehat{f}_1^{(k)}(x,+0)-\widehat{f}_2^{(k)}(x,+0)$ on ∂B. Therefore $\|\widetilde{\varphi}_k\|^2_{\mathbf{H}^3(\partial B)}\le C\|\widehat{f}_1^{(k)}-\widehat{f}_2^{(k)}\|^2_{\mathbf{H}^3(\partial B\times\{0\})}$. Taking into account this estimate and (2.31), we find that

$$\sum_{|\gamma|\le 3}\|D^\gamma\widehat{\varphi}\|^2_{\mathbf{L}^2(\partial B)}\le C\widehat{\varepsilon}^2,\tag{2.32}$$

where

$$\widehat{\varepsilon}^2=\sum_{k=1}^{2}\left(\|\widehat{f}_1^{(k)}-\widehat{f}_2^{(k)}\|^2_{\mathbf{H}^3(\partial B\times\{0\})}+\|\widetilde{\tau}_k\|^2_{\mathbf{H}^5(\partial B)}\right).$$

Taking into account inequalities (2.29), (2.30) and (2.32), we find

$$\|\widehat{\varphi}\|^2_{\mathbf{H}^3(B)}\le C\left(q_0^2\sum_{k=1}^{2}\|\widetilde{\varphi}_k\|^2_{\mathbf{H}^2(B)}+\widetilde{\varepsilon}^2\right),\tag{2.33}$$

where $\widetilde{\varepsilon}^2=\varepsilon^2+\widehat{\varepsilon}^2$. Therefore, for sufficiently small q_0, noting that $\widetilde{\varphi}_2=\widetilde{\varphi}_1-\widehat{\varphi}$, we can absorb $q_0^2\|\widehat{\varphi}\|^2_{H^2(B)}$ into the left hand side, so that we have

$$\|\widehat{\varphi}\|^2_{\mathbf{H}^3(B)}\le C\left(q_0^2\|\widetilde{\varphi}_1\|^2_{\mathbf{H}^2(B)}+\widetilde{\varepsilon}^2\right).\tag{2.34}$$

Then from (2.22), (2.24) and (2.27), it follows that

$$\max_{k=1,2}\|\widetilde{\tau}_k\|^2_{\mathbf{H}^3(B)}\le C\left(\|\widetilde{\varphi}_1\|^2_{\mathbf{H}^2(B)}+\widetilde{\varepsilon}^2\right),\tag{2.35}$$

$$\|\widetilde{q}\|^2_{\mathbf{L}^2(B)}\le C\left(\|\widetilde{\varphi}_1\|^2_{\mathbf{H}^2(B)}+\widetilde{\varepsilon}^2\right),\quad \|\widetilde{c}\|^2_{\mathbf{H}^2(B)}\le C\left(\|\widetilde{\varphi}_1\|^2_{\mathbf{H}^2(B)}+\widetilde{\varepsilon}^2\right).\tag{2.36}$$

Hence, for completing the proof, it is sufficient to estimate $\|\widetilde{\varphi}_1\|^2_{\mathbf{H}^2(B)}$ through the data of the inverse problem. For this, use relations (2.25) and (2.27) again. By equations (2.27) we have

$$\nabla\widetilde{\tau}_1 \cdot \rho_1 - \nabla\widetilde{\tau}_2 \cdot \rho_2 = 0. \tag{2.37}$$

We can reqrite equations (2.25) in the form

$$\begin{aligned} \nabla\widetilde{\varphi}_1 \cdot \rho_1 + \nabla\widetilde{\tau}_1 \cdot b_{31} + \Delta\widetilde{\tau}_1 = 0, \\ \nabla\widetilde{\varphi}_1 \cdot \rho_2 - \nabla\widehat{\varphi} \cdot \rho_2 + \nabla\widetilde{\tau}_2 \cdot b_{32} + \Delta\widetilde{\tau}_2 = 0. \end{aligned} \tag{2.38}$$

Applying the operator $\rho_1 \cdot \nabla$ to the first equation in (2.38) and the operator $\rho_2 \cdot \nabla$ to the second one and subtracting, we find the equation for $\widetilde{\varphi}_1$ in the form

$$L\widetilde{\varphi}_1 := \rho_1 \cdot \nabla(\nabla\widetilde{\varphi}_1 \cdot \rho_1) - \rho_2 \cdot \nabla(\nabla\widetilde{\varphi}_1 \cdot \rho_2) = h(x), \tag{2.39}$$

where

$$h(x) = \rho_2 \cdot \nabla(-\nabla\widehat{\varphi} \cdot \rho_2 + \nabla\widetilde{\tau}_2 \cdot b_{32} + \Delta\widetilde{\tau}_2) - \rho_1 \cdot \nabla(\nabla\widetilde{\tau}_1 \cdot b_{31} + \Delta\widetilde{\tau}_1). \tag{2.40}$$

Recall that $\rho_k := b_{2k} = \nabla(\tau_1 + \tau_2)(x, \nu^{(k)})$. By Lemma 1.1, we obtain

$$\|\nabla\tau(x,\nu) - \nu\|_{\mathbf{C}^{17}(B)} \le Cq_0.$$

One can easily prove a similar estimate for the function ρ_k, namely,

$$\|\rho_k - 2\nu^{(k)}\|_{\mathbf{C}^{17}(B)} \le Cq_0, \quad k = 1, 2. \tag{2.41}$$

Since $\nu^{(1)}$ and $\nu^{(2)}$ are linearly independent, it follows that ρ_1 and ρ_2 are also linearly independent in B if $q_0 > 0$ is small. Using relations (2.37) and (2.41), one can prove that

$$\|h\|^2_{\mathbf{H}^1(B)} \le C\left(\|\widehat{\varphi}\|^2_{\mathbf{H}^3(B)} + q_0^2 \max_{k=1,2} \|\widetilde{\tau}_k\|^2_{\mathbf{H}^3(B)}\right), \tag{2.42}$$

$$\left\| L\widetilde{\varphi}_1 - \left(|\rho_1|^2 \frac{\partial^2\widetilde{\varphi}_1}{\partial\rho_1^2} - |\rho_2|^2 \frac{\partial^2\widetilde{\varphi}_1}{\partial\rho_2^2}\right)\right\|_{\mathbf{H}^1(B)} \le Cq_0\|\widetilde{\varphi}_1\|_{\mathbf{H}^2(B)}. \tag{2.43}$$

Here $\frac{\partial}{\partial\rho_k}$, $k = 1, 2$, denotes the derivative along the direction ρ_k. The linear independence implies that the principal part of L is a second-order hyperbolic operator in $\mathbb{R}^2$. Therefore

$$\|\widetilde{\varphi}_1\|^2_{\mathbf{H}^2(B)} \le C\left(\|L\widetilde{\varphi}_1\|^2_{\mathbf{H}^1(B)} + \|\widetilde{\varphi}_1\|^2_{\mathbf{H}^2(\partial B)}\right). \tag{2.44}$$

Taking into account estimates (2.42) and (2.43), we obtain

$$\|\widetilde{\varphi}_1\|^2_{\mathbf{H}^2(B)} \leq C\Big(\|\widehat{\varphi}\|^2_{\mathbf{H}^3(\partial B)} + \|\widetilde{\varphi}_1\|^2_{\mathbf{H}^2(\partial B)} + q_0^2\big(\|\widetilde{\varphi}_1\|^2_{\mathbf{H}^2(B)} + \max_{k=1,2}\|\widetilde{\tau}_k\|^2_{\mathbf{H}^3(B)}\big)\Big). \tag{2.45}$$

Using relations (2.32), (2.34), (2.35) and the smallness of q_0, we obtain the final estimate

$$\|\widetilde{\varphi}_1\|^2_{\mathbf{H}^2(B)} \leq C\widetilde{\varepsilon}^2. \tag{2.46}$$

Then relations (2.36) lead to estimate (1.5). □

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