UTMS 2003–2 January 17, 2003

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CARLEMAN ESTIMATES FOR THE NON-STATIONARY LAMÉ SYSTEM AND THE APPLICATION TO AN INVERSE PROBLEM

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Abstract. In this paper, we establish Carleman estimates for the two dimensional isotropic non-stationary Lamé system with the zero Dirichlet boundary conditions. Using this estimate, we prove the uniqueness and the stability in determining spatially varying density and two Lamé coefficients by a single measurement of solution over $(0,T)\times\omega$, where $T > 0$ is a sufficiently large time interval and a subdomain ω satisfies a non-trapping condition.

§1. Introduction.

This paper is concerned with Carleman estimates for the two dimensional non-stationary isotropic Lamé system with the zero Dirichlet boundary conditions and an application to an inverse problem of determining spatially varying density and the Lamé coefficients by a single interior measurement of the solution. The Carleman estimate is a weighted L^2-estimate of solution to a partial differential equation and it has been fundamental for proving the uniqueness in a Cauchy problem for the partial differential equation or the unique continuation.

1991 *Mathematics Subject Classification.* 35B60, 35R25, 35R30, 73C02.
Key words and phrases. Carleman estimate, Lamé system, inverse problem.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

More precisely, we consider the two dimensional isotropic non-stationary Lamé system:

$$(P\mathbf{u})(x_0,x') \equiv \rho(x')\partial_{x_0}^2\mathbf{u}(x_0,x') - (L_{\lambda,\mu}\mathbf{u})(x_0,x') = \mathbf{f}(x_0,x'),$$
$$x \equiv (x_0,x') \in Q \equiv (0,T)\times\Omega, \tag{1.1}$$

where

$$(L_{\lambda,\mu}\mathbf{v})(x') \equiv \mu(x')\Delta\mathbf{v}(x') + (\mu(x')+\lambda(x'))\nabla_{x'}\mathrm{div}\,\mathbf{v}(x')$$
$$+(\mathrm{div}\,\mathbf{v}(x'))\nabla_{x'}\lambda(x') + (\nabla_{x'}\mathbf{v} + (\nabla_{x'}\mathbf{v})^T)\nabla_{x'}\mu(x'), \qquad x'\in\Omega. \tag{1.2}$$

Throughout this paper, $\Omega\subset\mathbb{R}^2$ is a bounded domain whose boundary $\partial\Omega$ is of class C^3, x_0 and $x'=(x_1,x_2)$ denote the time variable and the spatial variable respectively, and $\mathbf{u}=(u_1,u_2)^T$ where $\cdot^T$ denotes the transpose of matrices, E_k is the unit matrix of the size $k\times k$,

$$\partial_{x_j}\varphi = \varphi_{x_j} = \frac{\partial\varphi}{\partial x_j}, \quad j=0,1,2.$$

We set $\nabla_{x'}\mathbf{v} = (\partial_{x_k}v_j)_{1\le j,k\le 2}$ for a vector function $\mathbf{v}=(v_1,v_2)^T$ and $\nabla_{x'}\phi = (\partial_{x_1}\phi,\partial_{x_2}\phi)^T$ for a scalar function ϕ. Henceforth ∇ means $\nabla_x=(\partial_{x_0},\partial_{x_1},\partial_{x_2})$ if we do not specify.

Moreover the coefficients ρ, λ, μ satisfy

$$\rho,\lambda,\mu\in C^2(\overline{\Omega}), \quad \rho(x')>0,\ \mu(x')>0,\ \lambda(x')+\mu(x')>0 \quad \text{for } x'\in\overline{\Omega}. \tag{1.3}$$

The Carleman estimate is an essential technique not only for the unique continuation, but also for solving the exact controllability and stabilizability (e.g.,Bellassoued [B1]-[B3], Imanuvilov [Im1], Kazemi and Klibanov [KK], Tataru [Ta], and Lasiecka and Triggiani [LT] as related books) and the inverse problems (e.g., Bukhgeim [Bu], Bukhgeim and Klibanov [BuK], Klibanov [Kl]). Thus the first main purpose of this paper is to establish the Carleman estimates for system (1.1), (1.2).

Since the pioneering work [Ca] by Carleman, the theory of inequalities of Carleman's type has been rapidly developed and now many general results are available for a single partial differential equation (see [E1], [Hö], [Is2], [Is3], [Ta]), while for strongly coupled systems of partial differential equations, the situation is more complicated and much less understood. To our best knowledge, a most general result for systems of partial differential equations is Calderon's uniqueness theorem (see e.g., [E1], [Zui]). The technique developed by Calderon, reduces the system of partial differential equations to a system of pseudo-differential operators of the first order:

$$\frac{d\mathbf{u}}{dx_0} = M(x_0, x', D_{x'})\mathbf{u} + \mathbf{f},$$

where $M(x_0, x', D_{x'})$ is a matrix pseudo-differential operator. Then by some change of variables $\mathbf{u} = Q(x_0, x', D_{x'})\widetilde{\mathbf{u}}$, this matrix pseudo-differential operator M is reduced to $Q^{-1}MQ$ such that $Q^{-1}MQ$ consists of blocks of a small size located on the main diagonal and that in each block the principal symbols of all the operators located below the main diagonal are zero. In order to construct the matrix Q, the eigenvalues and eigenvectors of the matrix $M(x_0, x', \xi)$ should be smooth functions of the variables x_0, x' and $\xi \in \mathbb{R}^2$ and each eigenvalue should not change the multiplicity. This condition is restrictive, especially in the case where we are looking for a Carleman estimate near boundary, and therefore the choice for a variable x_0 is limited. For example, with the time variable x_0, the non-stationary Lamé system does not satisfy this condition, in general. On the other hand, for the stationary Lamé system, this method works well (see [DR]).

As long as the non-stationary Lamé system is concerned, it is known that thanks to the special structure of the system, the functions $\mathrm{div}\,\mathbf{u}$ and $\mathrm{rot}\,\mathbf{u}$ satisfy scalar wave equations (modulo lower order terms). The system of partial differential equations for functions $\mathbf{u}$, $\mathrm{div}\,\mathbf{u}$, $\mathrm{rot}\,\mathbf{u}$, is coupled via first order terms. This allows us to apply the Carleman estimate for a scalar hyperbolic equation in the case where the function $\mathbf{u}$ has a compact support (see [EINT], [IIY], [INY]).

On the other hand, the structure of our proof is in principle similar to the paper [Y]. That is, we work mainly with two hyperbolic equations depending on a parameter $s > 0$ for the functions $z_{\lambda+2\mu} \equiv e^{s\phi}\mathrm{div}\,\mathbf{u}$ and $z_\mu \equiv e^{s\phi}\mathrm{rot}\,\mathbf{u}$: $P_{\lambda+2\mu}(x_0, x', D, s)z_{\lambda+2\mu} = (\mathrm{div}\,\mathbf{f})e^{s\phi}$ and $P_\mu(x_0, x', D, s)z_\mu = (\mathrm{rot}\,\mathbf{f})e^{s\phi}$. The main difficulty one should overcome, is that there are no boundary conditions for these functions. This problem is solved in the following way: Outside an exceptional set in $T^*(Q)$, the operators $P_{\lambda+2\mu}$ and P_μ can be microlocally factorized into the product of two pseudo-differential operators of the first order:

$$P_\beta(x_0, x', D_{x'}, s) = P_{-,\beta}(x_0, x', D_{x'}, s)P_{+,\beta}(x_0, x', D_{x'}, s),$$

where $\beta = \lambda + 2\mu$ or $= \mu$, $P_{\pm,\beta} = D_{x_2} - \Gamma^{\pm}_{\beta}(x_0, x', D_{x'}, s)$, and x_2 is normal to the boundary $\partial\Omega$. Since the principal symbol of the operator $\Gamma^{-}_{\beta}(x_0, x', \xi, s)$ satisfies the inequality

$$-\mathrm{Im}\,\Gamma^{-}_{\beta}(x_0, x', \xi, s) \ge C|s|$$

with a constant $C > 0$, we have a priori estimates for $P_{+,\beta}(x_0, x', D_{x'}, s)z_\beta|_{x_2=0}$ in L^2. These estimates and the zero Dirichlet boundary condition yields the H^1

boundary estimates for z_β. The set on which we cannot factorize both the operators $P_\beta(x_0, x', D_{x'}, s)$ into a product of the first order operators, has to be discussed separately.

For the uniqueness and the stability in our inverse problem with the minimum number of observations, we are required to prove a Carleman estimate whose right hand side is estimated in H^{-1}-space. The Carleman estimate with right hand side in H^{-1}-space was proved by Imanuvilov [Im2], Ruiz [R], for a scalar hyperbolic equation and by Imanuvilov and Yamamoto [IY1] for a parabolic equation. In this paper, by a method in [IY1], we will derive an H^{-1}-Carleman estimate (Theorem 2.3) for (1.1) from a Carleman estimate (Theorem 2.1) with H^1-norm.

This paper is composed of eight sections and two appendices. In Section 2, we will show Carleman estimates (Theorems 2.1 - 2.3) for functions which do not have compact supports but satisfy the zero Dirichlet boundary condition on $(0,T)\times\partial\Omega$. Theorem 2.1 is a Carleman estimate whose right hand side is estimated in H^1-space. Theorems 2.2 and 2.3 are Carleman estimates respectively with right hand sides in L^2-space and in H^{-1}-space. In Section 3, we will apply the H^{-1}-Carleman estimate (Theorem 2.3), and prove the uniqueness and the conditional stability in the inverse problem with a single interior measurement. In Sections 4-7, we prove Theorem 2.1, while Theorems 2.2 and 2.3 are proved in Section 8.

§2. Carleman estimates for the two dimensional non-stationary Lamé system.

Let us consider the two dimensional Lamé system

$$P\mathbf{u}(x_0,x') \equiv \rho(x')\partial_{x_0}^2\mathbf{u}(x_0,x') - (L_{\lambda,\mu}\mathbf{u})(x_0,x') = \mathbf{f}(x_0,x') \quad \text{in } Q, \tag{2.1}$$

$$\mathbf{u}|_{(0,T)\times\partial\Omega} = 0, \quad \mathbf{u}(T,x') = \partial_{x_0}\mathbf{u}(T,x') = \mathbf{u}(0,x') = \partial_{x_0}\mathbf{u}(0,x') = 0, \tag{2.2}$$

where $\mathbf{u} = (u_1, u_2)^T, \mathbf{f} = (f_1, f_2)^T$ are vector-valued functions, and the partial differential operator $L_{\lambda,\mu}$ is defined by (1.2). The coefficients ρ, λ, $\mu \in C^2(\overline{\Omega})$ are assumed to satisfy (1.3).

Let $\omega \subset \Omega$ be an arbitrarily fixed subdomain. Denote by $\vec{n}(x') = (n_1(x'), n_2(x'))$ the outward unit normal vector to $\partial\Omega$ at x' and set $\frac{\partial v}{\partial \vec{n}} = \nabla_{x'} v \cdot \vec{n}$.

We set

$$Q_\omega = (0,T) \times \omega.$$

Let $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \xi_2)$. We set

$$\begin{cases} p_1(x,\xi) = \rho(x')\xi_0^2 - \mu(x')(|\xi_1|^2 + |\xi_2|^2), \\ p_2(x,\xi) = \rho(x')\xi_0^2 - (\lambda(x') + 2\mu(x'))(|\xi_1|^2 + |\xi_2|^2) \end{cases} \tag{2.3}$$

and $\nabla_\xi = (\partial_{\xi_0}, \partial_{\xi_1}, \partial_{\xi_2})$. For arbitrary smooth functions $\varphi(x,\xi)$ and $\psi(x,\xi)$, we define the Poisson bracket by the formula

$$\{\varphi, \psi\} = \sum_{j=0}^{2} (\partial_{\xi_j}\varphi)(\partial_{x_j}\psi) - (\partial_{\xi_j}\psi)(\partial_{x_j}\varphi).$$

We set $i = \sqrt{-1}$ and $< a, b >= \sum_{k=1}^3 a_k b_k$ for $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathbb{C}^3$.

We assume that the density ρ, the Lamé coefficients λ, μ and the domains Ω, ω satisfy the following condition (cf. [Hö]).

Condition 2.1. *There exists a function $\psi \in C^3(\overline{Q})$ such that*

$$\{p_k, \{p_k, \psi\}\}(x,\xi) > 0, \quad \forall k \in \{1, 2\} \tag{2.4}$$

if $(x,\xi) \in (\overline{Q \setminus Q_\omega}) \times (\mathbb{R}^3 \setminus \{0\})$ satisfies $p_k(x,\xi) =< \nabla_\xi p_k, \nabla\psi >= 0$, and

$$\{p_k(x, \xi - is\nabla\psi(x)), p_k(x, \xi + is\nabla\psi(x))\}/2is > 0, \quad \forall k \in \{1, 2\} \tag{2.5}$$

if $(x,\xi,s) \in (\overline{Q \setminus Q_\omega}) \times (\mathbb{R}^3 \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ *satisfies*

$$p_k(x,\xi + is\nabla\psi(x)) =< \nabla_\xi p_k(x,\xi + is\nabla\psi(x)), \nabla\psi(x) >= 0.$$

On the lateral boundary, we assume

$$p_1(x,\nabla\psi) < 0, \quad \forall x \in \overline{(0,T) \times \partial\Omega} \quad \textit{and} \quad \frac{\partial\psi}{\partial\vec{n}}\Big|_{(0,T)\times(\partial\Omega\setminus\partial\omega)} < 0. \tag{2.6}$$

Let $\psi(x)$ be the weight function in Condition 2.1. Using this function, we introduce the function $\phi(x)$ by

$$\phi(x) = e^{\tau\psi(x)}, \quad \tau > 1, \tag{2.7}$$

where the parameter $\tau > 0$ will be fixed below. Denote

$$\|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q)} = \int_Q \Bigg(\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 + s|\nabla \mathrm{rot}\, \mathbf{u}|^2 + s^3|\mathrm{rot}\, \mathbf{u}|^2$$
$$+s|\nabla \mathrm{div}\, \mathbf{u}|^2 + s^3|\mathrm{div}\, \mathbf{u}|^2 \Bigg) e^{2s\phi}dx,$$

where $\alpha = (\alpha_0,\alpha_1,\alpha_2)$, $\alpha_j \in \mathbb{N}\cup\{0\}$, $\partial_x^\alpha = \partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$.

Now we formulate our Carleman estimates as main results.

Theorem 2.1. *Let* $\mathbf{f} \in (H^1(Q))^2$, *and let the function* ϕ *satisfy Condition 2.1. Then there exists* $\widehat{\tau} > 0$ *such that for any* $\tau > \widehat{\tau}$, *there exists* $s_0 = s_0(\tau) > 0$ *such that for any solution* $\mathbf{u} \in (H^1(Q))^2 \cap L^2(0,T;(H^2(\Omega))^2)$ *to problem (2.1) - (2.2), the following estimate holds true:*

$$\|\mathbf{u}\|^2_{Y(\phi,Q)} \equiv \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q)} + s\left\|\frac{\partial \mathbf{u}}{\partial\vec{n}}e^{s\phi}\right\|^2_{(H^1((0,T)\times\partial\Omega))^2} + s\left\|\frac{\partial^2 \mathbf{u}}{\partial\vec{n}^2}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2}$$
$$+s^3\left\|\frac{\partial \mathbf{u}}{\partial\vec{n}}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega)))^2}$$
$$\le C_1(s^2\|\mathbf{f}e^{s\phi}\|^2_{(L^2(Q))^2} + \|(\nabla\mathbf{f})e^{s\phi}\|^2_{(L^2(Q))^2} + \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}), \quad \forall s \ge s_0(\tau), \tag{2.8}$$

where the constant $C_1 = C_1(\tau) > 0$ is independent of s.

Next we formulate Carleman estimates where norms of the function $\mathbf{f}$ are taken in $(L^2(Q))^2$ and $L^2(0,T;(H^{-1}(\Omega))^2)$. In particular, the latter Carleman estimate is essential for obtaining our sharp uniqueness result in the inverse problem.

In addition to Condition 2.1, we assume

$$\partial_{x_0}\phi(T,x') < 0, \quad \partial_{x_0}\phi(0,x') > 0, \qquad \forall x' \in \overline{\Omega}. \tag{2.9}$$

Theorem 2.2. *Let $\mathbf{f} \in (L^2(Q))^2$ and let the function ϕ satisfy (2.9) and Condition 2.1. Then there exists $\widehat{\tau} > 0$ such that for any $\tau > \widehat{\tau}$, there exists $s_0 = s_0(\tau) > 0$ such that for any solution $\mathbf{u} \in (H^1(Q))^2$ to problem (2.1) - (2.2), the following estimate holds true:*

$$\begin{aligned}&\int_Q (|\nabla \mathbf{u}|^2 + s^2|\mathbf{u}|^2)e^{2s\phi}dx \\ \le & C_1\left(\|\mathbf{f}e^{s\phi}\|^2_{(L^2(Q))^2} + \int_{Q_\omega}(|\nabla \mathbf{u}|^2 + s^2|\mathbf{u}|^2)e^{2s\phi}dx\right), \quad \forall s \ge s_0(\tau),\end{aligned} \tag{2.10}$$

where the constant $C_1 = C_1(\tau) > 0$ is independent of s.

Theorem 2.3. *Let $\mathbf{f} = \mathbf{f}_0 + \sum_{j=1}^2 \partial_{x_j}\mathbf{f}_j$ with $\mathbf{f}_0 \in L^2(0,T;(H^{-1}(\Omega))^2)$ and $\mathbf{f}_1, \mathbf{f}_2 \in (L^2(Q))^2$, and let the function ϕ satisfy (2.9) and Condition 2.1. Then there exists $\widehat{\tau} > 0$ such that for any $\tau > \widehat{\tau}$, there exists $s_0 = s_0(\tau) > 0$ such that for any solution $\mathbf{u} \in (L^2(Q))^2$ to problem (2.1) - (2.2), the following estimate holds true:*

$$\begin{aligned}&\int_Q |\mathbf{u}|^2 e^{2s\phi}dx \\ \le & C_1\left(\|\mathbf{f}_0e^{s\phi}\|^2_{L^2(0,T;(H^{-1}(\Omega))^2)} + \sum_{j=1}^2\|\mathbf{f}_je^{s\phi}\|^2_{(L^2(Q))^2} + \int_{Q_\omega}|\mathbf{u}|^2e^{2s\phi}dx\right), \quad \forall s \ge s_0(\tau),\end{aligned} \tag{2.11}$$

where the constant $C_1 = C_1(\tau) > 0$ is independent of s.

§3. Determination of the density and the Lamé coefficients by a single measurement.

Here we set recall that the differential operator $L_{\lambda,\mu}$ is defined by (1.2). We assume (1.3) for ρ, λ, μ. By $\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta)(x)$, we denote the sufficiently smooth solution to

$$\rho(x')(\partial_{x_0}^2 \mathbf{u})(x) = (L_{\lambda,\mu}\mathbf{u})(x), \quad x \in Q, \tag{3.1}$$

$$\mathbf{u}(x) = \eta(x), \qquad x \in (0,T) \times \partial\Omega, \tag{3.2}$$

$$\mathbf{u}(T/2, x') = \mathbf{p}(x'), \quad (\partial_{x_0}\mathbf{u})(T/2, x') = \mathbf{q}(x'), \quad x' \in \Omega, \tag{3.3}$$

where η, $\mathbf{p}$ and $\mathbf{q}$ are suitably given functions.

Let $\omega \subset \Omega$ be a suitably given subdomain. We consider the

Inverse Problem. Let $\mathbf{p}_j, \mathbf{q}_j, \eta_j$, $1 \le j \le \mathcal{N}$, be appropriately given. Then determine $\lambda(x')$, $\mu(x')$, $\rho(x')$, $x' \in \Omega$, by

$$\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}_j, \mathbf{q}_j, \eta_j)(x), \quad x \in Q_\omega \equiv (0,T) \times \omega. \tag{3.4}$$

Our formulation of the inverse problem is one with a finite number of observations (i.e., $\mathcal{N} < \infty$), and as for inverse problems for the non-stationary Lamé equation by infinitely many boundary observations (i.e., Dirichlet-to-Neumann map), we refer to Rachele [Ra], for example. Moreover see a monograph by Yahkno [Yak] for inverse problems for the Lamé system.

For the formulation with a finite number of observations, Bukhgeim and Klibanov [BuK] proposed a remarkable method based on a Carleman estimate and established the uniqueness for similar inverse problems for scalar partial differential equations. See also Bukhgeim [Bu], Bukhgeim, Cheng, Isakov and Yamamoto

[BCIY], Imanuvilov and Yamamoto [IY2], [IY3], [IY4], Isakov [Is1], [Is2], [Is3], Isakov and Yamamoto [IsY], Khaĭdarov [Kh1], [Kh2], Klibanov [Kl], Puel and Yamamoto [PY1], [PY2], Yamamoto [Ya] after Bukhgeim and Klibanov [BuK].

The Carleman estimate for the non-stationary Lamé equation was obtained for functions with compact supports, by Eller, Isakov, Nakamura and Tataru [EINT], Ikehata, Nakamura and Yamamoto [INY], Imanuvilov, Isakov and Yamamoto [IIY], Isakov [Is1], and, by the methodology by [BuK] or [IY2], several uniqueness results are available for the inverse problem for the Lamé system (3.1) - (3.3):

[Is1] Isakov established the uniqueness in determining a single coefficient $\rho(x')$, using four measurements.

Later [INY] reduced the number of measurements to three.

Recently [IIY] proved conditional stability and the uniqueness in the determination of the three functions $\lambda(x')$, $\mu(x')$, $\rho(x')$, $x' \in \Omega$, was proved with only two measurements.

In all the papers [Is1], [INY], [IIY], the authors have to assume that $\partial\omega \supset \partial\Omega$ because the basic Carleman estimates require that solutions under consideration have compact supports in Q.

In [Is1] and [INY], the key is a Carleman estimate where the right hand side is estimated in an L^2-space with the divergence, while in [IIY], the key is a Carleman estimate with L^2-right hand side without the divergence of $\mathbf{u}$. In [IIY], we need not take extra divergence for the Carleman estimate, and as its consequence, we can relax $\mathcal{N}$ for simultaneous determination of all the three functions λ, μ, ρ.

In this section, we will further apply a Carleman estimate (Theorem 2.3) whose right hand side is estimated in H^{-1} space to prove the conditional stability and the uniqueness with a single measurement: $\mathcal{N} = 1$. Thus the main achievements are

(1) the reduction of the number of observations, i.e., $\mathcal{N} = 1$. The previous paper [IIY] requires $\mathcal{N} = 2$.

(2) the reduction of the observation subdomain ω.

We notice that our results are true also in the three dimensional case.

In order to formulate our main result, we will introduce notations and an admissible set of unknown parameters λ, μ, ρ. Henceforth we set $(x', y') = \sum_{j=1}^2 x_j y_j$ for $x' = (x_1, x_2)$ and $y' = (y_1, y_2)$. Let a subdomain $\omega \subset \Omega$ satisfy

$$\partial\omega \supset \{x' \in \partial\Omega; ((x'-y'), \vec{n}(x')) \ge 0\} \equiv \Gamma \tag{3.5}$$

with some $y' \notin \overline{\Omega}$. Under condition (3.5) on ω, we can prove the observability inequality for the wave equation with constant coefficients (e.g., [Li2]).

Denote

$$d = (\sup_{x'\in\Omega} |x'-y'|^2 - \inf_{x'\in\Omega} |x'-y'|^2)^{\frac{1}{2}}. \tag{3.6}$$

Next we define an admissible set of unknown coefficients λ, μ, ρ. We introduce the conditions:

$$\begin{aligned} &\beta(x') \ge \theta_1 > 0, \quad x' \in \overline{\Omega}, \\ &\|\beta\|_{C^3(\overline{\Omega})} \le M_0, \quad \frac{(\nabla_{x'}\beta(x'), (x'-y'))}{2\beta(x')} \le 1 - \theta_0, \quad x' \in \overline{\Omega \setminus \omega} \end{aligned} \tag{3.7}$$

for any fixed constants $M_0 \ge 0$ and $0 < \theta_0 \le 1$, $\theta_1 > 0$. For fixed functions a, b, η on $\partial\Omega$ and $\mathbf{p}$, $\mathbf{q}$ in Ω, we set

$$\begin{aligned} \mathcal{W} = \mathcal{W}_{M_0,M_1,\theta_0,\theta_1,a,b} = \Bigg\{ &(\lambda, \mu, \rho) \in (C^3(\overline{\Omega}))^3; \lambda = a, \mu = b \quad \text{on } \partial\Omega, \\ \frac{\lambda+2\mu}{\rho}, \frac{\mu}{\rho} \text{ satisfy (3.7)}, &\|\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta)\|_{W^{7,\infty}(Q)} \le M_1 \Bigg\} \end{aligned} \tag{3.8}$$

where the constant M_1 is given. We choose $\theta > 0$ such that

$$\theta + \frac{M_0 d}{\sqrt{\theta_1}}\sqrt{\theta} < \theta_0\theta_1, \quad \theta_1 \inf_{x'\in\Omega} |x'-y'|^2 - \theta d^2 > 0. \tag{3.9}$$

Here we note that by $y' \notin \overline{\Omega}$, such $\theta > 0$ exists.

By $[\cdot]_1$, we denote the first component of the vector under consideration. Let (λ, μ, ρ) be an arbitrary element of $\mathcal{W}$.

Now we are ready to state

Theorem 3.1. *We assume that*

$$\Omega = \{(x_1, x_2);\ \gamma_0(x_2) < x_1 < \gamma_1(x_2),\ x_2 \in I\} \tag{3.10}$$

with some open interval I and $\gamma_0, \gamma_1 \in C^3(\overline{I})$. Moreover we assume that the functions $\mathbf{p} = (p_1, p_2)^T$ and $\mathbf{q} = (q_1, q_2)^T$ satisfy

$$det \begin{pmatrix} (L_{\lambda,\mu}\mathbf{p})(x') & (\mathrm{div}\,\mathbf{p}(x'))E_2 & (\nabla_{x'}\mathbf{p}(x') + (\nabla_{x'}\mathbf{p}(x'))^T)(x'-y') \\ (L_{\lambda,\mu}\mathbf{q})(x') & (\mathrm{div}\,\mathbf{q}(x'))E_2 & (\nabla_{x'}\mathbf{q}(x') + (\nabla_{x'}\mathbf{q}(x'))^T)(x'-y') \end{pmatrix} \ne 0, \forall x' \in \overline{\Omega}, \tag{3.11}$$

$$det \begin{pmatrix} (L_{\lambda,\mu}\mathbf{p})(x') & \nabla_{x'}\mathbf{p}(x') + (\nabla_{x'}\mathbf{p}(x'))^T & (\mathrm{div}\,\mathbf{p})(x'-y') \\ (L_{\lambda,\mu}\mathbf{q})(x') & \nabla_{x'}\mathbf{q}(x') + (\nabla_{x'}\mathbf{q}(x'))^T & (\mathrm{div}\,\mathbf{q})(x'-y') \end{pmatrix} \ne 0, \forall x' \in \overline{\Omega}, \tag{3.12}$$

and

$$\begin{aligned} & x_1 - y_1 \ne 0, \\ & [L_{\lambda,\mu}\mathbf{q}]_1(\partial_1 p_2 + \partial_2 p_1)(x') \ne [L_{\lambda,\mu}\mathbf{p}]_1(\partial_1 q_2 + \partial_2 q_1)(x'), \quad \forall x' \in \overline{\Omega} \end{aligned} \tag{3.13}$$

and that

$$T > \frac{2}{\sqrt{\theta}} d. \tag{3.14}$$

Then there exist constants $\kappa = \kappa(\mathcal{W}, \omega, \Omega, T, \lambda, \mu, \rho) \in (0,1)$ *and*

$C = C(\mathcal{W}, \omega, \Omega, T, \lambda, \mu, \rho) > 0$ *such that*

$$\begin{aligned}&\|\widetilde{\lambda} - \lambda\|_{L^2(\Omega)} + \|\widetilde{\mu} - \mu\|_{L^2(\Omega)} + \|\widetilde{\rho} - \rho\|_{H^{-1}(\Omega)}\\ \le & C\|\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta) - \mathbf{u}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}, \mathbf{p}, \mathbf{q}, \eta)\|^{\kappa}_{H^4(0,T;(L^2(\omega))^2)}\end{aligned}$$

for any $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}) \in \mathcal{W}$.

Our stability and uniqueness result requires only one measurement: $\mathcal{N} = 1$, but the conditions on the initial values $\mathbf{p}$, $\mathbf{q}$ are more restrictive.

Example of Ω, p, q meeting (3.11) - (3.13). We assume that λ, μ are positive constants and that $\{(x_1, x_2) \in \overline{\Omega};\ x_2 = y_2\}$ and $\{(x_1, x_2) \in \overline{\Omega};\ x_1 = y_1\}$ are empty. Moreover we take

$$\mathbf{p}(x') = \begin{pmatrix} 0 \\ (x_1 - y_1)(x_2 - y_2) \end{pmatrix}, \quad \mathbf{q}(x') = \begin{pmatrix} (x_2 - y_2)^2 \\ 0 \end{pmatrix}.$$

Then (3.11) - (3.13) are all satisfied.

Remark. In place of (3.10), let us assume

$$\Omega = \{(x_1, x_2);\ \widetilde{\gamma}_0(x_1) < x_2 < \widetilde{\gamma}_1(x_1),\ x_1 \in \widetilde{I}\} \tag{3.10'}$$

with some open interval $\widetilde{I}$. Then, after replacing (3.13) by

$$\begin{aligned}&x_2 - y_2 \ne 0,\\ &[L_{\lambda,\mu}\mathbf{q}]_2(\partial_1 p_2 + \partial_2 p_1)(x') \ne [L_{\lambda,\mu}\mathbf{p}]_2(\partial_1 q_2 + \partial_2 q_1)(x'), \quad x' \in \overline{\Omega},\end{aligned} \tag{3.13'}$$

the conclusion of Theorem 3.1 holds under conditions (3.11), (3.12) and (3.14). Moreover in the case when Ω is a more general smooth domain, we can prove the conditional stability in our inverse problem under other conditions of $\omega \subset \Omega$. We will omit the details, for the sake of compact description of the proof.

We set

$$\psi(x) = |x' - y'|^2 - \theta\left(x_0 - \frac{T}{2}\right)^2, \quad \phi(x) = e^{\tau\psi(x)}, \quad x = (x_0, x') \in Q. \qquad (3.15)$$

First we show

Lemma 3.1. *Let $(\lambda, \mu, \rho) \in \mathcal{W}$, and let us assume (3.9) and (3.14). Then, for sufficiently large $\tau > 0$, the function ψ given by (3.15) satisfies Condition 2.1 and (2.9). Therefore the conclusion of Theorem 2.3 holds and the constans $C_1(\tau)$, $\widehat{\tau}$ and $s_0(\tau)$ in (2.11) can be taken uniformly in $(\lambda, \mu, \rho) \in \mathcal{W}$.*

Proof. The conditions (2.9) and the second condition in (2.6) are directly verified by means of (3.5). The conditions (2.4) and (2.5) can be verified by the same way as in Imanuvilov and Yamamoto [IY4], for example. Finally we have to verify the first condition in (2.6). Without loss of generality, we may assume that $T = \frac{2d}{\sqrt{\theta}} + \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Because if Theorem 3.1 is proved for this value of T, then conclusion is true for any $\widetilde{T} > T$. Let $\beta = \frac{\lambda + 2\mu}{\rho}$ or $= \frac{\mu}{\rho}$. Then it suffices to verify

$$-(\theta(x_0 - T/2))^2 + \beta(x')|x' - y'|^2 > 0$$

for $x \in [0, T] \times \partial\Omega$. In fact, by means of the second inequality in (3.8) and (3.9), we have

$$4\beta(x')|x' - y'|^2 - 4\theta^2\left(x_0 - \frac{T}{2}\right)^2 \geq 4\theta_1 \inf_{x' \in \Omega}|x' - y'|^2 - \theta(\theta T^2)$$
$$\geq 4\theta_1 \inf_{x' \in \Omega}|x' - y'|^2 - \theta(2d + \varepsilon\sqrt{\theta})^2 > 0$$

because $\varepsilon > 0$ is sufficiently small. The uniformity of the constants $C_1(\tau)$, $\widehat{\tau}$ and $s_0(\tau)$ follows similarly to [IIY]. Thus the proof of Lemma 3.1 is complete. ■

Next we prove a Carleman estimate for a first order partial differential operator

$$(P_0 g)(x') = \sum_{j=1}^{2} p_{0,j}(x')\partial_{x_j} g(x').$$

Lemma 3.2. *We assume*

$$\sum_{j=1}^{2} p_{0,j}(x')\partial_{x_j}\phi(T/2, x') > 0, \quad x' \in \overline{\Omega}. \tag{3.16}$$

Then there exists a constant $\tau_0 > 0$ such that for all $\tau > \tau_0$, there exist $s_0 = s_0(\tau) > 0$ and $C = C(s_0, \tau_0, \Omega, \omega) > 0$ such that

$$\int_\Omega s^2|g|^2 e^{2s\phi(T/2,x')}dx' \le C\int_\Omega |P_0 g|^2 e^{2s\phi(T/2,x')}dx'$$

for all $s > s_0$ and $g \in H^1(\Omega)$ satisfying $g = 0$ on $\{x' \in \partial\Omega;\ \sum_{j=1}^2 p_{0,j}(x')n_j(x') \ge 0\}$.

Lemma 3.3. *We assume*

$$\sum_{j=1}^{2} p_{0,j}(x')\partial_{x_j}\phi(T/2, x') \ne 0, \quad x' \in \overline{\Omega}.$$

Then the conclusion of Lemma 3.2 is true for all $s > s_0$ and $g \in H_0^1(\Omega)$.

Proof of Lemma 3.2. For simplicity, we set $\phi_0(x') = \phi(T/2, x')$ and $w = e^{s\phi_0}g$, $Q_0 w = e^{s\phi_0}P_0(e^{-s\phi_0}w)$. Then

$$\int_\Omega |P_0 g|^2 e^{2s\phi(T/2,x')}dx' = \int_\Omega |Q_0 w|^2 dx'.$$

We have

$$Q_0 w = P_0 w - sq_0 w,$$

where $q_0(x') = \sum_{j=1}^2 p_{0,j}(x')\partial_{x_j}\phi_0(x')$. Therefore, by (3.16) and integration by parts, we obtain

$$\|Q_0 w\|^2_{L^2(\Omega)} = \|P_0 w\|^2_{L^2(\Omega)} + s^2\|q_0 w\|^2_{L^2(\Omega)} - 2s\int_\Omega \sum_{j=1}^2 p_{0,j}(\partial_{x_j} w)q_0 w dx'$$

$$\ge s^2\int_\Omega q_0(x')^2 w^2(x')dx' - s\int_\Omega \sum_{j=1}^2 p_{0,j}q_0\partial_{x_j}(w^2)dx'$$

$$\ge C_0 s^2\int_\Omega w^2(x')dx' - s\int_{\partial\Omega}\sum_{j=1}^2 p_{0,j}q_0 w^2 n_j dS + s\int_\Omega \sum_{j=1}^2 \partial_{x_j}(p_{0,j}q_0)w^2 dx'$$

$$\ge (C_0 s^2 - C_1 s)\int_\Omega w^2 dx' - s\int_{\partial\Omega\cap\{\sum_{j=1}^2 p_{0,j}n_j\le 0\}}\left(\sum_{j=1}^2 p_{0,j}n_j\right) q_0 w^2 dS.$$

By (3.16), we have $q_0 \ge 0$ on $\partial\Omega$, so that the right hand side is greater than or equal to $(C_0 s^2 - C_1 s)\int_\Omega w^2 dx'$. Thus by taking $s > 0$ sufficiently large, the proof of Lemma 3.2 is complete. ■

The proof of Lemma 3.3 is similar, thanks to the fact that the integral on $\partial\Omega$ vanishes by $g \in H^1_0(\Omega)$.

Now we proceed to

Proof of Theorem 3.1. The proof is similar to Isakov, Imanuvilov and Yamamoto [IIY], Imanuvilov and Yamamoto [IY2] - [IY4] and the new ingredient is an H^{-1}-Carleman estimate (Lemma 3.1) . Henceforth, for simplicity, we set

$$\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta), \quad \mathbf{v} = \mathbf{u}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}, \mathbf{p}, \mathbf{q}, \eta)$$

and

$$\mathbf{y} = \mathbf{u} - \mathbf{v}, \quad f = \rho - \widetilde{\rho}, \quad g = \lambda - \widetilde{\lambda}, \quad h = \mu - \widetilde{\mu}.$$

In (3.13), without loss of generality, we may assume that

$$x_1 - y_1 > 0, \qquad (x_1, x_2) \in \overline{\Omega}.$$

Then we set

$$F(x_1,x_2)=\int_{\gamma_1(x_2)}^{x_1} f(\xi,x_2)d\xi, \quad (x_1,x_2)\in\Omega. \tag{3.17}$$

If $x_1-y_1<0$ for $(x_1,x_2)\in\overline{\Omega}$, then it is sufficient to replace (3.17) by $F(x_1,x_2)=\int_{\gamma_0(x_2)}^{x_1} f(\xi,x_2)d\xi$, $(x_1,x_2)\in\Omega$. Then

$$\widetilde{\rho}\partial_{x_0}^2\mathbf{y}=L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{y}+G\mathbf{u} \quad \text{in } Q \tag{3.18}$$

and

$$\mathbf{y}\left(\frac{T}{2},x'\right)=\partial_{x_0}\mathbf{y}\left(\frac{T}{2},x'\right)=0, \qquad x'\in\Omega \tag{3.19}$$

and

$$\mathbf{y}=0 \qquad \text{in } (0,T)\times\partial\Omega. \tag{3.20}$$

Here we set

$$G\mathbf{u}(x)=-\partial_{x_1}F(x')\partial_{x_0}^2\mathbf{u}(x)+(g+h)(x')\nabla_{x'}(\operatorname{div}\mathbf{u})(x)+h(x')\Delta\mathbf{u}(x)$$

$$+(\operatorname{div}\mathbf{u})(x)\nabla_{x'}g(x')+(\nabla_{x'}\mathbf{u}(x)+(\nabla_{x'}\mathbf{u}(x))^T)\nabla h(x'). \tag{3.21}$$

By (3.14), we have the inequality $\frac{\theta T^2}{4}>d^2$. Therefore, by (3.6) and definition (3.15) of the function ϕ, we have

$$\phi(T/2,x')\ge d_1, \quad \phi(0,x')=\phi(T,x')<d_1, \qquad x'\in\overline{\Omega}$$

with $d_1=\exp(\tau\inf_{x'\in\Omega}|x'-y'|^2)$. Thus, for given $\varepsilon>0$, we can choose a sufficiently small $\delta=\delta(\varepsilon)>0$ such that

$$\phi(x)\ge d_1-\varepsilon, \quad x\in\left[\frac{T}{2}-\delta,\frac{T}{2}+\delta\right]\times\overline{\Omega} \tag{3.22}$$

and

$$\phi(x)\le d_1-2\varepsilon, \quad x\in([0,2\delta]\cup[T-2\delta,T])\times\overline{\Omega}. \tag{3.23}$$

In order to apply Lemma 3.1, it is necessary to introduce a cut-off function χ satisfying $0 \le \chi \le 1$, $\chi \in C^\infty(\mathbb{R})$ and

$$\chi = \begin{cases} 0 & \text{on } [0,\delta] \cup [T-\delta, T] \\ 1 & \text{on } [2\delta, T-2\delta]. \end{cases} \tag{3.24}$$

Henceforth $C > 0$ denotes generic constants depending on s_0, τ, M_0, M_1, θ_0, θ_1, η, Ω, T, y', ω, χ and $\mathbf{p}$, $\mathbf{q}$, ε, δ, but independent of $s > s_0$.

Setting $\mathbf{z}_1 = \chi\partial_{x_0}^2\mathbf{y}$, $\mathbf{z}_2 = \chi\partial_{x_0}^3\mathbf{y}$ and $\mathbf{z}_3 = \chi\partial_{x_0}^4\mathbf{y}$, we have

$$\begin{cases} \widetilde{\rho}\partial_{x_0}^2\mathbf{z}_1 = L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_1 + \chi G(\partial_{x_0}^2\mathbf{u}) + 2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^3\mathbf{y} + \widetilde{\rho}(\partial_{x_0}^2\chi)\partial_{x_0}^2\mathbf{y}, \\ \widetilde{\rho}\partial_{x_0}^2\mathbf{z}_2 = L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_2 + \chi G(\partial_{x_0}^3\mathbf{u}) + 2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^4\mathbf{y} + \widetilde{\rho}(\partial_{x_0}^2\chi)\partial_{x_0}^3\mathbf{y}, \\ \widetilde{\rho}\partial_{x_0}^2\mathbf{z}_3 = L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_3 + \chi G(\partial_{x_0}^4\mathbf{u}) + 2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^5\mathbf{y} + \widetilde{\rho}(\partial_{x_0}^2\chi)\partial_{x_0}^4\mathbf{y} \quad \text{in } Q. \end{cases} \tag{3.25}$$

Henceforth we set

$$\mathcal{E} = \int_{Q_\omega} (|\partial_{x_0}^2\mathbf{y}|^2 + |\partial_{x_0}^3\mathbf{y}|^2 + |\partial_{x_0}^4\mathbf{y}|^2)e^{2s\phi}dx.$$

Noting $\mathbf{u} \in W^{7,\infty}(Q)$, in view of (3.24), we apply Lemma 3.1 to (3.25), so that

$$\begin{aligned} &\sum_{j=2}^4 \int_Q |\partial_{x_0}^j\mathbf{y}|^2\chi^2 e^{2s\phi}dx \le C(\|Fe^{s\phi}\|^2_{L^2(Q)} + \|ge^{s\phi}\|^2_{L^2(Q)} + \|he^{s\phi}\|^2_{L^2(Q)}) \\ &+C\sum_{j=3}^5 \|(\partial_{x_0}\chi)(\partial_{x_0}^j\mathbf{y})e^{s\phi}\|^2_{L^2(0,T;(H^{-1}(\Omega))^2)} \\ &+C\sum_{j=2}^4 \|(\partial_{x_0}^2\chi)(\partial_{x_0}^j\mathbf{y})e^{s\phi}\|^2_{L^2(0,T;(H^{-1}(\Omega))^2)} + C\mathcal{E} \\ &\le C(\|Fe^{s\phi}\|^2_{L^2(Q)} + \|ge^{s\phi}\|^2_{L^2(Q)} + \|he^{s\phi}\|^2_{L^2(Q)}) + Ce^{2s(d_1-2\varepsilon)} + C\mathcal{E} \end{aligned} \tag{3.26}$$

for all large $s > 0$.

On the other hand,

$$
\begin{aligned}
&\int_\Omega |(\partial_{x_0}^2 \mathbf{y})(T/2,x')|^2 e^{2s\phi(T/2,x')} dx' \\
=&\int_0^{T/2} \frac{\partial}{\partial x_0}\left(\int_\Omega |(\partial_{x_0}^2 \mathbf{y})(x_0,x')|^2 \chi(x_0)^2 e^{2s\phi} dx'\right) dx_0 \\
=&\int_0^{T/2}\int_\Omega 2((\partial_{x_0}^3 \mathbf{y})\cdot(\partial_{x_0}^2 \mathbf{y}))\chi^2 e^{2s\phi} dx \\
+&2s\int_0^{T/2}\int_\Omega |\partial_{x_0}^2 \mathbf{y}|^2 \chi^2 (\partial_{x_0}\phi) e^{2s\phi} dx + \int_0^{T/2}\int_\Omega |\partial_{x_0}^2 \mathbf{y}|^2 (\partial_{x_0}(\chi^2)) e^{2s\phi} dx \\
\le& C\int_Q s\chi^2(|\partial_{x_0}^3 \mathbf{y}|^2 + |\partial_{x_0}^2 \mathbf{y}|^2) e^{2s\phi} dx + Ce^{2s(d_1-2\varepsilon)}.
\end{aligned}
$$

Therefore (3.26) yields

$$
\begin{aligned}
&\int_\Omega |(\partial_{x_0}^2 \mathbf{y})(T/2,x')|^2 e^{2s\phi(T/2,x')} dx' \\
\le& Cs\int_Q (|F|^2+|g|^2+|h|^2) e^{2s\phi} dx + Cse^{2s(d_1-2\varepsilon)} + Cs\mathcal{E}
\end{aligned}
\tag{3.27}
$$

for all large $s>0$. Similarly we can estimate $\int_\Omega |(\partial_{x_0}^3 \mathbf{y})(T/2,x')|^2 e^{2s\phi(T/2,x')} dx'$ to obtain

$$
\begin{aligned}
&\int_\Omega (|(\partial_{x_0}^2 \mathbf{y})(T/2,x')|^2 + |(\partial_{x_0}^3 \mathbf{y})(T/2,x')|^2) e^{2s\phi(T/2,x')} dx' \\
\le& Cs\int_Q (|F|^2+|g|^2+|h|^2) e^{2s\phi} dx + Cse^{2s(d_1-2\varepsilon)} + Cs\mathcal{E}
\end{aligned}
\tag{3.28}
$$

for all large $s>0$.

On the other hand, by (3.18), (3.19) and $\mathbf{u},\mathbf{v}\in W^{7,\infty}(Q)$, we have

$$
\widetilde{\rho}\partial_{x_0}^2 \mathbf{y}\left(\frac{T}{2},x'\right) = G\mathbf{u}\left(\frac{T}{2},x'\right), \quad \widetilde{\rho}\partial_{x_0}^3 \mathbf{y}\left(\frac{T}{2},x'\right) = G\partial_{x_0}\mathbf{u}\left(\frac{T}{2},x'\right). \tag{3.29}
$$

Then, setting

$$\begin{cases} -\dfrac{1}{\rho}L_{\lambda,\mu}\mathbf{p} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \quad -\dfrac{1}{\rho}L_{\lambda,\mu}\mathbf{q} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}, \\ \operatorname{div}\mathbf{p} = b_1, \qquad \operatorname{div}\mathbf{q} = b_2, \\ \nabla_{x'}\mathbf{p} + (\nabla_{x'}\mathbf{p})^T = \begin{pmatrix} c_1 & d_1 \\ d_1 & e_1 \end{pmatrix}, \quad \nabla_{x'}\mathbf{q} + (\nabla_{x'}\mathbf{q})^T = \begin{pmatrix} c_2 & d_2 \\ d_2 & e_2 \end{pmatrix}, \\ \widetilde{\rho}\partial_{x_0}^2\mathbf{y}\left(\dfrac{T}{2}, x'\right) - (g+h)\nabla_{x'}(\operatorname{div}\mathbf{p}) - h\Delta\mathbf{p} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \\ \widetilde{\rho}\partial_{x_0}^3\mathbf{y}\left(\dfrac{T}{2}, x'\right) - (g+h)\nabla_{x'}(\operatorname{div}\mathbf{q}) - h\Delta\mathbf{q} = \begin{pmatrix} G_3 \\ G_4 \end{pmatrix}, \end{cases} \tag{3.30}$$

we rewrite (3.29) as

$$\begin{pmatrix} a_{11} & b_1 & 0 \\ a_{21} & 0 & b_1 \\ a_{12} & b_2 & 0 \\ a_{22} & 0 & b_2 \end{pmatrix} \begin{pmatrix} \partial_{x_1}F \\ \partial_{x_1}g \\ \partial_{x_2}g \end{pmatrix} = \begin{pmatrix} G_1 - c_1\partial_{x_1}h - d_1\partial_{x_2}h \\ G_2 - d_1\partial_{x_1}h - e_1\partial_{x_2}h \\ G_3 - c_2\partial_{x_1}h - d_2\partial_{x_2}h \\ G_4 - d_2\partial_{x_1}h - e_2\partial_{x_2}h \end{pmatrix}. \tag{3.31}$$

Because linear system (3.31) possesses a solution $(\partial_{x_1}F, \partial_{x_1}g, \partial_{x_2}g)$, the coefficient matrix must satisfy

$$\det \begin{pmatrix} a_{11} & b_1 & 0 & G_1 - c_1\partial_{x_1}h - d_1\partial_{x_2}h \\ a_{21} & 0 & b_1 & G_2 - d_1\partial_{x_1}h - e_1\partial_{x_2}h \\ a_{12} & b_2 & 0 & G_3 - c_2\partial_{x_1}h - d_2\partial_{x_2}h \\ a_{22} & 0 & b_2 & G_4 - d_2\partial_{x_1}h - e_2\partial_{x_2}h \end{pmatrix} = 0,$$

that is,

$$(\partial_{x_1}h)\det \begin{pmatrix} a_{11} & b_1 & 0 & c_1 \\ a_{21} & 0 & b_1 & d_1 \\ a_{12} & b_2 & 0 & c_2 \\ a_{22} & 0 & b_2 & d_2 \end{pmatrix} + (\partial_{x_2}h)\det \begin{pmatrix} a_{11} & b_1 & 0 & d_1 \\ a_{21} & 0 & b_1 & e_1 \\ a_{12} & b_2 & 0 & d_2 \\ a_{22} & 0 & b_2 & e_2 \end{pmatrix}$$
$$= \det \begin{pmatrix} a_{11} & b_1 & 0 & G_1 \\ a_{21} & 0 & b_1 & G_2 \\ a_{12} & b_2 & 0 & G_3 \\ a_{22} & 0 & b_2 & G_4 \end{pmatrix}, \tag{3.32}$$

by the linearity of the determinant. Under condition (3.11), taking into consideration $h = \mu - \widetilde{\mu} = 0$ on $\partial\Omega$ and considering (3.32) as a first order partial differential operator in h, we apply Lemma 3.3, so that

$$
\begin{aligned}
&s^2\int_\Omega |h|^2 e^{2s\phi(T/2,x')}dx' \le C\left\|\det\begin{pmatrix} a_{11} & b_1 & 0 & G_1\\ a_{21} & 0 & b_1 & G_2\\ a_{12} & b_2 & 0 & G_3\\ a_{22} & 0 & b_2 & G_4\end{pmatrix} e^{s\phi(T/2,\cdot)}\right\|^2_{L^2(\Omega)}\\
\le& C\int_\Omega\left(\left|\partial^2_{x_0}\mathbf{y}\left(\frac{T}{2},x'\right)\right|^2+\left|\partial^3_{x_0}\mathbf{y}\left(\frac{T}{2},x'\right)\right|^2\right)e^{2s\phi(T/2,x')}dx'\\
+&C\int_\Omega(|g|^2+|h|^2)e^{2s\phi(T/2,x')}dx', \qquad (3.33)
\end{aligned}
$$

in view of (3.30). We rewrite (3.29) as

$$
\begin{pmatrix} a_{11} & c_1 & d_1\\ a_{21} & d_1 & e_1\\ a_{12} & c_2 & d_2\\ a_{22} & d_2 & e_2\end{pmatrix}\begin{pmatrix}\partial_{x_1}F\\ \partial_{x_1}h\\ \partial_{x_2}h\end{pmatrix}=\begin{pmatrix} G_1-b_1\partial_{x_1}g\\ G_2-b_1\partial_{x_2}g\\ G_3-b_2\partial_{x_1}g\\ G_4-b_2\partial_{x_2}g\end{pmatrix}
$$

and, using (3.12), we can similarly deduce

$$
\begin{aligned}
&s^2\int_\Omega |g|^2 e^{2s\phi(T/2,x')}dx' \le C\int_\Omega\left(\left|\partial^2_{x_0}\mathbf{y}\left(\frac{T}{2},x'\right)\right|^2+\left|\partial^3_{x_0}\mathbf{y}\left(\frac{T}{2},x'\right)\right|^2\right)e^{2s\phi(T/2,x')}dx'\\
+&C\int_\Omega(|g|^2+|h|^2)e^{2s\phi(T/2,x')}dx' \qquad (3.34)
\end{aligned}
$$

for all large $s>0$. By (3.33) and (3.34), for sufficiently large $s>0$, we have

$$
\begin{aligned}
&s^2\int_\Omega(|g|^2+|h|^2)e^{2s\phi(T/2,x')}dx'\\
\le& C\int_\Omega\left(\left|\partial^2_{x_0}\mathbf{y}\left(\frac{T}{2},x'\right)\right|^2+\left|\partial^3_{x_0}\mathbf{y}\left(\frac{T}{2},x'\right)\right|^2\right)e^{2s\phi(T/2,x')}dx'. \qquad (3.35)
\end{aligned}
$$

Moreover, eliminating $\partial_{x_2}h$ in the first and the third rows in (3.31) and using (3.13), we have

$$
\begin{aligned}
&\partial_{x_1}\left(F+\frac{d_2b_1-d_1b_2}{d_2a_{11}-d_1a_{12}}g+\frac{d_2c_1-d_1c_2}{d_2a_{11}-d_1a_{12}}h\right)\\
=&\frac{d_2G_1-d_1G_3}{d_2a_{11}-d_1a_{12}}+g\partial_{x_1}\left(\frac{d_2b_1-d_1b_2}{d_2a_{11}-d_1a_{12}}\right)+h\partial_{x_1}\left(\frac{d_2c_1-d_1c_2}{d_2a_{11}-d_1a_{12}}\right).
\end{aligned}
$$

By (3.10) and (3.17), if $n_1(x') \ge 0$, then $x_1 = \gamma_1(x_2)$, that is, we have $F(x_1, x_2) = 0$ if $n_1(x') \ge 0$. Therefore, noting $g = h = 0$ on $\partial\Omega$ and setting $p_{0,1} = 1$, $p_{0,2} = 0$ in Lemma 3.2, we can apply the lemma. Thus, in view of (3.35) and (3.30), we obtain

$$\begin{aligned} &s^2 \int_\Omega |F|^2 e^{2s\phi(T/2,x')} dx' \\ \le &C \int_\Omega \left(\left| \partial_{x_0}^2 \mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 + \left|\partial_{x_0}^3 \mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 \right) e^{2s\phi(T/2,x')} dx' \end{aligned} \tag{3.36}$$

for all large $s > 0$. Consequently, substituting (3.35) and (3.36) into (3.28) and using $\phi(T/2, x') \ge \phi(x_0, x')$ for $(x_0, x') \in Q$, we obtain

$$\begin{aligned} &\int_\Omega (|F|^2 + |g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx' \\ \le &\frac{CT}{s} \int_\Omega (|F|^2 + |g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx' + \frac{C}{s} e^{2s(d_1 - 2\varepsilon)} + \frac{C}{s}\mathcal{E} \end{aligned}$$

for all large $s > 0$. Taking $s > 0$ sufficiently large and noting $e^{2s\phi(T/2,x')} \ge e^{2sd_1}$ for $x' \in \overline{\Omega}$, we obtain

$$\int_\Omega (|F|^2 + |g|^2 + |h|^2) dx' \le Ce^{-4s\varepsilon} + Ce^{2sC} \int_{Q_\omega} (|\partial_{x_0}^2 \mathbf{y}|^2 + |\partial_{x_0}^3 \mathbf{y}|^2 + |\partial_{x_0}^4 \mathbf{y}|^2) dx \tag{3.37}$$

for all large $s > s_0$: a constant which is dependent on τ, but independent of s. Therefore we take $C > 0$ again dependently on $s_0 > 0$, so that (3.37) holds for all $s > 0$.

Now we choose $s > 0$ such that

$$e^{2sC} \int_{Q_\omega} (|\partial_{x_0}^2 \mathbf{y}|^2 + |\partial_{x_0}^3 \mathbf{y}|^2 + |\partial_{x_0}^4 \mathbf{y}|^2) dx = e^{-4s\varepsilon},$$

that is,

$$s = -\frac{1}{4\varepsilon + 2C} \log \int_{Q_\omega} (|\partial_{x_0}^2 \mathbf{y}|^2 + |\partial_{x_0}^3 \mathbf{y}|^2 + |\partial_{x_0}^4 \mathbf{y}|^2) dx.$$

Here we may assume that $\int_{Q_\omega}(|\partial_{x_0}^2\mathbf{y}|^2+|\partial_{x_0}^3\mathbf{y}|^2+|\partial_{x_0}^4\mathbf{y}|^2)dx<1$ and so $s>0$. Then it follows from (3.37) that

$$\begin{aligned}&\int_\Omega(|F|^2+|g|^2+|h|^2)dx'\\ \le&2C\left(\int_{Q_\omega}(|\partial_{x_0}^2\mathbf{y}|^2+|\partial_{x_0}^3\mathbf{y}|^2+|\partial_{x_0}^4\mathbf{y}|^2)dx\right)^{\frac{4\varepsilon}{4\varepsilon+2C}}.\end{aligned}$$

By definition (3.17) of F, we have

$$\int_\Omega frdx_1dx_2=\int_\Omega(\partial_{x_1}F)rdx_1dx_2=\int_\Omega F(\partial_{x_1}r)dx_1dx_2$$

for all $r\in H_0^1(\Omega)$ by integration by parts. Hence we can directly verify that $\|f\|_{H^{-1}(\Omega)}\le C\|F\|_{L^2(\Omega)}$, so that the proof of Theorem 3.1 is complete. ■

§4. Proof of Theorem 2.1 (the beginning).

Henceforth we set

$$D_{x_j}=\frac{1}{i}\partial_{x_j},\qquad j=0,1,2,\text{ etc.},$$

and $\overline{c}$ denotes the complex conjugate of $c\in\mathbb{C}$.

Without loss of generality, we may assume that $\rho\equiv1$. Otherwise we introduce new coefficients $\mu_1=\mu/\rho,\lambda_1=\lambda/\rho$ to argue similarly. We can directly verify that the functions $\mathrm{rot}\mathbf{u}\equiv\partial_{x_1}u_2-\partial_{x_2}u_1$ and $\mathrm{div}\mathbf{u}$ satisfy the equations

$$\partial_{x_0}^2\mathrm{rot}\,\mathbf{u}-\mu\Delta\mathrm{rot}\mathbf{u}=m_1,\quad\partial_{x_0}^2\mathrm{div}\mathbf{u}-(\lambda+2\mu)\Delta\mathrm{div}\mathbf{u}=m_2\qquad\text{in }Q,\tag{4.1}$$

where

$$m_1=K_1\mathrm{rot}\mathbf{u}+K_2\mathrm{div}\mathbf{u}+\mathcal{K}_1\mathbf{u}+\mathrm{rot}\,\mathbf{f},\quad m_2=K_3\mathrm{rot}\mathbf{u}+K_4\mathrm{div}\mathbf{u}+\mathcal{K}_2+\mathrm{div}\,\mathbf{f}$$

and K_j, $\mathcal{K}_k$ are first order differential operators with L^∞ coefficients.

Thanks to Condition 2.1 on the weight function ψ, there exists $\widehat{\tau}$ such that for all $\widetilde{\tau} > \widehat{\tau}$, we have

$$
\begin{aligned}
& s\|(\nabla \mathrm{rot}\, \mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2} + s\|(\nabla \mathrm{div}\, \mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2} \\
+ & s^3\|(\mathrm{rot}\, \mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2} + s^3\|(\mathrm{div}\, \mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2} \\
\le & C_1\left(s^2\|\mathbf{f}e^{s\phi}\|^2_{(L^2(Q))^2} + \|(\nabla \mathbf{f})e^{s\phi}\|^2_{(L^2(Q))^2} + s\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{(H^1((0,T)\times\partial\Omega))^2}\right. \\
+ & \left. s\left\|\frac{\partial^2 \mathbf{u}}{\partial \vec{n}^2}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2} + s^3\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2} + \|\mathbf{u}\|^2_{\mathcal{B}(Q_\omega)}\right), \quad \forall s \ge s_0(\widetilde{\tau}),
\end{aligned} \tag{4.2}
$$

where the constant C_1 is independent of s.

In order to estimate the $H^1(Q)$-norm of the function $\mathbf{u}$, we need the following proposition.

Proposition 4.1. *There exists $\widehat{\tau} > 1$ such that for any $\widetilde{\tau} > \widehat{\tau}$, there exists $s_0(\widetilde{\tau})$ such that*

$$
\begin{aligned}
& \int_Q \left(\frac{1}{s}\sum_{j,k=1}^{2} |\partial_{x_j}\partial_{x_k}\mathbf{u}|^2 + s|\nabla_{x'}\mathbf{u}|^2 + s^3|\mathbf{u}|^2 \right) e^{2s\phi}dx \\
\le & C_2\left(\|(rot\, \mathbf{u})e^{s\phi}\|^2_{H^1(Q)} + \|(div\, \mathbf{u})e^{s\phi}\|^2_{H^1(Q)} + \int_{Q_\omega} (s|\nabla \mathbf{u}|^2 + s^3|\mathbf{u}|^2)e^{2s\phi}dx \right), \\
& \forall s \ge s_0(\widetilde{\tau}),\ \mathbf{u} \in (H^1_0(Q))^2.
\end{aligned} \tag{4.3}
$$

Proof of Proposition 4.1. Denote $\mathrm{rot}\, \mathbf{u} = \mathbf{y}$ and $\mathrm{div}\, \mathbf{u} = \mathbf{w}$ and let $\mathrm{rot}^* v = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\right)$. Using a well-known formula: $\mathrm{rot}^*\mathrm{rot} = -\Delta_{x'} + \nabla_{x'}\mathrm{div}$, we obtain

$$
-\Delta_{x'}\mathbf{u} = -\mathrm{rot}^*\, \mathbf{y} - \nabla_{x'}\mathbf{w} \quad \text{in } \Omega, \quad \mathbf{u}|_{\partial\Omega} = 0.
$$

Then (4.3) follows from the Carleman estimate for an elliptic equations obtained by the first author in [Im1].■

By (4.2) and (4.3), we estimate $\sum_{|\alpha|=0,\alpha=(0,\alpha_1,\alpha_2)}^{2} \|(\partial_x^\alpha \mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}$ via the right hand side of (4.2). Next using this estimate and equation (1.1), we obtain the estimate for the norm $\|(\partial^2_{x_0}\mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}$ via the right hand side of (4.2). Finally we obtain the estimate for $\|(\partial_{x_0}\partial_{x_j}\mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}$ and $s^2\|(\partial_{x_0}\mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}$ by the interpolation argument. Therefore, combining these estimates with (4.2), we have

$$\|\mathbf{u}\|^2_{Y(\phi,Q)} \le C_3\Bigg(s^2\|\mathbf{f}e^{s\phi}\|^2_{(L^2(Q))^2} + \|(\nabla\mathbf{f})e^{s\phi}\|^2_{(L^2(Q))^2}$$
$$+ s\left\|\frac{\partial\mathbf{u}}{\partial\vec{n}}e^{s\phi}\right\|^2_{(H^1((0,T)\times\partial\Omega))^2} + s\left\|\frac{\partial^2\mathbf{u}}{\partial\vec{n}^2}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2}$$
$$+ s^3\left\|\frac{\partial\mathbf{u}}{\partial\vec{n}}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2} + \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}\Bigg), \quad \forall s \ge s_0(\widetilde{\tau}), \tag{4.4}$$

where the constant C_3 is independent of s.

Now we need to estimate the boundary integrals at the right hand side of (4.4). In order to do that, it is convenient to use another weight function φ such that $\varphi|_{\partial\Omega} = \phi|_{\partial\Omega}$ and $\varphi(x) < \phi(x)$ for all $x \in Q$. We construct such a function φ locally near the boundary $\partial\Omega$:

$$\varphi(x) = e^{\widetilde{\tau}\widetilde{\psi}(x)}, \quad \widetilde{\psi}(x) = \psi(x) - \widehat{\epsilon}\ell_1(x') + N\ell_1^2(x'),$$

where $\widehat{\epsilon} > 0$ is a small positive parameter, $N > 0$ is the large positive parameter, and $\ell_1 \in C^3(\overline{\Omega})$ is a function such that

$$\ell_1(x') > 0, \quad \forall x' \in \Omega, \quad \ell_1|_{\partial\Omega} = 0, \; \nabla_{x'}\ell_1|_{\partial\Omega} \ne 0.$$

Denote $\Omega_N = \{x' \in \Omega; \operatorname{dist}(x',\partial\Omega) \le \frac{1}{N^2}\}$. Obviously for any fixed $\widehat{\epsilon} > 0$, there exists $N_0(\widehat{\epsilon})$ such that

$$\varphi(x) < \phi(x), \quad \forall x \in [0,T]\times\Omega_N, \; N \in (N_0,\infty).$$

Now we will prove the following estimate: there exists $\widehat{\tau} > 0$ such that for all

$\widetilde{\tau} > \widehat{\tau}$, there exists $s_0(\widetilde{\tau})$ such that

$$\|\mathbf{u}\|^2_{Y(\varphi,Q)} + N\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|(\partial_x^\alpha \mathbf{u})e^{s\varphi}\|^2_{(L^2(Q))^2} \le C_4(s^2\|\mathbf{f}e^{s\varphi}\|^2_{(L^2(Q))^2}$$
$$+\|(\nabla \mathbf{f})e^{s\varphi}\|^2_{(L^2(Q))^2} + \|\mathbf{u}\|^2_{\mathcal{B}(\varphi,Q_\omega)}), \quad \forall s \ge s_0(\widetilde{\tau}, N),\ \mathrm{supp}\,\mathbf{u} \subset [0,T]\times\Omega_N, \tag{4.5}$$

where the constant C_4 is independent of s and N.

Proof of (4.5). First we note that, thanks to the large parameter N, it suffices to prove (4.5) only locally by assuming

$$\mathrm{supp}\,\mathbf{u} \subset B_\delta \cap ([0,T]\times\Omega_N),$$

where B_δ is the ball of the radius $\delta > 0$ centred at some point y^*. In the case of $B_\delta \cap ((0,T)\times\partial\Omega) = \emptyset$, we can prove in a usual way for a function with compact support (see e.g., [Hö]). Without loss of generality, we may assume that $y^* = (y_0^*, 0, 0)$. Moreover the parameter $\delta > 0$ can be chosen arbitrarily small. Assume that near $(0,0)$, the boundary $\partial\Omega$ is locally given by the equation $x_2 - \ell(x_1) = 0$. Furthermore, since the function $\widetilde{\mathbf{u}} = \mathcal{O}\mathbf{u}(x_0, \mathcal{O}^{-1}x')$ satisfies system (2.1) and (2.2) with $\widetilde{\mathbf{f}} = \mathcal{O}\mathbf{f}(x_0, \mathcal{O}^{-1}x')$ for any orthogonal matrix $\mathcal{O}$, we may assume that

$$\ell'(0) \equiv \frac{d\ell}{dx_1}(0) = 0. \tag{4.6}$$

Making the change of variables $y_1 = x_1$ and $y_2 = x_2 - \ell(x_1)$, we reduce equation (2.1) to the form

$$\begin{cases} \mathbb{P}_1\mathbf{u} = \dfrac{\partial^2 u_1}{\partial y_0^2} - \mu\left(\dfrac{\partial^2 u_1}{\partial y_1^2} - 2\ell'(y_1)\dfrac{\partial^2 u_1}{\partial y_1\partial y_2} + (1+|\ell'(y_1)|^2)\dfrac{\partial^2 u_1}{\partial y_2^2}\right) + \mu\ell''(y_1)\dfrac{\partial u_1}{\partial y_2} \\ -(\lambda+\mu)\dfrac{\partial}{\partial y_1}\left(\mathrm{div}\,\mathbf{u} - \dfrac{\partial u_1}{\partial y_2}\ell'\right) + (\lambda+\mu)\dfrac{\partial}{\partial y_2}\left(\mathrm{div}\,\mathbf{u} - \dfrac{\partial u_1}{\partial y_2}\ell'\right)\ell' + \widetilde{K}_1\mathbf{u} = f_1, \\ \mathbb{P}_2\mathbf{u} = \dfrac{\partial^2 u_2}{\partial y_0^2} - \mu\left(\dfrac{\partial^2 u_2}{\partial y_1^2} - 2\ell'(y_1)\dfrac{\partial^2 u_2}{\partial y_1\partial y_2} + (1+|\ell'(y_1)|^2)\dfrac{\partial^2 u_2}{\partial y_2^2}\right) + \mu\ell''(y_1)\dfrac{\partial u_2}{\partial y_2} \\ -(\lambda+\mu)\dfrac{\partial}{\partial y_2}\left(\mathrm{div}\,\mathbf{u} - \dfrac{\partial u_1}{\partial y_2}\ell'\right) + \widetilde{K}_2\mathbf{u} = f_2, \end{cases} \tag{4.7}$$

where we use the same notations $\mathbf{u}, \mathbf{f}$ after the change of variables and $\widetilde{K}_1, \widetilde{K}_2$ are partial differential operators of the first order. We set $\mathbb{P} = (\mathbb{P}_1, \mathbb{P}_2)$. After the change of variables, equations (4.1) have the form

$$P_\mu z_1 = \frac{\partial^2 z_1}{\partial y_0^2} - \mu\left(\frac{\partial^2 z_1}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 z_1}{\partial y_1 \partial y_2} + (1+|\ell'(y_1)|^2)\frac{\partial^2 z_1}{\partial y_2^2}\right)$$
$$+\mu\ell''(y_1)\frac{\partial z_1}{\partial y_2} = m_1 \quad \text{in } \mathcal{G}_N \triangleq \mathbb{R}^2 \times \left[0, \frac{\widehat{\kappa}}{N^2}\right], \tag{4.8}$$

$$P_{\lambda+2\mu} z_2 = \frac{\partial^2 z_2}{\partial y_0^2} - (\lambda+2\mu)\left(\frac{\partial^2 z_2}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 z_2}{\partial y_1 \partial y_2} + (1+|\ell'(y_1)|^2)\frac{\partial^2 z_2}{\partial y_2^2}\right)$$
$$+(\lambda+2\mu)\ell''(y_1)\frac{\partial z_2}{\partial y_2} = m_2 \quad \text{in } \mathcal{G}_N. \tag{4.9}$$

Here we set

$$z_1 = \frac{\partial u_2}{\partial y_1} - \frac{\partial u_2}{\partial y_2}\ell'(y_1) - \frac{\partial u_1}{\partial y_2}, \quad z_2 = \frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2} - \frac{\partial u_1}{\partial y_2}\ell'(y_1),$$

we use the same notations m_1, m_2 after the change of variables and the constant $\widehat{\kappa} > 0$ is chosen sufficiently large such that the image of $[0, T] \times \Omega_N$ belongs to $\mathcal{G}_N$. Henceforth we write $(z_1, z_2) = R(y, D)\mathbf{u}$.

Now we claim that in order to prove estimate (4.5), it suffices to establish the following estimate for the function $\mathbf{w} = (w_1, w_2) = e^{s\varphi}(z_1, z_2) = e^{s\varphi}R(y, D)\mathbf{u}$:

$$\|\mathbf{w}\|_*^2 \equiv s\|\mathbf{w}\|^2_{(H^1(\mathcal{G}_N))^2} + s^3\|\mathbf{w}\|^2_{(L^2(\mathcal{G}_N))^2} + s\left\|\frac{\partial \mathbf{w}}{\partial y_2}\right\|^2_{(L^2(\partial\mathcal{G}_N))^2} + s\|\mathbf{w}\|^2_{(H^1(\partial\mathcal{G}_N))^2}$$
$$+s^3\|\mathbf{w}\|^2_{(L^2(\partial\mathcal{G}_N))^2} \le C_5(\|\mathbb{P}\mathbf{u}e^{s\varphi}\|^2_{(H^1(\mathcal{G}_N))^2} + s^2\|\mathbb{P}\mathbf{u}e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2} + s\|\mathbf{g}\|^2_{(L^2(\partial\mathcal{G}_N))^2}$$
$$+\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|(\partial^\alpha_{y'}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}), \quad \forall s \ge s_0(\widetilde{\tau}, N), \tag{4.10}$$

for all $\mathbf{u} \in H^2(\mathcal{G}_N)$ satisfying $\mathbf{u}|_{\partial\mathcal{G}_N} = 0$ and $\mathrm{supp}\mathbf{u} \subset B_\delta \cap \mathcal{G}_N$. Obviously the function $\mathbf{w}$ satisfies the boundary condition

$$\frac{\partial w_1}{\partial y_2} = \frac{\lambda+2\mu}{\mu}\frac{\partial w_2}{\partial y_1} + s\varphi_{y_2}(y^*)w_1 - s\frac{\lambda+2\mu}{\mu}\varphi_{y_1}(y^*)w_2 + g_1, \quad \text{on } \partial\mathcal{G}_N, \tag{4.11}$$

$$\frac{\partial w_2}{\partial y_2} = -\frac{\mu}{\lambda+2\mu}\frac{\partial w_1}{\partial y_1} + s\varphi_{y_2}(y^*)w_2 + s\frac{\mu}{\lambda+2\mu}\varphi_{y_1}(y^*)w_1 + g_2, \quad \text{on } \partial\mathcal{G}_N, \tag{4.12}$$

where the function $\mathbf{g} = (g_1, g_2)$ satisfies the estimate

$$s\|\mathbf{g}\|^2_{(L^2(\partial\mathcal{G}_N))^2} \le \epsilon(\delta)\left(s\left\|\frac{\partial \mathbf{w}}{\partial y_2}\right\|^2_{(L^2(\partial\mathcal{G}_N))^2} + s\|\mathbf{w}\|^2_{(H^1(\partial\mathcal{G}_N))^2}\right.$$
$$\left.+s^3\|\mathbf{w}\|^2_{(L^2(\partial\mathcal{G}_N))^2}\right) + C_6 s\|\mathbb{P}\mathbf{u}e^{s\varphi}\|^2_{(L^2(\partial\mathcal{G}_N))^2}, \tag{4.13}$$

and $\lim_{\delta\to 0}\epsilon(\delta) = 0$.

The boundary conditions (4.11) and (4.12) with property (4.13) follow from (4.8), (4.9) and the zero Dirichlet boundary condition for $\mathbf{u}$. In order to deduce (4.5) from estimate (4.10), we need

$$\|\mathbf{u}\|^2_{Y(\varphi,\mathcal{G}_N)} \le C_7(\|\mathbf{w}\|^2_* + \|\mathbb{P}\mathbf{u}e^{s\varphi}\|^2_{(H^1(\mathcal{G}_N))^2} + s^2\|\mathbb{P}\mathbf{u}e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}) \tag{4.14}$$

and the following proposition:

Proposition 4.2. *There exist $\widehat{\tau} > 1$ and $N_0 > 1$ such that for any $\widetilde{\tau} > \widehat{\tau}$ and $N > N_0(\widetilde{\tau})$, there exists $s_0(\widetilde{\tau}, N)$ such that*

$$N\int_{\mathcal{G}_N}\left(\frac{1}{s\varphi}\sum_{j,k=1}^{2}|\partial_{y_j}\partial_{y_k}\mathbf{u}|^2 + s\varphi|\nabla_{y'}\mathbf{u}|^2 + s^3\varphi^3|\mathbf{u}|^2\right)e^{2s\varphi}dy'$$
$$\le C_8(\|z_1\, e^{s\varphi}\|^2_{H^1(\mathcal{G}_N)} + \|z_2 e^{s\varphi}\|^2_{H^1(\mathcal{G}_N)}), \quad \forall \mathbf{u}\in (H^1_0(\mathcal{G}_N))^2,\ supp\,\mathbf{u}\subset B_\delta\cap\mathcal{G}_N, \forall s\ge s_0(\widetilde{\tau}, N),$$

where the constant C_8 is independent of N.

We give the proof of Proposition 4.2 in Appendix I.

Thanks to Proposition 4.2 and equations (4.7), we obtain

$$N\|(\partial^2_{y_0}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2} + \sum_{|\alpha|=0,\alpha=(0,\alpha_1,\alpha_2)}^{2} N s^{4-2|\alpha|}\|(\partial^\alpha_{y'}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}$$
$$\le C_9(\|\mathbf{u}\|^2_{Y(\varphi,\mathcal{G}_N)} + N\|\mathbb{P}\mathbf{u}e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}) \quad \forall s\ge s_0(\widetilde{\tau}, N). \tag{4.15}$$

By (4.13), (4.14) and (4.15), we obtain

$$N\|(\partial_{y_0}^2\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}+\sum_{|\alpha|=0,\alpha=(0,\alpha_1,\alpha_2)}^{2} Ns^{4-2|\alpha|}\|(\partial_{y'}^\alpha\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}+\|\mathbf{u}\|^2_{Y(\varphi,\mathcal{G}_N)}$$
$$\le C_{10}(\|\nabla(\mathbb{P}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}+s^2\|\mathbb{P}\mathbf{u}e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2})\quad\forall s\ge\max\{s_0(\widetilde{\tau},N),N\}. \tag{4.16}$$

Finally, combining (4.16) with the estimates

$$s^2\|(\partial_{y_0}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}\le C_{11}(\|(\partial_{y_0}^2\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}+s^4\|\mathbf{u}e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2})$$

and

$$\|(\partial_{y_0}\partial_{y_k}\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2}\le C_{11}\sum_{j=0}^{2}\|(\partial_{y_j}^2\mathbf{u})e^{s\varphi}\|^2_{(L^2(\mathcal{G}_N))^2},\quad k\in\{1,2\},$$

we obtain (4.5).

We set $P_{\mu,s}=e^{|s|\varphi}P_\mu e^{-|s|\varphi}$ and $P_{\lambda+2\mu,s}=e^{|s|\varphi}P_{\lambda+2\mu}e^{-|s|\varphi}$. By $\mathbf{p}(y,\xi_0,\xi_1,\xi_2)$ and $p_\beta(y,\xi_0,\xi_1,\xi_2)$ with $\beta=\mu$ or $\lambda+2\mu$, we denote the principal symbols of the operators $\mathbb{P}$ and P_β respectively. In order to prove the Carleman estimate (4.10) it is convenient for us to introduce a new variable σ and consider s as a dual variable to σ. Following [T1, Chapter 14], we consider the pseudo-differential operators defined by

$$\mathbf{P}_\beta(y,D_\sigma,D_{y_0},D_{y_1},D_{y_2})v$$
$$=\int_{\mathbb{R}^3}p_\beta(y,\xi_0+i|s|\varphi_{y_0},\xi_1+i|s|\varphi_{y_1},D_{y_2}+i|s|\varphi_{y_2})\widehat{v}(s,\xi_0,\xi_1,y_2)e^{i(<y',\xi'>+\sigma s)}d\sigma d\xi',$$
$$\mathbb{P}_\sigma(y,D_\sigma,D_{y_0},D_{y_1},D_{y_2})v$$
$$=\int_{\mathbb{R}^3}\mathbf{p}(y,\xi_0+i|s|\varphi_{y_0},\xi_1+i|s|\varphi_{y_1},D_{y_2}+i|s|\varphi_{y_2})\widehat{v}(s,\xi_0,\xi_1,y_2)e^{i(<y',\xi'>+\sigma s)}d\sigma d\xi',$$

where $\xi'=(\xi_0,\xi_1), y'=(y_0,y_1)$ and $\widehat{v}(s,\xi_0,\xi_1,y_2)$ is the Fourier transform of $v(\sigma,y_0,y_1,y_2)$ with respect to σ,y_0,y_1. Let $\mathbf{v}(\sigma,y)=(v_1(\sigma,y),v_2(\sigma,y))$ be a function with the domain $\mathcal{Q}=\mathbb{R}^1_+\times\mathbb{R}^3$. Henceforth $\mathcal{F}_\sigma$ denotes the Fourier transform with respect to the variable σ. Let $h(s) = (1+s^2)^{\frac{1}{4}}$, $\Sigma = \partial\mathcal{Q}$. Moreover we set

$\mathbf{g} = (g_1, g_2)$,

$$R_s(y,D)\mathcal{U} = e^{|s|\phi}R(y,D)e^{-|s|\phi}\mathcal{U}, \tag{4.17}$$

and

$$\begin{cases} B_1\mathbf{w} \triangleq -\dfrac{\partial w_1}{\partial y_2} + \dfrac{\lambda+2\mu}{\mu}\dfrac{\partial w_2}{\partial y_1} + |s|\varphi_{y_2}(y^*)w_1 - |s|\dfrac{\lambda+2\mu}{\mu}\varphi_{y_1}(y^*)w_2, \\ B_2\mathbf{w} \triangleq -\dfrac{\partial w_2}{\partial y_2} - \dfrac{\mu}{\lambda+2\mu}\dfrac{\partial w_1}{\partial y_1} + |s|\varphi_{y_2}(y^*)w_2 + |s|\dfrac{\mu}{\lambda+2\mu}\varphi_{y_1}(y^*)w_1, \quad \text{on } \Sigma \end{cases}$$

for $\mathbf{w} = (w_1, w_2)$, provided that the right hand sides are well-defined.

Then we claim that in order to prove (4.5), it suffices to establish the following estimate

$$\begin{aligned} |||\mathbf{v}|||^2 \triangleq \sum_{j=0}^{1} \|h(D_\sigma)^{3-2j}\mathbf{v}\|^2_{L^2(\mathbb{R}^1;(H^j(\mathcal{G}_N))^2)} + \|h(D_\sigma)^{3-2j}\mathbf{v}\|^2_{(H^j(\Sigma))^2} + \left\|h(D_\sigma)\frac{\partial \mathbf{v}}{\partial y_2}\right\|^2_{(L^2(\Sigma))^2} \\ \le C_{12}(\|\mathbb{P}_\sigma(y,D)\mathcal{F}_\sigma^{-1}\mathcal{U}\|^2_{(H^1(\mathcal{Q}))^2} + \|h(D_\sigma)\mathcal{F}_\sigma^{-1}\mathbf{g}\|^2_{(L^2(\Sigma))^2} + \|\mathcal{U}\|^2_{(H^2(\mathcal{Q}))^2}), \end{aligned} \tag{4.18}$$

if $\mathcal{U}$ and $\mathbf{v}$ satisfy supp$\mathcal{U} \subset \mathbb{R}^1 \times (B_\delta \cap \mathcal{G}_N)$, supp $\mathcal{F}_\sigma^{-1}\mathcal{U} \subset (-\sigma_0, \sigma_0) \times (B_\delta \cap \mathcal{G}_N)$ with arbitrarily small parameter $\sigma_0 > 0$, and

$$\begin{cases} R_s(y,D)\mathcal{U} = \mathcal{F}_\sigma\mathbf{v}, \quad \mathcal{U}|_\Sigma = 0 \\ B_1(\mathcal{F}_\sigma\mathbf{v}) = g_1, \quad B_2(\mathcal{F}_\sigma\mathbf{v}) = g_2 \quad \text{on } \Sigma. \end{cases}$$

We set

$$\mathcal{F}_\sigma\mathbf{v} = \mathbf{w}.$$

Then

$$(B_1\mathbf{w},\, B_2\mathbf{w}) = (g_1, g_2) \equiv \mathbf{g}. \tag{4.19}$$

This fact can be proved exactly in the same way as in [T1, Chapter 14, Section 2]. Consider the finite covering of the unit sphere $S^2 \equiv \{(s,\xi_0,\xi_1);\, s^2+\xi_0^2+\xi_1^2 = 1\}$:

$S^2 \subset \cup_{\zeta^* \in S^2}\{\zeta = (s, \xi_0, \xi_1) \in S^2; |\zeta - \zeta^*| < \delta_1\}$ and the partition of unity $\chi_\nu(\zeta)$: $\sum_{\nu=1}^{K(\delta_1)} \chi_\nu(\zeta) = 1$ for any $\zeta \in S^2$ and $\operatorname{supp}\chi_\nu \subset \{\zeta \in S^2; |\zeta - \zeta^*_\nu| < \delta_1\}$.

We extend the function χ_ν on the set $|\zeta| > 1$ as the homogeneous function of the order zero in such a way that

$$\operatorname{supp}\chi_\nu \subset \mathcal{O}(\delta_1) \equiv \left\{\zeta; \left|\frac{\zeta}{|\zeta|} - \zeta^*\right| < \delta_1\right\}.$$

We set $D' = (D_\sigma, D_{y_0}, D_{y_1})$, and consider the pseudo-differential operator $\chi_\nu(D')$ and the function $\chi_\nu(D')\mathbf{v}$. Obviously equalities (4.19) hold true with $\mathbf{w}$ and $\mathbf{g}$ replaced by $\mathbf{w}_\nu = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\chi_\nu(D')\mathbf{v}e^{-is\sigma}d\sigma$ and $\mathbf{g}_\nu = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\chi_\nu(D')\mathcal{F}_\sigma^{-1}\mathbf{g}e^{-is\sigma}d\sigma$.

Moreover we claim that instead of (4.18), it suffices to prove the following estimate

$$|||\chi_\nu(D')\mathbf{v}||| \le C_{13}(\|\mathbb{P}_\sigma\chi_\nu(D')\mathcal{F}_\sigma^{-1}\mathcal{U}\|_{(H^1(\mathcal{Q}))^2}$$
$$+\|h(D_\sigma)\chi_\nu(D')\mathcal{F}_\sigma^{-1}\mathbf{g}\|_{(L^2(\Sigma))^2} + \|\mathcal{U}\|_{(H^2(\mathcal{Q}))^2}), \tag{4.20}$$

where

$$R_s(y, D')\mathcal{U} = \mathcal{F}_\sigma\mathbf{v}, \quad \mathcal{U}|_\Sigma = 0, \quad \operatorname{supp}\mathcal{F}_\sigma^{-1}\mathcal{U} \subset (-\sigma_0, \sigma_0) \times (B_\delta \cap \mathcal{G}_N),$$

$$B_1(w_{1,\nu}, w_{2,\nu}) = g_{1,\nu}, \quad B_2(w_{1,\nu}, w_{2,\nu}) = g_{2,\nu} \tag{4.21}$$

and C_{13} is independent of N. In fact, assume that estimate (4.20) is already proved.

Then

$$
\begin{aligned}
&|||\mathbf{v}|||^2 \le \sum_{\nu=1}^{K(\delta_1)} |||\chi_\nu(D')\mathbf{v}|||^2\\
\le& C_{14}\sum_{\nu=1}^{K}(\Vert\mathbb{P}_\sigma(y,D)\chi_\nu\mathcal{F}_\sigma^{-1}\mathcal{U}\Vert^2_{(H^1(\mathcal{Q}))^2}+\Vert h(s)\mathbf{g}_\nu\Vert^2_{(L^2(\Sigma))^2}+\Vert\chi_\nu(D')\mathcal{F}_\sigma^{-1}\mathcal{U}\Vert^2_{(H^2(\mathcal{Q}))^2})\\
\le& C_{15}\sum_{\nu=1}^{K}(\Vert\chi_\nu(D')\mathbb{P}_\sigma(y,D)\mathcal{F}_\sigma^{-1}\mathcal{U}\Vert^2_{(H^1(\mathcal{Q}))^2}+\Vert[\chi_\nu(D'),\mathbb{P}_\sigma(y,D')]\mathcal{F}_\sigma^{-1}\mathcal{U}\Vert^2_{(L^2(\mathcal{Q}))^2}\\
&+\Vert h(s)\mathbf{g}_\nu\Vert^2_{(L^2(\Sigma))^2}+\Vert\chi_\nu(D')\mathcal{U}\Vert^2_{(H^2(\mathcal{Q}))^2})\\
\le& C_{16}(\Vert\mathbb{P}_\sigma(y,D)\mathcal{F}_\sigma^{-1}\mathcal{U}\Vert^2_{(H^1(\mathcal{Q}))^2}+\Vert h(s)\mathbf{g}\Vert^2_{(L^2(\Sigma))^2}+\Vert\mathcal{U}\Vert^2_{(H^2(\mathcal{Q}))^2}),
\end{aligned}
$$

where $K=K(\delta_1)$ and C_{16} are independent of N.

The rest of this section and Sections 5 - 7 is devoted to verification of (4.20).

The principal symbol of the operator $P_{\beta,s}$ has the form

$$
\begin{aligned}
&p_\beta(y,s,\xi_0,\xi_1)=-(\xi_0+i|s|\varphi_{y_0})^2+\beta[(\xi_1+i|s|\varphi_{y_1})^2-2\ell'(\xi_1+i|s|\varphi_{y_1})(\xi_2+i|s|\varphi_{y_2})\\
&+(\xi_2+i|s|\varphi_{y_2})^2|G|^2], \qquad (4.22)
\end{aligned}
$$

where $|G|^2=1+(\ell'(y_1))^2$. The roots of this polynomial with respect to the variable ξ_2, are

$$
\Gamma^\pm_\beta(y,s,\xi_0,\xi_1)=-i|s|\varphi_{y_2}(y)+\alpha^\pm_\beta(y,s,\xi_0,\xi_1), \qquad (4.23)
$$

$$
\alpha^\pm_\beta(y,s,\xi_0,\xi_1)=\frac{(\xi_1+i|s|\varphi_{y_1}(y))\ell'(y_1)}{|G|^2}\pm\sqrt{r_\beta(y,s,\xi_0,\xi_1)}, \qquad (4.24)
$$

$$
r_\beta(y,\zeta)=\frac{((\xi_0+i|s|\varphi_{y_0}(y))^2-\beta(\xi_1+i|s|\varphi_{y_1}(y))^2)|G|^2+\beta(\xi_1+i|s|\varphi_{y_1})^2(\ell')^2}{\beta|G|^4}, \qquad (4.25)
$$

where $\sqrt{r_\beta}$ is defined below.

Denote $\gamma=(y^*,\zeta^*)=(y^*,s^*,\xi_0^*,\xi_1^*)$. Suppose that $|r_\beta(\gamma)|\ge 2\widehat{\delta}>0$. Now we claim that there exists $\delta_0(\widehat{\delta})>0$ such that for all $\delta,\delta_1\in(0,\delta_0)$, there exists a constant $C_{20} > 0$, independent of s, such that for one of the roots of the polynomial (4.22), which we denote by Γ^-_β, we have

$$-\operatorname{Im}\Gamma^-_\beta(y,s,\xi_0,\xi_1) \ge C_{20}|s|, \quad \forall (y,s,\xi_0,\xi_1) \in B_\delta \times \mathcal{O}(\delta_1). \tag{4.26}$$

Proof of (4.26). If $Im\sqrt{r_\beta(\gamma)} \neq 0$, then the statement (4.26) is trivial. So it suffices to consider the case $Im\sqrt{r_\beta(\gamma)} = 0$. Let $\theta \in (0, \frac{1}{8})$ be constant. We may assume that $\operatorname{Re} r_\beta(\gamma) \ge (1-\theta)|r_\beta(\gamma)|$. Obviously there exists $\widetilde{\delta}(\theta)$ such that for all $\delta, \delta_1 \in (0, \widetilde{\delta}(\theta))$,

$$\operatorname{Re} r_\beta(y,\zeta) \ge (1-2\theta)|r_\beta(y,\zeta)|, \quad \forall (y,s,\xi_0,\xi_1) \in B_\delta \times \mathcal{O}(\delta_1).$$

Then

$$|\operatorname{Im} r_\beta(y,\zeta)| \le \frac{2\theta}{1-2\theta}\operatorname{Re} r_\beta(y,\zeta), \quad \forall (y,s,\xi_0,\xi_1) \in B_\delta \times \mathcal{O}(\delta_1).$$

We denote $b(y,\zeta) = \operatorname{Im} r_\beta(y,\zeta)$ and $a(y,\zeta) = \operatorname{Re} r_\beta(y,\zeta)$ with $\zeta = (s,\xi_0,\xi_1)$. First, if $\operatorname{Im}\sqrt{r_\beta(\gamma)} = 0$ we have $a(\gamma) > 0$ and $b(\gamma) = 0$. In that case we can define the function $\sqrt{r_\beta(y,\zeta)}$ by the infinite series

$$(1+x)^{\frac{1}{2}} = \sum_{n=0}^\infty c_n x^n, \quad |x| < 1,$$

where $c_n = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-(n-1))}{n!}$.

That is, assuming that $|\frac{b}{a}| < \frac{2\theta}{1-2\theta} < \frac{1}{2}$ for all $(y,s,\xi_0,\xi_1) \in B_\delta \times \mathcal{O}(\delta_1)$, we set

$$\sqrt{r_\beta(y,\zeta)} = \sqrt{a}\sum_{n=0}^\infty c_n \left(\frac{ib}{a}\right)^n = \sqrt{a} + \frac{i}{2}|s|\left(\frac{b}{|s|\sqrt{a}}\right) - |s|\left(\frac{b}{a}\right)\frac{b}{|s|\sqrt{a}}\sum_{n=0}^\infty c_{n+2}\left(\frac{ib}{a}\right)^n. \tag{4.27}$$

The first term in infinite series (4.27) is real, and the absolute value of the third term is $\left||s|\frac{b}{|s|\sqrt{a}}\right| O(\theta)$. The function $\frac{b}{|s|\sqrt{a}}$ is a continuous homogeneous function of the order zero in the variable ζ.

If $\frac{b(\gamma)}{|s^*|\sqrt{a(\gamma)}} \le 0$, then we take $\Gamma^-_\beta(y,\zeta) = -i|s|\frac{\partial\varphi}{\partial y_2} + \alpha^-_\beta(y,\zeta)$ where $\alpha^-_\beta(y,\zeta)$ equals the right hand side of (4.27) plus $(\xi_1+i|s|\varphi_{y_1})\ell'(y_1)/|G|^2$. Otherwise $\Gamma^-_\beta(y,\zeta) = -i|s|\frac{\partial\varphi}{\partial y_2} + \alpha^+_\beta(y,\zeta)$ where $\alpha^+_\beta(y,\zeta)$ equals the right hand side of (4.27) multiplied by -1 plus $(\xi_1+i|s|\varphi_{y_1})\ell'(y_1)/|G|^2$.

For $\frac{b}{|s^*|\sqrt{a}}(\gamma) \le 0$, we obtain that $\frac{b}{|s|\sqrt{a}}(\gamma) - \frac{1}{2}\varphi_{y_2}(y) < 0$ for all $(y,s,\xi_0,\xi_1) \in B_\delta \times \mathcal{O}(\delta_1)$ and for $\frac{b}{|s^*|\sqrt{a}}(\gamma) \ge 0$ we obtain that $-\frac{b}{|s|\sqrt{a}}(\gamma) - \frac{1}{2}\varphi_{y_2}(y) < 0$ for all $(y,s,\xi_0,\xi_1) \in B_\delta \times \mathcal{O}(\delta_1)$. These inequalities imply (4.26) provided that δ_0 taken sufficiently small.

Under some conditions, we can see that the operator $\mathbf{P}_\beta$ can be represented as a product of two first order pseudo-differential operators:

Proposition 4.3. *Let $\beta \in \{\mu, \lambda+2\mu\}$ and $|r_\beta(y,\zeta)| \ge \widehat{\delta} > 0$ for all $(y,\zeta) \in B_\delta \times \mathcal{O}(2\delta_1)$. Then we can factorize the operator $\mathbf{P}_\beta$ into the product of two first order pseudo-differential operators:*

$$\mathbf{P}_\beta\chi_\nu(D')V = \beta|G|^2(D_{y_2} - \Gamma^-_\beta(y,D'))(D_{y_2} - \Gamma^+_\beta(y,D'))\chi_\nu(D')V + T_\beta V, \quad (4.28)$$

where $\mathrm{supp}\, V \subset B_\delta \cap \mathcal{G}_N$ and T_β is a continuous operator:

$$T_\beta : L^2(0,1;H^1(\mathbb{R}^3)) \to L^2(0,1;L^2(\mathbb{R}^3)).$$

Let us consider the equation

$$(D_{y_2} - \Gamma^-_\beta(y,D'))\chi_\nu(D')V = q, \quad V|_{y_2=1} = 0, \quad \text{supp } V \subset B_\delta \cap \mathcal{G}_N.$$

For solutions of this problem, we have an a priori estimate:

Proposition 4.4. *Let $\beta \in \{\mu, \lambda+2\mu\}$ and $|r_\beta(y,\zeta)| \ge \widehat{\delta} > 0$ for all $(y,\zeta) \in B_\delta \times \mathcal{O}(2\delta_1)$. Then there exists a constant $C_{22} > 0$, which is independent of N,* *such that*

$$\|h(D_\sigma)\chi_\nu(D')V|_{y_2=0}\|_{L^2(\mathbb{R}^3)} \le C_{22}\|q\|_{L^2(\mathcal{Q})}. \tag{4.29}$$

Proof of Proposition 4.4. Taking the scalar product of q and $h^2(D_\sigma)\chi_\nu(D')V$ for fixed y_2, we obtain

$$2\mathrm{Re}\,(q(y_2), h^2(D_\sigma)\chi_\nu(D')V(y_2))_{L^2(\Sigma)}e^{2\widetilde{\kappa}y_2} = \frac{\partial}{\partial y_2}\left(e^{2\widetilde{\kappa}y_2}\|h(D_\sigma)\chi_\nu(D')V(y_2)\|^2_{L^2(\Sigma)}\right)$$
$$- 2\mathrm{Re}\,(i\Gamma^-_\beta(y, D')\chi_\nu(D')V + \widetilde{\kappa}\chi_\nu(D')V, h^2(D_\sigma)\chi_\nu(D')V)_{L^2(\Sigma)}e^{2\widetilde{\kappa}y_2}.$$

By (4.26) and Proposition 2.4.A in [T2], for sufficiently large positive κ, we have

$$\mathrm{Re}\,(i\Gamma^-_\beta(y, D')h^{-2}(D_\sigma)h^2(D_\sigma)\chi_\nu(D')V + \widetilde{\kappa}\chi_\nu(D')V, h^2(D_\sigma)\chi_\nu(D')V)_{L^2(\Sigma)}$$
$$\ge C_{23}\|h^2(D_\sigma)\chi_\nu(D')V\|^2_{L^2(\Sigma)}.$$

Thus

$$2\mathrm{Re}\,(q(y_2), h^2(D_\sigma)\chi_\nu(D')V(y_2))_{L^2(\Sigma)}e^{2\widetilde{\kappa}y_2}$$
$$\le\frac{\partial}{\partial y_2}\left(e^{2\widetilde{\kappa}y_2}\|h(D_\sigma)\chi_\nu(D')V(y_2)\|^2_{L^2(\Sigma)}\right) - C_{24}\|h^2(D_\sigma)\chi_\nu(D')V(y_2)\|^2_{L^2(\Sigma)}e^{2\widetilde{\kappa}y_2},$$

and (4.29) follows from Gronwall's inequality. ■

Let $\widetilde{w}(s, y)$ satisfy a scalar second order hyperbolic equation

$$P_{\beta,s}\widetilde{w} = q \quad \text{in } \mathcal{G}_N, \quad \frac{\partial\widetilde{w}}{\partial y_2}|_{y_2=1} = \widetilde{w}|_{y_2=1} = 0, \quad \mathrm{supp}\,\widetilde{w} \subset \mathbb{R}^1 \times (B_\delta \cap \mathcal{G}_N)$$

for almost all $s \in \mathbb{R}^1$. Let $P^*_{\beta,s}$ be the formally adjoint operator to $P_{\beta,s}$, where $\beta \in \{\mu, \lambda + 2\mu\}$. Set

$$L_{+,\beta} = \frac{P_{\beta,s} + P^*_{\beta,s}}{2}, \quad L_{-,\beta} = \frac{P_{\beta,s} - P^*_{\beta,s}}{2}.$$

One can easily check that the principal part operator $L_{-,\beta}$ is given by formula

$$L_{-,\beta}\widetilde{w} = -2|s|\varphi_{y_0}\frac{\partial\widetilde{w}}{\partial y_0} + \beta\Biggl(2|s|\varphi_{y_1}\frac{\partial\widetilde{w}}{\partial y_1} - 2|s|\ell'(y_1)\left(\varphi_{y_2}\frac{\partial\widetilde{w}}{\partial y_1} + \varphi_{y_1}\frac{\partial\widetilde{w}}{\partial y_2}\right)$$
$$+2|s|(1+(\ell'(y_1))^2)\varphi_{y_2}\frac{\partial\widetilde{w}}{\partial y_2}\Biggr).$$

Obviously $L_{+,\beta}\widetilde{w} + L_{-,\beta}\widetilde{w} = q$. For almost all $s \in \mathbb{R}^1$, the following equality holds true:

$$B_\beta + \|L_{-,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \|L_{+,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \mathrm{Re}\int_{\mathcal{G}_N}([L_{+,\beta}, L_{-,\beta}]\widetilde{w}, \widetilde{w})dy$$
$$= \|q\|^2_{L^2(\mathcal{G}_N)}, \tag{4.30}$$

where

$$B_\beta = \mathrm{Re}\int_{\partial\mathcal{G}_N}\widetilde{p}_\beta(y, \nabla\varphi, -\vec{e}_3)(|s|\widetilde{p}_\beta(y, \nabla\widetilde{w}) - |s|^3\widetilde{p}_\beta(y, \nabla\varphi, \nabla\varphi)\widetilde{w}^2)dy_0dy_1$$
$$+\mathrm{Re}\int_{\partial\mathcal{G}_N}\widetilde{p}_\beta(y, \nabla\widetilde{w}, -\vec{e}_3)\overline{L_{-,\beta}\widetilde{w}}dy_0dy_1, \tag{4.31}$$

$\vec{e}_3 = (0,0,1)$ and

$$\widetilde{p}_\beta(y,\xi,\widetilde{\xi}) = \xi_0\widetilde{\xi}_0 - \beta(\xi_1\widetilde{\xi}_1 - \ell'(y_1)(\xi_1\widetilde{\xi}_2 + \xi_2\widetilde{\xi}_1) + (1+|\ell'(y_1)|^2)\xi_2\widetilde{\xi}_2).$$

We note that $\phi_{y_k}|_\Sigma = \varphi_{y_k}|_\Sigma$ for $k \in \{0,1\}$ and $\varphi_{y_2}|_\Sigma = (\phi_{y_2} - \widehat{\epsilon}\tau(\partial_{y_2}\ell_1)\phi)|_\Sigma$. Therefore on Σ the function $\nabla\varphi$ is independent of N and $|\nabla\phi(y) - \nabla\varphi(y)| \le C_{25}\widehat{\epsilon}$ for all $y \in \Sigma$ where $C_{25} > 0$ is independent of $\widehat{\epsilon}$ and N. In particular, taking $\widehat{\epsilon}$ sufficiently small, we have (2.6) for the function φ. It is convenient for us to rewrite (4.31) in the form

$$B_\beta = B^{(1)}_\beta + B^{(2)}_\beta,$$

$$B^{(1)}_\beta \equiv \mathrm{Re}\int_{y_2=0}2|s|\beta\frac{\partial\widetilde{w}}{\partial y_2}\overline{\left(\beta\frac{\partial\widetilde{w}}{\partial y_1}\varphi_{y_1}(y^*) + \beta\frac{\partial\widetilde{w}}{\partial y_2}\varphi_{y_2}(y^*) - \frac{\partial\widetilde{w}}{\partial y_0}\varphi_{y_0}(y^*)\right)}dy_0dy_1$$
$$+\int_{y_2=0}|s|\beta\varphi_{y_2}(y^*)\Biggl\{\left|\frac{\partial\widetilde{w}}{\partial y_0}\right|^2 - \beta\left(\left|\frac{\partial\widetilde{w}}{\partial y_1}\right|^2 + \left|\frac{\partial\widetilde{w}}{\partial y_2}\right|^2\right)$$
$$-|s|^2(\varphi^2_{y_0}(y^*) - \beta(\varphi^2_{y_1}(y^*) + \varphi^2_{y_2}(y^*))|\widetilde{w}|^2\Biggr\}dy_0dy_1.$$

Then

$$|B^{(2)}_\beta| \le \epsilon_0\left(|s|\left\|\frac{\partial\widetilde w}{\partial y_2}\right\|^2_{L^2(\partial\mathcal G_N)} + |s|\|\widetilde w\|^2_{H^1(\partial\mathcal G_N)} + |s|^3\|\widetilde w\|^2_{L^2(\partial\mathcal G_N)}\right), \tag{4.32}$$

where $\epsilon_0 = \epsilon_0(\delta)\to 0$ as $|\delta|\to 0$. It is known (see e.g., [Im2]) that there exists a parameter $\widehat\tau > 1$ such that for any $\widetilde\tau > \widehat\tau$, there exists $s_0(\widetilde\tau)$ such that

$$\|L_{-,\beta}\widetilde w\|^2_{L^2(\mathcal G_N)} + \|L_{+,\beta}\widetilde w\|^2_{L^2(\mathcal G_N)} + \mathrm{Re}\int_{\mathcal G_N}([L_{+,\beta},L_{-,\beta}]\widetilde w,\widetilde w)dy$$
$$+C'_{26}|s|\|\widetilde w\|_{L^2(\partial\mathcal G_N)}\|\partial_{y_2}\widetilde w\|_{L^2(\partial\mathcal G_N)} \ge C_{26}(|s|\|\widetilde w\|^2_{H^1(\mathcal G_N)}+|s|^3\|\widetilde w\|^2_{L^2(\mathcal G_N)}),\quad \forall |s|\ge s_0(\widetilde\tau), \tag{4.33}$$

where $C_{26} > 0$ is independent of s. We also claim that the constant C_{26} is independent of N. The proof of this statement is given in Appendix II.

Set

$$\Xi_\beta = \int_{-\infty}^{\infty} B_\beta ds,\quad \Xi^{(j)}_\beta = \int_{-\infty}^{\infty} B^{(j)}_\beta ds,\quad j=1,2.$$

Therefore, integrating (4.33) with respect to s in $\mathbb R^1$, we have

$$C_{27}(\|h(s)\widetilde w\|^2_{H^1(\mathcal Q)} + \|h^3(s)\widetilde w\|^2_{L^2(\mathcal Q)}) + \Xi_\beta \le C_{26}|s|\int_{-\infty}^{\infty}\|\widetilde w\|_{L^2(\partial\mathcal G_N)}\|\partial_{y_2}\widetilde w\|_{L^2(\partial\mathcal G_N)}ds$$
$$+\|q\|^2_{L^2(\mathcal Q)} + \|\widetilde w\|^2_{H^1(\mathcal Q)}\quad \forall |s|\ge s_0(\tau) \tag{4.34}$$

with some constant $C_{27} > 0$ and by (4.32)

$$|\Xi^{(2)}_{\widetilde\beta}| + |s|\int_{-\infty}^{\infty}\|\widetilde w\|_{L^2(\partial\mathcal G_N)}\|\partial_{y_2}\widetilde w\|_{L^2(\partial\mathcal G_N)}ds \le \epsilon\left\|\left(\frac{\partial\widetilde w}{\partial y_2},\widetilde w\right)\right\|^2_X, \tag{4.35}$$

where we set

$$\left\|\left(\frac{\partial\widetilde w}{\partial y_2},\widetilde w\right)\right\|^2_X = \left\|h(s)\frac{\partial\widetilde w}{\partial y_2}\right\|^2_{L^2(\Sigma)} + \|h(s)\widetilde w\|^2_{L^2(\mathbb R^1;H^1(\mathbb R^2))} + \|h(s)\widetilde w\|^2_{L^2(\Sigma)}$$

and the parameter $\epsilon(\delta)\to +0$ as $\delta\to +0$.

We set

$$w_{1,\nu} = \mathcal{F}_\sigma \chi_\nu(D')v_1, \quad w_{2,\nu} = \mathcal{F}_\sigma \chi_\nu(D')v_2.$$

Later we will need to apply (4.34) and (4.35) to the functions $w_{1,\nu}$ and $w_{2,\nu}$, because we would like to take the advantage of (4.28). However it is directly impossible because the condition $\mathrm{supp}\, \chi_\nu(D')\mathbf{v} \subset B_\delta \times \mathbb{R}^1$ does not hold true, in general. On the other hand, using the fact that

$$\int_{\mathbb{R}^2\setminus B_{2\delta}} \int_{\mathbb{R}^1} h^4(s) \sum_{|\alpha|\le 2} |D^\alpha w_{j,\nu}|^2 dy_0 dy_1 ds \le C_{28}\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2},$$

we can modify (4.34) and (4.35):

$$\begin{aligned} &C_{29}(\|h(s)w_{j(\beta),\nu}\|^2_{H^1(\mathcal{Q})} + \|h^3(s)w_{j(\beta),\nu}\|^2_{L^2(\mathcal{Q})}) + \Xi_\beta \\ \le&\|P_{\beta,s}w_{j(\beta),\nu}\|^2_{L^2(\mathcal{Q})} + C_{30}\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2} + C_{30}|s|\int_{-\infty}^{\infty} \|\widetilde{w}_{j(\beta),\nu}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{y_2}\widetilde{w}_{j(\beta),\nu}\|_{L^2(\partial\mathcal{G}_N)}ds, \end{aligned} \tag{4.36}$$

where $C_{29} > 0$ is independent of s, N and we set $j(\beta) = 1$ if $\beta = \mu$ and $j(\beta) = 2$ if $\beta = \lambda + 2\mu$, and

$$\begin{aligned} &|\Xi^{(2)}_\beta| + |s|\int_{-\infty}^{\infty} \|\widetilde{w}_{j(\beta),\nu}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{y_2}\widetilde{w}_{j(\beta),\nu}\|_{L^2(\partial\mathcal{G}_N)}ds \\ \le&\epsilon\left\|\left(\frac{\partial w_{j(\beta),\nu}}{\partial y_2}, w_{j(\beta),\nu}\right)\right\|^2_X + C_{31}\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}. \end{aligned} \tag{4.37}$$

Now we will prove (4.20) separately in the cases: $r_\mu(\gamma) = 0$ (Section 5), $r_{\lambda+2\mu}(\gamma) = 0$ (Section 6) and $r_\mu(\gamma) \ne 0$, $r_{\lambda+2\mu}(\gamma) \ne 0$ (Section 7).

§5. The case $r_\mu(\gamma) = 0$.

In this section, we treat the case where $r_\mu(\gamma) = 0$ with $\gamma = (y^*, \zeta^*) \equiv (y^*, s^*, \xi_0^*, \xi_1^*) \in \Sigma \times S^2$. Let χ_ν be a member of the partition of unity such that

$$\mathrm{supp}\, \chi_\nu \subset \mathcal{O}(\delta_1) \equiv \left\{\zeta = (s, \zeta_0, \zeta_1);\ \left|\frac{\zeta}{|\zeta|} - \zeta^*\right| < \delta_1\right\}.$$

We note that by (4.36) and (4.37), there exist $C_1 > 0$ and $C_2 > 0$ such that

$$C_1(\|h(s)w_{1,\nu}\|^2_{H^1(\mathcal{Q})} + \|h^3(s)w_{1,\nu}\|^2_{L^2(\mathcal{Q})}) + \Xi^{(1)}_\mu$$
$$\le C_2(\|\mathbf{P}_\mu v_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|w_{1,\nu}\|^2_{H^1(\mathcal{Q})}) + \epsilon(\delta)\left\|\left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu}\right)\right\|^2_X, \tag{5.1}$$

and the parameter ϵ can be taken sufficiently small, if we decrease δ. Note that $\Xi^{(1)}_\mu$ can be written in the form

$$\Xi^{(1)}_\mu = \int_\Sigma \left(|s|\mu^2\varphi_{y_2}(y^*)\left|\frac{\partial w_{1,\nu}}{\partial y_2}\right|^2 + |s|^3\mu^2\varphi^3_{y_2}(y^*)|w_{1,\nu}|^2\right) d\Sigma$$
$$+\mathrm{Re}\int_\Sigma 2|s|\mu\frac{\partial w_{1,\nu}}{\partial y_2}\overline{\left(\mu\varphi_{y_1}(y^*)\frac{\partial w_{1,\nu}}{\partial y_1} - \varphi_{y_0}(y^*)\frac{\partial w_{1,\nu}}{\partial y_0}\right)}d\Sigma$$
$$+\int_\Sigma |s|\mu\varphi_{y_2}(y^*)(\xi_0^2 - \mu\xi_1^2 - s^2\varphi^2_{y_0}(y^*) + s^2\mu\varphi^2_{y_1}(y^*))|\widehat{v}_{1,\nu}|^2 d\Sigma$$
$$\equiv J_1 + J_2 + J_3. \tag{5.2}$$

Let us introduce the set $\mathcal{M}$ by formula

$$\mathcal{M} = \Bigg\{\zeta = (s,\xi_0,\xi_1) \in S^2;$$
$$\frac{\mu}{2}\varphi_{y_2}(y^*)\widehat{C}s^2 > 4\mu^2\frac{\varphi^2_{y_1}(y^*)}{|\varphi_{y_2}(y^*)|}\xi_1^2 + 4\frac{\varphi^2_{y_0}(y^*)}{|\varphi_{y_2}(y^*)|}\xi_0^2 + 2\mu^2\varphi_{y_2}(y^*)(|\xi_0|^2 + |\xi_1|^2)\Bigg\}, \tag{5.3}$$

where $\widehat{C} = -p_\mu(y^*, \nabla\varphi(y^*))$. From (2.6), it follows that $\widehat{C}$ is positive.

Next we introduce the set $\widetilde{\mathcal{M}}$ by formula

$$\widetilde{\mathcal{M}} = \Bigg\{\zeta = (s,\xi_0,\xi_1) \in S^2;$$
$$\frac{\mu}{4}\varphi_{y_2}(y^*)\widehat{C}s^2 < 4\mu^2\frac{\varphi^2_{y_1}(y^*)}{|\varphi_{y_2}(y^*)|}\xi_1^2 + 4\frac{\varphi^2_{y_0}(y^*)}{|\varphi_{y_2}(y^*)|}\xi_0^2 + 2\mu^2\varphi_{y_2}(y^*)(|\xi_0|^2 + |\xi_1|^2)\Bigg\}.$$

Then we can see that $S^2 \subset \mathcal{M}\cup\widetilde{\mathcal{M}}$. Therefore, taking the parameter δ_1 sufficiently small, we obtain either $\mathcal{O}(\delta_1) \subset \mathcal{M}$ or $\mathcal{O}(\delta_1) \subset \widetilde{M}$. Thus we need to consider two cases:

Case A. Assume that $\mathrm{supp}\,\hat{\mathbf{v}}_\nu \subset \mathcal{O}(\delta_1) \subset \mathcal{M}$.

Applying the Cauchy-Bunyakovskii inequality and using (5.3) and (2.6), we obtain that there exists a constant $C_3 > 0$ such that

$$\begin{aligned}
\Xi_\mu^{(1)} \ge & \int_\Sigma \left(|s|\mu^2 \varphi_{y_2}(y^*) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 - |s|^3 \mu \varphi_{y_2}(y^*) p_\mu(y^*, \nabla\varphi(y^*)) |w_{1,\nu}|^2 \right) d\Sigma \\
- & \int_\Sigma \left(\frac{1}{2}|s|\mu^2 \varphi_{y_2}(y^*) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + 4|s|\mu^2 \frac{\varphi_{y_1}^2(y^*)}{|\varphi_{y_2}(y^*)|} \left| \frac{\partial w_{1,\nu}}{\partial y_1} \right|^2 + 4|s| \frac{\varphi_{y_0}^2(y^*)}{|\varphi_{y_2}(y^*)|} \left| \frac{\partial w_{1,\nu}}{\partial y_0} \right|^2 \right) d\Sigma \\
- & \int_\Sigma |s|\mu^2 \varphi_{y_2}(y^*) \xi_1^2 |\widehat{v}_{1,\nu}|^2 d\Sigma \\
\ge & C_3 \int_\Sigma \left(\frac{1}{2}|s|\mu^2 \varphi_{y_2}(y^*) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + |s| \left| \frac{\partial w_{1,\nu}}{\partial y_1} \right|^2 \right. \\
+ & \left. |s| \left| \frac{\partial w_{1,\nu}}{\partial y_0} \right|^2 + \frac{1}{2}|s|^3 \mu \varphi_{y_2}(y^*) \widehat{C} |w_{1,\nu}|^2 \right) d\Sigma. \qquad (5.4)
\end{aligned}$$

We note that by (4.21), we have the equality

$$\frac{\partial w_{2,\nu}}{\partial y_2} - |s|\varphi_{y_2}(y^*) w_{2,\nu} = -\frac{\mu}{\lambda + 2\mu} \left(\frac{\partial w_{1,\nu}}{\partial y_1} - |s|\varphi_{y_1}(y^*) w_{1,\nu} \right) + g_{2,\nu}. \qquad (5.5)$$

Taking the L^2-norm of the left and right hand sides of this equality and using estimate (5.4), we obtain

$$\begin{aligned}
& \int_\Sigma \left(h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_2} \right|^2 + h^6(s) \varphi_{y_2}^2(y^*) |w_{2,\nu}|^2 \right) d\Sigma \le C_4 \left(\Xi_\mu^{(1)} + \|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2} \right. \\
& \left. + \epsilon(\sigma_0) \left\| \left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu} \right) \right\|^2_X + \int_\Sigma \left(\left| \frac{\partial w_{2,\nu}}{\partial y_2} \right|^2 + s^2 \varphi_{y_2}^2(y^*) |w_{2,\nu}|^2 \right) d\Sigma \right),
\end{aligned}$$

where $\epsilon(\sigma_0) \to 0$ as $\sigma_0 \to 0$. By (5.3) and (4.21),

$$\begin{aligned}
& \int_\Sigma h^2(s) \left(\left| \frac{\partial w_{2,\nu}}{\partial y_1} \right|^2 + \left| \frac{\partial w_{2,\nu}}{\partial y_0} \right|^2 \right) d\Sigma \\
\le & C_5 \left(\Xi_\mu^{(1)} + \|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2} + \int_\Sigma \left(\left| \frac{\partial w_{2,\nu}}{\partial y_2} \right|^2 + s^2 \varphi_{y_2}^2(y^*) w_{2,\nu}^2 \right) d\Sigma \right). \qquad (5.6)
\end{aligned}$$

If we apply (4.36) with $\beta = \lambda + 2\mu$, then (5.1), (5.4) and (5.6) imply (4.20).

Case B. Assume that $\operatorname{supp}\hat{\mathbf{v}}_\nu \subset \widetilde{\mathcal{M}}$.

By (4.23) - (4.25), there exists $C_6 > 0$ such that

$$|\xi_0^2 - s^2\varphi_{y_0}^2(y^*) - \mu\xi_1^2 + \mu s^2\varphi_{y_1}^2(y^*)| + |\xi_0 s\varphi_{y_0}(y^*) - \mu s\xi_1\varphi_{y_1}(y^*)|$$
$$\le \delta_1 C_6(|\xi_1|^2 + |\xi_0|^2 + s^2), \quad \forall\zeta \in \mathcal{O}(\delta_1). \tag{5.7}$$

Now we suppose that the parameter δ_1 is sufficiently small such that there exists a constant $C_7 > 0$ such that

$$|\xi_0|^2 \le C_7(|\xi_1|^2 + s^2), \quad \forall\zeta \in \mathcal{O}(\delta_1). \tag{5.8}$$

Then, by (5.7), we have

$$|J_3| \le \delta_1\mu\varphi_{y_2}(y^*)\left\|\left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu}\right)\right\|_X^2. \tag{5.9}$$

Moreover we claim that there exists $\delta_0 > 0$ such that if $\delta_1 \in (0, \delta_0)$, then there exists $C_8 > 0$ such that

$$|\xi_0| \le C_8|\xi_1|, \quad \forall\zeta \in \mathcal{O}(\delta_1). \tag{5.10}$$

Our proof is by contradiction. Suppose that (5.10) is not true. Then for the sequence $\delta_1(n) = \frac{1}{n}$, there exists a sequence $(\xi_0(n), \xi_1(n)) \to (\xi_0^*, \xi_1^*)$ such that $\xi_1(n)/\xi_0(n) \to 0$. Hence for ζ^* we have $r_\mu(y^*, \zeta^*) = 0$, and $\xi_1^* = 0, \xi_0^* \ne 0$ by the definition of the set $\widetilde{\mathcal{M}}$. Therefore $s^*\varphi_{y_0}(y^*) = 0$. If $s^* = 0$, then we obtain $(\xi_0^*)^2 = 0$ and if $\varphi_{y_0}(y^*) = 0$, then $(\xi_0^*)^2 + \mu\varphi_{y_1}^2(y^*)(s^*)^2 = 0$ by (4.25). Therefore in the both cases, we have the equality $\xi_0^* = 0$ which leads us to a contradiction.

Note that if $r_{\lambda+2\mu}(\gamma) = 0$, then

$$\varphi_{y_0}(y^*) = 0, \quad \varphi_{y_1}(y^*) = 0, \quad \xi_0^* = \xi_1^* = 0, \; s^* = 1.$$

and the conic neighborhood of ζ^* is in the set $\mathcal{M}$ provided that the parameter δ_1 is chosen sufficiently small. Therefore if $\gamma \in \widetilde{\mathcal{M}}$ and $r_\mu(\gamma) = 0$, then we have $r_{\lambda+2\mu}(\gamma) \neq 0$ and by Proposition 4.4 the decomposition (4.28) holds true. We set $V^+_{\lambda+2\mu} = (D_{y_2} - \Gamma^+_{\lambda+2\mu}(y, D'))v_{2,\nu}$. Then

$$\mathbf{P}_{\lambda+2\mu}v_{2,\nu} = \beta|G|^2(D_{y_2} - \Gamma^-_{\lambda+2\mu}(y, D'))V^+_{\lambda+2\mu} + T_{\lambda+2\mu}v_{2,\nu},$$

where $T_{\lambda+2\mu} \in \mathcal{L}(H^1(\mathcal{Q}), L^2(\mathcal{Q}))$. This decomposition and Proposition 4.4 immediately imply

$$\begin{aligned}&\|h(D_\sigma)(D_{y_2} - \Gamma^+_{\lambda+2\mu}(y, D'))v_{2,\nu}|_{y_2=0}\|_{L^2(\Sigma)}\\ \le& C_9(\|P_{\lambda+2\mu,s}w_{2,\nu}\|_{L^2(\mathcal{Q})} + \|\mathbf{v}\|_{(H^1(\mathcal{Q}))^2}).\end{aligned} \tag{5.11}$$

Now we need again obtain the estimate of $\Xi^{(1)}_\mu$. We start from the term J_2. By (4.21), we have

$$\begin{aligned}J_2 =& \,\mathrm{Re}\int_\Sigma 2|s|(\lambda+2\mu)\left(\frac{\partial w_{2,\nu}}{\partial y_1} - |s|\varphi_{y_1}(y^*)w_{2,\nu}\right)\\ &\times\overline{\left(\mu\frac{\partial w_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial w_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma\\ &+\mathrm{Re}\int_\Sigma 2|s|\mu(|s|\varphi_{y_2}(y^*)w_{1,\nu} + g_{1,\nu})\overline{\left(\mu\frac{\partial w_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial w_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma.\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}&-\frac{\mu}{\lambda+2\mu}\left(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu}\right) - i\alpha^+_{\lambda+2\mu}(y, D')v_{2,\nu}\\ =& iV^+_{\lambda+2\mu}(\cdot, 0) - \frac{\mu}{\lambda+2\mu}\mathcal{F}^{-1}_\sigma g_{2,\nu}.\end{aligned} \tag{5.13}$$

Here and henceforth $|D_\sigma|$ is the pseudo-differential operator with the symbol $|s|$.

First assume that $s^* = 0$. Then we can see by $|s^*|^2 + |\xi^*_0|^2 + |\xi^*_1|^2 = 1$ that $|\alpha^+_{\lambda+2\mu}(\gamma)| = |r_{\lambda+2\mu}(\gamma)| \neq 0$. Therefore, by Proposition 4.2.A from [T2, p.105], there exists a parametrix of the operator $\alpha^+_{\lambda+2\mu}(y,D')$ which we denote by $(\alpha^+_{\lambda+2\mu}(y,D'))^{-1}$. From (5.13) we obtain

$$v_{2,\nu} = -\frac{1}{i}(\alpha^+_{\lambda+2\mu}(y,D'))^{-1}\Bigg(\frac{\mu}{\lambda+2\mu}\left(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu}\right) + iV^+_{\lambda+2\mu}(\cdot,0) - \frac{\mu}{\lambda+2\mu}g_{2,\nu}\Bigg) + T_0 v_{2,\nu}, \tag{5.14}$$

where $T_0 \in \mathcal{L}(L^2(\Sigma), H^1(\Sigma))$. Using (5.14), we transform (5.12) to obtain

$$J_2 = \mathrm{Re}\int_\Sigma -\frac{2|D_\sigma|\mu}{i}\left(\frac{\partial}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)\right)(\alpha^+_{\lambda+2\mu}(y,D'))^{-1} \left(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu}\right)\overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma + \kappa_3, \tag{5.15}$$

where

$$\kappa_3 = \mathrm{Re}\int_\Sigma 2|D_\sigma|\mu(|D_\sigma|\varphi_{y_2}(y^*)v_{1,\nu} + g_{1,\nu})\overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma$$
$$+\mathrm{Re}\int_\Sigma 2|D_\sigma|(\lambda+2\mu)\left(\frac{\partial}{\partial y_1} - |s|\varphi_{y_1}(y^*)\right)$$
$$\times\left[-\frac{1}{i}(\alpha^+_{\lambda+2\mu}(y,D'))^{-1}\left(iV^+_{\lambda+2\mu}(\cdot,0) - \frac{\mu}{\lambda+2\mu}\mathcal{F}^{-1}_\sigma g_{2,\nu}\right) + T_0 v_{2,\nu}\right]$$
$$\times\overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma.$$

Then we have

$$|\kappa_3| \le \epsilon\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu\right)\right\|^2_X + C_{10}(\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2} + \|P_{\lambda+2\mu,s}w_{2,\nu}\|^2_{L^2(Q)}) \tag{5.16}$$

and ϵ can be chosen arbitrarily small by taking δ small enough.

Let us consider the pseudo-differential operator

$$b(y,D') \equiv \frac{1}{i}\left(\frac{\partial}{\partial y_1} - |s|\varphi_{y_1}(y^*)\right)(\alpha^+_{\lambda+2\mu}(y,D'))^{-1}.$$

By (5.7), for the principal symbol of this operator, we have

$$\begin{aligned} b(y^*,\zeta) &= \frac{1}{i}(i\xi_1 - |s|\varphi_{y_1}(y^*))(\alpha^+_{\lambda+2\mu}(y^*,\zeta))^{-1} \\ &\equiv -\operatorname{sign}(\xi_1^*)\sqrt{\frac{\lambda+\mu}{\lambda+2\mu}(y^*)}\frac{(i\xi_1 - |s|\varphi_{y_1}(y^*))}{\xi_1 + i|s|\varphi_{y_1}(y^*)} + \widetilde{b}(y^*,\zeta) \\ &= \frac{1}{i}\sqrt{\frac{\lambda+\mu}{\lambda+2\mu}(y^*)} + \widetilde{b}(y^*,\zeta), \end{aligned} \tag{5.17}$$

where $\widetilde{b}(y^*,\xi^*) = 0$. Therefore the operator $b(y,D')$ can be represented in the form

$$b(y,D') = \frac{1}{i}\sqrt{\frac{\lambda+\mu}{\lambda+2\mu}(y)} + \widetilde{b}(y,D'),$$

where $\widetilde{b}(y,D') \in \mathcal{L}(L^2(\Sigma),L^2(\Sigma))$ and

$$\|\widetilde{b}(y,D')\|_{\mathcal{L}(L^2(\Sigma),L^2(\Sigma))} \le \epsilon. \tag{5.18}$$

Using (5.17) in (5.15), we obtain

$$\begin{aligned} J_2 = \operatorname{Re}\int_\Sigma -2|D_\sigma|\mu\left(\frac{\operatorname{sign}(\xi_1^*)}{i}\sqrt{\frac{\lambda+\mu}{\lambda+2\mu}} + \widetilde{b}(y,D')\right)\left(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu}\right) \\ \overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma + \kappa_3 \\ = \operatorname{Re}\int_\Sigma -2|D_\sigma|\mu\widetilde{b}(y,D')\left(\frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu}\right) \\ \overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma + \operatorname{Re}\kappa_3. \end{aligned}$$

By (5.7), (5.16) and (5.18), taking the parameters δ, δ_1 sufficiently small, we obtain

$$\begin{aligned} |J_2| \le \epsilon\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu\right)\right\|^2_X \\ + C_{11}(\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2} + \|P_{\lambda+2\mu,s}w_{2,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}). \end{aligned} \tag{5.19}$$

Next assume that $s^* \ne 0$. Then we have

$$|\varphi_{y_1}(y^*)\xi_1 - \varphi_{y_0}(y^*)\xi_0| \le C\delta_1|\zeta|, \quad \forall \zeta \in \mathcal{O}(\delta_1)$$

and (5.19) follows immediately. Therefore, for any $s^* \in \mathbb{R}^1$, by (5.1), (5.2), (5.9) and (5.19), we have

$$\int_\Sigma \left(h^2(s)\mu^2\varphi_{y_2}(y^*) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + h^6(s)\mu^2\varphi^3_{y_2}(y^*)|w_{1,\nu}|^2 \right) d\Sigma$$
$$+C_{12}(\|h(s)w_{1,\nu}\|^2_{H^1(\mathcal{Q})} + \|h^3(s)w_{1,\nu}\|^2_{L^2(\mathcal{Q})}) \le C_{13}(\|P_{\lambda+2\mu,s}w_{2,\nu}\|^2_{L^2(\mathcal{Q})}$$
$$+\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}) + \epsilon \left\| (\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu) \right\|^2_X . \tag{5.20}$$

From (4.21), we obtain

$$\int_\Sigma \left(|s| \left| \frac{\partial w_{2,\nu}}{\partial y_1} \right|^2 + |s|^3\mu^2\varphi^2_{y_1}(y^*)|w_{2,\nu}|^2 \right) d\Sigma$$
$$\le C_{14} \int_\Sigma \left(|s|\mu^2\varphi_{y_2}(y^*) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + |s|^3\mu^2\varphi^3_{y_2}(y^*)|w_{1,\nu}|^2 \right) d\Sigma + C_{14}\|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2}. \tag{5.21}$$

Using (5.10), (5.21) and the definition of the set $\widetilde{\mathcal{M}}$, we obtain

$$\int_\Sigma \left(h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_1} \right|^2 + h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_0} \right|^2 + h^6(s)|w_{2,\nu}|^2 \right) d\Sigma$$
$$\le C_{15} \Bigg\{ \int_\Sigma \left(|s|\mu^2\varphi_{y_2}(y^*) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + |s|^3\mu^2\varphi^3_{y_2}(y^*)|w_{1,\nu}|^2 \right) d\Sigma$$
$$+\epsilon(\sigma_0) \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu \right) \right\|^2_X + \|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2} \Bigg\}. \tag{5.22}$$

From (5.11) and (5.22), we have

$$\int_\Sigma h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_2} \right|^2 d\Sigma$$
$$\le C_{16} \Bigg\{ \int_\Sigma \left(h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_1} \right|^2 + h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_0} \right|^2 + h^6(s)|w_{2,\nu}|^2 \right) d\Sigma$$
$$+\|V^+_{\lambda+2\mu}(\cdot,0)\|^2_{L^2(\Sigma)} + \epsilon(\sigma_0) \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu \right) \right\|^2_X + \|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2} \Bigg\}$$
$$\le C_{17} \Bigg\{ \int_\Sigma \left(h^2(s) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + h^6(s)|w_{1,\nu}|^2 \right) d\Sigma + \|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2}$$
$$+\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2} + \|P_{\lambda+2\mu,s}w_{2,\nu}\|^2_{L^2(\mathcal{Q})} + \epsilon(\sigma_0) \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu \right) \right\|^2_X \Bigg\}. \tag{5.23}$$

Finally (4.21), (5.8), (5.10) and (5.23) imply

$$\begin{aligned}&\int_\Sigma h^2(s)\left(\left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2+\left|\frac{\partial w_{1,\nu}}{\partial y_0}\right|^2\right)d\Sigma\\ \le& C_{18}\left\{\int_\Sigma\left(h^2(s)\left|\frac{\partial w_{1,\nu}}{\partial y_2}\right|^2+h^6(s)|w_{1,\nu}|^2\right)d\Sigma+\|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2}\right.\\ &\left.+\|\mathbf{v}\|^2_{(H^1(Q))^2}+\|P_{\lambda+2\mu,s}w_{2,\nu}\|^2_{L^2(Q)}+\epsilon(\sigma_0)\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_2},\mathbf{w}_\nu\right)\right\|^2_X\right\}.\end{aligned} \tag{5.24}$$

The inequalities (5.1), (5.20) - (5.24) imply

$$\begin{aligned}&\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_2},\mathbf{w}_\nu\right)\right\|^2_X+\|h(s)w_{1,\nu}\|^2_{H^1(Q)}+\|h^3(s)w_{1,\nu}\|^2_{L^2(Q)}\le\epsilon\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_2},\mathbf{w}_\nu\right)\right\|^2_X\\ &+C_{19}(\|\mathbf{v}\|^2_{(H^1(Q))^2}+\|h(s)\mathbf{g}_\nu\|^2_{(L^2(\Sigma))^2}+\|P_{\mu,s}w_{2,\nu}\|^2_{L^2(Q)}+\|P_{\lambda+2\mu,s}w_{2,\nu}\|^2_{L^2(Q)}).\end{aligned}$$

From this inequality and (4.34), (4.35) with $\beta=\lambda+2\mu$, we obtain (4.20).■

§6. The case $r_{\lambda+2\mu}(\gamma)=0$.

Let $\gamma=(y^*,\zeta^*)$ be a point on $\Sigma\times S^2$ such that $r_{\lambda+2\mu}(\gamma)=0$ and $\mathrm{supp}\chi_\nu\subset\mathcal{O}(\delta_1)\subset\widetilde{\mathcal{M}}$. We note that if $r_\mu(\gamma)=0$, then $s^*\ne 0$ and $\xi^*_0=\xi^*_1=\varphi_{y_0}(y^*)=\varphi_{y_1}(y^*)=0$. Consequently $\zeta^*\in\mathcal{M}$ and this case was treated in the previous section. Therefore, taking the parameters δ and δ_1 sufficiently small, we may assume that there exists a constant $\widehat{C}>0$ such that

$$|r_\mu(y,\zeta)|\ge\widehat{C}|\zeta|,\quad \forall(y,\zeta)\in B_\delta\times\mathcal{O}(\delta_1).$$

By (4.24) and (4.25), there exist $\delta_0>0$ and $C_1>0$ such that for all $\delta_1\in(0,\delta_0)$ we have

$$|\xi_0|^2\le C_1(\xi_1^2+s^2),\quad \forall\zeta\in\mathcal{O}(\delta_1). \tag{6.1}$$

We consider the following three cases.

Case A. Assume that $s^* = 0$ and $\lim_{\zeta\to\zeta^*} \mathrm{Im}\, r_\mu(y^*,\zeta)/|s| = 0$. In that case, there exists a constant $C_2 > 0$ such that

$$-\mathrm{Im}\,\Gamma_\mu^\pm(y,\zeta) \ge C_2|s|, \quad \forall (y,\zeta)\in B_\delta\times\mathcal{O}(\delta_1),$$

provided that $|\delta|+|\delta_1|$ is sufficiently small. Since $s^*=0$, we may assume that for some constant $C_3>0$,

$$|\xi_0|^2 + s^2 \le C_3\xi_1^2, \quad \forall\zeta\in\mathcal{O}(\delta_1), \tag{6.2}$$

taking a sufficiently small δ_1. We set $V_\mu^\pm = (D_{y_2} - \Gamma_\mu^\pm(y,D'))v_{1,\nu}$. Then, by Proposition 4.3,

$$\mathbf{P}_\mu v_{1,\nu} = |G|^2\beta(D_{y_2} - \Gamma_\mu^\mp(y,D'))V_\mu^\pm + T_\mu^\pm v_{1,\nu}, \tag{6.3}$$

where $T_\mu^\pm \in \mathcal{L}(H^1(\mathcal{Q}), L^2(\mathcal{Q}))$. This decomposition and Proposition 4.4 imply

$$\|h(D_\sigma)(D_{y_2} - \Gamma_\mu^\pm(y,D'))v_{1,\nu}|_{y_2=0}\|_{L^2(\Sigma)} \le C_4(\|\mathbf{P}_\mu v_{1,\nu}\|_{L^2(\mathcal{Q})} + \|\mathbf{v}\|_{(H^1(\mathcal{Q}))^2}). \tag{6.4}$$

We have

$$V_\mu^+(\cdot,0) - V_\mu^-(\cdot,0) = (\alpha_\mu^+(y,D') - \alpha_\mu^-(y,D'))v_{1,\nu} \quad \text{on } \Sigma. \tag{6.5}$$

Since $\alpha_\mu^+(y^*,\zeta^*) - \alpha_\mu^-(y^*,\zeta^*) = 2\sqrt{r_\mu(y^*,\zeta^*)} \ne 0$, we have

$$\begin{aligned}&\int_\Sigma \left(h^2(s)\left(\left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2 + \left|\frac{\partial w_{1,\nu}}{\partial y_0}\right|^2 \right) + h^6(s)|w_{1,\nu}|^2 \right) d\Sigma \\ \le & C_5(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(Q))^2})\end{aligned} \tag{6.6}$$

by (6.4), (6.5) and Garding's inequality.

From (6.6) and (6.4), we obtain

$$\int_\Sigma h^2(s)\left|\frac{\partial w_{1,\nu}}{\partial y_2}\right|^2 d\Sigma \le C_6(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(Q))^2}). \tag{6.7}$$

Finally, by (6.6), (6.7) combined with (4.21), we obtain

$$\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\right)\right\|_X^2 \le C_7(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(Q))^2} + \|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2}). \quad (6.8)$$

By (6.6) - (6.8), (4.36) and (4.37), we obtain (4.20).

Case B. Assume that $s^* = 0$ and $\lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(y^*,\zeta)/|s| \ne 0$. By $s^*=0$, we note that $\operatorname{Re} r_\mu(y^*,\zeta^*) > 0$. Set $I = \operatorname{sign} \lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(y^*,\zeta)/|s|$. For all $(y,\zeta) \in B_\delta \times \mathcal{O}(\delta_1)$, we have

$$\Gamma^+_\mu(y^*,\zeta^*) = I\sqrt{\operatorname{Re} r_\mu(y^*,\zeta^*)}.$$

Therefore

$$\Gamma^+_\mu(y^*,\zeta^*)(\mu\varphi_{y_1}(y^*)\xi_1^* - \varphi_{y_0}(y^*)\xi_0^*) > 0.$$

Taking the parameters $\delta > 0$ and $\delta_1 > 0$ sufficiently small, we obtain

$$\operatorname{Re}\Gamma^+_\mu(y^*,\zeta)(\mu\varphi_{y_1}(y^*)\xi_1 - \varphi_{y_0}(y^*)\xi_0) > 0, \quad \forall (y,\zeta) \in B_\delta \times \mathcal{O}(\delta_1). \quad (6.9)$$

Let us consider the estimate (5.1). Let us recall that J_1, J_2, J_3 are defined in (5.2). We have

$$\begin{aligned} J_2 = &\operatorname{Re}\int_\Sigma 2|s|\mu\frac{\partial w_{1,\nu}}{\partial y_2}\overline{\left(\mu\frac{\partial w_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial w_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma \\ =&\operatorname{Re}\int_\Sigma 2|D_\sigma|\mu i\Gamma^+_\mu(y,D')v_{1,\nu}\overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma \\ +&\operatorname{Re}\int_\Sigma 2|D_\sigma|\mu i V^+_\mu(\cdot,0)\overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma \\ =&\operatorname{Re}\int_\Sigma 2\mu(D_{y_1}\varphi_{y_1}(y^*) - D_{y_0}\varphi_{y_0}(y^*))\Gamma^+_\mu(y,D')|D_\sigma|^{\frac12}\widehat{v}_{1,\nu}\overline{|D_\sigma|^{\frac12}\widehat{v}_{1,\nu}}d\Sigma \\ +&\operatorname{Re}\int_\Sigma 2|D_\sigma|\mu i V^+_\mu(\cdot,0)\overline{\left(\mu\frac{\partial v_{1,\nu}}{\partial y_1}\varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma. \end{aligned} \quad (6.10)$$

By (6.9) and Garding's inequality, we obtain from (6.10)

$$\begin{aligned} J_2 \ge C_8 &\int_\Sigma \left(h^2(s)\left(\left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2 + \left|\frac{\partial w_{1,\nu}}{\partial y_0}\right|^2\right) + h^6(s)|w_{1,\nu}|^2\right)d\Sigma \\ &- C_9\epsilon(\delta,\delta_1)\left\|\left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu}\right)\right\|_X^2 - C_{10}(\delta,\delta_1)(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(Q))^2}). \end{aligned} \quad (6.11)$$

Now we will estimate J_3. By (4.23) and (4.24), there exists a constant $C_{11} > 0$ such that

$$\begin{aligned}&|\xi_0^2 - s^2\varphi_{y_0}^2(y^*) - (\lambda+2\mu)\xi_1^2 + (\lambda+2\mu)s^2\varphi_{y_1}^2(y^*)|\\ \le& C_{11}\delta_1(|\xi_0|^2 + |\xi_1|^2 + s^2), \quad \forall\zeta \in \mathcal{O}(\delta_1).\end{aligned} \tag{6.12}$$

Using (6.12), we obtain

$$\begin{aligned}&\xi_0^2 - \mu\xi_1^2 - s^2\varphi_{y_0}^2(y^*) + s^2\mu\varphi_{y_1}^2(y^*)\\ =&(\lambda+\mu)(\xi_1^2 - s^2\varphi_{y_1}^2(y^*)) + (\xi_0^2 - (\lambda+2\mu)\xi_1^2 - s^2\varphi_{y_0}^2(y^*) + s^2(\lambda+2\mu)\varphi_{y_1}^2(y^*))\\ \ge&(\lambda+\mu)(\xi_1^2 - s^2\varphi_{y_1}^2(y^*)) - C_{12}\delta_1(|\xi_0|^2 + |\xi_1|^2 + s^2).\end{aligned}$$

Therefore, for all sufficiently small δ_1, there exists $C_{13} > 0$ such that

$$\xi_0^2 - \mu\xi_1^2 - s^2\varphi_{y_0}^2(y^*) + s^2\mu\varphi_{y_1}^2(y^*) \ge C_{13}\delta_1(|\xi_0|^2 + |\xi_1|^2 + s^2). \tag{6.13}$$

By (6.13), we see that $J_3 \ge 0$. Therefore $\Xi_\mu^{(1)} = J_1 + J_2 + J_3 \ge J_1 + J_2$, so that by (6.11) and (5.1), there exists a constant $C_{14} > 0$ such that

$$\Xi_\mu^{(1)} \ge C_{14}\left\|\left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu}\right)\right\|_X^2 - C_{10}(\delta,\delta_1)(\|P_{\mu,s}w_{1,\nu}\|_{L^2(\mathcal{Q})}^2 + \|\mathbf{v}\|_{(H^1(\mathcal{Q}))^2}^2).$$

This inequality and (4.21) implies

$$\begin{aligned}\Xi_\mu^{(1)} \ge& C_{15}\left\|\left(\frac{\partial \mathbf{w}_{\nu}}{\partial y_2}, \mathbf{w}_{\nu}\right)\right\|_X^2\\ &-C_{16}(\delta,\delta_1)(\|P_{\mu,s}w_{1,\nu}\|_{L^2(\mathcal{Q})}^2 + \|h(s)\mathbf{g}\|_{(L^2(\Sigma))^2}^2 + \|\mathbf{v}\|_{(H^1(\mathcal{Q}))^2}^2).\end{aligned} \tag{6.14}$$

From (6.14), (4.36) and (4.37), we obtain (4.20).

Case C. Assume that $s^* \ne 0$. If $\delta_1 > 0$ is small enough, then there exists a constant $C_{17} > 0$ such that

$$|\xi_0\varphi_{y_1}(y^*) - (\lambda+2\mu)\xi_1\varphi_{y_1}(y^*)|^2 \le \delta_1^2 C_{17}(|\xi_1|^2 + s^2). \tag{6.15}$$

By (4.36), there exists $C_{18} > 0$ such that

$$\Xi^{(1)}_{\lambda+2\mu} + C_{18}(\|h(s)w_{2,\nu}\|^2_{H^1(\mathcal{Q})} + \|h^3(s)w_{2,\nu}\|^2_{L^2(\mathcal{Q})})$$
$$\le C_{18}(\|\mathbf{P}_{\lambda+2\mu}v_2\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}) + \epsilon\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\right)\right\|^2_X. \tag{6.16}$$

Note that

$$\Xi^{(1)}_{\lambda+2\mu} = \int_\Sigma \left(|s|(\lambda+2\mu)^2\varphi_{y_2}(y^*)\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2 + |s|^3(\lambda+2\mu)^2\varphi^3_{y_2}(y^*)|w_{2,\nu}|^2\right)d\Sigma$$
$$+\mathrm{Re}\int_\Sigma 2|s|(\lambda+2\mu)\frac{\partial w_{2,\nu}}{\partial y_2}\overline{\left((\lambda+2\mu)\varphi_{y_1}(y^*)\frac{\partial w_{2,\nu}}{\partial y_1} - \varphi_{y_0}(y^*)\frac{\partial w_{2,\nu}}{\partial y_0}\right)}d\Sigma$$
$$+\int_\Sigma |s|(\lambda+2\mu)\varphi_{y_2}(y^*)(\xi_0^2 - (\lambda+2\mu)\xi_1^2 - s^2\varphi^2_{y_0}(y^*) + s^2(\lambda+2\mu)\varphi^2_{y_1}(y^*))|\widehat{v}_{2,\nu}|^2 d\Sigma$$
$$=\widetilde{J}_1 + \widetilde{J}_2 + \widetilde{J}_3. \tag{6.17}$$

By (6.12) and (6.15), we have

$$|\widetilde{J}_2 + \widetilde{J}_3| \le C_{19}\delta_1\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\right)\right\|^2_X. \tag{6.18}$$

By (6.18) we obtain from (6.17) that there exists a constant $C_{20} > 0$ such that

$$\Xi^{(1)}_{\lambda+2\mu} \ge -\epsilon\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\right)\right\|^2_X$$
$$+C_{20}\int_\Sigma\left(h^2(s)(\lambda+2\mu)^2\varphi_{y_2}(y^*)\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2 + h^6(s)(\lambda+2\mu)^2\varphi^3_{y_2}(y^*)|w_{2,\nu}|^2\right)d\Sigma. \tag{6.19}$$

From (4.21), we easily obtain

$$\left\|h(s)\left(\frac{\partial w_{2,\nu}}{\partial y_2} - s\varphi_{y_2}(y^*)w_{2,\nu} + g_{2,\nu}\right)\right\|^2_{L^2(\Sigma)}$$
$$=\frac{\mu^2}{(\lambda+2\mu)^2}\left(\left\|h(s)\frac{\partial w_{1,\nu}}{\partial y_1}\right\|^2_{L^2(\Sigma)} + \varphi^2_{y_1}(y^*)\|h^3(s)w_{1,\nu}\|^2_{L^2(\Sigma)}\right).$$

Hence (6.19) and this equality imply

$$\Xi^{(1)}_{\lambda+2\mu} \ge C_{21}\int_\Sigma\left(h^2(s)\left(\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2 + \left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2\right) + h^6(s)|w_{2,\nu}|^2\right)d\Sigma$$
$$-\epsilon\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\right)\right\|^2_X - C_{22}\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2}. \tag{6.20}$$

Now we claim that inequality (6.2) holds true for all sufficiently small δ_1. First we may assume that for all $\zeta \in \mathcal{O}(\delta_1)$ we have $s^2 \le C_{23}(\xi_0^2+\xi_1^2)$. In fact, if the last inequality is not true, then $\zeta^* \in \mathcal{M}$ and the case was treated in the previous section. Suppose that (6.2) is not true. In that case $\xi_1^* = 0$ and $\xi_0^* \ne 0, s^* \ne 0$. Therefore $\varphi_{y_0}(y^*) = 0$ by (4.23). However, this implies $(\xi_0^*)^2+(\lambda(y^*)+2\mu(y^*))\varphi_{y_1}^2(y^*)(s^*)^2 = 0$. Hence we arrived at a contradiction and the verification of (6.2) is complete.

The inequalities (6.2) and (6.20) imply

$$\Xi^{(1)}_{\lambda+2\mu} \ge C_{24}\int_\Sigma \left(h^2(s)\left(\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2+\left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2+\left|\frac{\partial w_{1,\nu}}{\partial y_0}\right|^2\right)+h^6(s)|\mathbf{w}_\nu|^2\right)d\Sigma$$
$$-\epsilon\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\right)\right\|^2_X - C_{22}\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2}. \tag{6.21}$$

From inequality (6.4) for $V_\mu^+(\cdot,0)$, we obtain the estimate

$$\left\|h(s)\frac{\partial w_{1,\nu}}{\partial y_2}\right\|^2_{L^2(\Sigma)} \le C_{25}\Bigg\{\int_\Sigma\left(h^2(s)\left(\left|\frac{\partial w_{1,\nu}}{\partial y_1}\right|^2+\left|\frac{\partial w_{1,\nu}}{\partial y_0}\right|^2\right)+h^6(s)|w_{1,\nu}|^2\right)d\Sigma$$
$$+\|\mathbf{P}_\mu v_{1,\nu}\|^2_{L^2(Q)}+\|\mathbf{v}\|^2_{(H^1(Q))^2}\Bigg\}. \tag{6.22}$$

The inequalities (6.21) and (6.22) imply

$$\Xi^{(1)}_{\lambda+2\mu} \ge C_{26}\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu\right)\right\|^2_X$$
$$-C_{27}(\delta,\delta_1)(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(Q)}+\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2}+\|\mathbf{v}\|^2_{(H^1(Q))^2}). \tag{6.23}$$

From (6.23), (4.36) and (4.37), we obtain (4.20). ■

§7. The case $r_\mu(\gamma) \ne 0$ and $r_{\lambda+2\mu}(\gamma) \ne 0$.

In this section, we consider the conic neighborhood $\mathcal{O}(\delta_1)$ of the point $\gamma \equiv (y^*,\zeta^*)$ such that

$$|r_\mu(y^*,\zeta^*)| \ne 0 \quad \text{and} \quad |r_{\lambda+2\mu}(y^*,\zeta^*)| \ne 0. \tag{7.1}$$

In that case, thanks to (7.1) and Proposition 4.3, decomposition (4.28) holds true for $\beta=\mu$ and $\beta=\lambda+2\mu$. Therefore we have

$$(D_{y_2}-\Gamma^+_\mu(y,D'))v_{1,\nu}|_{y_2=0}=V^+_\mu(\cdot,0), \tag{7.2}$$

$$(D_{y_2}-\Gamma^+_{\lambda+2\mu}(y,D'))v_{2,\nu}|_{y_2=0}=V^+_{\lambda+2\mu}(\cdot,0). \tag{7.3}$$

By Proposition 4.4, we have an a priori estimate

$$\begin{aligned}&\|h(D_\sigma)V^+_\mu(\cdot,0)\|^2_{L^2(\Sigma)}+\|h(D_\sigma)V^+_{\lambda+2\mu}(\cdot,0)\|^2_{L^2(\Sigma)}\\ \le& C_1(\|\mathbf{P}_{\lambda+2\mu}v_2\|^2_{L^2(\mathcal{Q})}+\|\mathbf{P}_\mu v_1\|^2_{L^2(\mathcal{Q})}+\|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}).\end{aligned} \tag{7.4}$$

Using (4.21), we may rewrite (7.2) and (7.3) as

$$\frac{\lambda+2\mu}{\mu}\left(\frac{\partial v_{2,\nu}}{\partial y_1}-|D_\sigma|\varphi_{y_1}(y^*)v_{2,\nu}\right)-i\alpha^+_\mu(y,D')v_{1,\nu}=V^+_\mu(\cdot,0)-\mathcal{F}^{-1}_\sigma g_{1,\nu}, \tag{7.5}$$

$$\frac{\mu}{\lambda+2\mu}\left(-\frac{\partial v_{1,\nu}}{\partial y_1}+|D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu}\right)-i\alpha^+_{\lambda+2\mu}(y,D')v_{2,\nu}=V^+_{\lambda+2\mu}(\cdot,0)-\mathcal{F}^{-1}_\sigma g_{2,\nu}. \tag{7.6}$$

Let $\mathbf{B}(y,D')$ be the matrix pseudo-differential operator with the symbol

$$\mathbf{B}(y,\zeta)=\begin{pmatrix}-i\alpha^+_\mu(y,\zeta) & \frac{\lambda+2\mu}{\mu}(i\xi_1-|s|\varphi_{y_1}(y))\\ \frac{\mu}{\lambda+2\mu}(-i\xi_1+|s|\varphi_{y_1}(y)) & -i\alpha^+_{\lambda+2\mu}(y,\zeta)\end{pmatrix}.$$

By (4.24) and (4.25), we see: If $\det\mathbf{B}(y^*,\zeta^*)=0$, then

$$\zeta^*\in\left\{\zeta\in\mathbb{R}^3;\,(\xi_1+i|s|\varphi_{y_1}(y^*))^2=\frac{(\xi_0+i|s|\varphi_{y_0}(y^*))^2}{(\lambda+3\mu)(y^*)}\right\}. \tag{7.7}$$

Now we consider two cases

Case A. $\det\mathbf{B}(\gamma)\ne0$.

In that case, there exists a parametrix of the operator $\mathbf{B}(y,D')$, which we denote by $\mathbf{B}^{-1}(y,D')$, such that

$$\begin{aligned}&(v_{1,\nu},v_{2,\nu})=\mathbf{B}^{-1}(y,D')(V^+_\mu(\cdot,0)-\mathcal{F}^{-1}_\sigma g_{1,\nu},V^+_{\lambda+2\mu}(\cdot,0)-\mathcal{F}^{-1}_\sigma g_{2,\nu})^T\\ &+K(v_{1,\nu},v_{2,\nu}),\end{aligned} \tag{7.8}$$

where $K : (L^2(\mathcal{Q}))^2 \to (H^1(\mathcal{Q}))^2$. By (7.4) and (7.8),

$$|\Xi_\mu| + |\Xi_{\lambda+2\mu}| \le C_2(\|\mathbf{P}_\mu v_1\|^2_{L^2(\mathcal{Q})} + \|\mathbf{P}_{\lambda+2\mu} v_2\|^2_{L^2(\mathcal{Q})}$$

$$+\|h(s)\mathbf{g}\|^2_{(L^2(\Sigma))^2} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}). \tag{7.9}$$

(Here and henceforth, for simplicity, we do not distinguish $\mathbf{a}^T$ from a vector $\mathbf{a}$.) By (7.9), (4.36) and (4.37), we obtain (4.20).

Case B. $\det \mathbf{B}(\gamma) = 0$.

We claim that this situation is possible in the two cases:

$$\text{(i)} \quad \varphi_{y_0}(y^*) = \varphi_{y_1}(y^*) = \xi_0^* = \xi_1^* = 0, \quad s^* = 1,$$

$$\text{(ii)} \quad \xi_0^* = 0, \quad s^*\varphi_{y_0}(y^*) = 0. \tag{7.10}$$

The first case was treated in Section 5. Let us consider the second case (7.10). Moreover we may assume that

$$\zeta^* \in \widetilde{\mathcal{M}}.$$

Otherwise, $\zeta^* \in \mathcal{M}$, so that the case was treated in Section 5. Moreover we may assume that

$$\operatorname{Im} \Gamma^+_\mu(\gamma) = \operatorname{Im} \Gamma^+_{\lambda+2\mu}(\gamma) \ge 0. \tag{7.11}$$

Really if

$$\operatorname{Im} \Gamma^+_\mu(\gamma) = \operatorname{Im} \Gamma^+_{\lambda+2\mu}(\gamma) < 0, \tag{7.12}$$

then the situation is simple since we have the decomposition

$$\mathbf{P}_\beta v_{j(\beta),\nu} = \beta |G|^2 (D_{y_2} - \Gamma^{\mp}_\beta(y, D'))V^{\pm}_\beta + T^{\pm}_\mu v_{j(\beta),\nu},$$

where $T^{\pm}_{\beta} \in \mathcal{L}(H^1(\mathcal{Q}), L^2(\mathcal{Q}))$, $\beta \in \{\mu, \lambda + 2\mu\}$, $j(\beta) = 1$ for $\beta = \mu$ and $j(\beta) = 2$ for $\beta = \lambda + 2\mu$. This decomposition, (7.12) and Proposition 4.3 imply

$$\begin{aligned} &\|h(D_\sigma)(D_{y_2} - \Gamma^{\pm}_{\beta}(y, D'))v_{j(\beta),\nu}|_{y_2=0}\|_{L^2(\Sigma)} \\ \le &C_3(\|\mathbf{P}_\beta v_{j(\beta),\nu}\|_{L^2(\mathcal{Q})} + \|\mathbf{v}\|_{(H^2(\mathcal{Q}))^2}). \end{aligned} \tag{7.13}$$

Obviously

$$V^{+}_{\beta}(\cdot, 0) - V^{-}_{\beta}(\cdot, 0) = (\alpha^{+}_{\beta}(y, D') - \alpha^{-}_{\beta}(y, D'))v_{1,\nu} \quad \text{on } \Sigma.$$

Since $\alpha^{+}_{\mu}(y^*, \zeta^*) - \alpha^{-}_{\mu}(y^*, \zeta^*) = 2\sqrt{r_\mu(y^*, \zeta^*)} \neq 0$, we have

$$\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_1}, \mathbf{w}_\nu\right)\right\|^2_X \le C_4(\|P_{\lambda+2\mu,s} w_{2,\nu}\|^2_{L^2(\mathcal{Q})} + \|P_{\mu,s} w_{1,\nu}\|^2_{L^2(\mathcal{Q})} + \|\mathbf{v}\|^2_{(H^1(\mathcal{Q}))^2}) \tag{7.14}$$

by (7.13) and Garding's inequality.

From (7.14), (4.36) and (4.37), we obtain (4.20) under condition (7.12).

In order to treat (7.10) under (7.11), we will use Calderon's method. First we introduce the new variables $U = (U_1, U_2)$ with four components, where

$$U_1 = \Lambda(D')\mathcal{F}^{-1}_{\sigma}\mathcal{U}, \quad U_2 = (D_2 + i|D_\sigma|\varphi_{y_2})\mathcal{F}^{-1}_{\sigma}\mathcal{U},$$

and Λ is the pseudodifferential operator with the symbol $(s^2 + \xi_1^2 + \xi_0^2 + 1)^{\frac{1}{2}}$. In the new notations, problem (4,6) and (4.7) can be written in the form

$$D_{y_2} U = M(y, D')U + F \quad \text{in } \mathbb{R}^3 \times [0, 1], \quad U_1(y)|_{y_2=0} = 0, \tag{7.15}$$

where $F = (0, \mathbb{P}_\sigma \mathcal{F}^{-1}_{\sigma}\mathcal{U})$. Here $M(y, D')$ is the matrix pseudo-differential operator with principal symbol $M_1(y, \zeta)$ is given by

$$M_1(y, \zeta) = \begin{pmatrix} 0 & \Lambda_1 E_2 \\ A^{-1} M_{21} \Lambda_1^{-1} & A^{-1} M_{22} \end{pmatrix} + i|s|\varphi_{y_2} E_4$$

(see [Y]). Here we set $\vec{\theta} = (\xi_1 + i|s|\varphi_{y_1}, 0)$, $G(y_1) = (-d\ell(y_1)/dy_1, 1)$, $\Lambda_1 = |\zeta|$, $M_{21}(y, \xi' + i|s|\nabla_{y'}\varphi(y)) = ((\xi_0 + i|s|\varphi_{y_0}(y))^2 - \mu(\xi' + i|s|\varphi_{y_1}(y))^2)E_2 - (\lambda+\mu)(y)\vec{\theta}^T\vec{\theta}$, $M_{22}(y, \xi') = -(\lambda+\mu)(y)\vec{\theta}^T G + G^T\vec{\theta} - 2\mu(\vec{\theta}, G)E_2$, $A = (\lambda+\mu)(y)G^T G + \mu(y)|G|^2 E_2$. The matrix $M_1(\gamma)$ has only two eigenvalues given by (4.23)-(4.25). Moreover it is known that the Jordan form of the matrix $M_1(\gamma)$ has two Jordan blocks of the form

$$M^{\pm} = \begin{pmatrix} \Gamma^{\pm}_{\mu}(\gamma) & 1 \\ 0 & \Gamma^{\pm}_{\mu}(\gamma) \end{pmatrix}.$$

Following [T1] and using the change of variables $W = S^{-1}(y, D')U$ which is constructed below, we can reduce the system (7.11) to the form

$$D_{y_2} W = \widetilde{M}(y, D')W + T(y, D')W + \widetilde{F}, \tag{7.16}$$

where the matrix $\widetilde{M}$ has the form

$$\widetilde{M}(y,\zeta) = \begin{pmatrix} M_+(y,\zeta) & 0 \\ 0 & M_-(y,\zeta) \end{pmatrix}, \quad M_{\pm} = \begin{pmatrix} \Gamma^{\pm}_{\lambda+2\mu}(y,\zeta) & m^{\pm}_{12}(y,\zeta) \\ 0 & \Gamma^{\pm}_{\mu}(y,\zeta) \end{pmatrix},$$

the operator T is in $L^\infty(0,1;\mathcal{L}((H^1(\Sigma))^4, (H^1(\Sigma))^4))$, $m^{\pm}_{12}(y, D')$ is a first order operator and

$$\|\widetilde{F}\|_{L^2(\mathbb{R}^1;(H^1(\Sigma))^2)} \le C_5(\|\mathbb{P}_\sigma \mathcal{F}^{-1}_\sigma \mathcal{U}\|_{(H^1(\mathcal{Q}))^2} + \|\mathcal{F}^{-1}_\sigma \mathcal{U}\|_{L^2(\mathbb{R}^1;(H^1(\Sigma))^2)}).$$

Now we describe the construction of the pseudo-differential operator S. We take the symbol S in the form $S = (s^+_1, s^+_2, s^-_1, s^-_2)$. Here

$$s^{\pm}_1 = \left((\vec{\theta} + \alpha^{\pm}_{\lambda+2\mu}G)\Lambda_1^{-1},\ \alpha^{\pm}_{\lambda+2\mu}(\vec{\theta} + \alpha^{\pm}_{\lambda+2\mu}G)\Lambda_1^{-1}\right)$$

are the eigenvectors of the matrix $M_1(y,\zeta)$ on the sphere $\zeta \in S^2$ which corresponds to the eigenvalue $\Gamma^{\pm}_{\lambda+2\mu}$ and the vectors $s^{\pm}_2$ are given by the formula

$$s^{\pm}_2 = E_{\pm}s^{\pm}, \quad E_{\pm} = \frac{1}{2\pi i}\int_{C^{\pm}} (z - M_1(y,\zeta))^{-1} dz,$$

where $C^{\pm}$ are small circles centered at $\Gamma^{\pm}_{\mu}(\gamma)$ and $s^{\pm}$ solves the equation $M_1(\gamma)s^{\pm} - \Gamma^{\pm}_{\mu}(\gamma)s^{\pm} = s^{\pm}_1$. Since $\zeta^* \in \widetilde{\mathcal{M}}$ and $\xi^*_0 = 0$ we have $\xi^*_1 \ne 0$. Therefore the circles $C^{\pm}$ may be taken such that the disks bounded by these circles do not intersect. Note that the vectors $s^{\pm}_j \in C^2(B_\delta \times \mathcal{O}_{\delta_1})$ are homogeneous functions of the order zero in (s, ξ_0, ξ_1). Now using a standard argument (see [Ku], p.241), we can estimate the last two components of W as follows

$$\|(W_3, W_4)\|_{(H^{\frac{3}{2}}(\Sigma))^2} \le C_6(\|\mathbb{P}_\sigma \mathcal{F}^{-1}_\sigma \mathcal{U}\|_{(H^1(\mathcal{Q}))^2} + \|\mathcal{U}\|_{(H^2(\mathcal{Q}))^2}),$$

where the constant C_6 is independent of N.

Now we need to estimate the first two components of the vector function W on Σ. Thanks to the zero boundary conditions for U_3 and U_4, we have

$$\begin{aligned} & S_{11}(y_0, y_1, 0, D')(W_1, W_2) \\ = & - S_{12}(y_0, y_1, 0, D')(W_3, W_4) + T_{-1}(y_0, y_1, 0, D')\mathcal{F}^{-1}_\sigma \mathcal{U}, \end{aligned} \tag{7.17}$$

where we set

$$S(y,\zeta) = \begin{pmatrix} S_{11}(y,\zeta) & S_{12}(y,\zeta) \\ S_{21}(y,\zeta) & S_{22}(y,\zeta) \end{pmatrix}, \quad T_{-1} : (H^1(\Sigma))^2 \to (H^2(\Sigma))^2.$$

The principal symbol of the pseudo-differential operator S_{11} is the 2×2 matrix such that the first column equals the last two coordinates of the vector s^+_1 and the second column equals the last two coordinates of the vector s^+_2. At the point γ, these vectors are given by the formulae

$$\vec{\eta} = (\xi^*_1 + is^*\varphi_{y_1}(y^*), i\text{sign}(\xi^*_1)(\xi^*_1 + is^*\varphi_{y_1}(y^*)))$$

$$s^+_1(\gamma) = \left(\vec{\eta},\, i\frac{\text{sign}(\xi^*_1)(\xi^*_1 + is^*\varphi_{y_1}(y^*))}{\sqrt{(\xi^*_1)^2 + (s^*)^2}}\vec{\eta}\right),$$

$$\vec{\varsigma}=\frac{-1}{\sqrt{(\xi_1^*)^2+(s^*)^2}}\frac{\lambda+3\mu}{2(\lambda+\mu)}(i\mathrm{sign}(\xi_1^*),1),$$

$$s_2^+(\gamma)=\left(\vec{\varsigma},\ \frac{1}{\sqrt{(\xi_1^*)^2+(s^*)^2}}(i\mathrm{sign}(\xi_1^*)(\xi_1^*+is^*\varphi_{y_1}(y^*))\vec{\varsigma}+\vec{\eta})\right).$$

Therefore $\det S_{11}(\gamma)\neq 0$. From (7.15), (7.16) and Garding's inequality, we obtain

$$\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_2},\mathbf{w}_\nu\right)\right\|_X\le C_7(\|\mathbb{P}_\sigma\mathcal{F}_\sigma^{-1}\mathcal{U}\|_{(H^1(\mathcal{Q}))^2}+\|\mathcal{U}\|_{(H^2(\mathcal{Q}))^2}), \tag{7.18}$$

where the constant C_7 is independent of N. By (7.9), (4.36) and (4.37), we obtain (4.20). ■

End of the proof of Theorem 2.1. Let us fix the parameter N such that (4.5) holds true. We take $\delta\in(0,\frac{1}{N^2})$ sufficiently small such that

$$\phi(x)>\varphi(x),\qquad \forall x\in\overline{\Omega_\delta\setminus\Omega_{\delta/2}}. \tag{7.19}$$

We consider a cut off function $\widetilde{\theta}\in C^3(\overline{\Omega}_\delta)$ such that $\widetilde{\theta}|_{\Omega_{\frac{\delta}{2}}}=1$ and $\widetilde{\theta}|_{\Omega_\delta\setminus\Omega_{\frac{3\delta}{4}}}=0$. The function $\widetilde{\theta}\mathbf{u}$ satisfies the equation

$$P(\widetilde{\theta}\mathbf{u})=\widetilde{\theta}\mathbf{f}+[P,\widetilde{\theta}]\mathbf{u},\quad \mathbf{u}|_{(0,T)\times\partial\Omega}=0,\quad \mathbf{u}(0,\cdot)=\mathbf{u}_{x_0}(0,\cdot)=\mathbf{u}(T,\cdot)=\mathbf{u}_{x_0}(T,\cdot)=0$$

Applying Carleman estimate (4.5) to this equation, we obtain

$$s\left\|\frac{\partial\mathbf{u}}{\partial\vec{n}}e^{s\phi}\right\|^2_{(H^1((0,T)\times\partial\Omega))^2}+s\left\|\frac{\partial^2\mathbf{u}}{\partial\vec{n}^2}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2}+s^3\left\|\frac{\partial\mathbf{u}}{\partial\vec{n}}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2}$$
$$\le C_8(s^2\|\mathbf{f}e^{s\varphi}\|^2_{(L^2(Q))^2}+\|(\nabla\mathbf{f})e^{s\varphi}\|^2_{(L^2(Q))^2}+s^2\|[P,\widetilde{\theta}]\mathbf{u}e^{s\varphi}\|^2_{(L^2(Q))^2}$$
$$+\|\nabla([P,\widetilde{\theta}]\mathbf{u})e^{s\varphi}\|^2_{(L^2(Q))^2}+\|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}),\quad \forall s\ge s_0(\widetilde{\tau}). \tag{7.20}$$

Since the supports of the coefficients of the commutator $[P,\theta]$ are in $\overline{\Omega_\delta\setminus\Omega_{\delta/2}}$ by (7.19), we have

$$s^2\|[P,\widetilde{\theta}]\mathbf{u}e^{s\varphi}\|^2_{(L^2(Q))^2}+\|\nabla([P,\widetilde{\theta}]\mathbf{u})e^{s\varphi}\|^2_{(L^2(Q))^2}+\|\mathbf{u}\|^2_{\mathcal{B}(\varphi,Q_\omega)}$$
$$\le C_9\left(\sum_{|\alpha|=0}^{2}s^{3-2|\alpha|}\|(\partial_x^\alpha\mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}+\|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}\right). \tag{7.21}$$

Combining (7.20) and (7.21), we obtain

$$s\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{(H^1((0,T)\times\partial\Omega))^2}+s\left\|\frac{\partial^2 \mathbf{u}}{\partial \vec{n}^2}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2}+s^3\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{(L^2((0,T)\times\partial\Omega))^2}$$
$$\le C_{10}\left(s^2\|\mathbf{f}e^{s\varphi}\|^2_{(L^2(Q))^2}+\|(\nabla\mathbf{f})e^{s\varphi}\|^2_{(L^2(Q))^2}+\sum_{|\alpha|=0}^{2}s^{3-2|\alpha|}\|(\partial_x^\alpha\mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}\right.$$
$$\left.+\|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}\right),\qquad \forall s\ge s_0(\widetilde{\tau}). \tag{7.22}$$

Finally we will estimate the surface integrals at the right hand side of (4.4) by the right hand side of (7.22). In the new inequality, the term

$$\sum_{|\alpha|=0}^{2}s^{3-2|\alpha|}\|(\partial_x^\alpha\mathbf{u})e^{s\phi}\|^2_{(L^2(Q))^2}$$

which appears at the right hand side, can be absorbed by $\|\mathbf{u}\|^2_{Y(\phi,Q)}$. Thus the proof of Theorem 2.1 is complete. ■

§8. Proofs of Theorems 2.2 and 2.3.

Proof of Theorem 2.2.

We introduce the Banach space $\mathcal{X}=(H^1(Q))^2$ with the norm $\|\mathbf{w}\|^2_{\mathcal{X}}=\int_Q(|\nabla\mathbf{w}|^2+s^2\mathbf{w}^2)dx$. In order to prove the theorem, we consider the following extremal problem

$$J(\mathbf{z},\mathbf{v}_1,\mathbf{v}_2)=\frac{1}{2}\|\mathbf{z}e^{-s\phi}\|^2_{(L^2(Q))^2}+\frac{1}{2}\|\mathbf{v}_1e^{-s\phi}\|^2_{(L^2(Q_\omega))^2}+\frac{1}{2s^2}\|\mathbf{v}_2e^{-s\phi}\|^2_{(L^2(Q_\omega))^2}$$
$$\to\inf, \tag{8.1}$$

$$P\mathbf{z}=\mathbf{u}e^{2s\phi}+\frac{\partial\mathbf{v}_1}{\partial x_0}+\mathbf{v}_2\quad\text{in } Q, \tag{8.2}$$

$$\operatorname{supp}\mathbf{v}_j\subset\overline{Q_\omega},\quad j\in\{1,2\},\quad \mathbf{z}|_{(0,T)\times\partial\Omega}=0,\mathbf{z}_{x_0}(0,x')=\mathbf{z}_{x_0}(T,x')=0. \tag{8.3}$$

Denote by $(\mathbf{z},\mathbf{v}_1,\mathbf{v}_2)$ the solution to extremal problem (8.1)-(8.3).

We have

Lemma 8.1. *Under the conditions of Theorem 2.2 for all* $\mathbf{u} \in (L^2(Q))^2$, *there exists a unique solution* $(\mathbf{z}, \mathbf{v}_1, \mathbf{v}_2) \in (L^2(Q))^2 \times (L^2(Q_\omega))^4$ *to problem (8.1) - (8.3). Moreover this solution satisfies the optimality system*

$$P^*\mathbf{p} + \mathbf{z}e^{-2s\phi} = 0 \quad \text{in } Q, \tag{8.4}$$

$$\mathbf{p}|_{(0,T)\times\partial\Omega} = 0, \quad \mathbf{p}_{x_0}(0,\cdot) = \mathbf{p}_{x_0}(T,\cdot) = \mathbf{p}(0,\cdot)|_\omega = \mathbf{p}(T,\cdot)|_\omega = 0, \tag{8.5}$$

$$\mathbf{p} = \frac{1}{s^2}\mathbf{v}_2 e^{-2s\phi} \quad \text{in } Q_\omega, \quad \frac{\partial \mathbf{p}}{\partial x_0} = -\mathbf{v}_1 e^{-2s\phi} \quad \text{in } Q_\omega, \tag{8.6}$$

$$P\mathbf{z} = \mathbf{u}e^{2s\phi} + \frac{\partial \mathbf{v}_1}{\partial x_0} + \mathbf{v}_2 \quad \text{in } Q, \tag{8.7}$$

$$\mathbf{z}|_{(0,T)\times\partial\Omega} = 0, \quad \mathbf{z}_{x_0}(0,\cdot) = \mathbf{z}_{x_0}(T,\cdot) = 0. \tag{8.8}$$

Here P^* denotes the formal adjoint operator to P. The proof of this lemma requires only the standard arguments (see e.g., [Li1]).

We extend the function $\mathbf{p}$ on the set $\widetilde{Q} = [-T, 2T]\times\Omega$ by the formula: $\mathbf{p}(x_0, x') = \mathbf{p}(-x_0, x')$ for $x \in [-T, 0]\times\Omega$ and $\mathbf{p}(x_0, x') = \mathbf{p}(2T - x_0, x')$ for $(x_0, x') \in [T, 2T]\times\Omega$. In the same way, we extend the function $-\mathbf{z}e^{-2s\phi}$ on the domain $\widetilde{Q}$ and denote the extended function by $\widetilde{\mathbf{f}}$. By (8.4), we have

$$P^*\mathbf{p} = \widetilde{\mathbf{f}} \quad \text{in } \widetilde{Q}. \tag{8.9}$$

Since we assume that $\frac{\partial\phi}{\partial x_0}(T, x') < 0$ for all $x' \in \overline{\Omega}$ and $\frac{\partial\phi}{\partial x_0}(0, x') > 0$ for all $x' \in \overline{\Omega}$, there exists $\delta > 0$ such that we can continue the function ϕ on $[-\delta, T+\delta]\times\Omega$ up to a C^3-function such that $\frac{\partial\phi(x)}{\partial x_0} < 0$ for all $x \in [T, T+\delta]\times\overline{\Omega}$ and $\frac{\partial\phi(x)}{\partial x_0} > 0$ for all $x \in [-\delta, 0]\times\overline{\Omega}$. Also Condition 2.1 for the function $\phi(x)$ holds true if we exchange the domains Q, Q_ω on $\widetilde{Q}, [-\delta, T+\delta]\times\omega$ respectively. Let $\chi_1 \in C_0^\infty[-\delta, T+\delta]$ be a cut-off function such that $\chi_1|_{[-\frac{\delta}{2}, T+\frac{\delta}{2}]} = 1$. Then

$$P^*\chi_1\mathbf{p} = \chi_1\widetilde{\mathbf{f}} - [\chi_1, P^*]\mathbf{p} \quad \text{in } \widetilde{Q}, \tag{8.10}$$

where supp $[\chi_1, P^*]\mathbf{p} \subset ([T+\frac{\delta}{2}, T+\delta]\times\overline{\Omega})\cup([-\delta, -\frac{\delta}{2}]\times\overline{\Omega})$. We will apply Carleman estimate (2.8) to equation (8.10).

We observe that

$$\begin{aligned} &\|\widetilde{\mathbf{f}}e^{s\phi}\|_{L^2(-\delta,T+\delta;(L^2(\Omega))^2)} \le C_1\|\mathbf{z}e^{-s\phi}\|_{(L^2(Q))^2}, \\ &\|[\chi_1, P^*]\mathbf{p}\|_{L^2(-\delta,T+\delta;(L^2(\Omega))^2)} \le \frac{C_2}{s}\|\mathbf{p}\|_{\mathcal{X}}. \end{aligned} \tag{8.11}$$

Moreover we can prove that at the right hand side of (2.8), we can exchange the integral over Q_ω by the following integral

$$\int_{Q_\omega}\left(\left|\frac{\partial^2\mathbf{u}}{\partial x_0^2}\right|^2 + s^2\left|\frac{\partial\mathbf{u}}{\partial x_0}\right|^2 + s^4|\mathbf{u}|^2\right)e^{2s\phi}dx.$$

Note that thanks to the choice of extension of the function ϕ, we have

$$\begin{aligned} &\int_{\widetilde{Q}_\omega}\left(\left|\frac{\partial^2(\chi_1\mathbf{p})}{\partial x_0^2}\right|^2 + s^2\left|\frac{\partial(\chi_1\mathbf{p})}{\partial x_0}\right|^2 + s^4|\chi_1\mathbf{p}|^2\right)e^{2s\phi}dx \\ \le& C_3\int_{Q_\omega}\left(\left|\frac{\partial^2\mathbf{p}}{\partial x_0^2}\right|^2 + s^2\left|\frac{\partial\mathbf{p}}{\partial x_0}\right|^2 + s^4|\mathbf{p}|^2\right)e^{2s\phi}dx. \end{aligned} \tag{8.12}$$

Using estimates (8.11) and (8.12), by Theorem 2.1, we obtain

$$\sum_{|\alpha|=2}\|(\partial_x^\alpha\mathbf{p})e^{s\phi}\|^2_{(L^2(Q))^2} + s^2\|\mathbf{p}e^{s\phi}\|^2_{\mathcal{X}} \le C_{14}J_1(\mathbf{z}, \mathbf{v}_1, \mathbf{v}_2), \tag{8.13}$$

where we set

$$J_1(\mathbf{z}, \mathbf{v}_1, \mathbf{v}_2) = \|\mathbf{z}e^{-s\phi}\|^2_{\mathcal{X}} + \int_{Q_\omega}\left(\left|\frac{\partial\mathbf{v}_1}{\partial x_0}\right|^2 + s^2|\mathbf{v}_1|^2 + |\mathbf{v}_2|^2\right)e^{-2s\phi}dx.$$

By (8.4), (8.5), (8.7), (8.8) and integration by parts, we have

$$\begin{aligned} &\left(\mathbf{u}e^{2s\phi} + \frac{\partial\mathbf{v}_1}{\partial x_0} + \mathbf{v}_2,\, \mathbf{p}\right)_{(L^2(Q))^2} \\ =&(P\mathbf{z}, \mathbf{p})_{(L^2(Q))^2} = (\mathbf{z}, P^*\mathbf{p})_{(L^2(Q))^2} = -(\mathbf{z}, \mathbf{z}e^{-2s\phi})_{(L^2(Q))^2}. \end{aligned}$$

Therefore, taking the scalar product of (8.7) and $\mathbf{p}$ in $(L^2(Q))^2$ and using (8.3) and (8.6), we obtain

$$2J(\mathbf{z},\mathbf{v}_1,\mathbf{v}_2) = -\frac{1}{2}\int_Q (\mathbf{u}e^{2s\phi},\mathbf{p})dx.$$

By (8.13), we obtain from this inequality

$$s^2 J(\mathbf{z},\mathbf{v}_1,\mathbf{v}_2) \le C_5\|\mathbf{u}e^{s\phi}\|_{(L^2(Q))^2} J_1(\mathbf{z},\mathbf{v}_1,\mathbf{v}_2)^{\frac{1}{2}}. \tag{8.14}$$

Next we differentiate equations (8.4) and (8.7) with respect to the variable x_0:

$$P^*\frac{\partial \mathbf{p}}{\partial x_0} = \frac{\partial}{\partial x_0}\widetilde{\mathbf{f}} \quad \text{in } Q, \tag{8.15}$$

$$P\frac{\partial \mathbf{z}}{\partial x_0} = \frac{\partial(\mathbf{u}e^{2s\phi})}{\partial x_0} + \frac{\partial^2 \mathbf{v}_1}{\partial x_0^2} + \frac{\partial \mathbf{v}_2}{\partial x_0} \quad \text{in } Q. \tag{8.16}$$

Taking the scalar product of (8.16) and $\frac{\partial \mathbf{p}}{\partial x_0}$ in $(L^2(Q))^2$ and integrating by parts, we similarly obtain

$$2J\left(\frac{\partial \mathbf{z}}{\partial x_0},\frac{\partial \mathbf{v}_1}{\partial x_0},\frac{\partial \mathbf{v}_2}{\partial x_0}\right)$$
$$= \int_Q \left(\left(\mathbf{u}e^{2s\phi},\frac{\partial^2\mathbf{p}}{\partial x_0^2}\right) + 2s\phi_{x_0}\left(\frac{\partial \mathbf{z}}{\partial x_0},\mathbf{z}\right) + 2s\phi_{x_0}\left(\frac{\partial \mathbf{v}_1}{\partial x_0},\mathbf{v}_1\right) + \frac{2\phi_{x_0}}{s}\left(\frac{\partial \mathbf{v}_2}{\partial x_0},\mathbf{v}_2\right)\right)dx.$$

This equality and (8.13), (8.14) imply

$$J(\partial_{x_0}\mathbf{z},\partial_{x_0}\mathbf{v}_1,\partial_{x_0}\mathbf{v}_2) \le C_6\|\mathbf{u}e^{s\phi}\|_{(L^2(Q))^2} J_1(\mathbf{z},\mathbf{v}_1,\mathbf{v}_2)^{\frac{1}{2}}. \tag{8.17}$$

Let $\widetilde{L}$ denote the part of first order of $L_{\lambda,\mu}$, that is, $(\widetilde{L}\mathbf{v})(x') = \operatorname{div}\mathbf{v}(x')\nabla_{x'}\lambda(x') + (\nabla_{x'}\mathbf{v} + (\nabla_{x'}\mathbf{v})^T)\nabla_{x'}\mu(x')$. Taking the scalar product of (8.7) with $\mathbf{z}e^{-2s\phi}$ in $(L^2(Q))^2$, we obtain

$$\int_Q (\mu|\nabla_{x'}\mathbf{z}|^2 + (\lambda+\mu)(\operatorname{div}\mathbf{z})^2)e^{-2s\phi}dx - \int_Q (\widetilde{L}\mathbf{z},\mathbf{z}e^{-2s\phi})dx$$
$$= \int_Q \left(\left|\frac{\partial \mathbf{z}}{\partial x_0}\right|^2 - 2s\phi_{x_0}\left(\frac{\partial \mathbf{z}}{\partial x_0},\mathbf{z}\right)\right)e^{-2s\phi}dx$$
$$+ \int_Q \left(2\mu s\sum_{k=1}^2(\partial_{x_k}\mathbf{z},(\partial_{x_k}\phi)\mathbf{z}) + 2(\lambda+\mu)(\operatorname{div}\mathbf{z})(\nabla_{x'}\phi,\mathbf{z})\right)e^{-2s\phi}dx$$
$$- \int_Q (s^2\mathbf{u}e^{2s\phi},\mathbf{z})e^{-2s\phi}dx + \int_Q \left(\frac{\partial \mathbf{v}_1}{\partial x_0} + \mathbf{v}_2,\mathbf{z}e^{-2s\phi}\right)dx.$$

We note that $|\partial_{x_j} z_k||z_\ell| \le \frac{\varepsilon}{2}|\partial_{x_j} z_k|^2 + \frac{1}{2\varepsilon}|z_\ell|^2$ for any $\varepsilon > 0$. Therefore if we take sufficiently small $\varepsilon > 0$ and sufficiently large $s > 0$, then by (8.13) and (8.17), we obtain

$$\|\mathbf{z}e^{-s\phi}\|^2_{\mathcal{X}} + \left\|\frac{\partial \mathbf{v}_1}{\partial x_0}e^{-s\phi}\right\|^2_{(L^2(Q_\omega))^2} + \|\mathbf{v}_2 e^{-s\phi}\|^2_{(L^2(Q_\omega))^2} \le C_7\|\mathbf{u}e^{s\phi}\|^2_{(L^2(Q))^2}. \tag{8.18}$$

Finally, taking the scalar product of (1.1) with $\mathbf{z}$ in $(L^2(Q))^2$ and integrating by parts, we obtain the equality

$$\|\mathbf{u}e^{s\phi}\|^2_{(L^2(Q))^2} = \int_Q (\mathbf{f}, \mathbf{z})dx - \int_{Q_\omega}\left(\mathbf{u}, \frac{\partial \mathbf{v}_1}{\partial x_0} + \mathbf{v}_2\right)dx. \tag{8.19}$$

Applying (8.18) to this equality and using again an inequality $|ab| \le \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2$ for any $\varepsilon > 0$, we obtain

$$\begin{aligned}&\int_Q s^2|\mathbf{u}|^2 e^{2s\phi}dx\\ \le& C_1(\|\mathbf{f}e^{s\phi}\|^2_{(L^2(Q))^2} + \int_{Q_\omega}(|\nabla \mathbf{u}|^2 + s^2|\mathbf{u}|^2)e^{2s\phi}dx), \quad \forall s \ge s_0(\tau),\end{aligned} \tag{8.20}$$

In order to estimate the first derivatives for the function $\mathbf{u}$, we consider extremal problem (8.1)-(8.3) with $\frac{\partial \mathbf{u}}{\partial x_0}$ instead of $\mathbf{u}$. Using the same notations for solution of this extremal problem and repeating the previous arguments, we obtain an analogue of (8.18):

$$\|\mathbf{z}e^{-s\phi}\|^2_{\mathcal{X}} + \left\|\frac{\partial \mathbf{v}_1}{\partial x_0}e^{-s\phi}\right\|^2_{(L^2(Q_\omega))^2} + \|\mathbf{v}_2 e^{-s\phi}\|^2_{(L^2(Q_\omega))^2} \le C_8\left\|\frac{\partial \mathbf{u}}{\partial x_0}e^{s\phi}\right\|^2_{(L^2(Q))^2}. \tag{8.21}$$

Since the Lamé coefficients are independent of x_0, we have

$$P\frac{\partial \mathbf{u}}{\partial x_0} = \frac{\partial \mathbf{f}}{\partial x_0} \quad \text{in } Q, \quad \frac{\partial \mathbf{u}}{\partial x_0}|_{(0,T)\times\partial\Omega} = 0, \mathbf{u}_{x_0}(T, x') = \mathbf{u}_{x_0}(0, x') = 0, \tag{8.22}$$

Taking the scalar product of (8.22) with $\mathbf{z}$ in $(L^2(Q))^2$ and integrating by parts, we obtain the equality

$$\left\| \frac{\partial \mathbf{u}}{\partial x_0} e^{s\phi} \right\|^2_{(L^2(Q))^2} = \int_Q \left(\frac{\partial \mathbf{f}}{\partial x_0}, \mathbf{z} \right) dx - \int_{Q_\omega} \left(\frac{\partial \mathbf{u}}{\partial x_0}, \frac{\partial \mathbf{v}_1}{\partial x_0} + \mathbf{v}_2 \right) dx.$$

Applying (8.18) and the inequality $2|ab| \le \delta |a|^2 + \frac{1}{\delta}|b|^2$ to the second term at the right hand side of this equality, we obtain

$$\begin{aligned} &\int_Q \left(\left| \frac{\partial \mathbf{u}}{\partial x_0} \right|^2 + s^2 |\mathbf{u}|^2 \right) e^{2s\phi} dx \\ \le & C_9 (\|\mathbf{f} e^{s\phi}\|^2_{(L^2(Q))^2} + \int_{Q_\omega} (|\nabla \mathbf{u}|^2 + s^2 |\mathbf{u}|^2) e^{2s\phi} dx), \quad \forall s \ge s_0(\tau), \end{aligned} \tag{8.23}$$

Finally, taking the scalar product of (1.1) with $\mathbf{u}e^{2s\phi}$ in $(L^2(Q))^2$, we obtain

$$\begin{aligned} &\int_Q (\mu |\nabla \mathbf{u}|^2 + (\lambda + \mu)(\operatorname{div} \mathbf{u})^2) e^{2s\phi} dx = \int_Q \left(\left| \frac{\partial \mathbf{u}}{\partial x_0} \right|^2 + 2s\phi_{x_0} \left(\frac{\partial \mathbf{u}}{\partial x_0}, \mathbf{u} \right) \right) e^{2s\phi} dx \\ &- \int_Q \left(2\mu s \sum_{k=1}^2 (\partial_{x_k} \mathbf{u}, (\partial_{x_k} \phi) \mathbf{u}) + 2(\lambda + \mu)(\operatorname{div} \mathbf{z})(\nabla_{x'} \phi, \mathbf{u}) \right) e^{2s\phi} dx \\ &+ \int_Q (\widetilde{L} \mathbf{u}, \mathbf{u} e^{2s\phi}) dx + \int_Q (\mathbf{f}, \mathbf{u}) e^{2s\phi} dx. \end{aligned}$$

This equality and (8.23) imply (2.10), the conclusion of Theorem 2.2. ■

Proof of Theorem 2.3.

In order to complete the proof, it is sufficient to estimate $\int_Q (\mathbf{f}, \mathbf{z}) dx$ in (8.19) as follows:

$$\begin{aligned} &\left| \int_Q (\mathbf{f}_0, \mathbf{z}) dx \right| \le \|\mathbf{f}_0 e^{s\phi}\|_{L^2(0,T;(H^{-1}(\Omega))^2)} \|\mathbf{z} e^{-s\phi}\|_{L^2(0,T;(H_0^1(\Omega))^2)} \\ \le & \|\mathbf{f}_0 e^{s\phi}\|_{L^2(0,T;(H^{-1}(\Omega))^2)} \|\mathbf{z} e^{-s\phi}\|_{\mathcal{X}} \end{aligned}$$

and

$$\begin{aligned} &\left| \int_Q (\partial_{x_j} \mathbf{f}_j, \mathbf{z}) dx \right| = \left| \int_Q (\mathbf{f}_j, \partial_{x_j} \mathbf{z}) dx \right| \le \|\mathbf{f}_j e^{s\phi}\|_{(L^2(Q))^2} \|(\partial_{x_j} \mathbf{z}) e^{-s\phi}\|_{(L^2(Q))^2} \\ \le & C_{10} \|\mathbf{f}_j e^{s\phi}\|_{(L^2(Q))^2} (\|\nabla (\mathbf{z} e^{-s\phi})\|_{(L^2(Q))^2} + s \|\mathbf{z} e^{-s\phi}\|_{(L^2(Q))^2}) \\ \le & C_{11} \|\mathbf{f}_j e^{s\phi}\|_{(L^2(Q))^2} \|\mathbf{z} e^{-s\phi}\|_{\mathcal{X}}. \end{aligned}$$

Therefore

$$\left|\int_Q \left(\left(\mathbf{f}_0+\sum_{j=1}^2 \partial_{x_j}\mathbf{f}_j\right),\mathbf{z}\right)dx\right|$$
$$\le C_{12}\left(\|\mathbf{f}_0 e^{s\phi}\|_{L^2(0,T;(H^{-1}(\Omega))^2}+\sum_{j=1}^2\|\mathbf{f}_j e^{s\phi}\|_{(L^2(Q))^2}\right)\|\mathbf{z}e^{-s\phi}\|_{\mathcal{X}}.$$

Appendix I. Proof of Proposition 4.2.

In order to prove the proposition, it is convenient to use the coordinate x instead of y. Moreover it suffices to prove the estimate for an arbitrary but fixed $x_0\in[0,T]$. Therefore we should establish the estimate: There exist $\widehat{\tau}>1$ and $N_0>1$ such that for any $\widetilde{\tau}>\widehat{\tau}$ and $N>N_0$, there exists $s_0(\widetilde{\tau},N)$ such that

$$N\int_{\Omega_N}\left(\frac{1}{s\varphi}\sum_{j,k=1}^2|\partial_{x_j}\partial_{x_k}\mathbf{u}|^2+s\varphi|\nabla_{x'}\mathbf{u}|^2+s^3\varphi^3|\mathbf{u}|^2\right)e^{2s\varphi}dx'$$
$$\le C_0(\|\mathrm{rot}\,\mathbf{u}\,e^{s\varphi}\|^2_{H^1(\Omega_N)}+\|\mathrm{div}\,\mathbf{u}e^{s\varphi}\|^2_{H^1(\Omega_N)}),$$
$$\forall\mathbf{u}\in(H^1_0(\Omega_N))^2,\quad\forall s\ge s_0(\widetilde{\tau}),\quad\mathrm{supp}\,\mathbf{u}\subset B_\delta\cap\Omega_N,\tag{1}$$

where the constant C_0 is independent of N.

First we choose $N_0>0$ sufficiently large such that

$$\nabla_{x'}\psi(x)\ne0,\quad\forall x'\in\Omega_N,\ \forall x_0\in(0,T).$$

The existence of such N_0 follows from (2.6).

Denote $\mathrm{rot}\,\mathbf{u}\equiv\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}=\mathbf{y}$ and $\mathrm{div}\,\mathbf{u}\equiv\mathbf{w}$. Let $\mathrm{rot}^*v=(\frac{\partial v}{\partial x_2},-\frac{\partial v}{\partial x_1})$. Using a formula $\mathrm{rot}^*\mathrm{rot}=-\Delta_{x'}+\nabla_{x'}\mathrm{div}$, we obtain

$$-\Delta_{x'}\mathbf{u}=-\mathrm{rot}^*\,\mathbf{y}-\nabla_{x'}\mathbf{w}\quad\text{in }\Omega_N,\quad\mathbf{u}|_{\partial\Omega_N}=0.$$

The function $\widetilde{\mathbf{u}}=\mathbf{u}e^{s\varphi}$ satisfies the equation

$$L_1\widetilde{\mathbf{u}}+L_2\widetilde{\mathbf{u}}=\mathbf{q}_s\quad\text{in }\Omega_N,\ \widetilde{\mathbf{u}}|_{\partial\Omega_N}=0,\tag{2}$$

where $L_1\widetilde{\mathbf{u}} = -\Delta_{x'}\widetilde{\mathbf{u}} - s^2|\nabla_{x'}\varphi|^2\widetilde{\mathbf{u}}$, $L_2\widetilde{\mathbf{u}} = 2s\sum_{k=1}^2(\partial_{x_k}\widetilde{\mathbf{u}})\varphi_{x_k} + s(\Delta_{x'}\varphi)\widetilde{\mathbf{u}}$ and $\mathbf{q}_s = (-\mathrm{rot}^*\,\mathbf{y} - \nabla_{x'}\mathbf{w})e^{s\varphi}$. Taking the L^2 norms of the right and the left hand sides of equation (2), we obtain

$$\|L_1\widetilde{\mathbf{u}}\|^2_{(L^2(\Omega_N))^2} + \|L_2\widetilde{\mathbf{u}}\|^2_{(L^2(\Omega_N))^2} + 2(L_1\widetilde{\mathbf{u}}, L_2\widetilde{\mathbf{u}})_{(L^2(\Omega_N))^2} = \|\mathbf{q}_s\|^2_{(L^2(\Omega_N))^2}.$$

Therefore we can obtain the formula

$$(L_1\widetilde{\mathbf{u}}, L_2\widetilde{\mathbf{u}})_{(L^2(\Omega_N))^2} = \int_{\Omega_N}\Bigg(2s\sum_{k,j=1}^2(\partial_{x_j}\widetilde{\mathbf{u}})(\partial_{x_k}\widetilde{\mathbf{u}})\varphi_{x_jx_k} + s^3(\mathrm{div}(|\nabla_{x'}\varphi|^2\nabla_{x'}\varphi)$$
$$-|\nabla_{x'}\varphi|^2\Delta_{x'}\varphi)|\widetilde{\mathbf{u}}|^2 - \frac{s}{2}\sum_{j=1}^2\frac{\partial^2\Delta_{x'}\varphi}{\partial x_j^2}|\widetilde{\mathbf{u}}|^2\Bigg)dx' - s\int_{\partial\Omega}\left|\frac{\partial\widetilde{\mathbf{u}}}{\partial\vec{n}}\right|^2(\nabla_{x'}\varphi, \vec{n})d\sigma. \qquad (3)$$

By (2.6), the last integral in (3) is nonnegative. Denote $\psi_1(x) = \psi(x) - \widehat{\epsilon}\ell_1(x)$. Then

$$\mathrm{div}(|\nabla_{x'}\varphi|^2\nabla_{x'}\varphi) - |\nabla\varphi_{x'}|^2\Delta_{x'}\varphi = 2\sum_{k,j=1}^2\varphi_{x_k}\varphi_{x_j}\varphi_{x_kx_j}$$
$$=2\varphi^3\sum_{k,j=1}^2\widetilde{\tau}^4(\partial_{x_k}\psi_1 + 2N\ell_1\partial_{x_k}\ell_1)^2(\partial_{x_j}\psi_1 + 2N\ell_1\partial_{x_j}\ell_1)^2$$
$$+\widetilde{\tau}^3(\partial_{x_k}\psi_1 + 2N\ell_1\partial_{x_k}\ell_1)(\partial_{x_j}\psi_1 + 2N\ell_1\partial_{x_j}\ell_1)(\partial_{x_j}\partial_{x_k}\psi_1 + 2N\partial_{x_k}\ell_1\partial_{x_j}\ell_1 + 2N\ell_1\partial_{x_k}\partial_{x_j}\ell_1).$$

Since $(\nabla_{x'}\psi_1, \nabla_{x'}\ell_1) > 0$ on $\partial\Omega$, there exists a constant $C_1 > 0$ which is independent of $N, \widetilde{\tau}, s$ such that

$$\mathrm{div}(|\nabla_{x'}\varphi|^2\nabla_{x'}\varphi) - |\nabla_{x'}\varphi|^2\Delta_{x'}\varphi \ge 2\varphi^3\widetilde{\tau}^4|\nabla_{x'}\psi_1|^4 + C_1N\widetilde{\tau}^3\varphi^3 + \varphi^2O(\widetilde{\tau}^3). \qquad (4)$$

On the other hand, by the definition of $\widetilde{\psi} = \psi - \widehat{\epsilon}\ell_1 + N\ell_1^2 = \psi_1 + N\ell_1^2$,

$$\sum_{k,j=1}^2(\partial_{x_j}\widetilde{\mathbf{u}})(\partial_{x_k}\widetilde{\mathbf{u}})\varphi_{x_jx_k} = \widetilde{\tau}^2(\nabla_{x'}\widetilde{\mathbf{u}}, \nabla_{x'}\widetilde{\psi})^2\varphi$$
$$+\widetilde{\tau}\sum_{k,j=1}^2(\partial_{x_j}\widetilde{\mathbf{u}})(\partial_{x_k}\widetilde{\mathbf{u}})(\partial_{x_j}\partial_{x_k}\psi_1 + 2N\ell_1\partial_{x_j}\partial_{x_k}\ell_1)\varphi + 2N\widetilde{\tau}(\nabla_{x'}\widetilde{\mathbf{u}}, \nabla_{x'}\ell_1)^2\varphi. \qquad (5)$$

Note that there exists a constant $C_2 > 0$, independent of N, such that

$$\|N\ell_1\partial^2_{x_ix_j}\ell_1\|_{C^0(\overline{\Omega_N})} \le C_2/N. \tag{6}$$

By (3)-(6), we obtain

$$\|L_1\widetilde{\mathbf{u}}\|^2_{(L^2(\Omega_N))^2} + \|L_2\widetilde{\mathbf{u}}\|^2_{(L^2(\Omega_N))^2} + \int_{\Omega_N}(2\varphi^3\widetilde{\tau}^4|\nabla_{x'}\psi_1|^4 + C_1N\widetilde{\tau}^3\varphi^3)|\widetilde{\mathbf{u}}|^2dx'$$
$$-s\widetilde{\tau}C_3\int_{\Omega_N}\varphi|\nabla_{x'}\widetilde{\mathbf{u}}|^2dx' \le \|\mathbf{q}_s\|^2_{(L^2(\Omega_N))^2}. \tag{7}$$

Multiplying equation (2) by $sN\varphi\widetilde{\mathbf{u}}$ and integrating by parts, we obtain

$$\int_{\Omega_N}(sN\varphi|\nabla_{x'}\widetilde{\mathbf{u}}|^2 + s^2N(\Delta_{x'}\varphi)\varphi|\widetilde{\mathbf{u}}|^2 - s^3\varphi^3|\nabla_{x'}\varphi|^2|\widetilde{\mathbf{u}}|^2 - \frac{sN}{2}\mathrm{div}\varphi|\widetilde{\mathbf{u}}|^2)dx'$$
$$= \int_{\Omega_N}\mathbf{q}_s sN\varphi\widetilde{\mathbf{u}}dx'. \tag{8}$$

Next we note that

$$\Delta_{x'}\varphi = (|\nabla_{x'}\widetilde{\psi}|^2\widetilde{\tau}^2 + \widetilde{\tau}\Delta_{x'}\psi_1 + 2\widetilde{\tau}N|\nabla_{x'}\ell_1|^2 + 2\widetilde{\tau}N\ell_1\Delta_{x'}\ell_1)\varphi \ge C_4\widetilde{\tau}N\varphi.$$

This inequality and (8) imply

$$\int_{\Omega_N}\{sN\varphi|\nabla_{x'}\widetilde{\mathbf{u}}|^2 + \frac{1}{2}s^2N(\Delta_{x'}\varphi)\varphi|\widetilde{\mathbf{u}}|^2 - s^3\varphi^3|\nabla_{x'}\varphi|^2|\widetilde{\mathbf{u}}|^2\}dx' \le C_4\|\mathbf{q}_s\|^2_{(L^2(\Omega_N))^2}. \tag{9}$$

From (7) and (9), we obtain

$$\|L_1\widetilde{\mathbf{u}}\|^2_{(L^2(\Omega_N))^2} + \|L_2\widetilde{\mathbf{u}}\|^2_{(L^2(\Omega_N))^2} + \int_{\Omega_N}\left(\frac{1}{2}\varphi^3\widetilde{\tau}^4|\nabla_{x'}\psi_1|^4 + C_1N\widetilde{\tau}^3\varphi^3\right)|\widetilde{\mathbf{u}}|^2dx'$$
$$+sN\int_{\Omega_N}\varphi|\nabla_{x'}\widetilde{\mathbf{u}}|^2dx' \le C_5\|\mathbf{q}_s\|^2_{(L^2(\Omega_N))^2}. \tag{10}$$

Let $\widetilde{\mathbf{u}} = \widetilde{\mathbf{u}}_1 + \widetilde{\mathbf{u}}_2$ where the functions $\widetilde{\mathbf{u}}_j$ are solutions to the initial value problems

$$-\Delta_{x'}\widetilde{\mathbf{u}}_1 = L_1\widetilde{\mathbf{u}} \quad \text{in} \quad \Omega_{N_0}, \quad \widetilde{\mathbf{u}}_1|_{\partial\Omega_{N_0}} = 0, \quad -\Delta_{x'}\widetilde{\mathbf{u}}_2 = s^2|\nabla_{x'}\varphi|^2\widetilde{\mathbf{u}}\,\text{in}\,\Omega_{N_0},\ \widetilde{\mathbf{u}}_2|_{\partial\Omega_{N_0}} = 0.$$

From a standard a priori estimate for the Laplace operator, we have

$$\|\widetilde{\mathbf{u}}_1\|_{(H^2(\Omega_N))^2} \le C_6 \|L_1 \widetilde{\mathbf{u}}\|_{(L^2(\Omega_N)^2}, \tag{11}$$

$$\frac{\sqrt{N}}{\sqrt{s}} \|\widetilde{\mathbf{u}}_2\|_{(H^2(\Omega_N))^2} \le C_7 \sqrt{N} \|s^{\frac{3}{2}} |\nabla_{x'}\varphi|^2 \widetilde{\mathbf{u}}\|_{(L^2(\Omega_N))^2}, \tag{12}$$

where the constants C_6 and C_7 are independent of N. Taking $s_0(\tau, N) \ge N$, we obtain (1) from (9) - (12). ■

Appendix II. Proof of the estimate (4.33).

We prove (4.33) for a more general hyperbolic operator. Denote $x = (x_0, x') = (x_0, x_1, ..., x_n)$, $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, ..., \xi_n)$ and $\mathcal{G}_N = \mathbb{R}^n \times [0, \frac{1}{N^2}]$.

Let a function $w \in H^1(\mathcal{G}_N)$ satisfy the equations

$$R(x', D)w = \frac{\partial^2 w}{\partial x_0^2} - \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x') \frac{\partial w}{\partial x_k}\right)$$
$$+ \sum_{j=0}^{n} b_j(x') \frac{\partial w}{\partial x_j} + c(x')w = g \text{ in } \mathcal{G}_N, \tag{1}$$

$$w|_{x_n = \frac{1}{N^2}} = \frac{\partial w}{\partial x_n}|_{x_n = \frac{1}{N^2}} = 0, \quad \operatorname{supp} w \subset B_\delta(x^*), \tag{2}$$

where x^* is an arbitrary point on ∂G_N and $B_\delta(x^*)$ is a ball of radius δ centered at x^*.

We assume that the coefficients of the linear operator R satisfy the conditions

$$a_{jk} \in C^1(\overline{\mathcal{G}_N}),\ a_{jk} = a_{kj}, \quad 1 \le j, k \le n, \quad b_\ell \in L^\infty(\mathcal{G}_N), \quad 0 \le \ell \le n, \quad c \in L^\infty(\mathcal{G}_N) \tag{3}$$

and the uniform ellipticity: there exists $\delta > 0$ such that

$$a(x', \xi, \xi) \equiv \sum_{j,k=1}^{n} a_{jk}(x') \xi_j \xi_k \ge \delta |\xi|^2, \quad \forall \xi \in \mathbb{R}^{n+1}, \quad \forall x \in \overline{\mathcal{G}_N}. \tag{4}$$

By $R(x',\xi)$, we denote the principal symbol of the operator R:

$$R(x',\xi) = -\xi_0^2 + \sum_{j,k=1}^{n} a_{jk}(x')\xi_j\xi_k,$$

and by $\widetilde{R}(x',\xi^1,\xi^2)$ the quadratic form

$$\widetilde{R}(x',\xi^1,\xi^2) = \xi_0^1\xi_0^2 - \sum_{j,k=1}^{n} a_{jk}(x')\xi_j^1\xi_k^2$$

with $\xi^1 = (\xi_0^1, ..., \xi_n^1)$ and $\xi^2 = (\xi_0^2, ..., \xi_n^2)$. Following [Hö], we introduce the notations:

$$R^{(j)}(x',\xi) = \frac{\partial R(x',\xi)}{\partial \xi_j}, \quad R^{(j,k)}(x',\xi) = \frac{\partial^2 R(x',\xi)}{\partial \xi_j \partial \xi_k}, \quad R_{(j)}(x',\xi) = \frac{\partial R(x',\xi)}{\partial x_j}.$$

We assume that there exists a function $\psi_1 \in C^2(\overline{\mathcal{G}_N})$ such that

$$\{R, \{R, \psi_1\}\}(x,\xi) > 0 \tag{5}$$

if $(x,\xi) \in (\overline{\mathcal{G}_N \setminus B_\delta(x^*)}) \times (\mathbb{R}^{n+1} \setminus \{0\})$ satisfies

$$R(x',\xi) =< \nabla_\xi R(x',\xi), \nabla\psi_1(x) >= 0,$$

and

$$\{R(x',\xi - is\nabla\psi_1(x)), R(x',\xi + is\nabla\psi_1(x))\}/2is > 0 \tag{6}$$

if $(x,\xi,s) \in (\overline{\mathcal{G}_N \setminus B_\delta(x^*)}) \times (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ satisfies

$$R(x',\xi + is\nabla\psi_1(x)) =< \nabla_\xi R(x',\xi + is\nabla\psi_1(x)), \nabla\psi_1(x) >= 0.$$

$$R(x, \nabla\psi_1) < 0.$$

Using the function ψ_1 and following [Hö], we construct the function φ by

$$\phi(x) = e^{\widetilde{\tau}\psi_1(x)}, \quad \widetilde{\tau} > 1. \tag{7}$$

It is known (see e.g., Theorem 8.6.2 [Hö,p.205]) that if the parameter $\widetilde{\tau}$ is sufficiently large, then:

$$\{R,\{R,\phi\}\}(x,\xi)>0 \tag{8}$$

if $(x,\xi)\in(\overline{\mathcal{G}_N\setminus B_\delta(x^*)})\times(\mathbb{R}^{n+1}\setminus\{0\})$ satisfies

$$R(x',\xi)=0, \tag{9}$$

and

$$\{R(x',\xi-is\nabla\phi(x)),R(x',\xi+is\nabla\phi(x))\}/2is>0$$

if $(x,\xi,s)\in(\overline{\mathcal{G}_N\setminus B_\delta(x^*)})\times(\mathbb{R}^{n+1}\setminus\{0\})\times(\mathbb{R}\setminus\{0\})$ satisfies

$$R(x',\xi+is\nabla\phi(x))=0.$$

Now we fix the parameter $\widetilde{\tau}$ such that inequalities (8) and (9) hold true. Let $\ell_1\in C^2(\mathcal{G}_N)$ be a function such that $\ell_1|_{x_n=0}=0$. Let $\widetilde{\psi}(x)=\psi_1(x)+N\ell_1^2(x)$ and $\varphi=e^{\widetilde{\tau}\widetilde{\psi}}$. Since $\varphi(x)=\phi(x)e^{\widetilde{\tau}N\ell_1^2(x)}$, using $\ell_1|_{x_n=0}=0$, we have

$$\varphi\to\phi\quad\text{in } C^1(\overline{\mathcal{G}_N})\ \text{ as }\ N\to+\infty. \tag{10}$$

Moreover

$$\begin{aligned}&\{R(x',\xi-is\nabla\varphi(x)),R(x',\xi+is\nabla\varphi(x))\}/2is\\&-2N\widetilde{\tau}\sum_{j,k=1}^{n}(\partial_{x_j}\ell_1(x))(\partial_{x_k}\ell_1(x))(R^{(j)}(x',\xi)R^{(k)}(x',\xi)+s^2R^{(j)}(x',\nabla\varphi)R^{(k)}(x',\nabla\varphi))\\&\longrightarrow\{R(x',\xi-is\nabla\phi(x)),R(x',\xi+is\nabla\phi(x))\}/2is\ \text{in}\ C(\mathcal{G}_N\times S^n)\quad\text{as}\quad N\to+\infty.\end{aligned} \tag{11}$$

By (8) - (11), there exists $N_0>0$ such that for any $N>N_0$, the following inequalities hold true:

$$\{R,\{R,\varphi\}\}(x,\xi)>0 \tag{12}$$

if $(x,\xi)\in(\overline{\mathcal{G}_N\setminus B_\delta(x^*)})\times(\mathbb{R}^{n+1}\setminus\{0\})$ satisfies $R(x,\xi)=0$, and

$$\{R(x',\xi-is\nabla\varphi(x)),R(x',\xi+is\nabla\varphi(x))\}/2is>C_1(|\xi|^2+Ns^2) \tag{13}$$

if $(x,\xi,s)\in(\overline{\mathcal{G}_N\setminus B_\delta(x^*)})\times(\mathbb{R}^{n+1}\setminus\{0\})\times(\mathbb{R}\setminus\{0\})$ satisfies $R(x',\xi+is\nabla\varphi(x))=0$, where the constant $C_1>0$ is independent of ξ,s,N.

Denote $\widetilde{w}(x)=w(x)e^{s\varphi}$. By (11), the following equality holds:

$$e^{s\phi}R(x',D)(e^{-s\varphi}\widetilde{w})=ge^{s\varphi}\quad\text{in } \mathcal{G}_N. \tag{14}$$

The short calculations give the equation

$$L_{2,\varphi}\widetilde{w}+L_{1,\varphi}\widetilde{w}=g_s \text{ in } \mathcal{G}_N, \tag{15}$$

where

$$L_{1,\varphi}\widetilde{w}=-\sum_{j=0}^{n}s\varphi_{x_j}R^{(j)}(x',\nabla\widetilde{w}),\quad L_{2,\varphi}\widetilde{w}=R\widetilde{w}+s^2R(x',\nabla\varphi)\widetilde{w},$$
$$g_s(x)=ge^{s\varphi}+\widetilde{w}R\varphi. \tag{16}$$

Taking the L_2-norms of the both sides of (15), we obtain

$$\|g_s\|^2_{L^2(\mathcal{G}_N)}=\|L_{2,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}+\|L_{1,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}+2(L_{1,\varphi}\widetilde{w},L_{2,\varphi}\widetilde{w})_{L^2(\mathcal{G}_N)}. \tag{17}$$

Denote

$$G_\phi(x,s,\widetilde{w})=\{R,\{R,\phi\}\}(x',\nabla\widetilde{w})+s^2\sum_{j,k=0}^{n}R_{(k)}(x',\nabla\phi)R^{(j)}(x',\nabla\phi)\widetilde{w}^2$$
$$+s^2\sum_{j,k=0}^{n}\phi_{x_jx_k}R^{(j)}(x',\nabla\phi)R^{(k)}(x',\nabla\phi)\widetilde{w}^2 \tag{18}$$

and $G_\varphi(x,s,\widetilde{w})$ is defined similarly.

Let us transform the last term at the right side of (17). In [Im2], one can find the following identity:

$$(L_{1,\varphi}\widetilde{w}, L_{2,\varphi}\widetilde{w})_{L^2(\mathcal{G}_N)} = \int_{\partial\mathcal{G}_N} \widetilde{R}(x', \vec{n}, \nabla\widetilde{w})L_{1,\varphi}\widetilde{w}\, d\Sigma + s\int_{\partial\mathcal{G}_N} \widetilde{R}(x', \nabla\varphi, \vec{n})R(x', \nabla\widetilde{w})d\Sigma$$
$$-s^3\int_{\partial\mathcal{G}_N} R(x', \nabla\varphi)\widetilde{R}(x', \vec{n}, \nabla\varphi)\widetilde{w}^2 d\Sigma + \int_{\mathcal{G}_N} sG_\varphi(x, s, \widetilde{w})\, dx$$
$$+\int_{\mathcal{G}_N} \frac{s}{2}\left(\sum_{j,k=0}^{n} R^{(k)}_{(k)}(x', \nabla\widetilde{w})\varphi_{x_j}R^{(j)}(x', \nabla\widetilde{w}) - \theta(R(x', \nabla\widetilde{w}) - s^2R(x', \nabla\varphi)\widetilde{w}^2)\right) dx, \tag{19}$$

where $\vec{n}$ is the unit outward normal vector to $\partial\mathcal{G}_N$ and

$$\theta(x) = \sum_{l,m=0}^{n} (\varphi_{x_l x_m}R^{(l,m)}(x', \nabla\widetilde{w}) + \varphi_{x_l}R^{(l,m)}_{(m)}(x', \nabla\widetilde{w})).$$

Now we need the following Lemma proved in [Im2].

Lemma 1. *Let $w \in H^1(\mathcal{G}_N)$ be a solution to (1) and (2).*

$$s\int_{\mathcal{G}_N} (|\nabla\widetilde{w}|^2 + s^2\widetilde{w}^2)dx \le C_2\int_{\mathcal{G}_N} sG_\phi(x, s, \widetilde{w})dx$$
$$+C_3\left(\frac{1}{s}\|L_{2,\phi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \frac{1}{s}\|L_{1,\phi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + s\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{x_n}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\right), \quad \forall s \ge s_0(\widetilde{\tau}), \tag{20}$$

where the constants C_2 and C_3 are independent of s, N.

We claim :

$$\left|\int_{\mathcal{G}_N} \frac{s}{2}\left(\sum_{j,k=0}^{n} R^{(k)}_{(k)}(x', \nabla\widetilde{w})\varphi_{x_j}R^{(j)}(x', \nabla\widetilde{w}) - \theta\{R(x', \nabla\widetilde{w}) - s^2R(x', \nabla\varphi)\widetilde{w}^2\}\right) dx\right|$$
$$\le \left|\frac{s}{2}\int_{\mathcal{G}_N} \sum_{j,k=0}^{n} R^{(k)}_{(k)}(x', \nabla\widetilde{w})\varphi_{x_j}R^{(j)}(x', \nabla\widetilde{w})\, dx\right| + \left|s\int_{\mathcal{G}_N} \theta(R(x', \nabla\widetilde{w}) - s^2R(x', \nabla\varphi)\widetilde{w}^2)dx\right|$$
$$\le \frac{\varepsilon s}{2}\int_{\mathcal{G}_N} (|\nabla\widetilde{w}|^2 + s^2\widetilde{w}^2)dx + C_4\left(\frac{1}{s\varepsilon}\|L_{1,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \frac{1}{s\varepsilon}\|L_{2,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}\right.$$
$$\left. + s\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{x_n}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\right). \tag{21}$$

In fact, by the Cauchy-Bunyakovskii inequality,

$$\left|\int_{\mathcal{G}_N} s\sum_{j,k=0}^{n} R^{(k)}_{(k)}(x',\nabla\widetilde{w})\varphi_{x_j}R^{(j)}(x',\nabla\widetilde{w})dx\right| \le \frac{\varepsilon s}{4}\|\widetilde{w}\|^2_{H^1(\mathcal{G}_N)} + \frac{C_5}{s\varepsilon}\|L_{1,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}. \tag{22}$$

Since the function θ is continuous, there exists $\theta_\varepsilon \in C^2(\overline{\mathcal{G}_N})$ such that $\|\theta - \theta_\varepsilon\|_{C(\overline{\mathcal{G}_N})} \le \frac{\varepsilon}{8}$. Taking the scalar product in $L^2(\mathcal{G}_N)$ of the functions $\theta_\varepsilon\widetilde{w}$ and $L_{2,\varphi}\widetilde{w}$, we obtain the equality

$$\int_{\mathcal{G}_N}\theta_\varepsilon(sR(x',\nabla\widetilde{w}) - s^3R(x',\nabla\varphi)\widetilde{w}^2)dx = -s\int_{\mathcal{G}_N}(L_{2,\varphi}\widetilde{w})\theta_\varepsilon\widetilde{w}dx$$
$$+s\int_{\mathcal{G}_N}\sum_{j,k=1}^{n}\left(\frac{\partial a_{jk}}{\partial x_j}\frac{\partial\widetilde{w}}{\partial x_k}\theta_\varepsilon\widetilde{w} - \widetilde{R}(x',\nabla\widetilde{w},\nabla\theta_\varepsilon)\widetilde{w}\right)dx + \int_{\partial\mathcal{G}_N} a(x,\vec{n},\nabla\widetilde{w})\theta_\epsilon\widetilde{w}d\Sigma.$$

Thus

$$\left|\int_{\mathcal{G}_N}\theta(sR(x',\nabla\widetilde{w}) - s^3R(x',\nabla\varphi)\widetilde{w}^2)dx\right|$$
$$\le\left|\int_{\mathcal{G}_N}(\theta-\theta_\varepsilon)(sR(x',\nabla\widetilde{w}) - s^3R(x',\nabla\varphi)\widetilde{w}^2)dx\right| + \left|\int_{\mathcal{G}_N}\theta_\varepsilon(sR(x',\nabla\widetilde{w}) - s^3R(x',\nabla\varphi)\widetilde{w}^2)dx\right|$$
$$\le\frac{\varepsilon s}{4}\int_{\mathcal{G}_N}(|\nabla\widetilde{w}|^2 + s^2\widetilde{w}^2)dx + C_6\Bigg(\frac{1}{s}\|L_{1,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \frac{1}{s}\|L_{2,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}$$
$$+s\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{x_n}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\Bigg). \tag{23}$$

The inequalities (22) and (23) imply (21).

By Lemma 1, we have

$$s\int_{\mathcal{G}_N}(|\nabla\widetilde{w}|^2 + s^2\widetilde{w}^2)dx + \int_{\mathcal{G}_N}4N\widetilde{\tau}\sum_{j,k=1}^{n}\partial_{x_j}\ell_1(x')\partial_{x_k}\ell_1(x')\{R^{(j)}(x',\nabla\widetilde{w})R^{(k)}(x',\nabla\widetilde{w})$$
$$+s^2R^{(j)}(x,\nabla\varphi)R^{(k)}(x',\nabla\varphi)\}dx \le \int_{\mathcal{G}_N}2sG_\varphi(x,s,\widetilde{w})dx + \int_{\mathcal{G}_N}\Bigg\{2sG_\phi(x,s,\widetilde{w}) - 2sG_\varphi(x,s,\widetilde{w})$$
$$+4N\widetilde{\tau}\sum_{j,k=1}^{n}\partial_{x_j}\ell_1(x')\partial_{x_k}\ell_1(x')\{R^{(j)}(x',\nabla\widetilde{w})R^{(k)}(x',\nabla\widetilde{w}) + s^2R^{(j)}(x',\nabla\varphi)R^{(k)}(x',\nabla\varphi)\}\Bigg\}dx$$
$$+C_8\left(\frac{1}{s}\|L_{2,\phi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \frac{1}{s}\|L_{1,\phi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + s\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{x_n}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\right). \tag{24}$$

Note that there exists a constant $C_9 > 0$, independent of N, such that

$$\int_{\mathcal{G}_N} 4N\widetilde{\tau} \sum_{j,k=1}^{n} \partial_{x_j}\ell_1(x')\partial_{x_k}\ell_1(x')\{R^{(j)}(x',\nabla\widetilde{w})R^{(k)}(x',\nabla\widetilde{w})$$

$$+s^2R^{(j)}(x',\nabla\varphi)R^{(k)}(x',\nabla\varphi)\}dx \geq C_9N\int_{\mathcal{G}_N}\widetilde{w}^2dx \tag{25}$$

for all sufficiently large N.

By (11), we have

$$\int_{\mathcal{G}_N}\Bigg(2sG_\varphi(x,s,\widetilde{w}) - 2sG_\phi(x,s,\widetilde{w})$$

$$-4N\widetilde{\tau}\sum_{j,k=1}^{n}\partial_{x_j}\ell_1(x')\partial_{x_k}\ell_1(x')\{R^{(j)}(x',\nabla\widetilde{w})R^{(k)}(x',\nabla\widetilde{w}) + s^2R^{(j)}(x',\nabla\varphi)R^{(k)}(x',\nabla\varphi)\}\Bigg)dx$$

$$\leq C_{10}(N)s\int_{\mathcal{G}_N}(|\nabla\widetilde{w}|^2 + s^2\widetilde{w}^2)dx, \tag{26}$$

where $C_{10}(N) \to 0$ as $N \to +\infty$. By (10), we obtain

$$\left|\frac{1}{s}\|L_{2,\phi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \frac{1}{s}\|L_{1,\phi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} - \frac{1}{s}\|L_{2,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} - \frac{1}{s}\|L_{1,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}\right|$$

$$\leq C_{11}(N)s\int_{\mathcal{G}_N}(|\nabla\widetilde{w}|^2 + s^2\widetilde{w}^2)dx, \tag{27}$$

where $C_{11}(N) \to 0$ as $N \to +\infty$. Using (25)-(27), from (24) we obtain

$$\frac{1}{C_7}s\int_{\mathcal{G}_N}(|\nabla\widetilde{w}|^2 + s^2\widetilde{w}^2)dx \leq \frac{1}{4}\|L_{1,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)} + \frac{1}{4}\|L_{2,\varphi}\widetilde{w}\|^2_{L^2(\mathcal{G}_N)}$$

$$+\int_{\mathcal{G}_N}2sG_\phi(x,s,\widetilde{w})dx + sC_9\|\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}\|\partial_{x_n}\widetilde{w}\|_{L^2(\partial\mathcal{G}_N)}, \qquad \forall s \geq s_0(\widetilde{\tau}). \tag{28}$$

Inequalities (21), (28) imply (4.33). The proof is finished. ■

Acknowlegement. Most of this paper has been written during the stays of the first named author at Graduate School of Mathematical Sciences of the University of Tokyo in July and January of 2002 and 2003. The author thanks the school for the hospitality. The authors are deeply indebted to Prof. Kazuhiro Yamamoto for the careful explanation of the results of the paper [Y]. The first named author was supported partially by the NSF Grant DMS-0205148. The second named author was supported partially by Grant-in-Aid for Scientific Research (14604005) from Japan Society for the Promotion of Science.

References

[B1] M. Bellassoued, *Distribution of resonances and decay of the local energy for the elastic wave equations*, Comm. Math. Phys. **215** (2000), 375-408.

[B2] M. Bellassoued, *Carleman estimates and decay rate of the local energy for the Neumann problem of elasticity*, Progr. Nonlinear Differential Equations Appl. **46** (2001), 15-36.

[B3] M. Bellassoued, *Unicité et contrôle pour le système de Lamé*, ESIAM. **6** (2001), 561-592.

[Bu] A.L. Bukhgeim, *Introduction to the Theory of Inverse Problems*, VSP, Utrecht, 2000.

[BCIY] A.L. Bukhgeim, J. Cheng, V. Isakov and M. Yamamoto, *Uniqueness in determining damping coefficients in hyperbolic equations*, in "Analytic Extension Formulas and their Applications" (2001), Kluwer, Dordrecht, 27-46.

[BuK] A.L. Bukhgeim and M.V. Klibanov, *Global uniqueness of a class of multidimensional inverse problems*, Soviet Math. Dokl. **24** (1981), 244–247.

[Ca] T. Carleman, *Sur un problème d'unicité pour les systèmes d'équations aux derivées partielles à deux variables independents*, Ark. Mat. Astr. Fys. **2B**, 1–9.

[DR] B. Dehman and L.Robbiano, *La propriété du prolongement unique pour un système elliptique. le système de Lamé*, J. Math. Pures Appl. **72** (1993), 475-492.

[E1] Yu. V. Egorov, *Linear Differential Equations of Principal Type* (1986), Consultants Bureau, New York.

[EINT] M. Eller, V. Isakov, G. Nakamura and D.Tataru, *Uniqueness and stability in the Cauchy problem for Maxwell's and the elasticity system*, in "Nonlinear Partial Differential Equations", Vol.14, Collège de France Seminar, Elsevier-Gauthier Villaras "Series in Applied Mathematics" **31** (2002), 329–350.

[Hö] L., Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.

[INY] M. Ikehata, G. Nakamura and M. Yamamoto, *Uniqueness in inverse problems for the isotropic Lamé system*, J. Math. Sci. Univ. Tokyo **5** (1998), 627–692.

[Im1] O. Imanuvilov, *Controllability of parabolic equations*, Mat. Sbornik **6** (1995), 109-132.

[Im2] O. Imanuvilov, *On Carleman estimates for hyperbolic equations*, to appear in Asymptotic Analysis.

[IIY] O. Imanuvilov, V. Isakov and M. Yamamoto, *An inverse problem for the dynamical Lamé system with two sets of boundary data*, to appear in Comm. Pure Appl. Math.

[IY1] O.Imanuvilov, M. Yamamoto, *Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations*, to appear in Publ. Res. Inst. Math. Sci.

[IY2] O. Imanuvilov and M. Yamamoto, *Global Lipschitz stability in an inverse hyperbolic problem by interior observations*, Inverse Problems **17** (2001), 717–728.

[IY3] O. Imanuvilov and M. Yamamoto, *Global uniqueness and stability in determining coefficients of wave equations*, Commun. in Partial Differential Equations **26** (2001), 1409–1425.

[IY4] O. Imanuvilov and M. Yamamoto, *Determination of a coefficient of the principal part in an acoustic equation with a single measurement*, to appear in Inverse Problems.

[Is1] V. Isakov, *A nonhyperbolic Cauchy problem for $\Box_b\Box_c$ and its applications to elasticity theory*, Comm. Pure and Applied Math. **39** (1986), 747–767.

[Is2] V. Isakov, *Inverse Source Problems*, American Mathematical Society, Providence, Rhode Island, 1990.

[Is3] V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, Berlin, 1998.

[IsY] V. Isakov and M. Yamamoto, *Carleman estimate with the Neumann boundary condition and its applications to the observability inequality and inverse hyperbolic problems*, Contem. Math. **268** (2000), 191–225.

[KK] M.A. Kazemi and M.V. Klibanov, *Stability estimates for ill-posed Cauchy problems involving hyperbolic equations and inequalities*, Appl. Anal. **50** (1993), 93-102.

[Kh1] A. Khaĭdarov, *Carleman estimates and inverse problems for second order hyperbolic equations*, Math. USSR Sbornik **58** (1987), 267-277.

[Kh2] A. Khaĭdarov, *On stability estimates in multidimensional inverse problems for differential equations*, Soviet Math.Dokl. **38** (1989), 614-617.

[Kl] M.V. Klibanov, *Inverse problems and Carleman estimates*, Inverse Problems **8** (1992), 575–596.

[Ku] H. Kumano-go, *Pseudo-differential Operators* (1981), MIT Press, Cambrige.

[LT] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, Cambridge University Press, Cambridge, 2000.

[Li1] J.L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin, 1971.

[Li2] J.L. Lions, *Contrôlabilité Exacte Perturbations et Stabilisation de Systèmes Distribués*, Masson, Paris, 1988.

[PY1] J.-P. Puel and M. Yamamoto, *On a global estimate in a linear inverse hyperbolic problem*, Inverse Problems **12** (1996), 995-1002.

[PY2] J.-P. Puel and M. Yamamoto, *Generic well-posedness in a multidimensional hyperbolic inverse problem*, J. Inverse Ill-posed Problems **5** (1997), 55–83.

[Ra] L. Rachele, *An inverse problem in elastodynamics : uniqueness of the wave speeds in the interior*, J. Differential Equations **162** (2000), 300–325.

[R] A. Ruiz, *Unique continuation for weak solutions of the wave equation plus a potential*, J. Math. Pures. Appl. **71** (1992), 455–467.

[Ta] D. Tataru, *Carleman estimates and unique continuation for solutions to boundary value problems*, J. Math. Pures. Appl. **75** (1996), 367-408.

[T1] M. Taylor, *Pseudodifferential Operators*, Princeton University Press, Princeton, New Jersey, 1981.

[T2] M. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Birkhäuser, Boston Basel Berlin, 1991.

[Yak] V.G. Yakhno, *Inverse Problems for Differential Equations of Elasticity*, Nauka, Novosibirsk, 1990.

[Y] K. Yamamoto, *Singularities of solutions to the boundary value problems for elastic and Maxwell's equations*, Japan J. Math. **14** (1988), no. 1, 119-163.

[Ya] M. Yamamoto, *Uniqueness and stability in multidimensional hyperbolic inverse problems*, J. Math. Pures Appl. **78** (1999), 65–98.

[Zui] C. Zuily, *Uniqueness and Non-uniqueness in the Cauchy Problem*, Birkhäuser, Boston Basel Berlin, 1983.

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