UTMS 2003–33 July 11, 2003

Principal series Whittaker functions on $SL(3, \mathbf{R})$

by

Hiroyuki MANABE, Taku ISHII,
and Takayuki ODA

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

Principal series Whittaker functions on $SL(3, \mathbf{R})$

Hiroyuki MANABE, Taku ISHII, and Takayuki ODA

Introduction

The origin of this paper is the master thesis [1] of the first named author Manabe, who obtained the holonomic systems for the A-radial part of (non-spherical) principal series Whittaker functions on $SL(3, \mathbf{R})$, and also the formal power series solutions of this holonomic system. The untimely disease made it impossible for him to develop this research further.

The last named author who has been the advisor of this thesis asked the second named author Ishii, to push forward this study to have integral expressions of the solutions for the holonomic systems mentioned above, and other related results.

The study of Whittaker models of algebraic groups over local fields has already some history. The Jacquet integral is named after the investigation of H.Jacquet [9]. Multiplicity free theorem by J.Shalika for quasi-split groups, was later enhanced for the case of the real field by N.Wallach. For reductive groups over the real field, this theme was investigated by M.Hashizume [6], B.Kostant, D. Vogan, H.Matsumoto, and the joint work of R.Goodman and N.Wallach [5].

More specifically $GL(n, \mathbf{R})$, explicit expressions for class 1 Whittaker functions are obtained, firstly for $n = 3$ by D.Bump [2]. The main contributor for the case of general n seems to be E.Stade. Other related results will be found in the references of the papers of him ([12],[13]).

Let us explain the outline of this paper. The purpose of the master thesis [1] refered above is to investigate the Whittaker functions belonging to the non-spherical principal series representations of $SL(3, \mathbf{R})$. The minimal K-type of such representations is 3-dimensional. So we have to consider vector-valued functions. The main results are, firstly, to obtain the holonomic system of the A-radial part of such Whittaker functions with minimal K-type explicitly (§4), and secondly to have 6 formal solutions (§5, Theorem (5.5)), which are considered as examples of confluent hypergeometric series of two variables. We also have integral expressions of these 6 solutions(§5, Theorem (5.6)). In the subsequent section, the Jacquet integral (so to say, the primary Whittaker function) is written as a sum of these 6 *secondary* Whittaker functions (§6-8). We believe these formulae obtained in this paper would be fundamental for deeper studies of these functions.

This year happens to be the centennial of the first edition of *A Course of Modern Analysis* [17] by Whittaker and Watson. We, the user of this fascinating book, are glad to be able to investigate the A_2-type Whittaker functions rather explicitly and exhaustively, together with the study BC_2-case by the authors [11], [8].

The authors thank the referee for careful reading to correct a number of typographic errors in an earlier draft of this paper.

1 Preliminaries. Basic terminology

1.1 Whittaker model

Given an irreducible admissible representation (π, H) of $G = SL(3, \mathbf{R})$, we consider its model or realization in the space of Whittaker functions. This means, for a non-degenerate unitary character ψ of a maximal unipotent subgroup $N = \{\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in G\}$ of G defined by

$$\psi(\begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}) = \exp\{2\pi\sqrt{-1}(c_1 x_{12} + c_2 x_{23})\}$$

with $c_1, c_2 \in \mathbf{R}$ being non-zero, we consider a smooth induction $C^\infty\text{-Ind}_N^G(\psi)$ to G, and the space of intertwining operators of smooth G-modules

$$\text{Hom}_G(H_\infty, C^\infty\text{-Ind}_N^G(\psi))$$

with H_∞ the subspace consisting of C^∞-vectors in H. Or more algebraically speaking, we might consider the corresponding space in the context of $(\mathfrak{g}, K)$-modules (with $\mathfrak{g} = \text{Lie}(G), K = SO(3)$):

$$\text{Hom}_{(\mathfrak{g},K)}(H_\infty, C^\infty\text{-Ind}_N^G(\psi)).$$

1.2 Principal series representations

Let P_0 be a minimal parabolic subgroup of G given by the upper triangular matrices in G, and $P_0 = MAN$ be a Langlands decomposition of P_0 with $M = K \cap \{\text{diagonals in } G\},\ A = \exp\mathfrak{a}$, with

$$\mathfrak{a} = \{\text{diag}(t_1, t_2, t_3) | t_i \in \mathbf{R}, t_1 + t_2 + t_3 = 0\}.$$

In order to define a principal series representation with respect to the minimal parabolic subgroup P_0 of G, we firstly fix a character σ of the finite abelian group M of type $(2,2)$ and a linear form $\nu \in \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C} = \text{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$. We write $\nu(\text{diag}(t_1, t_2, t_3)) = \nu_1 t_1 + \nu_2 t_2$. For such data, we can define a representation $\sigma \otimes e^\nu$ of MA, and extend this to P_0 by the identification $P_0/N \cong MA$. Then we set

$$\pi_{\sigma,\nu} = L^2\text{-Ind}_{P_0}^G(\sigma \otimes e^{\nu+\rho} \otimes 1_N).$$

Here ρ is the half-sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$ for P_0, given as follows. For $i < j$ $(1 \le i, j \le 3)$, we put $\eta_{ij}(a) = a_i/a_j$ for $a = \text{diag}(a_1, a_2, a_3)$ $(a_1 a_2 a_3 = 1)$. Then we have $a^{2\rho} = \prod_{i<j} a_i/a_j = a_1^2/a_3^2 = a_1^4 a_2^2$ by definition. Hence $a^\rho = a_1^2 a_2$.

Here the characters σ_j of M are identified as follows. The group M consisting of 4 elements is a finite abelian group of $(2,2)$ type, and its elements except for the unity is given by the matrices

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since M is commutative, all the irreducible unitary representations of it is 1-dimensional. For any $\sigma \in \widehat{M}$, we have $\sigma^2 = 1$. Therefore the set $\widehat{M}$ consisting of 4 characters $\{\sigma_j : j = 0, 1, 2, 3\}$, where each σ_j, except for the trivial character σ_0, is specified by the following table of values at the elements m_i.

	m_1	m_2	m_3
σ_1	1	-1	-1
σ_2	-1	1	-1
σ_3	-1	-1	1

Proposition (1.1) (i) *If σ is the trivial character of M, the representation $\pi_{\sigma,\nu}$ is spherical or class 1, i.e., it has a (unique) K-invariant vector in the representation space $H_{\sigma,\nu}$.*
(ii) *If σ is not trivial, then the minimal K-type of the restriction $\pi_{\sigma,\nu|K}$ to K is a 3-dimensional representation of $K = SO(3)$, which is isomorphic to the unique standard one (τ_2, V_2). The multiplicity of this minimal K-type is one:*

$$\dim_{\mathbf{C}} \mathrm{Hom}_K(\tau_2, H_{\sigma,\nu}) = 1,$$

namely there is a unique non-zero K-homomorphism

$$\iota : (\tau_2, V_2) \to (\pi_{\sigma,\nu|K}, H_{\sigma,\nu})$$

up to constant multiple.

2 Representations of $K = SO(3)$

2.1 The spinor covering

To describe the finite dimensional irreducible representations of $SO(3)$, the simplest way seems to utilize the double covering $s : SU(2) = Spin(3) \to SO(3)$, which is realized as follows.

The Hamilton quaternion algebra $\mathbf{H}$ is realized in $M_2(\mathbf{C})$ by

$$\mathbf{H} = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbf{C}) | a, b \in \mathbf{C}\}.$$

Then $SU(2)$ is the subgroup of the multiplicative group consisting of quaternions with reduced norm 1, i.e.,

$$SU(2) = \{x \in \mathbf{H} | \det x = 1\}.$$

Let $\mathbf{P} = \{x \in \mathbf{H} | \mathrm{tr} x = 0\}$ be the 3-dimensional real Euclidean space consisting of pure quaternions. Then for each $x \in SU(2)$, the map

$$p \in \mathbf{P} \mapsto x \cdot p \cdot x^{-1} \in \mathbf{P}$$

preserve the Euclid norm $p \mapsto \det p$ and the orientation, hence we have a homomorphism

$$s : SU(2) \to SO(\mathbf{P}, \det) = SO(3),$$

which is surjective, since the range is a connected group. The kernel of this homomorphism is given by $\{\pm 1_2\}$.

By the derivation of s $ds : \mathfrak{su}(2) \to \mathfrak{so}(3)$, the standard generators:

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

are mapped to $2K_1, 2K_2, 2K_3$ with

$$K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{k},$$

respectively. Here $\mathfrak{k}$ is the Lie algebra of K.

2.2 Representations of $SU(2)$

The set of equivalence classes of the finite dimensional continuous representations of $SU(2)$ is exhausted by the symmetric tensor products τ_l $(l = 0, 1, \dots ,)$ of the standard representation. These are realized as follows.

Let V_l be the subspace consisting of degree l homogeneous polynomials of two variables x, y in the polynomial ring $\mathbf{C}[x, y]$. For $g \in SU(2)$ with $g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, and $f(x, y) \in V_l$ we set

$$\tau_l(g)f(x, y) := f(ax + by, -\bar{b}x + \bar{a}y).$$

Passing to the Lie algebra $Lie(SU(2)) = \mathfrak{su}(2)$, the derivation of τ_l, denoted by the same symbol, is described as follows by using the standard basis $\{v_k = x^k y^{l-k}\ (0 \le k \le l)\}$ and the standard generators

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Namely we have

$$\tau_l(u_1)v_k = \sqrt{-1}(l - 2k)v_k, \quad \tau_l(X_+)v_k = (l - k)v_{k+1}, \quad \tau_l(X_-)v_k = -k \cdot v_{k-1}.$$

Here we put $X_+ = \frac{1}{2}(u_2 + \sqrt{-1}u_3),\ X_- = \frac{1}{2}(u_2 - \sqrt{-1}u_3)$.

The condition that τ_l defines a representation of $SO(3)$ by passing to the quotient with respect $s : SU(2) \to SO(3)$ is that $\tau_l(-1_2) = (-1)^l = +1$, i.e., l is even. Therefore the dimension of V_l, $l + 1$ is odd in this case.

The representation τ_2 of $SU(2)$ is equivalent to the spinor homomorphism. Hence passing to the quotient, τ_2 is equivalent to the tautological representation $SO(3) \to GL(3, \mathbf{C})$.

2.3 Irreducible components of $\tau_2 \otimes \tau_4$ and $\tau_2 \otimes Ad_{\mathfrak{p}}$

For our later use, we want to specify the standard basis of the unique irreducible constituent τ_2 in the tensor product $\tau_2 \otimes \tau_4$.

Lemma (2.1) *Let $\{v_i\ (i = 0, 1, 2)\}$ and $\{w_j\ (0 \le j \le 4)\}$ be the standard basis of (τ_2, V_2) and (τ_4, V_4), respectively. Then the elements*

$$\begin{aligned} v'_0 &= v_0 \otimes w_2 - 2v_1 \otimes w_1 + v_2 \otimes w_0, \\ v'_1 &= v_0 \otimes w_3 - 2v_1 \otimes w_2 + v_2 \otimes w_1, \\ v'_2 &= v_0 \otimes w_4 - 2v_1 \otimes w_3 + v_2 \otimes w_2 \end{aligned}$$

define a set of standard basis in $\tau_2 \subset \tau_2 \otimes \tau_4$, which is unique up to a common scalar multiple.

Proof. We have

$$\begin{aligned} X_- v'_0 &= X_- v_0 \otimes w_2 + v_0 \otimes X_- w_2 - 2X_- v_1 \otimes w_1 - 2v_1 \otimes X_- w_1 + X_- v_2 \otimes w_0 + v_2 \otimes X_- w_0 \\ &= 0 + v_0 \otimes (-2)w_1 + 2v_0 \otimes w_1 - 2v_1 \otimes (-1)w_0 + (-2)v_1 \otimes w_0 + 0 \\ &= 0. \end{aligned}$$

This means that v'_0 is the lowest weight vector of some irreducible subrepresentation in the tensor product. By a similar computation we have

$$X_+ v'_0 = 2v'_1, \quad X_+ v'_1 = v'_2, \text{ and } X_+ v'_2 = 0.$$

□

2.4 The K-module isomorphism between $\mathfrak{p}_{\mathbf{C}}$ and V_4

We denote by $\mathfrak{p}_{\mathbf{C}}$ the complexification of the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ with respect to the Killing form, on which the group K acts via the adjoint action $Ad_{\mathfrak{p}}$. We denote by E_{ij} the matrix unit with 1 at (i,j)-th entry and 0 at other entries. Then E_{ii} and $E_{ij}+E_{ji}$ are considered as elements in $\mathfrak{p}$. We set $H_{ij}=E_{ii}-E_{jj}$ for $i\neq j$.

Lemma (2.2) *Via the unique isomorphism V_4 and $\mathfrak{p}_{\mathbf{C}}$ as K-modules we have the identification*

$$\begin{aligned}
w_0 &= -2\{H_{23}-\sqrt{-1}(E_{23}+E_{32})\},\\
w_1 &= \sqrt{-1}\{(E_{12}+E_{21})-\sqrt{-1}(E_{13}+E_{31})\},\\
w_2 &= \tfrac{2}{3}(H_{12}+H_{13}),\\
w_3 &= \sqrt{-1}\{(E_{12}+E_{21})+\sqrt{-1}(E_{13}+E_{31})\},\\
w_4 &= -2\{H_{23}+\sqrt{-1}(E_{23}+E_{32})\}.
\end{aligned}$$

Proof. This can be confirmed by direct computation. The adjoint action of $\frac{1}{2}u_1$ on the right hand side of the third formula reads

$$[K_1, \frac{2}{3}\mathrm{diag}(2,-1,-1)] = 0.$$

Applying the adjoint actions of $X_{\pm}$ on these elements, we can confirm other identification. □

3 Principal series $(\mathfrak{g},K)$-modules

3.1 The case of the class one principal series

3.1.1 The Capelli elements

A set of generators for the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}=\mathfrak{sl}_3$ is obtained as Capelli elements, because $\mathfrak{sl}_3$ is of type A_2 (cf. [7, §11]).

Let

$$E'_{ii}=E_{ii}-\frac{1}{3}(\sum_{a=1}^{3}E_{aa}),\ E'_{ij}=E_{ij}\ (i\neq j).$$

Then $E'_{ij}\in\mathfrak{g}$. Define a matrix $\mathcal{C}$ of size 3 with entries in $\mathfrak{g}$ by

$$\mathcal{C}=\begin{pmatrix}E'_{11}&E'_{12}&E'_{13}\\E'_{21}&E'_{22}&E'_{23}\\E'_{31}&E'_{32}&E'_{33}\end{pmatrix}+\mathrm{diag}(-1,0,1).$$

Then for

$$\mathcal{A}=(A_{ij})_{1\le i,j\le 3}=x\cdot 1_3-\mathcal{C}\in M_3(\mathfrak{g}[x])\subset M_3(U(\mathfrak{g})[x]),$$

we define its *vertical* determinant by

$$\det\downarrow(\mathcal{A})=\sum_{\sigma\in\mathfrak{S}_3}\mathrm{sgn}(\sigma)A_{1\sigma(1)}A_{2\sigma(2)}A_{3\sigma(3)}.$$

Then it is written in the form $x^3+Cp_2x-Cp_3\in U(\mathfrak{g})[x]$ with some elements Cp_2 and Cp_3 in $Z(\mathfrak{g})$.

Proposition (3.1) *The set $\{Cp_2,Cp_3\}$ is a system of independent generators of $Z(\mathfrak{g})$. Here are explicit formulae of Cp_2 and Cp_3:*

$$\begin{aligned}
Cp_2 &= (E'_{11}-1)E'_{22}+E'_{22}(E'_{33}+1)+(E'_{11}-1)(E'_{33}+1)\\
&\quad -E_{23}E_{32}-E_{13}E_{31}-E_{12}E_{21},
\end{aligned}$$

$$\begin{aligned}
Cp_3 = &\ (E'_{11}-1)E'_{22}(E'_{33}+1)+E_{12}E_{23}E_{31}+E_{13}E_{21}E_{32}\\
&-(E'_{11}-1)E_{23}E_{32}-E_{13}E'_{22}E_{31}-E_{12}E_{21}(E'_{33}+1).
\end{aligned}$$

3.1.2 Reduction of Capelli elements

For the class one principal series, $\sigma = \sigma_0$ is the trivial character of M. Let f_0 be the element generating the minimal K-type in $H_{\sigma_0,\nu}$ normalized such that $f_0|K \equiv 1$.

To compute the infinitesimal character of our $\pi_{\sigma_0,\nu}$, we investigate the action of the Capelli elements Cp_2, Cp_3 on f_0. By simple computation, we have

$$\begin{array}{rcl} Cp_2 & = & (E'_{11}-1)E'_{22} + E'_{22}(E'_{33}+1) + (E'_{11}-1)(E'_{33}+1) \\ & & -E_{23}^2 - E_{13}^2 - E_{12}^2 - \frac{1}{2}E_{23}u_1 + \frac{1}{2}E_{13}u_2 - \frac{1}{2}E_{12}u_3. \end{array}$$

Therefore we have

$$\begin{array}{rcl} Cp_2 & \equiv & (E'_{11}-1)E'_{22} + E'_{22}(E'_{33}+1) + (E'_{11}-1)(E'_{33}+1) \\ & & -E_{23}^2 - E_{13}^2 - E_{12}^2 \ (\mathrm{mod} U(\mathfrak{g})\mathfrak{k}). \end{array}$$

By similar computation we have

$$\begin{array}{rl} Cp_3 \equiv & (E'_{11}-1)E'_{22}(E'_{33}+1) + E_{12}E_{23}E_{13} + E_{13}E_{12}E_{23} - E_{13}^2 \\ & -E_{23}^2(E'_{11}-1) - E_{13}^2E'_{22} - E_{12}^2(E'_{33}+1) \ (\mathrm{mod} U(\mathfrak{g})\mathfrak{k}). \end{array}$$

3.1.3 Eigenvalues of $Cp_2,\ Cp_3$

We want to compute the value $Cp_2f_0(e)$ and $Cp_3f_0(e)$. For elements X in $\mathfrak{n} = \mathrm{Lie}(N)$ we have $Xf_0(e) = 0$. Hence it suffices to check the first terms of Cp_2 and Cp_3. We have

$$\begin{array}{rcl} E'_{11}f_0(e) & = & \frac{1}{3}(2E_{11} - E_{22} - E_{33})f_0(e) \\ & = & \frac{1}{3}(H_{12} + H_{13})f_0(e) \\ & = & \frac{1}{3}\{2(\nu_1+\rho_1) - (\nu_2+\rho_2)\} \\ & = & \frac{1}{3}(2\nu_1 - \nu_2 + 3) = \frac{1}{3}(2\nu_1 - \nu_2) + 1. \end{array}$$

Similarly

$$E'_{22}f_0(e) = \frac{1}{3}(2\nu_2 - \nu_1) \text{ and } E'_{33}f_0(e) = -\frac{1}{3}(\nu_1 + \nu_2 + 3) = -\frac{1}{3}(\nu_1+\nu_2) - 1.$$

Therefore

$$Cp_2f_0(e) = S_2(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2))$$

and

$$Cp_3f_0(e) = S_3(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)).$$

Here $S_2(a,b,c) = ab + bc + ca$ and $S_3(a,b,c) = abc$ are the elementary symmetric functions of three variables of degree 2 and 3, respectively.

Now summing up the above computation, we have the following.

Proposition (3.2) *The infinitesimal character of $\pi_{\sigma_0,\nu}$ is given by*

$$Cp_2f_0 = S_2(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), -\frac{1}{3}(\nu_1 + \nu_2))f_0$$

and

$$Cp_3f_0 = S_3(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), -\frac{1}{3}(\nu_1 + \nu_2))f_0.$$

3.2 $(\mathfrak{g}, K)$-module structure of non-spherical principal series at the minimal K-type

3.2.1 Construction of K-equivariant differential operators

Lemma (3.3) *Let $\{f_i\ (i=0,1,2)\}$ be the set of the standard basis of the minimal K-type $\tau \subset \pi_{\sigma,\nu}$ of a non-spherical principal series representation $\pi_{\sigma,\nu} = \pi$. Define another three C^∞-elements $\{\varphi_i\ (i=0,1,2)\}$ by the formulae:*

$$\begin{aligned}
\varphi_0 &= \tfrac{2}{3}\pi(2E_{11}-E_{22}-E_{33})f_0 \\
&\quad -2\sqrt{-1}\pi(E_{12}+E_{21}-\sqrt{-1}(E_{13}+E_{31}))f_1 \\
&\quad -2\pi(E_{22}-E_{33}-\sqrt{-1}(E_{23}+E_{32}))f_2, \\
\varphi_1 &= \sqrt{-1}\pi(E_{12}+E_{21}+\sqrt{-1}(E_{13}+E_{31}))f_0 \\
&\quad -\tfrac{4}{3}\pi(2E_{11}-E_{22}-E_{33})f_1 \\
&\quad +\sqrt{-1}\pi(E_{12}+E_{21}-\sqrt{-1}(E_{13}+E_{31}))f_2, \\
\varphi_2 &= -2\pi(E_{22}-E_{33}+\sqrt{-1}(E_{23}+E_{32}))f_0 \\
&\quad -2\sqrt{-1}\pi(E_{12}+E_{21}+\sqrt{-1}(E_{13}+E_{31}))f_1 \\
&\quad +\tfrac{2}{3}\pi(2E_{11}-E_{22}-E_{33})f_2.
\end{aligned}$$

Then $(\varphi_0, \varphi_1, \varphi_2)$ is a constant multiple of (f_0, f_1, f_2).

Proof. Let $\iota : \tau_2 \subset \pi$ be the injective K-homomorphism unique up to constant multiple. Since the canonical surjection

$$\pi(\mathfrak{p}_{\mathbf{C}}) \otimes \mathrm{Im}(\iota) \to \pi(\mathfrak{p}_{\mathbf{C}}) \cdot \mathrm{Im}(\iota)$$

is a K-homomorphism, the target space is contained in the sum of τ_2, τ_4 and τ_6 isotypic subspaces of π. By the previous Lemmata (2.1) and (2.2), the 3 elements $\{\varphi_i\ (i=0,1,2)\}$ corresponds to the image of the standard basis in the τ_2-isotypic component in the tensor product $\pi(\mathfrak{p}_{\mathbf{C}}) \otimes \mathrm{Im}(\iota)$. Since the K-type τ_2 occurs with multiplicity one in π, we have the conclusion of our lemma. □

3.2.2 Computation of eigenvalues

The previous lemma tells that there exist a scalar $\lambda(\sigma,\nu)$ depending on σ and ν such that $\varphi_i = \lambda(\sigma,\nu) f_i\ (i=0,1,2)$. We determine this eigenvalue $\lambda(\sigma,\nu)$ by using explicit models of the principal series $\pi_{\sigma,\nu}$.

To do this, we have to find functions in

$$L^2\text{-}\mathrm{Ind}_M^K(\sigma_i) = L^2_{M,\sigma_i}(K) = \{f \in L^2(K) | f(mk) = \sigma_i(m)f(k) \text{ for all } m \in M, k \in K\}$$

corresponding to the standard basis in the minimal K-type for each i.

In the larger space $L^2(K)$, the τ_2-isotypic component is generated by the 9 matrix elements $s_{ij}(k)$ $(1 \le i,j \le 3)$ of the tautological representation

$$k \in K \mapsto S(k) = (s_{ab}(k))_{1\le a,b\le 3} \in SO(3).$$

It is directly confirmed that $s_{ib}(k)$ $(b=0,1,2)$ belong to the subspace $L^2_{M,\sigma_i}(K)$ for each i.

Diagonalizing the action of u_1, we find that s_{i1} corresponds to v_1 for each i. And finally we find that the standard basis is given by

$$v_0 = \sqrt{-1}(s_{i2} - \sqrt{-1}s_{i3}),\ v_1 = s_{i1}, \ \text{ and } v_2 = \sqrt{-1}(s_{i2} + \sqrt{-1}s_{i3}).$$

We need the values of these standard functions $f_a(k) = v_a$ $(a=0,1,2)$ at the identity $e \in K$.

Lemma (3.4) *The values of the standard functions at $e \in K$ is given as follows.*

1. *If* $\sigma = \sigma_1$, $(f_0(e), f_1(e), f_2(e)) = (0, 1, 0)$.
2. *If* $\sigma = \sigma_2$, $(f_0(e), f_1(e), f_2(e)) = (\sqrt{-1}, 0, \sqrt{-1})$.
3. *If* $\sigma = \sigma_3$, $(f_0(e), f_1(e), f_2(e)) = (1, 0, -1)$.

Now we can proceed to the computation of the value $\lambda(\sigma_i, \nu)$.

Lemma (3.5)

$$\lambda(\sigma_1, \nu) = -\frac{4}{3}(2\nu_1 - \nu_2), \ \lambda(\sigma_2, \nu) = \frac{4}{3}(\nu_1 - 2\nu_2), \ \lambda(\sigma_3, \nu) = \frac{4}{3}(\nu_1 + \nu_2).$$

Proof. If $\sigma = \sigma_1$, the values of the standard functions at $e \in K$ is given by $(f_0(e), f_1(e), f_2(e)) = (0, 1, 0)$. Hence to determine $\lambda(\sigma, \nu)$ it suffices to compute the value $\varphi_1(e) = \lambda(\sigma, \nu)$. Since the principal series is the induced representation $\mathrm{Ind}_{P_0}^G(\sigma \otimes e^{\nu+\rho} \otimes 1_N)$, for any element $X \in \mathfrak{n}$ the values $X f_i(e)$ is equal to 0. Hence $\pi(E_{ij} + E_{ji}) f_i(e) = \pi(E_{ji} - E_{ij}) f_i(e)$ for $i < j$. Thus

$$\begin{aligned}
\varphi_1(e) &= \sqrt{-1}\pi(E_{21} - E_{12} + \sqrt{-1}(E_{31} - E_{13}))f_0(e) - \tfrac{4}{3}\pi(H_{12} + H_{13})f_1(e) \\
&\quad + \sqrt{-1}\pi(E_{21} - E_{12} + \sqrt{-1}(E_{31} - E_{13})f_2(e) \\
&= \tfrac{1}{2}\sqrt{-1}\pi(u_3 - \sqrt{-1}u_2)f_0(e) + \tfrac{1}{2}\sqrt{-1}\pi(u_3 + \sqrt{-1}u_2)f_2(e) \\
&\quad - \tfrac{4}{3}\pi(H_{12} + H_{13})f_1(e) \\
&= \pi(X_+)f_0(e) - \pi(X_-)f_2(e) - \tfrac{4}{3}\pi(H_{12} + H_{13})f_1(e) \\
&= 2f_1(e) + 2f_1(e) - \tfrac{4}{3}\{2(\nu_1 + \rho_1) - (\nu_2 + \rho_2)\}f_1(e) \\
&= -\tfrac{4}{3}(2\nu_1 - \nu_2)f_1(e).
\end{aligned}$$

Therefore $\lambda(\sigma_1, \nu) = -\frac{4}{3}(2\nu_1 - \nu_2)$.

If $\sigma = \sigma_2$ we have $(f_0(e), f_1(e), f_2(e)) = \sqrt{-1}(1, 0, 1)$. Thus by similar computation, we have

$$\begin{aligned}
\varphi_0(e) &= \tfrac{2}{3}\pi(H_{12} + H_{13})f_0(e) \\
&\quad - 2\sqrt{-1}\pi(E_{21} - E_{12} - \sqrt{-1}(E_{31} - E_{13}))f_1(e) \\
&\quad - 2\pi(H_{23} - \sqrt{-1}(E_{32} - E_{23}))f_2(e)) \\
&= \tfrac{2}{3}\pi(H_{12} + H_{13})f_0(e) + 2\pi(X_-)f_1(e) - 2\pi(H_{23} - \tfrac{1}{2}\sqrt{-1}u_1)f_2(e) \\
&= \tfrac{2}{3}\pi(H_{12} + H_{13})f_0(e) + (-2)f_0(e) + 2f_2(e) + (-2)\pi(H_{23})f_2(e) \\
&= \tfrac{2}{3}\{2(\nu_1 + \rho_1) - (\nu_2 + \rho_2)\}f_0(e) + (-2)f_0(e) + (-2)\{(\nu_2 + \rho_2) - 1\}f_2(e) \\
&= \tfrac{2}{3}(2\nu_1 - \nu_2)f_0(e) + (-2)\nu_2 f_2(e).
\end{aligned}$$

Since $f_0(e) = f_2(e)$ in this case, we have $\varphi_0(e) = \frac{4}{3}(\nu_1 - 2\nu_2)f_0(e)$, i.e., $\lambda(\sigma_2, \nu) = \frac{4}{3}(\nu_1 - 2\nu_2)$.

If $\sigma = \sigma_3$, the initial values are given by

$$(f_0(e), f_1(e), f_2(e)) = (1, 0, -1).$$

Quite similarly as the case σ_2, we have the equality

$$\varphi_0(e) = \frac{2}{3}(2\nu_1 - \nu_2)f_0(e) + (-2)\nu_2 f_2(e).$$

Input $f_0(e) = 1$ and $f_2(e) = -1$ to get

$$\varphi_0(e) = \frac{4}{3}\nu_1 - \frac{2}{3}\nu_2 + 2\nu_2 = \frac{4}{3}(\nu_1 + \nu_2).$$

□

Summing up the lemmata in this section, we have the following.

Proposition (3.6) *Let $\{f_i\ (i=0,1,2)\}$ be the set of the standard basis of the minimal K-type $\tau \subset \pi_{\sigma,\nu}$ of a non-spherical principal series representation $\pi_{\sigma,\nu} = \pi$. Define another three C^∞-elements $\{\varphi_i\ (i=0,1,2)\}$ by the formulae:*

$$\begin{aligned}
\varphi_0 &= \tfrac{2}{3}\pi(H_{12}+H_{13})f_0 \\
&\quad -2\sqrt{-1}\pi(E_{12}+E_{21}-\sqrt{-1}(E_{13}+E_{31}))f_1 \\
&\quad -2\pi(H_{23}-\sqrt{-1}(2E_{23}+\tfrac{1}{2}u_1)f_2, \\
\varphi_1 &= \sqrt{-1}\pi(E_{12}+E_{21}+\sqrt{-1}(E_{13}+E_{31}))f_0 \\
&\quad -\tfrac{4}{3}\pi(H_{12}+H_{13})f_1 \\
&\quad +\sqrt{-1}\pi(E_{12}+E_{21}-\sqrt{-1}(E_{13}+E_{31}))f_2, \\
\varphi_2 &= -2\pi(H_{23}+\sqrt{-1}(2E_{23}+\tfrac{1}{2}u_1)f_0 \\
&\quad -2\sqrt{-1}\pi(E_{12}+E_{21}+\sqrt{-1}(E_{13}+E_{31}))f_1 \\
&\quad +\tfrac{2}{3}\pi(H_{12}+H_{13})f_2.
\end{aligned}$$

Then we have

$$(\varphi_0, \varphi_1, \varphi_2) = \lambda(\sigma_i,\nu)(f_0, f_1, f_2)$$

with eigenvalue $\lambda(\sigma_i,\nu)$ given by

$$\lambda(\sigma_1,\nu) = -\frac{4}{3}(2\nu_1-\nu_2),\ \lambda(\sigma_2,\nu) = \frac{4}{3}(\nu_1-2\nu_2),\ \lambda(\sigma_3,\nu) = \frac{4}{3}(\nu_1+\nu_2).$$

In the next section, we consider the Whittaker realization of the equation of the above proposition. Then we need the following Iwasawa decomposition of standard elements of $\mathfrak{g}$.

Lemma (3.7) *We have the following decomposition of standard generators of $\mathfrak{g}$ with respect to the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$. For $H_{ij} \in \mathfrak{a}$ we have*

$$H_{ij} = 0 + H_{ij} + 0.$$

Since $E_{ij}+E_{ji} = 2E_{ij} - (E_{ij}-E_{ji})$, we have

$$E_{12}+E_{21} = 2E_{12}+0+K_3, \quad E_{13}+E_{31} = 2E_{13}+0+(-K_2), \quad E_{23}+E_{23} = 2E_{23}+0+K_1.$$

4 The holonomic system for the A-radial part of the principal series Whittaker functions

4.1 The case of the class one principal series

Let I be a non-zero Whittaker functional from the class one principal series $\pi_{\sigma_0,\nu}$ to $C^\infty\text{-Ind}_N^G(\psi)$. Let F be the restriction of the image $I(f_0)$ of the K-fixed vector f_0 to A. We write here the holonomic system for F with respect to the variables $y_1 = \eta_{12}(a) = a_1/a_2,\ y_2 = \eta_{23}(a) = a_2/a_3 = a_1a_2^2$.

Proposition (4.1) *Put $F(y_1,y_2) = y_1y_2G(y_1,y_2)$ (note $a^\rho = y_1y_2$). Then $G(y_1,y_2)$ satisfies the partial differential eqautions:*

$$\Delta_2 G = \frac{1}{3}(\nu_1^2+\nu_2^2-\nu_1\nu_2)G$$

and

$$\{\partial_1(\partial_1-\partial_2)\partial_2 + 4\pi^2c_2^2y_2^2\partial_1 - 4\pi^2c_1^2y_1^2\partial_2\}G = -\frac{1}{27}(2\nu_1-\nu_2)(2\nu_2-\nu_1)(\nu_1+\nu_2)G.$$

Here ∂_i is the Euler operator $y_i\frac{\partial}{\partial y_i}$ for $i=1,2$. and we write

$$\Delta_2 = (\partial_1^2+\partial_2^2-\partial_1\partial_2) - 4\pi^2(c_1^2y_1^2+c_2^2y_2^2).$$

Proof. By definition the action of any element X in $\mathfrak{k}$ on F is zero. We have two lemmata.

Lemma (4.2) *The A-radial part of the operators E_{12}, E_{13}, E_{23} are the multiplication operators of the functions*

$$2\pi\sqrt{-1}c_1y_1, \quad 0, \quad 2\pi\sqrt{-1}c_2y_2,$$

respectively.

Proof of Lemma. The action of $\exp(tE_{12})$ on the restriction F' of some element in the Whittaker model to A is given by

$$\begin{aligned} F'(a\cdot\exp(tE_{12})) &= F'(a\exp(tE_{12})a^{-1}\cdot a) \\ &= \psi(a\exp(tE_{12})a^{-1})F'(a) \\ &= \exp(2\pi\sqrt{-1}c_1y_1)F'(a), \end{aligned}$$

for $a\exp(tE_{12})a^{-1}\in N$. The derivation of this in t at $t=0$ yields $E_{12}F' = 2\pi\sqrt{-1}c_1y_1F'$. The cases E_{13}, E_{23} are done similarly. □

Lemma (4.3) *For $F'=F'(y)$ on A which is the restriction of some element in the Whittaker model, the action of H_{ij} are given by*

$$\begin{aligned} H_{12}F' &= (2\partial_1-\partial_2)F'(y), \\ H_{13}F' &= (\partial_1+\partial_2)F'(y), \\ H_{23}F' &= (-\partial_1+2\partial_2)F'(y). \end{aligned}$$

Moreover for E'_{ii} we have

$$E'_{11}F' = \partial_1F', \quad E'_{22}F' = (-\partial_1+\partial_2)F', \quad E'_{33}F' = -\partial_2F'.$$

Proof of Lemma. By one parameter subgroup $\exp(tH_{12})$ ($t\in\mathbf{R}$), (y_1,y_2) is mapped to $(y_1\exp(2t), y_2\exp(-t))$. Then

$$\begin{aligned} H_{12}F' &= \frac{d}{dt}\{F'(y_1\exp(2t), y_2\exp(-t)\}\}|_{t=0} \\ &= 2y_1\frac{\partial}{\partial y_1}F' - y_2\frac{\partial}{\partial y_2}F'. \end{aligned}$$

The other two H_{ij} are settled in the same way.

The latter statement follows immediately from the former, since

$$E'_{11} = \frac{1}{3}(H_{12}+H_{13}),\ E'_{22} = \frac{1}{3}(-H_{12}+H_{23}),\ E'_{33} = \frac{1}{3}(-H_{13}-H_{23}).$$

□

Now let us go back to the proof of Proposition above. Since the Capelli element Cp_2 is equal to

$$(E'_{11}-1)E'_{22} + E'_{22}(E'_{33}+1) + (E'_{11}-1)(E'_{33}+1) - E_{23}^2 - E_{13}^2 - E_{12}^2,$$

its action on the A-radial part F of the K-invariant element in the Whittaker model is given by the operator

$$(\partial_1-1)(-\partial_1+\partial_2)+(-\partial_1+\partial_2)(-\partial_2+1)+(\partial_1-1)(-\partial_2+1)-(2\pi\sqrt{-1}c_1y_1)^2-(2\pi\sqrt{-1}c_2y_2)^2.$$

By a^ρ-twist, it gives the operator

$$\partial_1(-\partial_1+\partial_2)+(-\partial_1+\partial_2)(-\partial_2)+\partial_1(-\partial_2)-(2\pi\sqrt{-1}c_1y_1)^2-(2\pi\sqrt{-1}c_2y_2)^2$$

on G. Therefore the first equation follows from the first equation of Proposition (3.2).

The second equation is obtained similarly from the A-radial part

$$\rho_A(Cp_3)=(\partial_1-1)(-\partial_1+\partial_2)(-\partial_2+1)-(2\pi c_2y_2\sqrt{-1})^2(\partial_1-1)-(2\pi c_1y_1\sqrt{-1})^2(-\partial_2+1).$$

By a^ρ-twist of this, we have the second equation of the proposition above.

Remark From these equations for the monodromy exponents α_1,α_2 at the origin $y_1=0,\ y_2=0$, we have an equality of sets of complex numbers:

$$\{\alpha_1,-\alpha_1+\alpha_2,-\alpha_2\}=\{\frac{1}{3}(2\nu_1-\nu_2),\frac{1}{3}(2\nu_2-\nu_1),-\frac{1}{3}(\nu_1+\nu_2)\}.$$

4.2 The holonomic system for the A-radial part of non-spherical Whittaker functions

Let I be a non-zero Whittaker functional from the principal series $\pi_{\sigma_i,\nu}$. For the set $\{f_i|(i=0,1,2)\}$ of standard functions, we put $F_i=I(f_i)$.

Theorem (4.4) *Let $F(a)={}^t(F_0(a),F_1(a),F_2(a))=(y_1y_2)^t(G_0(y),G_1(y),G_2(y))$ be the vector of the A-radial part of the standard Whittaker functions with minimal K-type of the principal series representation $\pi_{\sigma,\nu}$ with non-trivial $\sigma=\sigma_i$. Then it satisfies the following partial differential equations:*

(i):

$$\begin{pmatrix}\partial_1 & 4\pi c_1y_1 & \partial_1-2\partial_2-4\pi c_2y_2\\ -2\pi c_1y_1 & -2\partial_1 & -2\pi c_1y_1\\ \partial_1-2\partial_2+4\pi c_2y_2 & 4\pi c_1y_1 & \partial_1\end{pmatrix}\begin{pmatrix}G_0(y)\\G_1(y)\\G_2(y)\end{pmatrix}=\frac{1}{2}\lambda_i\begin{pmatrix}G_0(y)\\G_1(y)\\G_2(y)\end{pmatrix},$$

(ii):

$$\Delta_2\cdot 1_3\cdot\begin{pmatrix}G_0(y)\\G_1(y)\\G_2(y)\end{pmatrix}-2\pi c_2y_2\begin{pmatrix}G_0(y)\\0\\-G_2(y)\end{pmatrix}+2\pi c_1y_1\begin{pmatrix}G_1(y)\\\frac{1}{2}(G_0(y)+G_2(y))\\G_1(y)\end{pmatrix}=\frac{1}{3}\mu\begin{pmatrix}G_0(y)\\G_1(y)\\G_2(y)\end{pmatrix}.$$

Moreover the eigenvalues λ_i and μ depending on the representation $\pi_{\sigma,\nu}$ are given by

$$\begin{cases}\lambda_1=-\frac{4}{3}(2\nu_1-\nu_2) & (\sigma=\sigma_1)\\ \lambda_2=\frac{4}{3}(\nu_1-2\nu_2) & (\sigma=\sigma_2)\\ \lambda_3=\frac{4}{3}(\nu_1+\nu_2) & (\sigma=\sigma_3)\end{cases}\quad \text{and } \mu=\nu_1^2+\nu_2^2-\nu_1\nu_2.$$

Remark We can write the differential equations (i) and (ii) of the above Theorem as

$$\text{(i)}:\quad \mathcal{D}_1G=\lambda_iG\qquad \text{(ii)}:\quad \mathcal{D}_2G=\mu G,$$

with $\mathcal{D}_i$ $(i=1,2)$ 3 by 3 matrix-valued differential operators. Then we have

$$\mathcal{D}_1\cdot\mathcal{D}_2-\mathcal{D}_2\cdot\mathcal{D}_1=0.$$

Proof of Theorem. (i): This equation is obtained from Proposition (3.6), i.e., it is the

Whittaker realization of this proposition. The main ingredient to compute this realization is the Iwasawa decomposition of the standard elements in $\mathfrak{g}$, i.e., Lemma (3.7). After that it suffices to apply Lemmata (4.2) and (4.3).

(ii): The Capelli element Cp_2 is written as

$$\begin{aligned} Cp_2 &= (E'_{11}-1)E'_{22}+E'_{22}(E'_{33}+1)+(E'_{11}-1)(E'_{33}+1) \\ &\quad -E_{23}^2-E_{13}^2-E_{12}^2-E_{23}K_1+E_{13}K_2-E_{12}K_3. \end{aligned}$$

Since E_{13} belongs to the commutator of $\mathfrak{n}$ we can drop the terms E_{13}^2 and $E_{13}K_2$. Then the Whittaker realization of the first 5 terms are the same as the case of class one:

$$-\{(\partial_1^2+\partial_2^2-\partial_1\partial_2)-4\pi^2(c_1^2y_1^2+c_2^2y_2^2)\}1_3.$$

We need to compute only the sum of the remaining two terms $-E_{23}K_1-E_{12}K_3$. Since $-E_{23}$ and $-E_{12}$ are the multiplication operators $-2\pi\sqrt{-1}c_2y_2$ and $-2\pi\sqrt{-1}c_1y_1$ respectively, this sum is realized as

$$+2\pi c_2y_2\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}-2\pi c_1y_1\begin{pmatrix}0&1&0\\\frac{1}{2}&0&\frac{1}{2}\\0&1&0\end{pmatrix}.$$

4.3 The equations via the tautological basis

Let $k\in K\mapsto S(k)=(s_{ij}(k))_{1\le i,j\le 3}$ be the tautological representation of $K=SO(3)$. Let $I\in \mathrm{Hom}_{\mathfrak{g},K}(\pi_{\sigma_i,\nu},\mathrm{Ind}_N^G(\psi))$ be a Whittaker functional and define function T_{ij} on A by

$$I(s_{ij})_{|A}=y_1y_2T_{ij}(y)\quad (1\le i,j,\le 3).$$

Then

$$\begin{pmatrix}G_0\\G_1\\G_2\end{pmatrix}=\begin{pmatrix}0&\sqrt{-1}&1\\1&0&0\\0&\sqrt{-1}&-1\end{pmatrix}\begin{pmatrix}T_{i1}\\T_{i2}\\T_{i3}\end{pmatrix}.$$

Then for each i, the equation (1) of the above theorem is transformed to

$$\begin{pmatrix}-\partial_1&-2\pi\sqrt{-1}c_1y_1&0\\-2\pi\sqrt{-1}c_1y_1&\partial_1-\partial_2&-2\pi\sqrt{-1}c_2y_2\\0&-2\pi\sqrt{-1}c_2y_2&\partial_2\end{pmatrix}\begin{pmatrix}T_{i1}\\T_{i2}\\T_{i3}\end{pmatrix}=\frac{1}{2}\lambda_i\begin{pmatrix}T_{i1}\\T_{i2}\\T_{i3}\end{pmatrix},$$

and the equation (ii) to

$$\left[\Delta_2\cdot 1_3+\begin{pmatrix}0&2\pi\sqrt{-1}c_1y_1&0\\-2\pi\sqrt{-1}c_1y_1&0&2\pi\sqrt{-1}c_2y_2\\0&-2\pi\sqrt{-1}c_2y_2&0\end{pmatrix}\right]\begin{pmatrix}T_{i1}\\T_{i2}\\T_{i3}\end{pmatrix}=\frac{1}{3}\mu\begin{pmatrix}T_{i1}\\T_{i2}\\T_{i3}\end{pmatrix}.$$

5 Power series solutions at the origin

We determine 6 linearly independent formal power series at the origin $(y_1, y_2) = (0, 0)$ for generic parameter ν in this section. These formal solutions converges because the singularity at the origin is a regular singularity. These solutions do not have exponential decay at infinity, different from the unique 'good' solution given by Jacquet integral. We refer to these solutions as *secondary* Whittaker functions sometimes.

5.1 The case of the class one principal series

This case is more or less discussed in the paper of Bump [2], up to some difference of notations.

Theorem (5.1) *Assume that* $\frac{1}{4}(\lambda_k - \lambda_l) \notin \mathbf{Z}$. *Let* $\{e_1, e_2, e_3\}$ *be a permutation of the three complex numbers* $\{-\frac{1}{3}(2\nu_1 - \nu_2), -\frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)\} = \{\frac{1}{4}\lambda_1, \frac{1}{4}\lambda_2, \frac{1}{4}\lambda_3\}$. *Then the power series*

$$\Phi(y_1, y_2) := y_1^{-e_1} y_2^{e_2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(\frac{e_2-e_1}{2}+1)_{n_1+n_2} (\pi c_1 y_1)^{2n_1} (\pi c_2 y_2)^{2n_2}}{n_1! n_2! (\frac{e_2-e_1}{2}+1)_{n_1} (\frac{e_2-e_1}{2}+1)_{n_2} (\frac{e_3-e_1}{2}+1)_{n_1} (\frac{e_2-e_3}{2}+1)_{n_2}}.$$

Here the symbol $(\alpha)_m$ *means* $\Gamma(\alpha+m)/\Gamma(\alpha)$.

An integral expression of this power series solution was found by Stade ([12, Lemma 3.10], [14, Theorem 2]) as an analogue of an integral formula for Jacquet integral by Vinogradov and Takhadzhyan [15].

Theorem (5.2) *For* $\mathrm{Re}(e_2 - e_1) > 2$,

$$\Phi(y_1, y_2) = \Gamma(\tfrac{e_2-e_1}{2}+1)\Gamma(\tfrac{e_3-e_1}{2}+1)\Gamma(\tfrac{e_2-e_3}{2}+1)(\pi c_1 y_1)^{\frac{e_3}{2}} (\pi c_2 y_2)^{-\frac{e_3}{2}} (\pi c_1)^{e_1} (\pi c_2)^{-e_2}$$
$$\cdot \frac{1}{2\pi\sqrt{-1}} \int_{|u|=1} I_{\frac{e_2-e_1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{e_2-e_1}{2}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{4}e_3} \frac{du}{u}.$$

5.2 The case of the non-spherical principal series

In this case also, the holonomic system obtained in Theorem (4.4) has regular singularities at the origin $(y_1, y_2) = (0, 0)$. The rank of this system is 6, i.e., the order of the Weyl group of $SL(3, \mathbf{R})$, for generic values of parameter ν. We want to determine the characteristic indices and the convergent formal power series solutions at $y = 0$. Here to abridge the notation, we write the set of variables (y_1, y_2) as y collectively.

Also after some computation, by inspection we find that it is convenient to introduce scalar functions $\Phi_i(y_1, y_2)$ $(i = 0, 1, 2)$ by

$$F(y) = y_1 y_2 G(y) = y_1 y_2 \{\Phi_0(y) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \Phi_1(y) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \Phi_2(y) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \}.$$

5.3 The holonomic system for $\Phi_i(y)$

Now we can rewrite the holonomic system for G_i to that for Φ_i.

Proposition (5.3) *The holonomic system in Theorem (4.4) is equivalent to the following system for* $\Phi_i = \Phi_i(y_1, y_2)$ $(i = 0, 1, 2)$.

(1) (i) $[\partial_1 + \frac{1}{4}\lambda_i]\Phi_0 + (2\pi c_1 y_1)\Phi_1 = 0$,

(ii) $[\partial_1 - \partial_2 - \frac{1}{4}\lambda_i]\Phi_1 + (2\pi c_1 y_1)\Phi_0 + (2\pi c_2 y_2)\Phi_2 = 0$,

(iii) $[\partial_2 - \frac{1}{4}\lambda_i]\Phi_2 - (2\pi c_2 y_2)\Phi_1 = 0$,

(2) (i) $[\Delta_2 - \frac{1}{3}\mu]\Phi_0 + (2\pi c_1 y_1)\Phi_1 = 0$,

(ii) $[\Delta_2 - \frac{1}{3}\mu]\Phi_1 + (2\pi c_1 y_1)\Phi_0 - (2\pi c_2 y_2)\Phi_2 = 0$,

(iii) $[\Delta_2 - \frac{1}{3}\mu]\Phi_2 - (2\pi c_2 y_2)\Phi_1 = 0$.

5.4 The characteristic indices at the origin $(y_1, y_2) = (0, 0)$ and the recurrence formulae.

Let

$$\Phi_k(y) = y_1^{-e_1} y_2^{e_2} \sum_{n_1, n_2 \geq 0} c_{k;n_1,n_2} (\pi c_1 y_1)^{n_1} (\pi c_2 y_2)^{n_2}, \quad (k = 0, 1, 2)$$

be a system of formal power series solutions at the origin $y = 0$. Then the first task is to compute the characteristic indices $(-e_1, e_2)$ of the system at the origin and to determine the first coefficients $c_{k;0,0}$. Moreover we need the recurrence relations between the coefficients.

Now we can determine the 6 pairs $(-e_1, e_2)$ of characteristic indices, and the corresponding initial values conditions for F or Φ_i.

Lemma (5.4) *When $\sigma = \sigma_i$ for $i = 1,\ 2$ or 3, we have the following*:

(1) *The characteristic indices take the six values*:

$$(-e_1, e_2) = (-\tfrac{1}{4}\lambda_k, \tfrac{1}{4}\lambda_l) \quad (1 \leq k \neq l \leq 3).$$

(2) *For each case, the set of first coefficients, or the initial values at the origin are given as follows*:

(i) *If* $(-e_1, e_2) = (-\frac{1}{4}\lambda_i, \frac{1}{4}\lambda_k) \quad (k \neq i)$,

$(y_1^{e_1} y_2^{-e_2} G)(0,0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, *i.e.*, $(y_1^{e_1} y_2^{-e_2}\Phi_0)(0,0) = 1$, *and* $(y_1^{e_1} y_2^{-e_2}\Phi_j)(0,0) = 0$ *for other* j.

(ii) *If* $(-e_1, e_2) = (-\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_l) \quad (k \neq i, l \neq i, k \neq l)$,

$(y_1^{e_1} y_2^{-e_2} G)(0,0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, *i.e.*, $(y_1^{e_1} y_2^{-e_2}\Phi_1)(0,0) = 1$, *and* $(y_1^{e_1} y_2^{-e_2}\Phi_j)(0,0) = 0$ *for other* j.

(iii) *If* $(-e_1, e_2) = (-\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_i) \quad (k \neq i)$,

$(y_1^{e_1} y_2^{-e_2} G)(0,0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, *i.e.*, $(y_1^{e_1} y_2^{-e_2}\Phi_2)(0,0) = 1$, *and* $(y_1^{e_1} y_2^{-e_2}\Phi_j)(0,0) = 0$ *for other* j.

(3) *We have the following recurrence relations for the coefficients*:

(i) $(n_1 - e_1 + \frac{1}{4}\lambda_i)c_{0;n_1,n_2} + 2c_{1;n_1-1,n_2} = 0$;

(ii) $(n_1 - n_2 - e_1 - e_2 - \frac{1}{4}\lambda_i)c_{1;n_1,n_2} + 2c_{0;n_1-1,n_2} + 2c_{2;n_1,n_2-1} = 0$;

(iii) $(n_2 + e_2 - \frac{1}{4}\lambda_i)c_{2;n_1,n_2} - 2c_{1;n_1,n_2-1} = 0$.

Proof. (1): From the holonomic system we have

$$\begin{pmatrix} -e_1+\frac{1}{4}\lambda_i & & \\ & -e_1-e_2-\frac{1}{4}\lambda_i & \\ & & e_2-\frac{1}{4}\lambda_i \end{pmatrix} \begin{pmatrix} c_{0;0,0} \\ c_{1;0,0} \\ c_{2;0,0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$((-e_1)^2+e_2^2-(-e_1)(e_2)-\frac{1}{3}\mu)1_3 \begin{pmatrix} c_{0;0,0} \\ c_{1;0,0} \\ c_{2;0,0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence we have

$$(-e_1+\frac{1}{4}\lambda_i)(-e_1-e_2-\frac{1}{4}\lambda_i)(e_2-\frac{1}{4}\lambda_i)=0,$$

$$(e_1)^2+e_2^2-(-e_1)(e_2)-\frac{1}{3}\mu=0.$$

By the first equation, we have either

$$-e_1=-\frac{1}{4}\lambda_i, \quad e_2=\frac{1}{4}\lambda_i, \text{ or } -e_1-e_2=\frac{1}{4}\lambda_i.$$

Input either of these three values, then the second equations give both $-e_1$ and e_2.
(2): This is a just exercise of linear algebra.
(3): Direct computation yields the relations. □

5.5 Power series solutions at the origin

Now we can show the following formulae for the power series solutions.

Theorem (5.5) *Assume that* $\frac{1}{4}(\lambda_k-\lambda_l)\notin \mathbf{Z}$. *Then we have the following.*

(I) *When* $\sigma=\sigma_1$ *we have the following six independent solutions.*

$${}^t(\Phi_0^{1,\mathrm{I}},\Phi_1^{1,\mathrm{I}},\Phi_2^{1,\mathrm{I}}) = y_1^{-\frac{\lambda_1}{4}} y_2^{\frac{\lambda_2}{4}}$$

$$\cdot \begin{pmatrix} \displaystyle\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+m_2}}{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2}}{m_1!m_2!(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_2-\lambda_3}{8}+1)_{m_2}} \\ \displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1}(\pi c_2 y_2)^{2m_2}}{m_1!m_2!(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_2-\lambda_3}{8}+1)_{m_2}} \\ \displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_2+1}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1}(\pi c_2 y_2)^{2m_2+1}}{m_1!m_2!(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_2-\lambda_3}{8}+1)_{m_2}} \end{pmatrix},$$

$${}^t(\Phi_0^{1,\mathrm{III}},\Phi_1^{1,\mathrm{III}},\Phi_2^{1,\mathrm{III}}) = y_1^{-\frac{\lambda_2}{4}} y_2^{\frac{\lambda_3}{4}}$$

$$\cdot \begin{pmatrix} \displaystyle\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1+m_2}}{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_3-\lambda_2}{8}+1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1}(\pi c_2 y_2)^{2m_2}}{m_1!m_2!(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_2}} \\ \displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1+m_2}}{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_3-\lambda_2}{8}+1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2}}{m_1!m_2!(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_2}} \\ \displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1+m_2}}{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_3-\lambda_2}{8}+1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2+1}}{m_1!m_2!(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_2+1}} \end{pmatrix},$$

$${}^t(\Phi_0^{1,\mathrm{V}},\Phi_1^{1,\mathrm{V}},\Phi_2^{1,\mathrm{V}}) = y_1^{-\frac{\lambda_2}{4}} y_2^{\frac{\lambda_1}{4}}$$

$$\cdot\left(\begin{array}{l} -\sum_{m_1,m_2\geq 0}\frac{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_2+1}}\cdot\frac{(\pi c_1y_1)^{2m_1+1}(\pi c_2y_2)^{2m_2+1}}{m_1!m_2!(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})_{m_2+1}} \\ \sum_{m_1,m_2\geq 0}\frac{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_2+1}}\cdot\frac{(\pi c_1y_1)^{2m_1}(\pi c_2y_2)^{2m_2+1}}{m_1!m_2!(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})_{m_2+1}} \\ \sum_{m_1,m_2\geq 0}\frac{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+m_2}}{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_2}}\cdot\frac{(\pi c_1y_1)^{2m_1}(\pi c_2y_2)^{2m_2}}{m_1!m_2!(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})_{m_2}} \end{array}\right),$$

and other three solutions $\Phi_i^{1,\mathrm{II}}, \Phi_i^{1,\mathrm{IV}}$ *and* $\Phi_i^{1,\mathrm{VI}}$ *are given by exchanging the role of* λ_2 *and* λ_3 *in the expression for* $\Phi_i^{1,\mathrm{I}}, \Phi_i^{1,\mathrm{III}}$ *and* $\Phi_i^{1,\mathrm{V}}$*, respectively.*

(II) *When* $\sigma = \sigma_2$*, exchange* λ_1 *and* λ_2 *in the part* (I).
(III) *When* $\sigma = \sigma_3$*, exchange* λ_1 *and* λ_3 *in the part* (I).

Proof. The third statement of the previous lemma, i.e., the recurrence relations between coefficients determines the coefficients $c_{k;n_1,n_2}$ recursively from the initial coefficients $c_{k;0,0}$. So the necessary task is to check that our formulae are compatible with these relations and the other relations coming from the system of equations (ii) in Theorem (4.4).

Case 1: When $(-e_1, e_2) = \frac{1}{4}(-\lambda_1, \lambda_2)$ (or $(-e_1, e_2) = \frac{1}{4}(-\lambda_1, \lambda_3)$, resp.). This means that we consider $\Phi_i^{1,\mathrm{I}}$ (or $\Phi_i^{1,\mathrm{II}}$, resp.).
The assumption $\frac{1}{4}(\lambda_k - \lambda_l) \notin \mathbf{Z}$ implies

(i) $c_{1;n_1-1,n_2} = 0 \Rightarrow c_{0;n_1,n_2} = 0$,

(ii) $c_{1;n_1,n_2-1} = 0 \Rightarrow c_{2;n_1,n_2} = 0$,

(iii) $c_{0;n_1-1,n_2} = c_{2;n_1,n_2-1} = 0 \Rightarrow c_{1;n_1,n_2} = 0$.

Since $\begin{pmatrix} c_{0;0,0} \\ c_{1;0,0} \\ c_{2;0,0}\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0\end{pmatrix}$, the $c_{0;n_1,n_2}, c_{1;n_1,n_2}, c_{2;n_1,n_2}$ are zero unless $(n_1, n_2) \in \{(\mathrm{even}, \mathrm{even})\}$, $(n_1, n_2) \in \{(\mathrm{odd}, \mathrm{even})\}$, $(n_1, n_2) \in \{(\mathrm{odd}, \mathrm{odd})\}$, respectively.

For $\Phi_i^{1,\mathrm{I}}$ the remaining recurrence relations are

Sublemma 1.

(1) (i) $2m_1c_{0;2m_1,2m_2} + 2c_{1;2m_1-1,2m_2} = 0$;
(ii) $(2m_1 - 2m_2 + \frac{1}{4}(\lambda_3 - \lambda_1) + 1)c_{1;2m_1+1,2m_2} + 2c_{0;2m_1,2m_2} + 2c_{2;2m_1+1,2m_2-1} = 0$;
(iii) $(2m_2 + \frac{1}{4}(\lambda_2 - \lambda_1) + 1)c_{2;2m_1+1,2m_2+1} - 2c_{1;2m_1+1,2m_2} = 0$.

(2) (i) $[(2m_1-\frac{1}{4}\lambda_1)^2+(2m_2+\frac{1}{4}\lambda_2)^2-(2m_1-\frac{1}{4}\lambda_1)(2m_2+\frac{1}{4}\lambda_2)-\frac{1}{32}(\lambda_1^2+\lambda_2^2+\lambda_3^2)]c_{0;2m_1,2m_2} - 4c_{0;2m_1-2,2m_2} - 4c_{0;2m_1,2m_2-2} + 2c_{1;2m_1-1,2m_2} = 0$;
(ii) $[(2m_1 - \frac{1}{4}\lambda_1 + 1)^2 + (2m_2 + \frac{1}{4}\lambda_2)^2 - (2m_1 - \frac{1}{4}\lambda_1 + 1)(2m_2 + \frac{1}{4}\lambda_2) - \frac{1}{32}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)]c_{1;2m_1+1,2m_2} - 4c_{1;2m_1-1,2m_2} - 4c_{1;2m_1+1,2m_2-2} + 2c_{0;2m_1,2m_2} - 2c_{1;2m_1+1,2m_2-1} = 0$;
(iii) $[(2m_1 - \frac{1}{4}\lambda_1 + 1)^2 + (2m_2 + \frac{1}{4}\lambda_2 + 1)^2 - (2m_1 - \frac{1}{4}\lambda_1 + 1)(2m_2 + \frac{1}{4}\lambda_2 + 1) - \frac{1}{32}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)]c_{2;2m_1+1,2m_2+1} - 4c_{2;2m_1-1,2m_2+1} - 4c_{2;2m_1+1,2m_2-1} - 2c_{1;2m_1+1,2m_2} = 0$.

And up to constant multiple, the unique system of non-zero solutions of these recurrence relations is given by

(i) $c_{0;2m_1,2m_2} = \frac{(p)_{m_1+m_2}}{(p)_{m_1}(p)_{m_2}}\frac{1}{m_1!m_2!(q)_{m_1}(r)_{m_2}}$,

(ii) $c_{1;2m_1+1,2m_2} = -\frac{(p)_{m_1+m_2+1}}{(p)_{m_1+1}(p)_{m_2}} \frac{1}{m_1!m_2!(q)_{m_1+1}(r)_{m_2}}$,

(iii) $c_{2;2m_1+1,2m_2+1} = -\frac{(p)_{m_1+m_2+1}}{(p)_{m_1+1}(p)_{m_2+1}} \frac{1}{m_1!m_2!(q)_{m_1+1}(r)_{m_2}}$.

with

$$p = \frac{1}{8}(\lambda_2 - \lambda_1) + \frac{1}{2}, \quad q = \frac{1}{8}(\lambda_3 - \lambda_1) + \frac{1}{2}, \quad r = \frac{1}{8}(\lambda_2 - \lambda_3) + 1 \quad (-p + q + r = 1).$$

Proof. These are obtained by direct computation.

Case 2: When $(-e_1, e_2) = \frac{1}{4}(-\lambda_2, \lambda_1)$, or $(-e_1, e_2) = \frac{1}{4}(-\lambda_3, \lambda_1)$, i.e., we consider $\Phi_i^{1,\mathrm{V}}$ or $\Phi^{1,\mathrm{VI}}$.

Similarly as Case 1, the initial condition Since $\begin{pmatrix} c_{0;0,0} \\ c_{1;0,0} \\ c_{2;0,0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, implies that the $c_{0;n_1,n_2}$, $c_{1;n_1,n_2}$, $c_{2;n_1,n_2}$ are zero unless $(n_1, n_2) \in \{(\text{odd,odd})\}$, $(n_1, n_2) \in \{(\text{even,odd})\}$, $(n_1, n_2) \in \{(\text{even,even})\}$.

The remaining recurrence relations are

Sublemma 2.

(1) (i) $(2m_1 + \frac{1}{4}(\lambda_1 - \lambda_2) + 1)c_{0;2m_1+1,2m_2+1} + 2c_{1;2m_1,2m_2+1} = 0$;

(ii) $(2m_1 - 2m_2 + \frac{1}{4}(\lambda_3 - \lambda_1) - 1)c_{1;2m_1,2m_2+1} + 2c_{0;2m_1-1,2m_2+1} + 2c_{2;2m_1,2m_2} = 0$;

(iii) $2m_2 c_{2;2m_1,2m_2} - 2c_{1;2m_1,2m_2-1} = 0$.

(2) (i) $[(2m_1 - \frac{1}{4}\lambda_2 + 1)^2 + (2m_2 + \frac{1}{4}\lambda_1 + 1)^2 - (2m_1 - \frac{1}{4}\lambda_2 + 1)(2m_2 + \frac{1}{4}\lambda_1 + 1) - \frac{1}{32}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)]c_{0;2m_1+1,2m_2+1} - 4c_{0;2m_1-1,2m_2+1} - 4c_{0;2m_1+1,2m_2-1} + 2c_{1;2m_1,2m_2+1} = 0$;

(ii) $[(2m_1 - \frac{1}{4}\lambda_2)^2 + (2m_2 + \frac{1}{4}\lambda_1 + 1)^2 - (2m_1 - \frac{1}{4}\lambda_2)(2m_2 + \frac{1}{4}\lambda_1 + 1) - \frac{1}{32}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)]c_{1;2m_1,2m_2+1} - 4c_{1;2m_1-2,2m_2+1} - 4c_{1;2m_1,2m_2-1} + 2c_{0;2m_1-1,2m_2+1} - 2c_{2;2m_1,2m_2} = 0$;

(iii) $[(2m_1 - \frac{1}{4}\lambda_2)^2 + (2m_2 + \frac{1}{4}\lambda_1)^2 - (2m_1 - \frac{1}{4}\lambda_2)(2m_2 + \frac{1}{4}\lambda_1) - \frac{1}{32}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)]c_{2;2m_1,2m_2} - 4c_{2;2m_1-2,2m_2+1} - 4c_{2;2m_1,2m_2-2} - 2c_{1;2m_1,2m_2-1} = 0$.

And up to constant multiple, the unique system of non-zero solutions of these recurrence relations is given by

(i) $c_{0;2m_1,2m_2} = -\frac{(p)_{m_1+m_2+1}}{(p)_{m_1+1}(p)_{m_2+1}} \frac{1}{m_1!m_2!(q)_{m_1}(r)_{m_2+1}}$,

(ii) $c_{1;2m_1+1,2m_2} = \frac{(p)_{m_1+m_2+1}}{(p)_{m_1}(p)_{m_2+1}} \frac{1}{m_1!m_2!(q)_{m_1}(r)_{m_2+1}}$,

(iii) $c_{2;2m_1+1,2m_2+1} = \frac{(p)_{m_1+m_2}}{(p)_{m_1}(p)_{m_2}} \frac{1}{m_1!m_2!(q)_{m_1}(r)_{m_2}}$.

with

$$p = \frac{1}{8}(\lambda_1 - \lambda_2) + \frac{1}{2}, \quad q = \frac{1}{8}(\lambda_3 - \lambda_2) + 1, \quad r = \frac{1}{8}(\lambda_1 - \lambda_3) + \frac{1}{2}.$$

Proof. These are obtained by direct computation.

Case 3: When $(-e_1, e_2) = \frac{1}{4}(-\lambda_2, \lambda_3)$, or $(-e_1, e_2) = \frac{1}{4}(-\lambda_3, \lambda_2)$.

In this case, the $c_{0;n_1,n_2}, c_{1;n_1,n_2}, c_{2;n_1,n_2}$ are zero unless $(n_1, n_2) \in \{(\text{odd,even})\}, (n_1, n_2) \in \{(\text{even,even})\}(n_1, n_2) \in \{(\text{even,odd})\}$.

The remaining recurrence relations are

Sublemma 3.

(1) (i) $(2m_1 + \frac{1}{4}(\lambda_1 - \lambda_2) + 1)c_{0;2m_1+1,2m_2} + 2c_{1;2m_1,2m_2} = 0;$

(ii) $(2m_1 - 2m_2)c_{1;2m_1,2m_2} + 2c_{0;2m_1-1,2m_2} + 2c_{2;2m_1,2m_2-1} = 0;$

(iii) $(2m_2 + \frac{1}{4}(\lambda_3 - \lambda_1) + 1)c_{2;2m_1,2m_2+1} - 2c_{1;2m_1,2m_2} = 0.$

(2) (i) $[(2m_1 - \frac{1}{4}\lambda_2 + 1)^2 + (2m_2 + \frac{1}{4}\lambda_3)^2 - (2m_1 - \frac{1}{4}\lambda_2 + 1)(2m_2 + \frac{1}{4}\lambda_3) - \frac{1}{32}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)]c_{0;2m_1+1,2m_2} - 4c_{0;2m_1-1,2m_2} - 4c_{0;2m_1+1,2m_2-2} + 2c_{1;2m_1,2m_2} = 0;$

(ii) $[(2m_1-\frac{1}{4}\lambda_2)^2+(2m_2+\frac{1}{4}\lambda_3)^2-(2m_1-\frac{1}{4}\lambda_2)(2m_2+\frac{1}{4}\lambda_3)-\frac{1}{32}(\lambda_1^2+\lambda_2^2+\lambda_3^2)]c_{1;2m_1,2m_2} - 4c_{1;2m_1-2,2m_2} - 4c_{1;2m_1,2m_2-2} + 2c_{0;2m_1-1,2m_2} - 2c_{2;2m_1,2m_2-1} = 0;$

(iii) $[(2m_1 - \frac{1}{4}\lambda_2)^2 + (2m_2 + \frac{1}{4}\lambda_3 + 1)^2 - (2m_1 - \frac{1}{4}\lambda_2)(2m_2 + \frac{1}{4}\lambda_3 + 1) - \frac{1}{32}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)]c_{2;2m_1,2m_2+1} - 4c_{2;2m_1-2,2m_2+1} - 4c_{2;2m_1,2m_2-1} - 2c_{1;2m_1,2m_2} = 0.$

And up to constant multiple, the unique system of non-zero solutions of these recurrence relations is given by

(i) $c_{0;2m_1+1,2m_2} = \frac{(p)_{m_1+m_2}}{(p)_{m_1}(p)_{m_2}} \frac{1}{m_1!m_2!(q)_{m_1+1}(r)_{m_2}},$

(ii) $c_{1;2m_1,2m_2} = -\frac{(p)_{m_1+m_2}}{(p)_{m_1}(p)_{m_2}} \frac{1}{m_1!m_2!(q)_{m_1}(r)_{m_2}},$

(iii) $c_{2;2m_1,2m_2+1} = -\frac{(p)_{m_1+m_2}}{(p)_{m_1}(p)_{m_2}} \frac{1}{m_1!m_2!(q)_{m_1}(r)_{m_2+1}}.$

with

$$p = \frac{1}{8}(\lambda_3 - \lambda_2) + 1, \quad q = \frac{1}{8}(\lambda_1 - \lambda_2) + \frac{1}{2}, \quad r = \frac{1}{8}(\lambda_3 - \lambda_1) + \frac{1}{2} \quad (p = q + r).$$

Proof. These are obtained by direct computation. □

5.6 Integral representations of the secondary Whittaker functions

In this subsection, we rewrite the power series solutions of the previous subsection by integral expressions.

Theorem (5.6) (I) *When $\sigma = \sigma_1$ we have*

$$ {}^t(\Phi_0^{1,\mathrm{I}}, \Phi_1^{1,\mathrm{I}}, \Phi_2^{1,\mathrm{I}}) = (\pi c_1 y_1)^{\frac{\lambda_3}{8}+\frac{1}{2}} (\pi c_2 y_2)^{-\frac{\lambda_3}{8}+\frac{1}{2}}$$

$$\cdot (2\pi\sqrt{-1})^{-1}\Gamma(\tfrac{\lambda_2-\lambda_1}{8} + \tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8} + \tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8} + 1)(\pi c_1)^{\frac{\lambda_1}{4}}(\pi c_2)^{-\frac{\lambda_2}{4}}$$

$$\cdot \begin{pmatrix} \int_{|u|=1} I_{\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}}(2\pi c_1 y_1\sqrt{1+1/u}) I_{\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3+\frac{1}{4}} \frac{du}{u} \\ (-1)\int_{|u|=1} I_{\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2}}(2\pi c_1 y_1\sqrt{1+1/u}) I_{\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3-\frac{1}{4}}(1+u)^{\frac{1}{2}} \frac{du}{u} \\ (-1)\int_{|u|=1} I_{\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2}}(2\pi c_1 y_1\sqrt{1+1/u}) I_{\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3-\frac{1}{4}} \frac{du}{u} \end{pmatrix}$$

for $\mathrm{Re}(\frac{\lambda_2-\lambda_1}{8}) > \frac{3}{2}$,

$$ {}^t(\Phi_0^{1,\mathrm{III}}, \Phi_1^{1,\mathrm{III}}, \Phi_2^{1,\mathrm{III}}) = (\pi c_1 y_1)^{\frac{\lambda_1}{8}} (\pi c_2 y_2)^{-\frac{\lambda_1}{8}}$$

$$\cdot (2\pi\sqrt{-1})^{-1}\Gamma(\tfrac{\lambda_1-\lambda_2}{8} + \tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8} + \tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8} + 1)(\pi c_1)^{\frac{\lambda_2}{4}}(\pi c_2)^{-\frac{\lambda_3}{4}}$$

$$
\cdot\begin{pmatrix}
(\pi c_1 y_1)\displaystyle\int_{|u|=1} I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_1 y_1\sqrt{1+1/u}) I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_2 y_2\sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3-\frac{1}{2}}\frac{du}{u} \\
(-1)\displaystyle\int_{|u|=1}\big[\pi c_1 y_1\sqrt{1+1/u}\, I_{\frac{\lambda_3-\lambda_2}{8}-1}(2\pi c_1 y_1\sqrt{1+1/u}) + (\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2}) \\
\cdot I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_1 y_1\sqrt{1+1/u})\big] I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_2 y_2\sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3-\frac{1}{2}}\frac{du}{u} \\
(-1)(\pi c_2 y_2)\displaystyle\int_{|u|=1} I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_1 y_1\sqrt{1+1/u}) I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_2 y_2\sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3+\frac{1}{2}}\frac{du}{u}
\end{pmatrix}
$$

for $\mathrm{Re}(\frac{\lambda_3-\lambda_2}{8}) > 1$,

$$
{}^t(\Phi_0^{1,\mathrm{V}}, \Phi_1^{1,\mathrm{V}}, \Phi_2^{1,\mathrm{V}}) = (\pi c_1 y_1)^{\frac{\lambda_3}{8}+\frac{1}{2}}(\pi c_2 y_2)^{-\frac{\lambda_3}{8}+\frac{1}{2}}
$$

$$
\cdot(2\pi\sqrt{-1})^{-1}\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+1)(\pi c_1)^{\frac{\lambda_2}{4}}(\pi c_2)^{-\frac{\lambda_1}{4}}
$$

$$
\cdot\begin{pmatrix}
(-1)\displaystyle\int_{|u|=1} I_{\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2}}(2\pi c_1 y_1\sqrt{1+1/u}) I_{\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2}}(2\pi c_2 y_2\sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3+\frac{1}{4}}\frac{du}{u} \\
\displaystyle\int_{|u|=1} I_{\frac{\lambda_1-\lambda_2}{8}-\frac{1}{2}}(2\pi c_1 y_1\sqrt{1+1/u}) I_{\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2}}(2\pi c_2 y_2\sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3-\frac{1}{4}}(1+u)^{\frac{1}{2}}\frac{du}{u} \\
\displaystyle\int_{|u|=1} I_{\frac{\lambda_1-\lambda_2}{8}-\frac{1}{2}}(2\pi c_1 y_1\sqrt{1+1/u}) I_{\frac{\lambda_1-\lambda_2}{8}-\frac{1}{2}}(2\pi c_2 y_2\sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3-\frac{1}{4}}\frac{du}{u}
\end{pmatrix}
$$

for $\mathrm{Re}(\frac{\lambda_1-\lambda_2}{8}) > \frac{3}{2}$.

To have the integral expression for $\Phi_i^{1,\mathrm{II}}, \Phi_i^{1,\mathrm{IV}}$ *and* $\Phi_i^{1,\mathrm{VI}}$*, we have to exchange the role of* λ_2 *and* λ_3 *in the expression for* $\Phi_i^{1,\mathrm{I}}, \Phi_i^{1,\mathrm{III}}$ *and* $\Phi_i^{1,\mathrm{V}}$*, respectively.*

(II) *When* $\sigma = \sigma_2$*, exchange* λ_1 *and* λ_2 *in* (I).
(III) *When* $\sigma = \sigma_3$*, exchange* λ_1 *and* λ_3 *in* (I).

5.7 Proof of Theorem (5.6)

Firstly we need the following.

Lemma (5.7) *For* $\mathrm{Re}\,(x+y+1) > 0$,

$$
\frac{1}{2\pi\sqrt{-1}}\int_{|u|=1} I_\alpha(2\pi c_1 y_1\sqrt{1+1/u}) I_\beta(2\pi c_2 y_2\sqrt{1+u}) u^{\frac{\alpha}{2}-x}(1+u)^{x+y-\frac{\alpha+\beta}{2}}\frac{du}{u}
$$
$$
= \sum_{k_1,k_2\ge 0}\frac{(\pi c_1 y_1)^{2k_1+\alpha}(\pi c_2 y_2)^{2k_2+\beta}}{k_1!k_2!\,\Gamma(k_1+\alpha+1)\Gamma(k_2+\beta+1)}\frac{\Gamma(k_1+k_2+x+y+1)}{\Gamma(k_1+x+1)\Gamma(k_2+y+1)}
$$

Proof of Lemma. Utilize the formula ([14, Theorem 1])

$$
\frac{1}{2\pi\sqrt{-1}}\int_{|u|=1}(1+1/u)^{x-1}(1+u)^{y-1}\frac{du}{u} = \frac{\Gamma(x+y-1)}{\Gamma(x)\Gamma(y)} \qquad \text{for } \mathrm{Re}(x+y-1) > 0.
$$

Then (LHS) is

$$
\frac{1}{2\pi\sqrt{-1}}\int_{|u|=1}\sum_{k_1=0}^{\infty}\frac{(\pi c_1 y_1\sqrt{1+1/u})^{2k_1+\alpha}}{k_1!\,\Gamma(k_1+\alpha+1)}\sum_{k_2=0}^{\infty}\frac{(\pi c_2 y_2\sqrt{1+u})^{2k_2+\beta}}{k_2!\,\Gamma(k_2+\beta+1)}
$$
$$
\cdot u^{\frac{\alpha}{2}-x}(1+u)^{x+y-\frac{\alpha+\beta}{2}}\frac{du}{u}.
$$

Interchanging the order of integration and summation, the last formula equals to

$$\sum_{k_1,k_2\ge 0}\frac{(\pi c_1y_1)^{2k_1+\alpha}(\pi c_2y_2)^{2k_2+\beta}}{k_1!k_2!\,\Gamma(k_1+\alpha+1)\Gamma(k_2+\beta+1)}$$
$$\cdot\frac{1}{2\pi\sqrt{-1}}\int_{|u|=1}(1+1/u)^{k_1+\frac{\alpha}{2}}(1+u)^{k_2+\frac{\beta}{2}}u^{\frac{\alpha}{2}-x}(1+u)^{x+y-\frac{\alpha+\beta}{2}}\frac{du}{u}.$$

Here in view of the equality $u=\frac{1+u}{1+1/u}$, the last integrand is equal to

$$(1+u)^{k_2+y}(1+1/u)^{k_1+x}.$$

Therefore the last integral equals to

$$\frac{\Gamma(k_1+k_2+x+y+1)}{\Gamma(k_1+x+1)\Gamma(k_2+y+1)}$$

as desired. □

Proof of Theorem (5.6). We consider only the case $\sigma=\sigma_1$. Other cases are settled completely similarly by symmetry.

Case 1: For $\Phi_i^{1,\mathrm{I}}$ or $\Phi_i^{1,\mathrm{II}}$. Also by symmetry $\lambda_2\leftrightarrow\lambda_3$, it suffices to consider $\Phi_i^{1,\mathrm{I}}$. Then characteristic indices are $(-\frac{\lambda_1}{4},\frac{\lambda_2}{4})$. Rewrite the power series expression of $\Phi_0^{1,\mathrm{I}}$ as

$$\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8}+1)(\pi c_1)^{\frac{\lambda_1}{4}}(\pi c_2)^{-\frac{\lambda_2}{4}}$$
$$\sum_{m_1,m_2\ge 0}\frac{(\pi c_1y_1)^{2m_1-\frac{1}{4}\lambda_1}(\pi c_2y_2)^{2m_2+\frac{1}{4}\lambda_2}}{m_1!m_2!\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2}+m_1)\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2}+m_2)}\frac{\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2}+m_1+m_2)}{\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2}+m_1)\Gamma(\frac{\lambda_2-\lambda_3}{8}+1+m_2)}.$$

Apply the previous lemma with

$$\alpha=\beta=\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2},\ x=\frac{\lambda_3-\lambda_1}{8}-\frac{1}{2},\ y=\frac{\lambda_2-\lambda_3}{8},$$

then we have the formula in the theorem. For $\Phi_1^{1,\mathrm{I}}$ and $\Phi_2^{1,\mathrm{I}}$ the factor before summation should be taken as

$$-\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8}+1).$$

And to apply the lemma, we set

$$\alpha=\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2},\ \beta=\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2},\ x=\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2},\ y=\frac{\lambda_2-\lambda_3}{8},$$

and

$$\alpha=\beta=\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2},\ x=\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2},\ y=\frac{\lambda_2-\lambda_3}{8},$$

respectively. We have done the case 1.

Case 2: $\Phi_i^{1,\mathrm{V}}$ is considered, i.e., the case where the characteristic indices is given by $(-\frac{\lambda_2}{4},\frac{\lambda_1}{4})$. The case $\Phi_i^{1,\mathrm{VI}}$ is similar.

Here we note the symmetry with the case $\Phi_i^{1,\mathrm{I}}$. Say, $\Phi_0^{1,\mathrm{V}}$ is obtained from $\Phi_2^{1,\mathrm{I}}$ by replacement of variables and parameters:

$$y_1\leftrightarrow y_2,\quad (\lambda_1,\lambda_2,\lambda_3)=(-\lambda_1,-\lambda_2,-\lambda_3),\ \text{and } u\to 1/u \text{ in the integral.}$$

Case 3: $\Phi_i^{1,\mathrm{III}}$ and $\Phi_i^{1,\mathrm{IV}}$. It suffices to show the former case, where the characteristic indices are $(-\frac{\lambda_2}{4}, \frac{\lambda_3}{4})$. For $\Phi_0^{1,\mathrm{III}}$ and $\Phi_2^{1,\mathrm{III}}$ firstly we pull out the gamma factor

$$\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})$$

before the summation symbol. After that we apply the previous lemma with

$$\alpha=\beta=\frac{\lambda_3-\lambda_2}{8},\ x=\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2},\ y=\frac{\lambda_3-\lambda_1}{8}-\frac{1}{2},$$

and

$$\alpha=\beta=\frac{\lambda_3-\lambda_2}{8},\ x=\frac{\lambda_1-\lambda_2}{8}-\frac{1}{2},\ y=\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2},$$

respectively.

The remaining case $\Phi_1^{1,\mathrm{III}}$ is slightly different from other cases. We start from $\Phi_1^{1,\mathrm{III}} = -\frac{1}{2\pi c_1 y_1}(\partial_1+\frac{\lambda_1}{4})\Phi_0^{1,\mathrm{III}}$. The derivation in the integrand shows our formula.

Now we have finished the proof of our Theorem. □

6 Evaluation of Jacquet integrals

In this section, we give explicit descriptions of Jacquet integrals for non-spherical principal series Whittaker functions. The procedures of evaluation is similar to the class one case ([15]).

6.1 Jacquet integrals

Let us denote by $g = n(g)a(g)k(g)$ the Iwasawa decomposition of $g \in G$. We define *Jacquet integral* J_{ij} for $\sigma_i \in \widehat{M}$ $(1 \le i, j \le 3)$ as

$$J_{ij}(g) = \int_N \psi(n)^{-1} a(s_0^{-1}ng)^{\nu+\rho} s_{ij}(k(s_0^{-1}ng))dn$$

for $1 \le j \le 3$. Here

$$s_0 = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}$$

the longest element in the Weyl group of $SL(3,\mathbf{R})$ and $s_{ij}(k)$ is the element of the tautological representation of K (cf. [5, (7.1)]).

Since

$$v_0 = \sqrt{-1}(s_{i2}-\sqrt{-1}s_{i3}),\ v_1 = s_{i1},\ v_2 = \sqrt{-1}(s_{i2}+\sqrt{-1}s_{i3})$$

(§3.2.2) and

$$\Phi_0 = G_1,\ 2\Phi_1 = G_0+G_2,\ 2\Phi_2 = G_0-G_2,$$

(§5.2) the vector of integrals ${}^t(J_{i1}, \sqrt{-1}J_{i2}, J_{i3})$ has the same K-type as ${}^t(\Phi_0,\Phi_1,\Phi_2)$.

Lemma (6.1) *If we use the coordinate $(y_1,y_2) = (a_1/a_2, a_1a_2^2)$ for $a = \mathrm{diag}(a_1,a_2,1/(a_1a_2)) \in A$, the Iwasawa decomposition $s_0^{-1}na = n(s_0^{-1}na)a(s_0^{-1}na)k(s_0^{-1}na)$ is described as follows.*

$$a(s_0^{-1}na) = \left(\frac{y_1^{\frac{1}{3}}y_2^{\frac{2}{3}}}{\sqrt{\Delta_1}}, \left(\frac{y_2}{y_1}\right)^{\frac{1}{3}}\sqrt{\frac{\Delta_1}{\Delta_2}}\right),$$

$$n(s_0^{-1}na) = \begin{pmatrix} 1 & (y_1^2 n_2 + n_1(n_1n_2 - n_3))/\Delta_1 & n_3/\Delta_2 \\ 0 & 1 & n_1(y_2^2 + n_3)/\Delta_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and $k(s_0^{-1}na)$ *is*

$$\frac{1}{\sqrt{\Delta_1\Delta_2}} \begin{pmatrix} \sqrt{\Delta_2}(n_3 - n_1n_2) & \sqrt{\Delta_2}y_1n_2 & -\sqrt{\Delta_2}y_1y_2 \\ y_1y_2^2n_1 + y_1n_2n_3 & n_3(n_1n_2 - n_3) - y_1^2y_2^2 & y_2n_1(n_3 - n_1n_2) - y_1^2y_2n_2 \\ -\sqrt{\Delta_1}y_1y_2 & -\sqrt{\Delta_1}y_2n_1 & -\sqrt{\Delta_1}n_3 \end{pmatrix},$$

with

$$\Delta_1 = y_1^2y_2^2 + y_1^2n_2^2 + (n_1n_2 - n_3)^2, \quad \Delta_2 = y_1^2y_2^2 + y_2^2n_1^2 + n_3^2.$$

Proof. Direct computations. □

Under the symbol of the lemma

$$J_{ij}(y) = y_1^{(2\nu_1-\nu_2)/3+1} y_2^{(\nu_1+\nu_2)/3+1} \cdot \int_{\mathbf{R}^3} \Delta_1^{(\nu_2-\nu_1-1)/2} \Delta_2^{(-\nu_2-1)/2} k_{ij} \exp\bigl(-2\pi\sqrt{-1}(c_1n_1 + c_2n_2)\bigr)\, dn_1dn_2dn_3.$$

Here $(k_{ij})_{1\leq i,j\leq 3} = k(s_0^{-1}na)$.

6.2 Integral representations of Jacquet integrals

To write down our results, we use the following notation.

Notation.

$$K(\alpha,\beta,\gamma,\delta\,;y) := 4\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}(y_1y_2)(\pi|c_1|y_1)^{\frac{\lambda_2}{8}}(\pi|c_2|y_2)^{-\frac{\lambda_2}{8}} \cdot \int_0^\infty K_{\frac{\lambda_3-\lambda_1}{8}+\alpha}(2\pi|c_1|y_1\sqrt{1+1/v}) K_{\frac{\lambda_3-\lambda_1}{8}+\beta}(2\pi|c_2|y_2\sqrt{1+v})\, v^{-\frac{3}{16}\lambda_2+\gamma}(1+v)^{\delta}\,\frac{dv}{v}$$

with $K_\nu(z)$ the K-Bessel function.

6.2.1 The case of the class one principal series

In the case of class one, the Jacquet integral $J_0(y)$ is

$$J_0(y) = y_1^{(2\nu_1-\nu_2)/3+1} y_2^{(\nu_1+\nu_2)/3+1} \cdot \int_{\mathbf{R}^3} \Delta_1^{(\nu_2-\nu_1-1)/2} \Delta_2^{(-\nu_2-1)/2} \exp\bigl(-2\pi\sqrt{-1}(c_1n_1 + c_2n_2)\bigr)\, dn_1dn_2dn_3.$$

Theorem (6.2) ([15]) *For* $\mathrm{Re}(\lambda_2 - \lambda_1) > 0$, $\mathrm{Re}(\lambda_3 - \lambda_2) > 0$,

$$J_0(y) = \frac{1}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})} K(0,0,0,0\,;y).$$

6.2.2 The case of the non-spherical principal series

Theorem (6.3) *For* $\mathrm{Re}(\lambda_2-\lambda_1)>0$, $\mathrm{Re}(\lambda_3-\lambda_2)>0$, *the Jacquet integrals* J_{ij} *can be written as follows.*

$$\begin{pmatrix} J_{11}(y) \\ J_{12}(y) \\ J_{13}(y) \end{pmatrix} = \frac{(\pi|c_1|)^{\frac12}(\pi|c_2|)^{\frac12}(\pi|c_1|y_1)^{\frac12}(\pi|c_2|y_2)^{\frac12}}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac12)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \cdot \begin{pmatrix} \varepsilon_1\varepsilon_2 K(-\frac12,-\frac12,\frac14,0\,;y) \\ -\sqrt{-1}\,\varepsilon_2 K(\frac12,-\frac12,-\frac14,\frac12\,;y) \\ -K(\frac12,\frac12,-\frac14,0\,;y) \end{pmatrix},$$

$$\begin{pmatrix} J_{21}(y) \\ J_{22}(y) \\ J_{23}(y) \end{pmatrix} = \frac{1}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac12)} \cdot \begin{pmatrix} -\sqrt{-1}\,\varepsilon_1 K(0,0,-\frac12,0\,;y) \\ -K(0,0,\frac12,-1\,;y) \\ \sqrt{-1}\,\varepsilon_2 K(0,0,\frac12,0\,;y) \end{pmatrix},$$

$$\begin{pmatrix} J_{31}(y) \\ J_{32}(y) \\ J_{33}(y) \end{pmatrix} = \frac{(\pi|c_1|)^{\frac12}(\pi|c_2|)^{\frac12}(\pi|c_1|y_1)^{\frac12}(\pi|c_2|y_2)^{\frac12}}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac12)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \cdot \begin{pmatrix} -K(\frac12,\frac12,\frac14,0\,;y) \\ \sqrt{-1}\,\varepsilon_1 K(-\frac12,\frac12,-\frac14,\frac12\,;y) \\ \varepsilon_1\varepsilon_2 K(-\frac12,-\frac12,-\frac14,0\,;y) \end{pmatrix}.$$

Here ε_i $(i=1,2)$ *means* 1 *if* $c_i>0$ *and* -1 *if* $c_i<0$.

6.3 Preparation for the proof of theorem

Let us start the most computational subsection in this paper. We want to prove the following lemma.

Lemma (6.4) *Let*

$$\begin{aligned} I(\alpha,\beta,\gamma) &= I(\alpha,\beta,\gamma\,;\nu,\mu\,;y) \\ &:= \int_{\mathbf{R}^3} n_1^{\alpha} n_2^{\beta} n_3^{\gamma} \Delta_1^{\nu}\Delta_2^{\mu} \exp\bigl(-2\pi\sqrt{-1}(c_1n_1+c_2n_2)\bigr)\, dn_1dn_2dn_3, \end{aligned}$$

with $\alpha,\beta,\gamma\in\mathbf{Z}$, $\nu,\mu\in\mathbf{C}$ *and put*

$$\begin{aligned} A(\alpha,\beta,\gamma,\delta\,;y) &= \frac{4\pi^{\frac32}(\pi^2|c_1c_2|)^{-\nu-\mu-1}(y_1y_2)^{\nu+\mu}}{\Gamma(-\nu)\Gamma(-\mu)\Gamma(-\nu-\mu-\frac12)} \\ &\cdot \int_0^\infty K_{\nu+\mu+\alpha}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\nu+\mu+\beta}(2\pi|c_2|y_2\sqrt{1+v})\, v^{\frac{\nu-\mu}{2}+\gamma}(1+v)^{\delta}\,\frac{dv}{v}. \end{aligned}$$

For $\mathrm{Re}(\nu)<-\frac12$ *and* $\mathrm{Re}(\mu)<0$,

(1) $I(0,0,0)=y_1y_2\,A(1,1,0,0\,;y)$,

(2) $I(1,0,0)=-\sqrt{-1}\,\varepsilon_1\,y_1^2y_2\,A(2,1,-\frac12,\frac12\,;y)$,

(3) $I(0,1,0)=-\sqrt{-1}\,\varepsilon_2\,y_1y_2^2\,A(1,2,0,\frac12\,;y)$,

(4) $I(0,0,1)=-\varepsilon_1\varepsilon_2\,y_1^2y_2^2\,A(2,2,-\frac12,0\,;y)$,

(5) $I(1,1,0)=-\varepsilon_1\varepsilon_2\,y_1^2y_2^2\,A(2,2,-\frac12,1\,;y)$,

(6) $I(1,0,1)=\sqrt{-1}\,\varepsilon_2\bigl\{(\pi|c_1|)^{-1}(\nu+\mu+\frac32)y_1^2y_2^2\,A(2,2,-\frac12,0\,;y)+y_1^3y_2^2\,A(1,2,-1,\frac12\,;y)\bigr\}$,

(7) $I(0,1,1)=-\sqrt{-1}\,\varepsilon_1\bigl\{(2\pi|c_2|)^{-1}y_1^2y_2^2\,A(2,2,-\frac12,0\,;y)-y_1^2y_2^3\,A(2,3,-\frac12,\frac12\,;y)\bigr\}$,

(8) $I(2,1,0)=\sqrt{-1}\,\varepsilon_2\bigl\{(\pi|c_1|)^{-1}(\nu+\mu+\frac32)y_1^2y_2^2\,A(2,2,-\frac12,1\,;y)+y_1^3y_2^2\,A(1,2,-1,\frac32\,;y)\bigr\}$,

(9) $I(0,0,2)-I(1,1,1)=(2\pi|c_2|)^{-1}y_1^3y_2^2\,A(1,2,0,-\frac12\,;y)-y_1^3y_2^3\,A(1,3,0,0\,;y)$
$\qquad -(\pi|c_1|)^{-1}(\nu+\mu+\frac32)y_1^2y_2^3\,A(2,3,\frac12,-\frac12\,;y)$.

Proof. Our calculation is similar to [15]. We first collect some Fourier transforms which will be used in the proof of lemma (cf. [4]).

Auxiliary formulae *For* $a > 0, b > 0, c \in \mathbf{R}$ *and* $\nu \in \mathbf{C}$,

$$\text{(i)} \int_{\mathbf{R}} (ax^2+b)^{\nu} \exp(-2\pi\sqrt{-1}cx)\, dx = \frac{2a^{\frac{2\nu-1}{4}} b^{\frac{2\nu+1}{4}} |c|^{-\nu-\frac{1}{2}}}{\pi^{\nu}\,\Gamma(-\nu)} K_{\nu+\frac{1}{2}}\Big(2\pi|c|\sqrt{\frac{b}{a}}\Big),$$

([4, chap I, 1.3 (7)])

$$\text{(ii)} \int_{\mathbf{R}} x(ax^2+b)^{\nu} \exp(-2\pi\sqrt{-1}cx)\, dx = \frac{2a^{\frac{2\nu-3}{4}} b^{\frac{2\nu+3}{4}} |c|^{-\nu-\frac{1}{2}}}{\sqrt{-1}\,\pi^{\nu}\,\Gamma(-\nu)} \operatorname{sgn}(c) K_{\nu+\frac{3}{2}}\Big(2\pi|c|\sqrt{\frac{b}{a}}\Big),$$

([4, chap II, 2.3 (11)])

$$\text{(iii)} \int_{\mathbf{R}} \exp(-ax^2+2\pi\sqrt{-1}cx)\, dx = \Big(\frac{\pi}{a}\Big)^{\frac{1}{2}} \exp\Big(-\frac{c^2\pi^2}{a}\Big),$$

$$\text{(iv)} \int_{\mathbf{R}} x\exp(-ax^2+2\pi\sqrt{-1}cx)\, dx = \sqrt{-1}\,c\Big(\frac{\pi}{a}\Big)^{\frac{3}{2}} \exp\Big(-\frac{c^2\pi^2}{a}\Big),$$

$$\text{(v)} \int_{\mathbf{R}} x^2\exp(-ax^2+2\pi\sqrt{-1}cx)\, dx = \frac{a\pi^{\frac{1}{2}}-2c^2\pi^{\frac{5}{2}}}{2a^{\frac{5}{2}}} \exp\Big(-\frac{c^2\pi^2}{a}\Big).$$

Here the conditions on the parameters are (i) $\mathrm{Re}(\nu) < 0$ *and* (ii) $\mathrm{Re}(\nu) < -\frac{1}{2}$.

Now we return to the proof of lemma. The change of variables $(n_1, n_3) \mapsto (n_1y_1, n_3y_1y_2)$, induces the replacement

$$\begin{cases} \Delta_1 \mapsto y_1^2(1+n_1^2)\big(n_2 - \frac{y_2n_1n_3}{1+n_1^2}\big)^2 + \frac{y_1^2y_2^2(1+n_1^2+n_3^2)}{1+n_1^2}, \\ \Delta_2 \mapsto y_1^2y_2^2(1+n_1^2+n_3^2). \end{cases}$$

Then $I(\alpha,\beta,\gamma)$ is equal to

$$\int_{\mathbf{R}^3} (n_1y_1)^{\alpha}(n_3y_1y_2)^{\gamma}\big(y_1^2y_2^2(1+n_1^2+n_3^2)\big)^{\mu} y_1^2y_2 \exp\Big(-2\pi\sqrt{-1}\big(c_1y_1n_1 + \frac{c_2y_2n_1n_3}{1+n_1^2}\big)\Big)$$
$$\cdot\Big(n_2 + \frac{y_2n_1n_3}{1+n_1^2}\Big)^{\beta}\Big(y_1^2(1+n_1^2)n_2^2 + \frac{y_1^2y_2^2(1+n_1^2+n_3^2)}{1+n_1^2}\Big)^{\nu} \exp(-2\pi\sqrt{-1}c_2n_2)dn_1dn_2dn_3.$$

After the integration with respect to n_2 by using (i) and (ii) of Auxiliary formulae, we get

$$I(\alpha,0,\gamma) = \frac{2\,|c_2|^{-\nu-\frac{1}{2}}}{\pi^{\nu}\Gamma(-\nu)}\, y_1^{2\nu+2\mu+2+\alpha+\gamma} y_2^{\nu+2\mu+\frac{3}{2}+\gamma}$$
$$\cdot\int_{\mathbf{R}^2} n_1^{\alpha}n_3^{\gamma}(1+n_1^2)^{-\frac{1}{2}}(1+n_1^2+n_3^2)^{\mu+\frac{2\nu+1}{4}} \exp\Big(-2\pi\sqrt{-1}\big(c_1y_1n_1 + \frac{c_2y_2n_1n_3}{1+n_1^2}\big)\Big)$$
$$\cdot K_{\nu+\frac{1}{2}}\Big(2\pi|c_2|y_2\frac{\sqrt{1+n_1^2+n_3^2}}{1+n_1^2}\Big)dn_1dn_3,$$

$$I(\alpha,1,\gamma) = \frac{2\,|c_2|^{-\nu-\frac{1}{2}}}{\pi^{\nu}\Gamma(-\nu)}\, y_1^{2\nu+2\mu+2+\alpha+\gamma} y_2^{\nu+2\mu+\frac{5}{2}+\gamma}$$
$$\cdot\int_{\mathbf{R}^2} n_1^{\alpha}n_3^{\gamma}(1+n_1^2)^{-\frac{3}{2}} \exp\Big(-2\pi\sqrt{-1}\big(c_1y_1n_1 + \frac{c_2y_2n_1n_3}{1+n_1^2}\big)\Big)$$
$$\cdot\Big[n_1n_3(1+n_1^2+n_3^2)^{\mu+\frac{2\nu+1}{4}} K_{\nu+\frac{1}{2}}\Big(2\pi|c_2|y_2\frac{\sqrt{1+n_1^2+n_3^2}}{1+n_1^2}\Big)$$

$$-\sqrt{-1}\,\varepsilon_2(1+n_1^2+n_3^2)^{\mu+\frac{2\nu+3}{4}}K_{\nu+\frac{3}{2}}\Big(2\pi|c_2|y_2\frac{\sqrt{1+n_1^2+n_3^2}}{1+n_1^2}\Big)\Big]\,dn_1dn_3.$$

for $\mathrm{Re}(\nu)<-\frac{1}{2}$. We modify the K-Bessel function in the above. In view of

$$K_\nu(ct)=\frac{c^\nu}{2}\int_0^\infty \exp\Big(-(x+\frac{c^2}{x})\frac{t}{2}\Big)x^{-\nu}\frac{dx}{x}\qquad (c,t>0),$$
$$c^{-\nu}=\frac{1}{\Gamma(\nu)}\int_0^\infty \exp(-cx)\,x^{\nu}\frac{dx}{x}\qquad (c>0,\ \mathrm{Re}(\nu)>0),$$

we find

$$\begin{aligned}
&(1+n_1^2+n_3^2)^pK_q\Big(\sqrt{1+n_1^2+n_3^2}\cdot\frac{2\pi|c_2|y_2}{1+n_1^2}\Big)\\
&=\frac{1}{2}\Big(\frac{1+n_1^2}{\pi|c_2|y_2}\Big)^{p+\frac{q}{2}}\Big(\pi|c_2|y_2\frac{1+n_1^2+n_3^2}{1+n_1^2}\Big)^{p+\frac{q}{2}}\int_0^\infty \exp\Big(-\frac{\pi|c_2|y_2}{1+n_1^2}(t_1+\frac{1+n_1^2+n_3^2}{t_1})\Big)\,t_1^{-q}\frac{dt_1}{t_1}\\
&=\frac{1}{2\,\Gamma(-p-\frac{q}{2})}\Big(\frac{1+n_1^2}{\pi|c_2|y_2}\Big)^{p+\frac{q}{2}}\\
&\qquad\cdot\int_0^\infty\!\!\int_0^\infty \exp\Big(-\pi|c_2|y_2t_2\frac{1+n_1^2+n_3^2}{1+n_1^2}-\frac{\pi|c_2|y_2}{1+n_1^2}(t_1+\frac{1+n_1^2+n_3^2}{t_1})\Big)\,t_1^{-q}t_2^{-p-\frac{q}{2}}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\\
&=\frac{1}{2\,\Gamma(-p-\frac{q}{2})}\Big(\frac{1+n_1^2}{\pi|c_2|y_2}\Big)^{p+\frac{q}{2}}\\
&\qquad\cdot\int_0^\infty\!\!\int_0^\infty \exp\Big(-\pi|c_2|y_2(t_2+\frac{1}{t_1}+\frac{t_1}{1+n_1^2})-\frac{\pi|c_2|y_2}{1+n_1^2}(t_2+\frac{1}{t_1})n_3^2\Big)\,t_1^{-q}t_2^{-p-\frac{q}{2}}\frac{dt_1}{t_1}\frac{dt_2}{t_2}.
\end{aligned}$$

for $\mathrm{Re}(p+\frac{q}{2})<0$. We apply this formula for $(p,q)=(\mu+\frac{2\nu+1}{4},-\nu-\frac{1}{2}),(\mu+\frac{2\nu+3}{4},-\nu-\frac{3}{2})$ (note that $K_\nu(z)=K_{-\nu}(z)$). Then, for $\mathrm{Re}(\mu)<0$,

$$\begin{aligned}
I(\alpha,0,\gamma)=&\frac{\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-\frac{1}{2}}}{\Gamma(-\nu)\,\Gamma(-\mu)}\,y_1^{2\nu+2\mu+2+\alpha+\gamma}y_2^{\nu+\mu+\frac{3}{2}+\gamma}\int_{\mathbf{R}^2}\int_0^\infty\!\!\int_0^\infty n_1^\alpha(1+n_1^2)^{\mu-\frac{1}{2}}\\
&\cdot\exp\Big(-\pi|c_2|y_2(t_2+\frac{1}{t_1}+\frac{t_1}{1+n_1^2})-2\pi\sqrt{-1}c_1y_1n_1\Big)\,n_3^\gamma\\
&\cdot\exp\Big(-\frac{\pi|c_2|y_2}{1+n_1^2}(t_2+\frac{1}{t_1})n_3^2-2\pi\sqrt{-1}\,\frac{c_2y_2n_1n_3}{1+n_1^2}\Big)\,t_1^{\nu+\frac{1}{2}}t_2^{-\mu}\frac{dt_1}{t_1}\frac{dt_2}{t_2}dn_1dn_3,
\end{aligned}$$

$$\begin{aligned}
I(\alpha,1,\gamma)=&\frac{\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-\frac{1}{2}}}{\Gamma(-\nu)\,\Gamma(-\mu)}\,y_1^{2\nu+2\mu+2+\alpha+\gamma}y_2^{\nu+\mu+\frac{5}{2}+\gamma}\int_{\mathbf{R}^2}\int_0^\infty\!\!\int_0^\infty n_1^\alpha(1+n_1^2)^{\mu-\frac{3}{2}}\\
&\cdot\exp\Big(-\pi|c_2|y_2(t_2+\frac{1}{t_1}+\frac{t_1}{1+n_1^2})-2\pi\sqrt{-1}c_1y_1n_1\Big)\,n_3^\gamma(n_1n_3-\sqrt{-1}\,\varepsilon_2t_1)\\
&\cdot\exp\Big(-\frac{\pi|c_2|y_2}{1+n_1^2}(t_2+\frac{1}{t_1})n_3^2-2\pi\sqrt{-1}\,\frac{c_2y_2n_1n_3}{1+n_1^2}\Big)\,t_1^{\nu+\frac{1}{2}}t_2^{-\mu}\frac{dt_1}{t_1}\frac{dt_2}{t_2}dn_1dn_3.
\end{aligned}$$

Now we consider the integration with respect to n_3. If we define P_δ ($\delta=0,1,2$) by

$$\int_{\mathbf{R}} n_3^\delta\exp\Big(-\frac{\pi|c_2|y_2}{1+n_1^2}(t_2+\frac{1}{t_1})n_3^2-2\pi\sqrt{-1}\,\frac{c_2y_2n_1n_3}{1+n_1^2}\Big)dn_3=P_\delta\,\exp\Big(-\frac{\pi n_1^2|c_2|y_2}{(1+n_1^2)(t_2+1/t_1)}\Big),$$

then (iii) (iv) and (v) of Auxiliary formulae lead

$$\begin{cases}
P_0=(|c_2|y_2)^{-\frac{1}{2}}(1+n_1^2)^{\frac{1}{2}}(t_2+1/t_1)^{-\frac{1}{2}},\\
P_1=-\sqrt{-1}\varepsilon_2(|c_2|y_2)^{-\frac{1}{2}}n_1(1+n_1^2)^{\frac{1}{2}}(t_2+1/t_1)^{-\frac{3}{2}},\\
P_2=(2\pi)^{-1}(|c_2|y_2)^{-\frac{3}{2}}(1+n_1^2)^{\frac{3}{2}}(t_2+1/t_1)^{-\frac{3}{2}}-(|c_2|y_2)^{-\frac{1}{2}}n_1^2(1+n_1^2)^{\frac{1}{2}}(t_2+1/t_1)^{-\frac{5}{2}}.
\end{cases}$$

Therefore we find the following.

$$I(\alpha,0,0) = \frac{\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+2+\alpha} y_2^{\nu+\mu+1}$$
$$\cdot \int_{\mathbf{R}}\int_0^\infty\int_0^\infty n_1^\alpha (1+n_1^2)^\mu (t_2+1/t_1)^{-\frac{1}{2}} t_1^{\nu+\frac{1}{2}} t_2^{-\mu}$$
$$\cdot \exp\Big(-\pi|c_2|y_2\big(t_2+\frac{1}{t_1}+\frac{1}{t_2+1/t_1}+\frac{1}{1+n_1^2}\frac{t_1t_2}{t_2+1/t_1}\big) - 2\pi\sqrt{-1}c_1y_1n_1\Big)\frac{dt_1}{t_1}\frac{dt_2}{t_2}dn_1,$$

$$I(\alpha,0,1) = -\frac{\sqrt{-1}\,\varepsilon_2\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+3+\alpha} y_2^{\nu+\mu+2}$$
$$\cdot \int_{\mathbf{R}}\int_0^\infty\int_0^\infty n_1^{\alpha+1} (1+n_1^2)^\mu (t_2+1/t_1)^{-\frac{3}{2}} t_1^{\nu+\frac{1}{2}} t_2^{-\mu}$$
$$\cdot \exp\Big(-\pi|c_2|y_2\big(t_2+\frac{1}{t_1}+\frac{1}{t_2+1/t_1}+\frac{1}{1+n_1^2}\frac{t_1t_2}{t_2+1/t_1}\big) - 2\pi\sqrt{-1}c_1y_1n_1\Big)\frac{dt_1}{t_1}\frac{dt_2}{t_2}dn_1,$$

$$I(\alpha,1,0) = \frac{\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+2+\alpha} y_2^{\nu+\mu+2} \int_{\mathbf{R}}\int_0^\infty\int_0^\infty n_1^\alpha (1+n_1^2)^{\mu-1}$$
$$\cdot \varepsilon_2\big[-\sqrt{-1}n_1^2(t_2+1/t_1)^{-\frac{3}{2}} t_1^{\nu+\frac{1}{2}} - \sqrt{-1}(t_2+1/t_1)^{-\frac{1}{2}} t_1^{\nu+\frac{3}{2}}\big] t_2^{-\mu}$$
$$\cdot \exp\Big(-\pi|c_2|y_2\big(t_2+\frac{1}{t_1}+\frac{1}{t_2+1/t_1}+\frac{1}{1+n_1^2}\frac{t_1t_2}{t_2+1/t_1}\big) - 2\pi\sqrt{-1}c_1y_1n_1\Big)\frac{dt_1}{t_1}\frac{dt_2}{t_2}dn_1$$
$$= -\frac{\sqrt{-1}\,\varepsilon_2\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+2+\alpha} y_2^{\nu+\mu+2}$$
$$\cdot \int_{\mathbf{R}}\int_0^\infty\int_0^\infty n_1^\alpha (1+n_1^2)^{\mu-1} (t_2+1/t_1)^{-\frac{3}{2}} t_1^{\nu+\frac{1}{2}} t_2^{-\mu}(1+n_1^2+t_1t_2)$$
$$\cdot \exp\Big(-\pi|c_2|y_2\big(t_2+\frac{1}{t_1}+\frac{1}{t_2+1/t_1}+\frac{1}{1+n_1^2}\frac{t_1t_2}{t_2+1/t_1}\big) - 2\pi\sqrt{-1}c_1y_1n_1\Big)\frac{dt_1}{t_1}\frac{dt_2}{t_2}dn_1,$$

$$I(\alpha,1,1) = \frac{\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+3+\alpha} y_2^{\nu+\mu+3} \int_{\mathbf{R}}\int_0^\infty\int_0^\infty n_1^\alpha$$
$$\cdot\Big[(2\pi|c_2|y_2)^{-1} n_1(1+n_1^2)^\mu (t_2+1/t_1)^{-\frac{3}{2}} t_1^{\nu+\frac{1}{2}}$$
$$-n_1^3(1+n_1^2)^{\mu-1}(t_2+1/t_1)^{-\frac{5}{2}} t_1^{\nu+\frac{1}{2}} - n_1(1+n_1^2)^{\mu-1}(t_2+1/t_1)^{-\frac{3}{2}} t_1^{\nu+\frac{3}{2}}\Big] t_2^{-\mu}$$
$$\cdot \exp\Big(-\pi|c_2|y_2\big(t_2+\frac{1}{t_1}+\frac{1}{t_2+1/t_1}+\frac{1}{1+n_1^2}\frac{t_1t_2}{t_2+1/t_1}\big) - 2\pi\sqrt{-1}c_1y_1n_1\Big)\frac{dt_1}{t_1}\frac{dt_2}{t_2}dn_1.$$

We change the variables $(t_1,t_2)\mapsto(v_1,v_2)$ by

$$v_1 = t_2+\frac{1}{t_1},\quad v_2 = \frac{t_1t_2}{1+n_1^2}.$$

Then

$$t_1 = \frac{1+(1+n_1^2)v_2}{v_1},\quad t_2 = \frac{(1+n_1^2)v_1v_2}{1+(1+n_1^2)v_2}\quad\text{and}\quad \frac{dt_1dt_2}{t_1t_2} = \frac{dv_1dv_2}{v_1v_2}.$$

Further we integrate with respect to v_1 by using

$$\int_0^\infty v_1^{-p}\exp\Big(-\pi|c_2|y_2\big(v_1+\frac{1+v_2}{v_1}\big)\Big)\frac{dv_1}{v_1} = 2(1+v_2)^{-\frac{p}{2}}K_p(2\pi|c_2|y_2\sqrt{1+v_2}).$$

Therefore,

$$
\begin{aligned}
I(\alpha,0,0) &= \frac{\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+2+\alpha} y_2^{\nu+\mu+1} \\
&\quad \cdot \int_{\mathbf{R}}\int_0^\infty\int_0^\infty n_1^{\alpha} v_2^{-\mu}\big(1+(1+n_1^2)v_2\big)^{\nu+\mu+\frac{1}{2}} \\
&\qquad \cdot v_1^{-\nu-\mu-1}\exp\Big(-\pi|c_2|y_2\big(v_1+\frac{1+v_2}{v_1}\big)-2\pi\sqrt{-1}c_1y_1n_1\Big)\frac{dv_1}{v_1}\frac{dv_2}{v_2}dn_1 \\
&= \frac{2\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+2+\alpha} y_2^{\nu+\mu+1} \\
&\quad \cdot \int_{\mathbf{R}}\int_0^\infty n_1^{\alpha} v_2^{-\mu}\big(1+(1+n_1^2)v_2\big)^{\nu+\mu+\frac{1}{2}}(1+v_2)^{\frac{-\nu-\mu-1}{2}} \\
&\qquad \cdot K_{\nu+\mu+1}(2\pi|c_2|y_2\sqrt{1+v_2})\exp(-2\pi\sqrt{-1}c_1y_1n_1)\frac{dv_2}{v_2}dn_1,
\end{aligned}
$$

$$
\begin{aligned}
I(\alpha,0,1) &= -\frac{\sqrt{-1}\,\varepsilon_2\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+3+\alpha} y_2^{\nu+\mu+2} \\
&\quad \cdot \int_{\mathbf{R}}\int_0^\infty\int_0^\infty n_1^{\alpha+1} v_2^{-\mu}\big(1+(1+n_1^2)v_2\big)^{\nu+\mu+\frac{1}{2}} \\
&\qquad \cdot v_1^{-\nu-\mu-2}\exp\Big(-\pi|c_2|y_2\big(v_1+\frac{1+v_2}{v_1}\big)-2\pi\sqrt{-1}c_1y_1n_1\Big)\frac{dv_1}{v_1}\frac{dv_2}{v_2}dn_1 \\
&= -\frac{2\sqrt{-1}\,\varepsilon_2\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+3+\alpha} y_2^{\nu+\mu+2} \\
&\quad \cdot \int_{\mathbf{R}}\int_0^\infty n_1^{\alpha+1} v_2^{-\mu}\big(1+(1+n_1^2)v_2\big)^{\nu+\mu+\frac{1}{2}}(1+v_2)^{\frac{-\nu-\mu-2}{2}} \\
&\qquad \cdot K_{\nu+\mu+2}(2\pi|c_2|y_2\sqrt{1+v_2})\exp(-2\pi\sqrt{-1}c_1y_1n_1)\frac{dv_2}{v_2}dn_1,
\end{aligned}
$$

$$
\begin{aligned}
I(\alpha,1,0) &= -\frac{\sqrt{-1}\,\varepsilon_2\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+2+\alpha} y_2^{\nu+\mu+2} \\
&\quad \cdot \int_{\mathbf{R}}\int_0^\infty\int_0^\infty n_1^{\alpha} v_2^{-\mu}(1+v_2)\big(1+(1+n_1^2)v_2\big)^{\nu+\mu+\frac{1}{2}} \\
&\qquad \cdot v_1^{-\nu-\mu-2}\exp\Big(-\pi|c_2|y_2\big(v_1+\frac{1+v_2}{v_1}\big)-2\pi\sqrt{-1}c_1y_1n_1\Big)\frac{dv_1}{v_1}\frac{dv_2}{v_2}dn_1 \\
&= -\frac{2\sqrt{-1}\,\varepsilon_2\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+2+\alpha} y_2^{\nu+\mu+2} \\
&\quad \cdot \int_{\mathbf{R}}\int_0^\infty n_1^{\alpha} v_2^{-\mu}\big(1+(1+n_1^2)v_2\big)^{\nu+\mu+\frac{1}{2}}(1+v_2)^{\frac{-\nu-\mu}{2}} \\
&\qquad \cdot K_{\nu+\mu+2}(2\pi|c_2|y_2\sqrt{1+v_2})\exp(-2\pi\sqrt{-1}c_1y_1n_1)\frac{dv_2}{v_2}dn_1,
\end{aligned}
$$

$$
\begin{aligned}
I(\alpha,1,1) &= \frac{\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+3+\alpha} y_2^{\nu+\mu+3} \\
&\quad \cdot \int_{\mathbf{R}}\int_0^\infty\int_0^\infty n_1^{\alpha+1}\Big[(2\pi|c_2|y_2)^{-1}v_1^{-\nu-\mu-2}v_2^{-\mu}\big(1+(1+n_1^2)v_2\big)^{\nu+\mu+\frac{1}{2}} \\
&\qquad\qquad -v_1^{-\nu-\mu-3}v_2^{-\mu}(1+v_2)\big(1+(1+n_1^2)v_2\big)^{\nu+\mu+\frac{1}{2}}\Big] \\
&\qquad \cdot\exp\Big(-\pi|c_2|y_2\big(v_1+\frac{1+v_2}{v_1}\big)-2\pi\sqrt{-1}c_1y_1n_1\Big)\frac{dv_1}{v_1}\frac{dv_2}{v_2}dn_1
\end{aligned}
$$

$$= \frac{2\pi^{-\nu-\mu}|c_2|^{-\nu-\mu-1}}{\Gamma(-\nu)\,\Gamma(-\mu)}\, y_1^{2\nu+2\mu+3+\alpha} y_2^{\nu+\mu+3}$$
$$\cdot \int_{\mathbf{R}} \int_0^\infty n_1^{\alpha+1} v_2^{-\mu} \big(1 + (1+n_1^2)v_2\big)^{\nu+\mu+\frac{1}{2}}$$
$$\cdot \Big[(2\pi|c_2|y_2)^{-1}(1+v_2)^{\frac{-\nu-\mu-2}{2}} K_{\nu+\mu+2}(2\pi|c_2|y_2\sqrt{1+v_2}) - (1+v_2)^{\frac{-\nu-\mu-1}{2}}$$
$$\cdot K_{\nu+\mu+3}(2\pi|c_2|y_2\sqrt{1+v_2}) \Big] \exp(-2\pi\sqrt{-1}c_1y_1n_1)\frac{dv_2}{v_2}dn_1.$$

Finally, we integrate with respect to n_1. By means of (i) and (ii) of Auxiliary formulae,

$$Q_\delta = \int_{\mathbf{R}} n_1^\delta \big(1+(1+n_1^2)v_2\big)^{\nu+\mu+\frac{1}{2}} \exp(-2\pi\sqrt{-1}c_1y_1n_1)dn_1$$

is as follows.

$$Q_0 = \frac{2\pi^{-\nu-\mu-\frac{1}{2}}(|c_1|y_1)^{-\nu-\mu-1}}{\Gamma(-\nu-\mu-\frac{1}{2})}\, v_2^{\frac{\nu+\mu}{2}} (1+v_2)^{\frac{\nu+\mu+1}{2}} K_{\nu+\mu+1}(2\pi|c_1|y_1\sqrt{1+1/v_2}),$$
$$Q_1 = -\frac{2\sqrt{-1}\,\varepsilon_1\pi^{-\nu-\mu-\frac{1}{2}}(|c_1|y_1)^{-\nu-\mu-1}}{\Gamma(-\nu-\mu-\frac{1}{2})}\, v_2^{\frac{\nu+\mu-1}{2}} (1+v_2)^{\frac{\nu+\mu+2}{2}} K_{\nu+\mu+2}(2\pi|c_1|y_1\sqrt{1+1/v_2}),$$
$$Q_2 = \frac{2\pi^{-\nu-\mu-\frac{3}{2}}(|c_1|y_1)^{-\nu-\mu-2}}{\Gamma(-\nu-\mu-\frac{3}{2})}\, v_2^{\frac{\nu+\mu-1}{2}} (1+v_2)^{\frac{\nu+\mu+2}{2}} K_{\nu+\mu+2}(2\pi|c_1|y_1\sqrt{1+1/v_2})$$
$$- \frac{2\pi^{-\nu-\mu-\frac{1}{2}}(|c_1|y_1)^{-\nu-\mu-1}}{\Gamma(-\nu-\mu-\frac{1}{2})}\, v_2^{\frac{\nu+\mu-2}{2}} (1+v_2)^{\frac{\nu+\mu+3}{2}} K_{\nu+\mu+1}(2\pi|c_1|y_1\sqrt{1+1/v_2}).$$

Thus we complete the proof of lemma. □

6.4 Completion of the proof of theorem

The formulae for J_{1j} and J_{3j} are immediate from the lemma. From (2) and (7) of lemma

$$J_{21}(y) = y_1^{(2\nu_1-\nu_2)/3+1} y_2^{(\nu_1+\nu_2)/3+1}\, \frac{4\pi^{\frac{3}{2}}(\pi^2|c_1c_2|)^{\frac{\nu_1}{2}+1}(y_1y_2)^{-\frac{\nu_1}{2}-2}}{\Gamma(\frac{\nu_1-\nu_2+2}{2})\Gamma(\frac{\nu_2+2}{2})\Gamma(\frac{\nu_1+3}{2})} (\sqrt{-1}\,\varepsilon_1 y_1^3 y_2^3)$$
$$\cdot \int_0^\infty K_{-\frac{\nu_1}{2}}(2\pi|c_1|y_1\sqrt{1+1/v}) \Big[-(2\pi|c_2|y_2)^{-1} K_{-\frac{\nu_1}{2}}(2\pi|c_2|y_2\sqrt{1+v})$$
$$+(1+v)^{\frac{1}{2}} \Big(-K_{-\frac{\nu_1}{2}-1}(2\pi|c_2|y_2\sqrt{1+v}) + K_{-\frac{\nu_1}{2}+1}(2\pi|c_2|y_2\sqrt{1+v}) \Big) \Big] v^{\frac{-\nu_1+2\nu_2-2}{4}} \frac{dv}{v}$$

By using the relation

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z) \qquad (*)$$

for the bracket [ ], it becomes

$$\frac{-\nu_1-1}{2\pi|c_2|y_2} K_{-\frac{\nu_1}{2}}(2\pi|c_2|y_2\sqrt{1+v}) = -\frac{\nu_1+1}{2}\frac{1}{\pi|c_2|y_2} K_{\frac{\nu_1}{2}}(2\pi|c_2|y_2\sqrt{1+v}).$$

Thus

$$J_{21}(y) = -\frac{4\sqrt{-1}\,\varepsilon_1\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\nu_1}{2}+1}(\pi|c_2|)^{\frac{\nu_1}{2}}}{\Gamma(\frac{\nu_1-\nu_2+2}{2})\Gamma(\frac{\nu_2+2}{2})\Gamma(\frac{\nu_1+1}{2})}\, y_1^{\frac{\nu_1-2\nu_2}{6}+2} y_2^{\frac{-\nu_1+2\nu_2}{6}+1}$$

$$\cdot \int_0^\infty K_{\frac{\nu_1}{2}}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\nu_1}{2}}(2\pi|c_2|y_2\sqrt{1+v})v^{\frac{-\nu_1+2\nu_2-2}{4}}\frac{dv}{v}.$$

In a similar way, we can prove for $J_{23}(y)$.

Finally we treat $J_{22}(y)$. From (1) and (9) of the lemma, $J_{22}(y)$ is equal to

$$-\frac{4\pi^{\frac{5}{2}}(\pi|c_1|)^{\frac{\nu_1}{2}}(\pi|c_2|)^{\frac{\nu_1}{2}}}{\Gamma(\frac{\nu_1-\nu_2+2}{2})\Gamma(\frac{\nu_2+2}{2})\Gamma(\frac{\nu_1+3}{2})}y_1^{\frac{\nu_1-2\nu_2}{6}+1}y_2^{\frac{-\nu_1+2\nu_2}{6}+1}(\pi|c_1c_2|y_1y_2)$$
$$\cdot \int_0^\infty \Big[-K_{\frac{\nu_1}{2}+1}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\nu_1}{2}+1}(2\pi|c_2|y_2\sqrt{1+v})$$
$$-\frac{1}{2\pi|c_2|y_2\sqrt{1+v}}K_{\frac{\nu_1}{2}+1}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\nu_1}{2}}(2\pi|c_2|y_2\sqrt{1+v})$$
$$-\frac{\nu_1+1}{2\pi|c_1|y_1\sqrt{1+1/v}}K_{\frac{\nu_1}{2}}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\nu_1}{2}-1}(2\pi|c_2|y_2\sqrt{1+v})$$
$$+K_{\frac{\nu_1}{2}+1}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\nu_1}{2}-1}(2\pi|c_2|y_2\sqrt{1+v})\Big]v^{\frac{-\nu_1+2\nu_2}{4}}\frac{dv}{v}.$$

If we use $(*)$ for the integrand, the term in the bracket [ ] is written as

$$-\frac{\nu_1+1}{2\pi|c_2|y_2\sqrt{1+v}}K_{\frac{\nu_1}{2}+1}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\nu_1}{2}}(2\pi|c_2|y_2\sqrt{1+v})$$
$$-\frac{\nu_1+1}{2\pi|c_1|y_1\sqrt{1+1/v}}K_{\frac{\nu_1}{2}}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\nu_1}{2}-1}(2\pi|c_2|y_2\sqrt{1+v}).$$

Then $J_{22}(y)$ becomes

$$-\frac{4\pi^{\frac{5}{2}}(\pi|c_1|)^{\frac{\nu_1}{2}}(\pi|c_2|)^{\frac{\nu_1}{2}}}{\Gamma(\frac{\nu_1-\nu_2+2}{2})\Gamma(\frac{\nu_2+2}{2})\Gamma(\frac{\nu_1+1}{2})}y_1^{\frac{\nu_1-2\nu_2}{6}+1}y_2^{\frac{-\nu_1+2\nu_2}{6}+1}$$
$$\cdot\Big[(|c_1|y_1)\int_0^\infty K_{\frac{\nu_1}{2}+1}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\nu_1}{2}}(2\pi|c_2|y_2\sqrt{1+v})v^{\frac{-\nu_1+2\nu_2}{4}}(1+v)^{-\frac{1}{2}}\frac{dv}{v}$$
$$+(|c_2|y_2)\int_0^\infty K_{\frac{\nu_1}{2}}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\nu_1}{2}-1}(2\pi|c_2|y_2\sqrt{1+v})v^{\frac{-\nu_1+2\nu_2+2}{4}}(1+v)^{-\frac{1}{2}}\frac{dv}{v}\Big]$$
$$=-\frac{4\pi^{\frac{5}{2}}(\pi|c_1|)^{\frac{\nu_1}{2}}(\pi|c_2|)^{\frac{\nu_1}{2}}}{\Gamma(\frac{\nu_1-\nu_2+2}{2})\Gamma(\frac{\nu_2+2}{2})\Gamma(\frac{\nu_1+1}{2})}y_1^{\frac{\nu_1-2\nu_2}{6}+1}y_2^{\frac{-\nu_1+2\nu_2}{6}+1}$$
$$\cdot\int_0^\infty K_{\frac{\nu_1}{2}}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\nu_1}{2}}(2\pi|c_2|y_2\sqrt{1+v})v^{\frac{-\nu_1+2\nu_2+2}{4}}(1+v)^{-1}\frac{dv}{v}.$$

Here we use the formula

$$\int_0^\infty \big\{x_1\sqrt{1+1/v}\,K_{\nu+1}(x_1\sqrt{1+1/v})K_\nu(x_2\sqrt{1+v})$$
$$+x_2\sqrt{1+v}\,K_\nu(x_1\sqrt{1+1/v})K_{\nu-1}(x_2\sqrt{1+v})\big\}v^\gamma(1+v)^{-1}\frac{dv}{v}$$
$$=\int_0^\infty K_\nu(x_1\sqrt{1+1/v})K_\nu(x_2\sqrt{1+v})v^\gamma(1+v)^{-1}\frac{dv}{v},$$

which can be verified by considering the Mellin transform of both sides (cf. Lemma (7.1)). This completes the proof of theorem. □

7 Integral expression of Mellin-Barnes type

As in [12], we consider the Mellin-Barnes integral expression for $J_{ij}(y)$ to find linear relations between Jacquet integrals J_{ij} and power series solutions $\Phi_k^{i,*}$.

Lemma (7.1) *For* $p, q \in \mathbf{C}$,

$$(\pi|c_1|y_1)^p(\pi|c_2|y_2)^q \int_0^\infty K_\alpha(2\pi|c_1|y_1\sqrt{1+1/v})K_\beta(2\pi|c_2|y_2\sqrt{1+v})\, v^\gamma(1+v)^\delta \,\frac{dv}{v}$$
$$= \frac{1}{2^4(2\pi\sqrt{-1})^2}\int_{\rho_1-\sqrt{-1}\infty}^{\rho_1+\sqrt{-1}\infty}\int_{\rho_2-\sqrt{-1}\infty}^{\rho_2+\sqrt{-1}\infty} V_0(s_1,s_2)(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2}ds_1ds_2,$$

with

$$V_0(s_1,s_2) = \frac{\Gamma(\frac{s_1+p+\alpha}{2})\Gamma(\frac{s_1+p-\alpha}{2})\Gamma(\frac{s_1+p+2\gamma}{2})\Gamma(\frac{s_2+q+\beta}{2})\Gamma(\frac{s_2+q-\beta}{2})\Gamma(\frac{s_2+q-2\gamma-2\delta}{2})}{\Gamma(\frac{s_1+s_2+p+q}{2}-\delta)}.$$

Here the lines of integration are taken as to the right of all poles of the integrand.

Proof. Mellin transform of the left hand side is

$$\int_0^\infty \left(\int_0^\infty K_\alpha(2\pi|c_1|y_1\sqrt{1+1/v})(\pi|c_1|y_1)^{p+s_1}\frac{dy_1}{y_1}\right)$$
$$\cdot\left(\int_0^\infty K_\beta(2\pi|c_2|y_2\sqrt{1+v})(\pi|c_2|y_2)^{q+s_1}\frac{dy_2}{y_1}\right)v^\gamma(1+v)^\delta\frac{dv}{v}$$
$$= 2^{-4}\Gamma\left(\tfrac{s_1+p+\alpha}{2}\right)\Gamma\left(\tfrac{s_1+p-\alpha}{2}\right)\Gamma\left(\tfrac{s_2+q+\beta}{2}\right)\Gamma\left(\tfrac{s_2+q-\beta}{2}\right)$$
$$\cdot\int_0^\infty v^{\gamma+(s_1+p)/2}(1+v)^{\delta-(s_1+s_2+p+q)/2}\frac{dv}{v}.$$

Here we use the formula

$$\int_0^\infty K_\nu(ax)\,x^s\frac{dx}{x} = 2^{s-2}a^{-s}\Gamma(\frac{s+\nu}{2})\Gamma(\frac{s-\nu}{2})$$

($\mathrm{Re}(s) > |\mathrm{Re}(\nu)|$, $a > 0$) and then the Mellin inversion formula implies the assertion. □

Proposition (7.2) *Let*

$$M(a_1,a_2,a_3\,;b_1,b_2,b_3\,;c\,;y)$$
$$= \frac{1}{(2\pi\sqrt{-1})^2}\int_{\rho_1-\sqrt{-1}\infty}^{\rho_1+\sqrt{-1}\infty}\int_{\rho_2-\sqrt{-1}\infty}^{\rho_2+\sqrt{-1}\infty} V(s_1,s_2)(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2}ds_1ds_2,$$

with

$$V(s_1,s_2) = \frac{\Gamma(\frac{s_1+a_1-\lambda_1}{2})\Gamma(\frac{s_1+a_2-\lambda_2}{2})\Gamma(\frac{s_1+a_3-\lambda_3}{2})\Gamma(\frac{s_1+b_1+\lambda_1}{2})\Gamma(\frac{s_1+b_2+\lambda_2}{2})\Gamma(\frac{s_1+b_3+\lambda_3}{2})}{\Gamma(\frac{s_1+s_2+c}{2})}.$$

Here the lines of integration are taken as to the right of all poles of the integrand. Then

$$\begin{pmatrix} J_{11}(y)\\ J_{12}(y)\\ J_{13}(y)\end{pmatrix} = \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)}\cdot\begin{pmatrix}\varepsilon_1\varepsilon_2 M(0,1,1\,;1,0,0\,;1\,;y)\\ -\sqrt{-1}\,\varepsilon_2 M(1,0,0\,;1,0,0\,;0\,;y)\\ -M(1,0,0\,;0,1,1\,;1\,;y)\end{pmatrix},$$

$$\begin{pmatrix} J_{21}(y)\\ J_{22}(y)\\ J_{23}(y)\end{pmatrix} = \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})}\cdot\begin{pmatrix}-\sqrt{-1}\,\varepsilon_1 M(1,0,1\,;0,1,0\,;1\,;y)\\ -M(0,1,0\,;0,1,0\,;0\,;y)\\ \sqrt{-1}\,\varepsilon_2 M(0,1,0\,;1,0,1\,;1\,;y)\end{pmatrix},$$

$$\begin{pmatrix} J_{31}(y) \\ J_{32}(y) \\ J_{33}(y) \end{pmatrix} = \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \cdot \begin{pmatrix} -M(1,1,0\,;0,0,1\,;1\,;y) \\ \sqrt{-1}\,\varepsilon_1 M(0,0,1\,;0,0,1\,;0\,;y) \\ \varepsilon_1\varepsilon_2 M(0,0,1\,;1,1,0\,;1\,;y) \end{pmatrix}.$$

Proof. It is obvious from Lemma (7.1). □

Remark. In view of this proposition, we can see the following symmetry for J_{ij} with respect to the parameter $(\lambda_1,\lambda_2,\lambda_3)$. This is natural but is not immediately seen from the formulae for J_{ij} (Theorem 6.3). We denote

$$\tilde{J}_i(\lambda_1,\lambda_2,\lambda_3) = \left(\frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+p_i)\Gamma(\frac{\lambda_3-\lambda_2}{8}+q_i)\Gamma(\frac{\lambda_3-\lambda_1}{8}+r_i)}\right)^{-1} {}^t(J_{i1}(y),J_{i2}(y),J_{i3}(y))$$

with $(p_i,q_i,r_i)=(1,\frac{1}{2},1)\ (i=1),\ (1,1,\frac{1}{2})\ (i=2),\ (\frac{1}{2},1,1)\ (i=3)$. Then

$$\tilde{J}_2(\lambda_1,\lambda_2,\lambda_3) = (-\sqrt{-1})\,\varepsilon_2\tilde{J}_1(\lambda_2,\lambda_1,\lambda_3), \quad \tilde{J}_3(\lambda_1,\lambda_2,\lambda_3) = -\varepsilon_1\varepsilon_2\tilde{J}_1(\lambda_3,\lambda_2,\lambda_1).$$

8 Relation between Jacquet integrals and power series solutions.

8.1 The case of the class one principal series

Theorem (8.1) ([12]) *Let*

$$\Phi^0(\lambda_1,\lambda_2,\lambda_3) = y_1^{-\frac{\lambda_1}{4}}y_2^{\frac{\lambda_3}{4}}\sum_{m_1,m_2\ge 0}\frac{(\frac{\lambda_3-\lambda_1}{8}+1)_{m_1+m_2}(\pi c_1y_1)^{2m_1}(\pi c_2y_2)^{2m_2}}{(\frac{\lambda_3-\lambda_1}{8}+1)_{m_1}(\frac{\lambda_3-\lambda_1}{8}+1)_{m_2}(\frac{\lambda_2-\lambda_1}{8}+1)_{m_1}(\frac{\lambda_3-\lambda_2}{8}+1)_{m_2}}$$

(*Theorem* (5.1)). *Then*

$$J_0(y) = \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})}$$
$$\cdot\sum_{w\in W} w\Big((\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_1-\lambda_2}{8})\Gamma(\tfrac{\lambda_1-\lambda_3}{8})\Gamma(\tfrac{\lambda_3-\lambda_2}{8})\Phi^0(\lambda_1,\lambda_2,\lambda_3)\Big).$$

Here $W\cong\mathfrak{S}_3$ *is the Weyl group of* $SL(3,\boldsymbol{R})$ *and whose elements permute* $(\lambda_1,\lambda_2,\lambda_3)$.

Remark. Hashizume ([6]) studied the linear relation between the Jacquet integrals and power series solutions for class one principal series for arbitrary semisimple Lie group inspired by the work of Harish-Chandra for spherical functions.

8.2 The case of the non-spherical principal series

In the same way of [12] for class one case, we move the lines of Mellin-Barnes integral expression in Proposition (7.2) to the left and sum up the residues at the poles. Then we obtain the following.

Theorem (8.2)

$${}^t(J_{11}(y),J_{12}(y),J_{13}(y)) = \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)}$$
$$\cdot\Big[\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8})\,{}^t(\Phi_0^{1,\mathrm{I}},\Phi_1^{1,\mathrm{I}},\Phi_2^{1,\mathrm{I}})$$

$$+\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8})\,{}^t(\Phi_0^{1,\mathrm{II}},\Phi_1^{1,\mathrm{II}},\Phi_2^{1,\mathrm{II}})$$
$$-\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8})\,{}^t(\Phi_0^{1,\mathrm{III}},\Phi_1^{1,\mathrm{III}},\Phi_2^{1,\mathrm{III}})$$
$$-\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8})\,{}^t(\Phi_0^{1,\mathrm{IV}},\Phi_1^{1,\mathrm{IV}},\Phi_2^{1,\mathrm{IV}})$$
$$-(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8})\,{}^t(\Phi_0^{1,\mathrm{V}},\Phi_1^{1,\mathrm{V}},\Phi_2^{1,\mathrm{V}})$$
$$-(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8})\,{}^t(\Phi_0^{1,\mathrm{VI}},\Phi_1^{1,\mathrm{VI}},\Phi_2^{1,\mathrm{VI}})\Big],$$

$${}^t(J_{21}(y),J_{22}(y),J_{23}(y))=\frac{-\sqrt{-1}\,\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})}$$
$$\cdot\Big[\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8})\,{}^t(\Phi_0^{2,\mathrm{I}},\Phi_1^{2,\mathrm{I}},\Phi_2^{2,\mathrm{I}})$$
$$+\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8})\,{}^t(\Phi_0^{2,\mathrm{II}},\Phi_1^{2,\mathrm{II}},\Phi_2^{2,\mathrm{II}})$$
$$-(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8})\,{}^t(\Phi_0^{2,\mathrm{III}},\Phi_1^{2,\mathrm{III}},\Phi_2^{2,\mathrm{III}})$$
$$-(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8})\,{}^t(\Phi_0^{2,\mathrm{IV}},\Phi_1^{2,\mathrm{IV}},\Phi_2^{2,\mathrm{IV}})$$
$$-\varepsilon_2(\pi|c_1|)^{\frac{-\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8})\,{}^t(\Phi_0^{2,\mathrm{V}},\Phi_1^{2,\mathrm{V}},\Phi_2^{2,\mathrm{V}})$$
$$-\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8})\,{}^t(\Phi_0^{2,\mathrm{VI}},\Phi_1^{2,\mathrm{VI}},\Phi_2^{2,\mathrm{VI}})\Big],$$

$${}^t(J_{31}(y),J_{32}(y),J_{33}(y))=\frac{-\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)}$$
$$\cdot\Big[(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8})\,{}^t(\Phi_0^{3,\mathrm{I}},\Phi_1^{3,\mathrm{I}},\Phi_2^{3,\mathrm{I}})$$
$$+(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8})\,{}^t(\Phi_0^{3,\mathrm{II}},\Phi_1^{3,\mathrm{II}},\Phi_2^{3,\mathrm{II}})$$
$$-\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8})\,{}^t(\Phi_0^{3,\mathrm{III}},\Phi_1^{3,\mathrm{III}},\Phi_2^{3,\mathrm{III}})$$
$$-\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8})\,{}^t(\Phi_0^{3,\mathrm{IV}},\Phi_1^{3,\mathrm{IV}},\Phi_2^{3,\mathrm{IV}})$$
$$-\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8})\,{}^t(\Phi_0^{3,\mathrm{V}},\Phi_1^{3,\mathrm{V}},\Phi_2^{3,\mathrm{V}})$$
$$-\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8})\,{}^t(\Phi_0^{3,\mathrm{VI}},\Phi_1^{3,\mathrm{VI}},\Phi_2^{3,\mathrm{VI}})\Big].$$

Proof. By Proposition (7.2), poles of the integrand of J_{11} are

$$\{(s_1,s_2)=(\tfrac{\lambda_1}{4}-2k_1,-\tfrac{\lambda_2}{4}-2k_2),\ (\tfrac{\lambda_1}{4}-2k_1,-\tfrac{\lambda_3}{4}-2k_2),$$
$$(\tfrac{\lambda_2}{4}-1-2k_1,-\tfrac{\lambda_3}{4}-2k_2),\ (\tfrac{\lambda_3}{4}-1-2k_1,-\tfrac{\lambda_2}{4}-2k_2),$$
$$(\tfrac{\lambda_2}{4}-1-2k_1,-\tfrac{\lambda_1}{4}-1-2k_2),\ (\tfrac{\lambda_3}{4}-1-2k_1,-\tfrac{\lambda_1}{4}-1-2k_2)\mid k_1,k_2\in\mathbf{Z}_{\geq 0}\}.$$

The residue at $(s_1,s_2)=(\frac{\lambda_1}{4}-2k_1,-\frac{\lambda_2}{4}-2k_2)$ is $(\pi|c_1|y_1)^{2k_1-\frac{\lambda_1}{4}}(\pi|c_2|y_2)^{2k_2+\frac{\lambda_2}{4}}$ times

$$\frac{(-1)^{k_1+k_2}}{k_1!k_2!}\frac{\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2}-k_1)\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2}-k_1)\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2}-k_2)\Gamma(\frac{\lambda_3-\lambda_2}{8}-k_2)}{\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2}-k_1-k_2)}.$$

By using

$$\Gamma(a-k)=\Gamma(a)\frac{(-1)^k}{(1-a)_k},\quad \Gamma(a+\tfrac{1}{2}-k)=\Gamma(a+\tfrac{1}{2})\frac{(-1)^k}{(\frac{1}{2}-a)_k},$$

the above is equal to

$$\frac{\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8})\cdot(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{k_1+k_2}}{k_1!k_2!(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{k_1}(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{k_2}(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{k_1}(\frac{\lambda_2-\lambda_3}{8}+1)_{k_2}}.$$

Thus $\sum_{k_1,k_2\geq 0}\mathrm{Res}_{(s_1,s_2)=(\frac{\lambda_1}{4}-2k_1,-\frac{\lambda_2}{4}-2k_2)}$ is

$$(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8})\,\Phi_0^{1,\mathrm{I}}.$$

We see the pole $(s_1,s_2)=(\frac{\lambda_2}{4}-1-2k_1,-\frac{\lambda_3}{4}-2k_2)$ similarly. Since the residue is $(\pi|c_1|y_1)^{2k_1-\frac{\lambda_2}{4}+1}(\pi|c_2|y_2)^{2k_2+\frac{\lambda_3}{4}}$ times

$$\begin{aligned}&\frac{(-1)^{k_1+k_2}}{k_1!k_2!}\frac{\Gamma(\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}-k_1)\Gamma(\frac{\lambda_2-\lambda_3}{8}-k_1)\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2}-k_2)\Gamma(\frac{\lambda_2-\lambda_3}{8}-k_2)}{\Gamma(\frac{\lambda_2-\lambda_3}{8}-k_1-k_2)}\\&=\frac{\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8})\cdot(-1)(\frac{\lambda_3-\lambda_2}{8}+1)_{k_1+k_2}}{k_1!k_2!(\frac{\lambda_3-\lambda_2}{8}+1)_{k_1}(\frac{\lambda_3-\lambda_2}{8}+1)_{k_2}(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{k_1+1}(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{k_2}}\end{aligned}$$

(note $\Gamma(a-\frac{1}{2}-k)=\Gamma(a+\frac{1}{2})(-1)^{k+1}/(\frac{1}{2}-a)_{k+1}$), then $\sum_{k_1,k_2\geq 0}\mathrm{Res}_{(s_1,s_2)=(\frac{\lambda_2}{4}-1-2k_1,-\frac{\lambda_3}{4}-2k_2)}$ is

$$\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8})\,\Phi_1^{1,\mathrm{I}}.$$

Also, the residue at $(s_1,s_2)=(\frac{\lambda_2}{4}-1-2k_1,-\frac{\lambda_1}{4}-1-2k_2)$ is $(\pi|c_1|y_1)^{2k_1-\frac{\lambda_2}{4}+1}(\pi|c_2|y_2)^{2k_2+\frac{\lambda_1}{4}+1}$ times

$$\begin{aligned}&\frac{(-1)^{k_1+k_2}}{k_1!k_2!}\frac{\Gamma(\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}-k_1)\Gamma(\frac{\lambda_2-\lambda_3}{8}-k_1)\Gamma(\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}-k_2)\Gamma(\frac{\lambda_3-\lambda_1}{8}-\frac{1}{2}-k_2)}{\Gamma(\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}-k_1-k_2)}\\&=\frac{\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8})\cdot(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{k_1+k_2+1}}{k_1!k_2!(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{k_1+1}(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{k_2+1}(\frac{\lambda_3-\lambda_2}{8}+1)_{k_1}(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})_{k_2+1}}.\end{aligned}$$

Then $\sum_{k_1,k_2\geq 0}\mathrm{Res}_{(s_1,s_2)=(\frac{\lambda_2}{4}-1-2k_1,-\frac{\lambda_1}{4}-1-2k_2)}$ is

$$\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8})\,\Phi_2^{1,\mathrm{I}}.$$

Therefore we get the relation between J_{11} and $\Phi_k^{1,*}$ and the others can be shown in the same way. □

References

[1] H. Manabe, On Whittaker functions for the principal series representations of $SL(3,\mathbf{R})$ (in Japanese), Master thesis, 2002 March, the University of Tokyo.

[2] D. Bump, Automorphic Forms on $GL(3,\mathbf{R})$, Lecture Notes in Mathematics **1083**, Springer-Verlag Berlin Heidelberg New York Tokyo 1984.

[3] D. Bump and J. Huntley, Unramified Whittaker functions for $GL(3,\mathbf{R})$, J. d'Analyse Mathématique, **65** (1995), 19–44.

[4] A. Erdélyi et al., Table of integral transforms, vol. I, McGraw-Hill, New York, (1954).

[5] R. Goodman and N. R. Wallach, Whittaker vectors and canonical vectors, J. Funct. Anal., **39** (1980), 199-279.

[6] M. Hashizume, Whittaker functions on semisimple Lie groups, Hiroshima Math. J., **12** (1982), 259-293.

[7] R. Howe, T. Umeda, The Capelli identity, the double commutant theorem, and multiplicity-free actions, Math. Ann., **290** (1991), 565-619.

[8] T. Ishii, On principal series Whittaker functions on $Sp(2, \mathbf{R})$, preprint (2002).

[9] H. Jacquet, Fonctions de Whittaker associées aux groupes de Chevalley, Bull. Soc. Math. France, **95** (1967), 243-309.

[10] B. Kostant, On Whittaker Vectors and Representation Theory, Inventions Math. **48** (1978), 101-184.

[11] T. Miyazaki and T. Oda, Principal series Whittaker functions on $Sp(2, \mathbf{R})$, – Explicit formulae of differential equations–, Proc. of the 1993 Workshop, Automorphic Forms and Related Topics, The Pyungsan Institute for Math. Sci., 59-92.

[12] E.Stade, Poincaré series for $GL(3, \mathbf{R})$-Whittaker functions, Duke J. Math., **58** (1989), 695-729.

[13] ———, On explicit formulas for $GL(n, \mathbf{R})$-Whittaker functions, Duke J. Math., **60** (1990), 313-362.

[14] ———, The reciprocal of the beta function and $GL(n, \mathbf{R})$ Whittaker functions. Ann. Inst. Fourier (Grenoble), **44** (1994), no. 1, 93-108.

[15] Vinogradov, A. I. and Tahtadžjan, L. A. , Theory of the Eisenstein series for the group SL(3, R) and its application to a binary problem. I. Fourier expansion of the highest Eisenstein series. (Russian) Analytic number theory and the theory of functions. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **76** (1978), 5-52, 216. =J.Soviet Math. **18** (1982), 294-324.

[16] N. Wallach, Asymptotic expansions of generalized matrix entries of representations of real reductive groups, Lect. Notes in Math. **1024** (1984), 287-369.

[17] E.T. Whittaker and G.N. Watson, A course of modern analysis, the 4-th edition, Cambridge University Press, 1927.

Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-Ku
Tokyo 153-8914, Japan
ishii@ms.u-tokyo.ac.jp
takayuki@ms.u-tokyo.ac.jp

UTMS

2003–22 Koji Nuida: *On centralizers of parabolic subgroups in Coxeter groups.*

2003–23 Sungwhan Kim: *Recovery of a support of a source term in an elliptic equation.*

2003–24 J. Cheng and M. Yamamoto: *Uniqueness in the inverse scattering problem within polygonal obstacles by a single incoming wave.*

2003–25 Mourad Choulli and Masahiro Yamamoto: *Conditional stability in determining a heat source.*

2003–26 Shushi Harashita: *The a-number stratification on the moduli space of supersingular abelian varieties.*

2003–27 Xu Bin: *Derivatives of spectral function and Sobolev norms of eigenfunctions on a closed Riemannian manifold.*

2003–28 Li Shumin and Masahiro Yamamoto: *Inverse source problem for Maxwell's equations in anisotropic media.*

2003–29 Igor Trooshin and Masahiro Yamamoto: *Identification problem for one-dimensional vibrating equation.*

2003–30 Xu Bin: *The degree of symmetry of certain compact smooth manifolds II.*

2003–31 Miki Hirano and Takayuki Oda: *Integral switching engine for special Clebsch-Gordan coefficients for the representations of* $\mathfrak{gl}_3$ *with respect to Gelfand-Zelevinsky basis.*

2003–32 Akihiro Shimomura: *Scattering theory for the coupled Klein-Gordon-Schrödinger equations in two space dimensions.*

2003–33 Hiroyuki Manabe, Taku IshiI, and Takayuki Oda: *Principal series Whittaker functions on* $SL(3, R)$.

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012