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1 Introduction

It is important to compute expectations of diffusion processes numerically, in the case when we apply mathematical finance to practical problems. There are a lot of works in this field (cf.. Ballay and Talay [1], Kloeden and Platen [2]). The author gave a new method in [3] and some related works have already appeared (Lyons and Victoir [7], Ninomiya [8]).

In the present paper, we refine and extend the idea in [3] by using notions in [9]. We use the notation in [9] for free Lie algebra. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{(B^1(t), \ldots, B^d(t); t \in [0,\infty)\}$ be a d-dimensional Brownian motion. Let $B^0(t) = t$, $t \in [0,\infty)$. Let $V_0, V_1, \ldots, V_d \in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$. Here $C_b^\infty(\mathbf{R}^N;\mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined in $\mathbf{R}^N$ whose devivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$ as vector fields on $\mathbf{R}^N$.

Now let $X(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, be the solution to the Stratonovich stochastic integral equation

$$X(t,x) = x + \sum_{i=0}^{d} \int_0^t V_i(X(s,x)) \circ dB^i(s). \tag{1}$$

Then there is a unique solution to this equation. Moreover we may assume that with probability one $X(t,x)$ is continuous in t and smooth in x.

Let $A = A_d = \{v_0, v_1, \ldots, v_d\}$, be an alphabet, a set of letters, and A^* be the set of words consisting of A including the empty word which is denoted by 1. For $u = u^1 \cdots u^k \in A^*$, $u^j \in A$, $j = 1, \ldots, k$, $k \geq 0$, we denote by $n_i(u)$, $i = 0, \ldots, d$, the cardinal of $\{j \in \{1,\ldots,k\}; u^j = v_i\}$. Let $|u| = n_0(u) + \ldots + n_d(u)$, a length of u, and $\| u \| = |u| + n_0(u)$ for $u \in A^*$. Let $\mathbf{R}\langle A\rangle$ be the $\mathbf{R}$-algebra of noncommutative polynomials on A, $\mathbf{R}\langle\langle A\rangle\rangle$ be the $\mathbf{R}$-algebra of noncommutative formal series on A, $\mathcal{L}(A)$ be the free Lie algebra over $\mathbf{R}$ on the set A, and $\mathcal{L}((A))$ be the $\mathbf{R}$ Lie algebra of free Lie series on the set A.

Let ι denotes the left normed bracketing operator, i.e.,

$$\iota(v_{i_1} \cdots v_{i_n}) = [\ldots[v_{i_1}, v_{i_2}], \ldots, v_{i_n}].$$

For any $w_i = \sum_{u \in A^*} a_{iu} u, \in \mathbf{R}\langle A\rangle$, $i = 1,2$, let us define an inner product $\langle w_1, w_2\rangle$ and a norm $\| w_1 \|_2$ by

$$\langle w_1, w_2\rangle = \sum_{u \in A^*} a_{1u} a_{2u} \in \mathbf{R} \text{ and } \| w_1 \|_2 = (\langle w_1, w_1\rangle)^{1/2}.$$

We can regard vector fields $V_0, V_1, \dots, V_d$ as first differential operators over $\mathbf{R}^N$. Let $\mathcal{DO}(\mathbf{R}^N)$ denotes the set of smooth differential operators over $\mathbf{R}^N$. Then $\mathcal{DO}(\mathbf{R}^N)$ is a noncommutative algebra over $\mathbf{R}$. Let $\Phi : \mathbf{R}\langle A\rangle \to \mathcal{DF}(\mathbf{R}^N)$ be a homomorphism given by

$$\Phi(1) = Identity, \qquad \Phi(v_{i_1}\cdots v_{i_n}) = V_{i_1}\cdots V_{i_n}, \qquad n \geq 1,\ i_1,\dots,i_n = 0,1,\dots,d.$$

Then we see that

$$\Phi(\iota(v_{i_1}\cdots v_{i_n})) = [\cdots[V_{i_1}, V_{i_2}],\cdots, V_{i_n}], \qquad n \geq 2,\ i_1,\dots,i_n = 0,1,\dots,d.$$

Let $B(t;u)$, $t \in [0,\infty)$, $u \in A^*$, be inductively defined by

$$B(t;1) = 1, \qquad B(t;v_i) = B^i(t),\ i = 0,1,\dots,d,$$

and

$$B(t;uv_i) = \int_0^t B(s;u)\circ dB^i(s) \qquad u \in A^*,\ i = 0,\dots,d.$$

Also we define $B(t;w)$ $t \in [0,\infty)$, $w \in \mathbf{R}\langle A\rangle$ by

$$B(t;\sum_{u\in A^*} a_u u) = \sum_{u\in A^*} a_u B(t;u).$$

Let $A^*_m = \{u \in A^*;\ \| u \| = m\}$, $m \geq 0$, and let $\mathbf{R}\langle A\rangle_m = \sum_{u\in A^*_m} \mathbf{R}u$, and $\mathbf{R}\langle A\rangle_{\leq m}$ $= \sum_{k=0}^m \mathbf{R}\langle A\rangle_k$, $m \geq 0$. Let $j_m : \mathbf{R}\langle\langle A\rangle\rangle \to \mathbf{R}\langle A\rangle_{\leq m}$ be a natural sujective linear map such that $j_m(u) = u$, $u \in A^*$, $\| u \| \leq m$, and $j_m(u) = 0$, $u \in A^*$, $\| u \| \geq m+1$. Let $\mathcal{L}(A)_m$ $= \mathcal{L}(A) \cap \mathbf{R}\langle A\rangle_m$, and $\mathcal{L}(A)_{\leq m} = \mathcal{L}(A) \cap \mathbf{R}\langle A\rangle_{\leq m}$, $m \geq 1$. Let $A^{**} = \{u \in A^*;\ u \neq 1, v_0\}$, and $A^{**}_{\leq m} = \{u \in A^{**};\ \| u \| \leq m\}$, $m \geq 1$.

Let $\Psi_s : \mathbf{R}\langle\langle A\rangle\rangle \to \mathbf{R}\langle\langle A\rangle\rangle$, $s > 0$, be given by

$$\Psi_s(\sum_{m=0}^\infty x_m) = \sum_{m=0}^\infty s^{m/2} x_m, \qquad x_m \in \mathbf{R}\langle A\rangle_m, \quad m \geq 0.$$

Now we introduce a condition (UFG) on the family of vector field $\{V_0, V_1, \dots, V_d\}$ as follows.

(UFG) There are an integer ℓ and $\varphi_{u,u'} \in C_b^\infty(\mathbf{R}^N)$, $u \in A^{**}$, $u' \in A^{**}_{\leq \ell}$, satisfying the following.

$$\Phi(\iota(u)) = \sum_{u'\in A^{**}_{\leq\ell}} \varphi_{u,u'}\Phi(\iota(u')), \qquad u \in A^{**}.$$

Let us define a semi-norm $\| \cdot \|_{V,n}$, $n \geq 1$, on $C_b^\infty(\mathbf{R}^N;\mathbf{R})$ by

$$\| f \|_{V,n} = \sum_{k=1}^n \sum_{u_1,\dots,u_k\in A^{**}, \|u_1\cdots u_k\| = n} \| \Phi(\iota(u_1)\cdots\iota(u_k))f \|_\infty .$$

Here $\| f \|_\infty = \sup\{|f(x)|;\ x \in \mathbf{R}^N\}$.

Now let us define a semigroup of linear operators $\{P_t\}_{t\in[0,\infty)}$ by

$$(P_t f)(x) = E[f(X(t,x))], \qquad t \in [0,\infty),\ f \in C_b^\infty(\mathbf{R}^N).$$

Let us think of a family $\{Q_{(s)};\ s \in (0,1]\}$ of linear operators in $C_b(\mathbf{R}^N)$.

Definition 1 *We say that* $Q_{(s)}$, $s \in (0,1]$, *is* m*-similar,* $m \geq 1$, *if there are a constant* $C > 0$ *and* $M \geq m+1$ *such that*

$$\| P_s f - Q_{(s)} f \|_\infty \leq C(\sum_{k=m+1}^{M} s^{k/2} \| f \|_{V,k} + s^{(m+1)/2} \| \nabla f \|_\infty),$$

$$\| Q_{(s)} f - P_s f \|_\infty \leq C s^{1/2} \| \nabla f \|_\infty,$$

and

$$\| Q_{(s)} f \|_\infty \leq \exp(Cs) \| f \|_\infty$$

for any $s \in (0,1]$, *and* $f \in C_b^\infty(\mathbf{R}^N;\mathbf{R})$.

Definition 2 (1) *We say that an* $\mathcal{L}((A))$*-valued random variable* ξ *is* $L^{\infty-}$, *if*

$$E[\| j_n(\xi) \|_2^n] < \infty \qquad \text{for any } n \geq 1.$$

(2) *We say that an* $\mathcal{L}((A))$*-valued random variable* ξ *is* m*-*$\mathcal{L}$*-moment similar,* $m \geq 2$, *if* $j_m(\xi)$ *is* $L^{\infty-}$,

$$\langle \xi, v_0 \rangle = 1 \quad a.s.,$$

and if

$$E[j_m(\exp(\xi))] = E[j_m(X(1))].$$

Our main results are the following.

Theorem 3 *Let* $m \geq 1$ *and* ξ *be an* $\mathcal{L}((A))$*-valued* m*-*$\mathcal{L}$*-moment similar random variable. Also, let* $Y : (0,1] \times \mathbf{R}^N \times \Omega \to \mathbf{R}^N$ *be a measurable map such that* $Y(s,\cdot,\omega) : \mathbf{R}^N \to \mathbf{R}^N$ *is continuous for any* $s \in (0,1]$ *and* $\omega \in \Omega$, *and*

$$\sup_{s\in(0,1],x\in\mathbf{R}^N} s^{-(m+1)/2} E[|Y(s,x)|] < \infty.$$

Let us define linear operators $Q_{(s)}$, $s > 0$, *in* $C_b(\mathbf{R}^N)$ *by*

$$(Q_{(s)}f)(x) = E[f(\exp(\Phi(j_m(\Psi_s(\xi))))(x) + Y(s,x))], \qquad f \in C_b(\mathbf{R}^N).$$

Then $\{Q_{(s)};\ s \in (0,1]\}$ *is* m*-similar.*

Theorem 4 *Assume that the family of vector fields satisfies the condition* (UFG). *Let* $m \geq 1$ *and* $Q_{(s)}$, $s > 0$, *be an* m*-similar family of linear operators in* $C_b(\mathbf{R}^N)$. *Also, let* $T > 0$ *and* $\gamma > 0$, $t_k = t_k^{(n)} = \dfrac{k^\gamma T}{n^\gamma}$, $n \geq 1$, $k = 0,1,\ldots,n$, *and let* $s_k = s_k^{(n)} = t_k - t_{k-1}$, $k = 1,\ldots,n$. *Then we have the following.*
For $\gamma \in (0, m-1)$, *there is a constant* $C > 0$ *such that*

$$\| P_T f - Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} f \|_\infty \leq C n^{-\gamma/2} \| \nabla f \|_\infty, \quad f \in C_b^\infty(\mathbf{R}^N),\ n \geq 1.$$

For $\gamma = m-1$, *there is a constant* $C > 0$ *such that*

$$\| P_T f - Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} f \|_\infty \leq C n^{-\frac{m-1}{2}} \log(n+1) \| \nabla f \|_\infty,$$

$$f \in C_b^\infty(\mathbf{R}^N),\ n \geq 1.$$

For $\gamma > m-1$, *there is a constant* $C > 0$ *such that*

$$\| P_T f - Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} f \|_\infty \leq C n^{-\frac{m-1}{2}} \| \nabla f \|_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geq 1.$$

2 Proof of Theorem 4

First, note the following (cf. [4]).

Theorem 5 *Assume that the family of vector fields satisfies the condition* (UFG). *Then for any* $n \geq 2$ *there is a constant* $C > 0$ *such that*

$$\| P_t f \|_{V,n} \leq \frac{C}{t^{(n-1)/2}} \| \nabla f \|_\infty, \quad f \in C_b^\infty(\mathbf{R}^N),\ t \in (0,1].$$

Now let us prove Theorem 4. Note that for $k = 2, \ldots, n$, and $\ell \geq m+1$,

$$\frac{s_k^{\ell/2}}{t_{k-1}^{(\ell-1)/2}} = T^{1/2} \frac{(\int_{k-1}^k \gamma s^{\gamma-1} ds)^{\ell/2}}{n^{\gamma/2}(k-1)^{(\ell-1)\gamma/2}} \leq T^{1/2}\gamma^\ell n^{-\gamma/2}(k-1)^{(\gamma-\ell)/2}((\frac{k}{k-1})^{\gamma-1} \vee 1).$$

So we have

$$\| P_T f - Q_{(s_n)} \cdots Q_{(s_1)} f \|_\infty$$
$$\leq \sum_{k=1}^n \| Q_{(s_n)} \cdots Q_{(s_{k+1})} P_{t_k} f - Q_{(s_n)} \cdots Q_{(s_k)} P_{t_{k-1}} f \|_\infty$$
$$\leq e^{CT} \sum_{k=1}^n \| P_{s_k} P_{t_{k-1}} f - Q_{(s_k)} P_{t_{k-1}} f \|_\infty$$
$$\leq Ce^{CT}(\sum_{k=2}^n(\sum_{\ell=m+1}^M s_k^{\ell/2} \| P_{t_{k-1}} f \|_{V,\ell} + s_k^{(m+1)/2} \| \nabla P_{t_{k-1}} f \|_\infty) + s_1^{(m+1)/2} \| \nabla f \|_\infty)$$
$$\leq C_1(\sum_{k=2}^n(\sum_{\ell=m+1}^M \frac{s_k^{\ell/2}}{t_{k-1}^{(\ell-1)/2}}) + \sum_{k=1}^n s_k^{(m+1)/2}) \| \nabla f \|_\infty$$
$$\leq C_2(n^{-\gamma/2} \sum_{k=2}^n (k-1)^{(\gamma-(m+1))/2} + n^{-(m+1)/2}) \| \nabla f \|_\infty .$$

So we have the assertions in Theorem 4.

3 Algebraic Structure of itterated integrals

We define a metric function dis over $\mathbf{R}\langle\langle A \rangle\rangle$ by

$$dis(w_1, w_2) = \sum_{u \in A^*} (d+2)^{-|u|}(1 \wedge |a_{1,u} - a_{2,u}|)$$

for $w_i = \sum_{u \in A^*} a_{i,u} u$, $i = 1, 2$, $a_{i,u} \in \mathbf{R}$, $u \in A^*$. Then $\mathbf{R}\langle\langle A \rangle\rangle$ becomes a Polish space. Let $\mathcal{B}(\mathbf{R}\langle\langle A \rangle\rangle)$ be a Borel algebra over $\mathbf{R}\langle\langle A \rangle\rangle$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. One can define $\mathbf{R}\langle\langle A \rangle\rangle$-valued random variables and their expectaions etc. naturally. Let $\{\mathcal{F}_t\}_{t \in [0,\infty)}$ be a filtration satisfying a usual hypothesis, $(B^1(t), \ldots, B^d(t))$, $t \in [0, \infty)$, be a d-dimensional $\{\mathcal{F}_t\}_{t \in [0,\infty)}$-Brownian motion, and $B^0(t) = t$, $t \in [0, \infty)$. We say that $X(t)$ is an $\mathbf{R}\langle\langle A \rangle\rangle$-valued continuous semimartingale, if there are continuous semimartingales X_u, $u \in A^*$, such that $X(t) =$

$\sum_{u\in A^*} X_u(t)u$. For $\mathbf{R}\langle\langle A\rangle\rangle$-valued continuous semimartingale $X(t), Y(t)$, we can define $\mathbf{R}\langle\langle A\rangle\rangle$-valued continuous semimartingales $\int_0^t X(s)\circ dY(s)$ and $\int_0^t \circ dX(s)Y(s)$ by

$$\int_0^t X(s)\circ dY(s) = \sum_{u,w\in A^*} (\int_0^t X_u(s)\circ dY_w(s))uw,$$

$$\int_0^t \circ dX(s)Y(s) = \sum_{u,w\in A^*} (\int_0^t Y_w(s)\circ dX_u(s))uw,$$

where

$$X(t) = \sum_{u\in A^*} X_u(t)u, \qquad Y(t) = \sum_{w\in A^*} Y_w(t)w.$$

Then we have

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s)\circ dY(s) + \int_0^t \circ dX(s)Y(s).$$

Since $\mathbf{R}$ is regarded a vector subspace in $\mathbf{R}\langle\langle A\rangle\rangle$, we can define $\int_0^t X(s)\circ dB^i(s)$, $i = 0, 1, \dots, d$, naturally.

Now let us consider the following SDE on $\mathbf{R}\langle\langle A\rangle\rangle$

$$X(t) = 1 + \sum_{i=0}^d \int_0^t X(s)v_i \circ dB^i(s), \qquad t \geq 0. \tag{2}$$

One can easily solve this SDE and obtains

$$X(t) = \sum_{u\in A^*} B(t;u)u.$$

We also have the following.

Proposition 6 $\log X(t) \in \mathcal{L}((A))$, $t \geq 0$, *with probability one.*

Proof. Note that

$$\delta(X(t)) = 1\otimes 1 + \sum_{i=0}^d \int_0^t \delta(X(s))(v_i\otimes 1 + 1\otimes v_i)\circ dB^i(s),$$

and

$$X(t)\otimes X(t) = 1\otimes 1 + \int_0^t \circ d(X(s)\otimes 1)(1\otimes X(s)) + \int_0^t (X(s)\otimes 1)\circ d(1\otimes X(s))$$

$$= 1\otimes 1 + \sum_{i=0}^d \int_0^t X(s)\otimes X(s)(v_i\otimes 1 + 1\otimes v_i)\circ dB^i(s).$$

Here δ is the coproduct (see [9] p.19). Since one can easily see the uniqueness of such SDE on $\mathbf{R}\langle\langle A\rangle\rangle$, we have

$$\delta(X(t)) = X(t)\otimes X(t).$$

Then we have our assertion from [9] Theorem 3.2. ∎

Proposition 7 *For any $m, n \geq 1$, and $x \in \mathcal{L}((A))$ with $\langle x, 1\rangle = 0$,*

$$j_m(\pi_n \exp(x)) = \pi_n(j_m \exp(x)).$$

Here π_n is the canonical projection (see [9] p.57-61).

Proof. Let $x \in \mathcal{L}((A))$ with $\langle x, 1\rangle = 0$. Then there are $x_k \in \mathcal{L}(A)_k$, $k = 1, 2, \ldots$, such that $x = \sum_{k=0}^{\infty} x_k$. Then we see that

$$\exp(x) = 1 + \sum_{\ell=1}^{\infty} \frac{1}{(\ell!)^2} \sum_{k_1,\ldots k_\ell} \sum_{\sigma \in S_\ell} x_{k_{\sigma(1)}} \cdots x_{k_{\sigma(\ell)}}.$$

One can easiely see that

$$j_m(\pi_n(\sum_{\sigma\in S_\ell} x_{k_{\sigma(1)}} \cdots x_{k_{\sigma(\ell)}})) = \pi_n(j_m(\sum_{\sigma\in S_\ell} x_{k_{\sigma(1)}} \cdots x_{k_{\sigma(\ell)}})).$$

So we have our assertion. ∎

Let $E_m = \mathcal{L}(A)_{\leq m} \cap (\sum_{u\in A^{**}} \mathbf{R}u)$, $m \geq 1$, and let $\Phi_m : E_m \to \mathbf{R}\langle A\rangle_{\leq m}$, $m \geq 2$, be an algebraic map given by

$$\Phi_m(x) = j_m(\exp(x + v_0)), \qquad x \in E_m.$$

Then by Proposition 7, we see that

$$\pi_1(\Phi_m(x)) = x + v_0, \qquad x \in E_m.$$

So we see that Φ_m is an immersion and $\Phi_m(E_m)$ is a closed manifold in $\mathbf{R}\langle A\rangle_{\leq m}$ of dimensions *dim* E_m.

Lemma 8 *The distribution of $j_m(\log X(1) - v_0)$ on E_m is absolutely continuous and its density is smooth for any $m \geq 2$.*

Proof. This lemma is somehow well-known in Malliavin calculus, so we give a sketch of a proof only. Let $Y = j_m(\log X(1) - v_0)$. Let H be the Cameron-Martin space of d-dimensional Wiener process, that is, H is the Hilbert space consisting of $h = (h^1, \ldots, h^d) : [0, \infty) \to \mathbf{R}$ such that $h^i(t)$, $i = 1, \ldots, d$, are absolutely continuous in t, and

$$\| h \|_H^2 = \sum_{i=1}^{d} \int_0^\infty |\frac{d}{dt} h^i(t)|^2 dt < \infty.$$

Then we see that for each $h \in H$

$$D(X(t))(h) = \sum_{i=0}^{d} \int_0^t D(X(s))(h) v_i \circ dB^i(s) + \sum_{i=1}^{d} \int_0^t X(s) v_i \frac{d}{ds} h^i(s) ds,$$

and so we have

$$D(X(t))(h) X(t)^{-1} = \sum_{i=1}^{d} \int_0^t X(s) v_i X(s)^{-1} \frac{d}{ds} h^i(s) ds, \qquad t \geq 0.$$

Note that for $w \in \mathbf{R}\langle A\rangle$

$$X(t)wX(t)^{-1} = w + \sum_{i=0}^{d} \int_0^t X(s)[v_i, w]X(s)^{-1} \circ dB^i(s), \qquad t \geq 0.$$

Then we have

$$j_m(D(X(T))(h)X(T)^{-1}) = \sum_{i=1}^{d} \int_0^T \Big(\sum_{u \in \mathbf{R}\langle A\rangle_{\leq m-1}} B(t;u)\ell(uv_i)\Big) \frac{d}{ds} h^i(t)dt, \qquad T \geq 0.$$

Here ℓ is an operator defined in [9] . Then by the usual argument (e.g. [4]), we see that

$$E[\inf\{\| \langle j_m(D(X(1))(\cdot)X(1)^{-1}), w\rangle \|_{H^*};\ w \in E_m, \langle w, w\rangle = 1\}^{-p}] < \infty, \qquad p \in (1, \infty).$$

Note that $j_m(X(1)) = \Phi_m(Y)$. So we have our assertion from Taniguchi [10]. ∎

4 Proof of Theorem 3

For any vector field $V \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$ on $\mathbf{R}^N$, let us think of ODE given by

$$\frac{d}{dt} x(t, x) = V(x(t, x)), \qquad t > 0,$$

$$x(0, x) = x \in \mathbf{R}^N,$$

and let us define a diffeomorphism $\exp(V) : \mathbf{R}^N \to \mathbf{R}^N$ by $\exp(V)(x) = x(1, x)$. Then we have

$$\frac{d}{dt} f(\exp(tV)(x)) = (Vf)(\exp(tV)(x))$$

for any $f \in C^\infty(\mathbf{R}^N)$.

So we have the following.

Proposition 9 *For any vector field $V \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$,*

$$f(\exp(tV)(x)) = \sum_{k=0}^{n} \frac{t^k}{k!} (V^k f)(x) + \int_0^t \frac{(t-s)^n}{n!} (V^{n+1} f)(\exp(sV)(x))ds,$$

for any $n \geq 1$, $t > 0$, $x \in \mathbf{R}^N$ and $f \in C^\infty(\mathbf{R}^N)$. In particular,

$$|f(\exp(V)(x)) - \sum_{k=0}^{n} \frac{1}{k!} (V^k f)(x)| \leq \frac{1}{(n+1)!} \| V^n f \|_\infty,$$

for any $n \geq 1$, $x \in \mathbf{R}^N$ and $f \in C^\infty(\mathbf{R}^N)$.

Corollary 10 *Let $z \in \mathcal{L}((A))$ and $n, m \geq 1$. Then we have*

$$|f(\exp(\Phi(j_m z))(x)) - \sum_{k=0}^{n} \frac{1}{k!} (\Phi((j_m z)^k) f)(x)| \leq \frac{1}{(n+1)!} \| \Phi((j_m z)^{n+1}) f \|_\infty,$$

for any $x \in \mathbf{R}^N$ and $f \in C^\infty(\mathbf{R}^N)$.

Then we have the following.

Lemma 11 *Let* $z_1, z_2 \in \mathcal{L}((A))$ *and* $m \geq 1$. *Then we have*

$$|f(\exp(\Phi(j_m z_1))(\exp(\Phi(j_m z_2))(x)) - \sum_{k+\ell \leq m} \frac{1}{k!\ell!} (\Phi((j_m z_2)^k (j_m z_1)^\ell f)(x)|$$

$$\leq \sum_{\ell=0}^{m} \frac{1}{\ell!(m+1-\ell)!} \parallel \Phi((j_m z)^{m+1-\ell} (j_m z_1)^\ell) f \parallel_\infty,$$

for any $x \in \mathbf{R}^N$ *and* $f \in C^\infty(\mathbf{R}^N)$.

Proof. Note that

$$|f(\exp(\Phi(j_m z_1))(x)) - \sum_{\ell=0}^{m} \frac{1}{\ell!} (\Phi((j_m z_1)^\ell) f)(x)| \leq \frac{1}{(m+1)!} \parallel \Phi((j_m z_1)^{m+1}) f \parallel_\infty,$$

and

$$|(\Phi((j_m z_1)^\ell) f)(\exp(\Phi j_m z_2)(x)) - \sum_{k=0}^{m-\ell} \frac{1}{k!} (\Phi((j_m z_2)^k (j_m z_1)^\ell) f)(x)|$$

$$\leq \frac{1}{(m+1-\ell)!} \parallel \Phi((j_m z_2)^{m+1-\ell} (j_m z_1)^\ell) f \parallel_\infty .$$

Thus we have our assertion. ∎

Corollary 12 *Let* $z_1, z_2 \in \mathcal{L}((A))$ *and* $m \geq 1$. *Then we have*

$$|f(\exp(\Phi(j_m z_1))(\exp(\Phi(j_m z_2))(x)) - (\Phi(j_m(\exp(j_m z_2)\exp(j_m z_1)))f)(x)|$$

$$\leq \sum_{2 \leq k+\ell \leq m+1} \frac{1}{\ell!k!} \parallel \Phi((j_{m^{m+1}} - j_m)((j_m z_2)^k (j_m z_1)^\ell)) f \parallel_\infty,$$

for any $x \in \mathbf{R}^N$ *and* $f \in C^\infty(\mathbf{R}^N)$.

Proof. Note that

$$j_m(\exp(j_m z_2)\exp(j_m z_1))$$

$$= \sum_{k+\ell \leq m} \frac{1}{k!\ell!} (j_m z_2)^k (j_m z_1)^\ell - \sum_{2 \leq k+\ell \leq m} \frac{1}{\ell!k!} (j_{m^{m+1}} - j_m)((j_m z_2)^k (j_m z_1)^\ell),$$

and

$$\sum_{\ell=0}^{m} \frac{1}{\ell!(m+1-\ell)!} (j_m z)^{(m+1-\ell)} (j_m z_1)^\ell$$

$$= \sum_{k+\ell=m+1} \frac{1}{\ell!k!} (j_{m^{m+1}} - j_m)((j_m z_2)^k (j_m z_1)^\ell).$$

So we have our assertion. ∎

Lemma 13 *For any $n \geq 1$, there is a $C_n > 0$ such that*

$$\| \Phi(j_n z) f \|_\infty \leq C_n \| j_n z \|_2 \| \nabla f \|_{C^{n-1}}$$

for any $z \in \mathcal{L}((A))$ and $f \in C^\infty(\mathbf{R}^N)$. Here

$$\| f \|_{C^n} = \| f \|_\infty + \sum_{k=1}^{n} \sum_{\alpha_1,\ldots,\alpha_k=1}^{N} \| \frac{\partial^k}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_k}} f \|_\infty, \quad n \geq 0.$$

Proof. For each $w \in A^* \setminus \{1\}$, there exists a $C_w > 0$ such that

$$\| \Phi(w) f \|_\infty \leq C_w \| \nabla f \|_{C^{|w|-1}}$$

for any $f \in C^\infty(\mathbf{R}^N)$. Then we have

$$\| \Phi(j_n z) f \|_\infty \leq \sum_{w \in A, 1 \leq \|w\| \leq n} C_w |\langle z, w \rangle| \, \| \nabla f \|_\infty .$$

This implies our assertion. ∎

Lemma 14 *For any $m \geq 1$, there is a $C_m > 0$ such that*

$$\begin{aligned}
&|f(\exp(\Phi(j_m \Psi_s z_1))(\exp(\Phi(j_m \Psi_s z_2))(x)) \\
&\qquad - f(\exp(\Phi(j_m(\log(\exp(j_m \Psi_s z_2) \exp(j_m \Psi_s z_1))))(x))| \\
&\leq C_m s^{(m+1)/2} (1 + \| j_m z_1 \|_2 + \| j_m z_2 \|_2)^{m^2(m+1)} \| \nabla f \|_{C^m}
\end{aligned}$$

for any $s \in (0, 1]$, $z_1, z_2 \in \mathcal{L}((A))$ and $f \in C^\infty(\mathbf{R}^N)$.

Proof. Let $w = \log(\exp(j_m z_2) \exp(j_m z_1))$. Then we have

$$\Psi_s w = \log(\exp(j_m \Psi_s z_2) \exp(j_m \Psi_s z_1))$$

and

$$j_m \exp(j_m \Psi_s w) = j_m(\exp(j_m \Psi_s z_2) \exp(j_m \Psi_s z_1)).$$

Then letting $z_1 = w$ and $z_2 = 0$ in Corollary 12, we have

$$|f(\exp(\Phi(j_m \Psi_s w))(x)) - (\Phi(j_m(\exp(j_m \Psi_s w))) f)(x)|$$

$$\leq \sum_{k=2}^{m+1} \frac{1}{k!} \| \Phi((j_{m^{m+1}} - j_m)((j_m w)^k)) f \|_\infty .$$

Therefore by Corollary 12, there is a $C > 0$

$$\begin{aligned}
&|f(\exp(\Phi(j_m \Psi_s z_1))(\exp(\Phi(j_m \Psi_s z_2))(x)) - f(\exp(\Phi(j_m \Psi_s w))(x))| \\
&\leq C(\sum_{2 \leq k+\ell \leq m+1} \| (j_{m^{m+1}} - j_m)((j_m \Psi_s z_2)^k (j_m \Psi_s z_1)^\ell) \|_2 \\
&\qquad + \sum_{k=2}^{m+1} \| (j_{m^{m+1}} - j_m)((j_m w)^k)) \|_2) \| \nabla f \|_{C^m}
\end{aligned}$$

for any $s \in (0,1]$ and $f \in C^\infty(\mathbf{R}^N)$. Note that

$$\| (j_{m^{m+1}} - j_m)((j_m\Psi_s z_2)^k (j_m\Psi_s z_1)^\ell) \|_2 \leq s^{(m+1)/2} \| (j_m z_2)^k (j_m z_1)^\ell \|_2$$
$$\leq s^{(m+1)/2} \| j_m z_2 \|_2^k \| j_m z_1 \|_2^\ell$$

and that

$$\| j_m w \|_2 = \| j_m (\sum_{i=1}^m \frac{(-1)^{i-1}}{i} (\sum_{1\leq k+\ell \leq m} \frac{1}{k!\ell!} (j_m z_2)^k (j_m z_1)^\ell))^i) \|_2$$
$$\leq \sum_{i=1}^m (\sum_{1\leq k+\ell\leq m} \| j_m z_2 \|_2^k \| j_m z_1 \|_2^\ell)^i.$$

These imply our assertion. ∎

Corollary 15 *Let ξ_1, ξ_2 be $\mathcal{L}((A))$-valued $L^{\infty-}$ random variable. Then for any $m \geq 1$ and $p \in [1,\infty)$, there is a $C > 0$ such that*

$$\| \exp(\Phi(j_m\Psi_s\xi_1))(\exp(\Phi(j_m\Psi_s\xi_2))(x))$$
$$- \exp(\Phi(j_m(\log(\exp(j_m\Psi_s\xi_2)\exp(j_m\Psi_s\xi_1))))(x) \|_{L^p} \leq Cs^{(m+1)/2}$$

for any $s \in (0,1]$ and $x \in \mathbf{R}^N$.

Proof. Let $f(x) = x^i$, $x = (x^1, \ldots, x^N) \in \mathbf{R}^N$. Then we have $\| \nabla f \|_{C^n} = 1$. Applying Lemma 14, we have our assertion. ∎

Proposition 16 (1) *For any $m \geq 1$ and $f \in C^\infty(\mathbf{R}^N)$,*

$$f(X(t,x)) = (\Phi(j_m X(t))f)(x)$$
$$+ \sum{}' \int_0^t \circ dB^{i_1}(s_1) \int_0^{s_1} \circ dB^{i_2}(s_2) \cdots \int_0^{s_{n-1}} \circ dB^{i_n}(s_n)(V_{i_n} \ldots V_{i_1} f)(X(s_n,x)).$$

Here $\sum'$ is the summation taken for $i_1, \ldots, i_n = 0, 1, \ldots, N$ such that $\| v_{i_{n-1}} v_{i_{n-2}} \ldots v_{i_1} \| \leq m$ and $\| v_{i_{n-1}} v_{i_{n-2}} \ldots v_{i_1} \| \geq m+1$.
(2) *For any $m \geq 1$ and $p \in [1,\infty)$, there is a $C > 0$ such that*

$$\| f(X(t,x)) - (\Phi(j_m(X(t)))f)(x) \|_{L^p} \leq Ct^{(m+1)/2} \| \nabla f \|_{C^{m+1}}$$

for any $t \in (0,1]$ and $f \in C^\infty(\mathbf{R}^N)$.

Proof. The assertion (1) is easy to prove by induction in m. The assertion (2) follows from the fact that

$$\int_0^{s_{n-1}} \circ dB^{i_n}(s_n)(V_{i_n} \ldots V_{i_1} f)(X(s_n,x))$$
$$= \int_0^{s_{n-1}} (V_{i_n} \ldots V_{i_1} f)(X(s_n,x)) dB^{i_n}(s_n) + \frac{1}{2} \sum_{j=1}^N \delta_{ji_n} \int_0^{s_{n-1}} (V_j V_{i_n} \ldots V_{i_1} f)(X(s_n,x)) ds_n.$$

This completes the proof. ∎

Corollary 17 *For any $m \geq 1$, there is a $C > 0$ such that*

$$|E[f(X(s,x))] - E[f(\exp(\Phi(j_m \Psi_s \log X(1))))(x)]| \leq C s^{(m+1)/2} \parallel \nabla f \parallel_\infty$$

for any $x \in \mathbf{R}^N$, $s \in (0,1]$ and $f \in C_b^\infty(\mathbf{R}^N)$.

Proof. Let $H(x) = x$, $x \in \mathbf{R}^N$. Then by Proposition 16 (2), there is a $C_1 > 0$ such that

$$\parallel X(s,x) - (\Phi(j_m(X(s)))H)(x) \parallel_{L^1} \leq C_1 s^{(m+1)/2}, \qquad x \in \mathbf{R}^N,\ s \in (0,1].$$

So we see that

$$|E[f(X(s,x))] - E[f((\Phi(j_m(X(s)))H)(x))]| \leq C_1 s^{(m+1)/2} \parallel \nabla f \parallel_\infty, \quad x \in \mathbf{R}^N,\ s \in (0,1].$$

Also by Corollary 12 we have

$$|\exp(\Phi(j_m \Psi_s \log X(1)))(x) - (\Phi(j_m(\Psi_s X(1)))H)(x)|$$

$$\leq \sum_{k=2}^{m+1} \frac{1}{k!} s^{(m+1)/2} \parallel \Phi((j_{m^{m+1}} - j_m)((j_m \log X(1))^k))H \parallel_\infty, \qquad x \in \mathbf{R}^N,\ s \in (0,1].$$

So we see that there is a $C_2 > 0$ such that

$$\parallel \exp(\Phi(j_m \Psi_s \log X(1)))(x) - (\Phi(j_m(\Psi_s X(1)))H)(x) \parallel_{L^1} \leq C_2 s^{(m+1)/2}$$

for any $x \in \mathbf{R}^N$, $s \in (0,1]$, which implies that

$$|E[f(\exp(\Phi(j_m \Psi_s \log X(1)))(x))] - E[f((\Phi(j_m(\Psi_s X(1)))H)(x))]| \leq C_2 s^{(m+1)/2} \parallel \nabla f \parallel_\infty$$

for any $x \in \mathbf{R}^N$, $s \in (0,1]$. Since $j_m(X(s))$ and $j_m \Psi_s X(1)$ has the same law, we have our assertion. ∎

Lemma 18 *Let $m \geq 2$ and ξ is a m-$\mathcal{L}$-similar $\mathcal{L}((A))$-valued random variable. Then there is a constant $C > 0$ such that*

$$|E[f(X(s,x))] - E[f(\exp(\Phi(j_m \Psi_s \xi))(x))]|$$

$$\leq C(\sum_{k=m+1}^{m^{m+1}} s^{k/2} \parallel f \parallel_{V,k} + s^{(m+1)/2} \parallel \nabla f \parallel_\infty)$$

for any $s \in (0,1]$ and $f \in C_b^\infty(\mathbf{R}^N)$.

Proof. Let $\eta_0 = \log(\exp(-v_0)X(1))$ and $\eta_1 = \log(\exp(-v_0)\exp(\xi))$. Then η_0 and η_1 are $\mathcal{L}((A))$-valued $L^{\infty-}$ random variable and we see that

$$E[j_m(\exp(\eta_0))] = E[j_m(\exp(-v_0) j_m(X(1)))]$$

$$= E[j_m(\exp(-v_0) j_m(\exp(\xi))] = E[j_m(\exp(\eta_1))].$$

Note that $j_m(\eta_i) \in \mathcal{L}(A) \cap (\sum_{w \in A^{**}} \mathbf{R}w)$, $i = 0, 1$. So there is a $C_1 > 0$ such that

$$\| \Phi((j_{m^{m+1}} - j_m)(j_m \Psi_s \eta_i)^\ell) f \|_\infty \le C_1 \sum_{k=m+1}^{m^{m+1}} s^{k/2} \| f \|_{V,k}$$

for any $i = 0, 1$, $s \in (0, 1]$ and $f \in C_b^\infty(\mathbf{R}^N)$. So we see that there is a $C_2 > 0$ such that

$$\| f(\exp(\Phi(j_m \Psi_s \eta_i))(y)) - (\Phi(j_m(\exp(j_m \Psi_s \eta_i)) f)(y) \|_{L^1} \le C_2 \sum_{k=m+1}^{m^{m+1}} s^{k/2} \| f \|_{V,k}$$

for any $i = 0, 1$, $s \in (0, 1]$, $y \in \mathbf{R}^N$, and $f \in C_b^\infty(\mathbf{R}^N)$. However, $E[\Phi(j_m(\exp(j_m \Psi_s \eta_i)) f)(y)]$, $i = 0, 1$ are coincident. So letting $y = \exp(\Phi(j_m \Psi_s v_0))(x)$, we have

$$\begin{aligned} &|E[f(\exp(\Phi(j_m \Psi_s \eta_0))(\exp(\Phi(j_m \Psi_s v_0))(x)))] \\ &\quad - E[f(\exp(\Phi(j_m \Psi_s \eta_1))(\exp(\Phi(j_m \Psi_s v_0))(x)))]| \le 2C_2 \sum_{k=m+1}^{m^{m+1}} s^{k/2} \| f \|_{V,k} \end{aligned}$$

for any $x \in \mathbf{R}^N$, and $f \in C_b^\infty(\mathbf{R}^N)$. Note that

$$j_m \log(\exp((j_m v_0))(\exp(j_m \eta_i))) = j_m \log(\exp(v_0)(\exp(\eta_i))), \qquad i = 0, 1.$$

Then by Corollaries 15 and 17, we have our assertion.

This completes the proof. ∎

Now Theorem 3 follows from Lemma 18, since

$$|E[f(\exp(\Phi(j_m \Psi_s \xi))(x))]| - E[f(\exp(\Phi(j_m \Psi_s \xi))(x) + Y(s, x))]| \le E[|Y(s, x)|] \| \nabla f \|_\infty .$$

This completes the proof of Theorem 3.

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