UTMS 2004–14 April 26, 2004

Universal characters, integrable chains and the Painlevé equations

by

Teruhisa Tsuda

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

Universal characters, integrable chains and the Painlevé equations

Teruhisa TSUDA*
Graduate School of Mathematical Sciences,
The University of Tokyo,
Komaba, Tokyo 153-8914, Japan.

Abstract

The universal character is a generalization of the Schur polynomial attached to a pair of partitions; see [8]. We prove that the universal character solves the Darboux chain. The N-periodic closing of the chain is equivalent to the Painlevé equation of type $A^{(1)}_{N-1}$. Consequently we obtain an expression of rational solutions of the Painlevé equations in terms of the universal characters.

Introduction

The universal character $S_{[\lambda,\mu]}$, defined by Koike [8], is a polynomial attached to a pair of partitions $[\lambda,\mu]$ and is a generalization of the Schur polynomial. The universal character is in fact the irreducible character of a rational representation of the general linear group $GL(n,\mathbb{C})$, while the Schur polynomial that of a polynomial representation.

The Darboux chain, given by (1.1) below, is a sequence of ordinary differential equations with quadratic nonlinearity. This is closely interconnected with the spectral theory of Schrödinger operators; in fact, governs a sequence of Schrödinger operators connected with the neighbours by the Darboux transformations; see [16, 17, 18, 24, 25].

The Painlevé equations can be derived from the Darboux chains with suitable boundary conditions; as is well known that the chains, (1.1), with periods of order three and four yield Painlevé equations P_{IV} and P_{V} respectively. In general, for an integer $N \geq 3$, the N-periodic closing of the chain coincides with the (higher order) Painlevé equation of type

*Current address: Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan.
E-mail address: tudateru@ms.u-tokyo.ac.jp, tudateru@math.kobe-u.ac.jp

$A^{(1)}_{N-1}$, proposed by Noumi–Yamada; see [1, 14, 15, 18, 24, 25].

The aim of the present article is to show certain relationships among the universal characters, the Darboux chains, and the Painlevé equations. We prove that the Darboux chain admits particular solutions expressed by means of the universal characters (see Theorems 1.4 and 1.5). We have a cycle of the universal characters connected with the action of vertex operators (see Lemma 2.2). Finally, these cycles, together with the fact that N-periodic closing of the chain is equivalent to the Painlevé equation of type $A^{(1)}_{N-1}$, yield a class of rational solutions of the equation in terms of the universal characters (see Theorems 2.4 and 2.6). Section 3 is devoted to the verification of Theorem 1.5.

1 Integrable chains

1.1 Darboux chain and its bilinear form

First we recall the definition of the Darboux chain and then review its bilinearization following [16, 25]. Consider the sequence of nonlinear differential equations for $v_n = v_n(x)$ of the form:

$$\dot{v}_n + \dot{v}_{n-1} = v_n^2 - v_{n-1}^2 + \alpha_n \quad (n \in \mathbb{Z}), \tag{1.1}$$

which is called the *Darboux chain*. Here $\dot{v}$ stands for the derivative of v with respect to x and $\alpha_n \in \mathbb{C}$ is a constant parameter.

Introduce the variable $u_n = u_n(x)$ defined by

$$u_n = v_n^2 - \dot{v}_n - c_n, \tag{1.2}$$

where $c_n \in \mathbb{C}$ is a constant such that

$$\alpha_n = c_{n-1} - c_n. \tag{1.3}$$

Equation (1.2) is considered as a Riccati equation for v_n and thus linearizable via the change of the variables

$$v_n = -\frac{\mathrm{d}}{\mathrm{d}x} \log \psi_n; \tag{1.4}$$

then (1.2) is transformed into the Shrödinger equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - u_n \right) \psi_n = c_n \psi_n. \tag{1.5}$$

It follows from (1.1) and (1.2) that the potential u_n $(n \in \mathbb{Z})$ satisfies the recursion relation:

$$u_{n+1} = u_n - 2(\log \psi_n)_{xx}. \tag{1.6}$$

Now let us define the function $\tau_n = \tau_n(x)$, called the τ-*function*, by

$$u_n = -2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\tau_n - \frac{\epsilon^2}{4}x^2 + n\epsilon. \tag{1.7}$$

Then we obtain from (1.5) the Hirota bilinear equation:

$$\left(D_x^2 + \epsilon x D_x + \epsilon k_n\right)\tau_n\cdot\tau_{n+1} = 0, \tag{1.8}$$

with

$$k_n = -\frac{c_n}{\epsilon} - n - \frac{1}{2}. \tag{1.9}$$

Here the symbol D_x denotes the Hirota differential with respect to the variable x; for instance,

$$D_x f\cdot g = \dot{f}g - f\dot{g}, \quad D_x^2 f\cdot g = \ddot{f}g - 2\dot{f}\dot{g} + f\ddot{g}.$$

Remark 1.1. Conversely a solution $\{v_n = v_n(x); n\in\mathbb{Z}\}$ of the chain (1.1) is expressible in terms of the τ-functions. Combining (1.4), (1.6) and (1.7), we obtain in fact

$$v_n = \frac{\epsilon}{2}x - \frac{\mathrm{d}}{\mathrm{d}x}\log\frac{\tau_{n+1}}{\tau_n}. \tag{1.10}$$

1.2 Universal characters and Schur polynomials

For a pair of sequences of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{l'})$, the *universal character* $S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y})$ is a polynomial in $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, x_2, \ldots, y_1, y_2, \ldots)$ defined as follows (see [8]):

$$S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}) = \det\begin{bmatrix} q_{\mu_{l'-i+1}+i-j}(\boldsymbol{y}), & 1\le i\le l' \\ p_{\lambda_{i-l'}-i+j}(\boldsymbol{x}), & l'+1\le i\le l+l' \end{bmatrix}_{1\le i,j\le l+l'}. \tag{1.11}$$

Here $p_n(\boldsymbol{x})$ and $q_n(\boldsymbol{y})$ are the elementary Schur polynomials defined by

$$\sum_{n=0}^{\infty} p_n(\boldsymbol{x})z^n = e^{\xi(\boldsymbol{x},z)}, \quad \xi(\boldsymbol{x}, z) = \sum_{n=1}^{\infty} x_n z^n, \tag{1.12}$$

and $p_{-n}(\boldsymbol{x}) = 0$ for $n > 0$; or

$$p_n(\boldsymbol{x}) = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{1.13}$$

Polynomial $q_n(\boldsymbol{y})$ is just the same as $p_n(\boldsymbol{x})$ except replacing x_i by y_i. The Schur polynomial $S_\lambda(\boldsymbol{x})$ (see *e.g.* [9]) is regarded as a special case of the universal character:

$$S_\lambda(\boldsymbol{x}) = \det\left(p_{\lambda_i-i+j}(\boldsymbol{x})\right) = S_{[\lambda,\emptyset]}(\boldsymbol{x}, \boldsymbol{y}). \tag{1.14}$$

If we count the degree of each variable x_n and y_n $(n = 1, 2, \ldots)$ as

$$\deg x_n = n, \quad \deg y_n = -n,$$

then $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ is a (weighted) homogeneous polynomial of degree $|\lambda|-|\mu|$, where we let $|\lambda| = \lambda_1+\cdots+\lambda_l$.

Introduce the operators $V_k = V_k(z;\boldsymbol{x},\boldsymbol{y})$ $(k\in\mathbb{Z})$ as follows:

$$V_k(z;\boldsymbol{x},\boldsymbol{y}) = e^{k\xi(\boldsymbol{x}-\widetilde{\partial}_{\boldsymbol{y}},z)}e^{-k\xi(\widetilde{\partial}_{\boldsymbol{x}},z^{-1})}, \tag{1.15}$$

where $\widetilde{\partial}_{\boldsymbol{x}}$ stands for $\left(\frac{\partial}{\partial x_1},\frac{1}{2}\frac{\partial}{\partial x_2},\frac{1}{3}\frac{\partial}{\partial x_3},\ldots\right)$. For $n\in\mathbb{Z}$, we define the vertex operators $X_n = X_n(\boldsymbol{x},\partial_{\boldsymbol{x}},\partial_{\boldsymbol{y}})$ and $Y_n = Y_n(\boldsymbol{y},\partial_{\boldsymbol{x}},\partial_{\boldsymbol{y}})$ by

$$V_1(z;\boldsymbol{x},\boldsymbol{y}) = \sum_{n\in\mathbb{Z}} X_n z^n, \quad V_1(w^{-1};\boldsymbol{y},\boldsymbol{x}) = \sum_{n\in\mathbb{Z}} Y_n w^{-n}. \tag{1.16}$$

Note that $[X_n, Y_m] = 0$.

Proposition 1.2 (see [19]). *The operators X_n and Y_n $(n\in\mathbb{Z})$ are raising operators for the universal characters in the sense that*

$$S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = X_{\lambda_1}\cdots X_{\lambda_l}Y_{\mu_1}\cdots Y_{\mu_{l'}}\cdot 1.$$

Example 1.3. We give a few examples of the universal characters below.

$$\begin{aligned}
S_{[\emptyset,\emptyset]}(\boldsymbol{x},\boldsymbol{y}) &= 1,\\
S_{[(1),\emptyset]}(\boldsymbol{x},\boldsymbol{y}) &= S_{(1)}(\boldsymbol{x}) = x_1,\\
S_{[(1),(1)]}(\boldsymbol{x},\boldsymbol{y}) &= x_1y_1-1,\\
S_{[(2,1),\emptyset]}(\boldsymbol{x},\boldsymbol{y}) &= S_{(2,1)}(\boldsymbol{x}) = \frac{x_1^3}{3}-x_3,\\
S_{[(2,1),(1)]}(\boldsymbol{x},\boldsymbol{y}) &= \left(\frac{x_1^3}{3}-x_3\right)y_1 - x_2^2,\\
S_{[(2,1),(2,1)]}(\boldsymbol{x},\boldsymbol{y}) &= \left(\frac{x_1^3}{3}-x_3\right)\left(\frac{y_1^3}{3}-y_3\right)-(x_1y_1-1)^2.
\end{aligned}$$

We also note that (see [19])

$$S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = S_\lambda(\boldsymbol{x}-\widetilde{\partial}_{\boldsymbol{y}})S_\mu(\boldsymbol{y}-\widetilde{\partial}_{\boldsymbol{x}})\cdot 1. \tag{1.17}$$

1.3 Similarity reduction of modified KP hierarchy

The Darboux chain (1.1) is closely related to the theory of infinite dimensional integrable systems. As seeing below, the chain (1.1), or (1.8), is regarded as a similarity reduction of the modified KP hierarchy.

Recall that the modified KP hierarchy is the following system of Hirota bilinear equations (see [3]):

$$\sum_{k=0}^{\infty} p_k(-2\boldsymbol{u})p_{k+2}(\widetilde{D}_{\boldsymbol{x}})\exp\left(\sum_{j\geq 1}u_jD_{x_j}\right)\sigma_n(\boldsymbol{x})\cdot\sigma_{n+1}(\boldsymbol{x}) = 0, \tag{1.18}$$

where $\boldsymbol{u} = (u_1, u_2, \ldots)$ are parameters and $\widetilde{D}_{\boldsymbol{x}} = \left(D_{x_1}, \frac{1}{2}D_{x_2}, \frac{1}{3}D_{x_3}, \ldots\right)$. We obtain in particular, from the constant term with respect to $\boldsymbol{u}$, the bilinear equation:

$$\left(D_{x_1}^2 + D_{x_2}\right) \sigma_n \cdot \sigma_{n+1} = 0. \tag{1.19}$$

Let the pair $(\sigma_n, \sigma_{n+1}) = (\sigma_n(x_1, x_2), \sigma_{n+1}(x_1, x_2))$ be a solution of (1.19) such that

$$\left(x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2}\right) \sigma_j(x_1, x_2) = d_j \sigma_j(x_1, x_2), \quad d_j \in \mathbb{C}. \tag{1.20}$$

Noticing $\sigma_j(cx_1, c^2x_2) = c^{d_j}\sigma_j(x_1, x_2)$ for any $c \in \mathbb{C}^\times$, we set

$$\tau_j(x) = \sigma_j\left(x, -\frac{1}{2\epsilon}\right). \tag{1.21}$$

It follows from (1.19) and (1.20) that the pair $\tau_n(x)$ and $\tau_{n+1}(x)$ satisfies the bilinear equation (see [14]):

$$\left(D_x^2 + \epsilon x D_x + \epsilon(d_{n+1} - d_n)\right) \tau_n \cdot \tau_{n+1} = 0, \tag{1.22}$$

which is exactly the bilinear form of the Darboux chain (1.8).

1.4 Schur polynomial solves the chain

The Schur polynomial $S_\lambda = S_{[\lambda,\emptyset]}$ gives a particular solution of the Darboux chain (1.1), or (1.8), in the sense of the following theorem.

Theorem 1.4. *For any integer k and sequence of integers λ, let $f = f(x)$ and $g = g(x)$ be the functions defined by*

$$f = S_\lambda(\boldsymbol{x}), \quad g = S_{(k,\lambda)}(\boldsymbol{x}), \tag{1.23}$$

with

$$x_1 = x, \quad x_2 = -\frac{1}{2\epsilon}, \quad x_n = 0 \quad (n \geq 3). \tag{1.24}$$

Then we have

$$\left(D_x^2 + \epsilon x D_x + \epsilon k\right) f \cdot g = 0. \tag{1.25}$$

Proof. A key is the following

Claim (see *e.g.* [4, 14])**.** *For any integer k and sequence of integers λ, the pair $\sigma_n(\boldsymbol{x}) = S_\lambda(\boldsymbol{x})$ and $\sigma_{n+1}(\boldsymbol{x}) = S_{(k,\lambda)}(\boldsymbol{x})$ solves the first modified KP hierarchy* (1.18); *in particular satisfies* (1.19).

As shown in the previous section, equation (1.25) is deduced from (1.19) via the similarity reduction; see (1.20) and (1.21). Since $S_\lambda(\boldsymbol{x})$ is homogeneous of degree $|\lambda|$, the theorem follows immediately from the above claim. ∎

1.5 Universal character solves the chain

Consider the change of the variables:

$$t = -\frac{\epsilon}{2}x^2, \tag{1.26}$$

and

$$\tilde{\tau}_n = t^{-\frac{1}{4}\left(a-\frac{1}{2}\right)\left(a+\frac{1}{2}\right)}\tau_n, \quad \tilde{\tau}_{n+1} = t^{-\frac{1}{4}\left(a+\frac{1}{2}\right)\left(a+\frac{3}{2}\right)}\tau_{n+1}, \tag{1.27}$$

the bilinear form of the Darboux chain (1.8) is converted to

$$\left(tD_t^2 - (t+a)D_t - \frac{1}{2}\left(k_n - a - \frac{1}{2}\right)\right)\tilde{\tau}_n \cdot \tilde{\tau}_{n+1} + \tilde{\tau}_n \frac{\mathrm{d}\tilde{\tau}_{n+1}}{\mathrm{d}t} = 0. \tag{1.28}$$

The universal character $S_{[\lambda,\mu]}$ also solves the Darboux chain:

Theorem 1.5. *For any integer k and pair of sequences of integers $[\lambda, \mu]$, let $f = f(t)$ and $g = g(t)$ be the functions defined as follows:*

$$f = S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}), \quad g = S_{[(k,\lambda),\mu]}(\boldsymbol{x}', \boldsymbol{y}'), \tag{1.29}$$

with

$$x_n = t + \frac{a}{n}, \quad y_n = -t + \frac{a}{n}, \quad x'_n = t + \frac{a+1}{n}, \quad y'_n = -t + \frac{a+1}{n}, \tag{1.30}$$

where $a \in \mathbb{C}$ being an arbitrary constant. Then we have

$$\left(tD_t^2 - (t+a)D_t - k\right) f \cdot g + f\frac{\mathrm{d}g}{\mathrm{d}t} = 0. \tag{1.31}$$

The proof of the theorem is given in Section 3 below.

Remark 1.6. Let λ^T denote the transpose of the partition λ; see [9]. Viewing the properties:

$$S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}) = \pm S_{[\lambda^T,\mu^T]}(-\boldsymbol{x}, -\boldsymbol{y}) \quad \text{and} \quad S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}) = S_{[\mu,\lambda]}(\boldsymbol{y}, \boldsymbol{x}),$$

we have from Theorem 1.5 that the pair

$$\tilde{f} = S_{[\lambda,\mu^T]}(\boldsymbol{x}', \boldsymbol{y}'), \quad \tilde{g} = S_{[\lambda,(k,\mu)^T]}(\boldsymbol{x}, \boldsymbol{y}), \tag{1.32}$$

satisfies

$$\left(tD_t^2 - (t-a-1)D_t - k\right) \tilde{f} \cdot \tilde{g} + \tilde{f}\frac{\mathrm{d}\tilde{g}}{\mathrm{d}t} = 0. \tag{1.33}$$

Remark 1.7. An extension of the KP hierarchy, called the *UC hierarchy*, is proposed in [19, 23]. Since the UC hierarchy admits all the universal characters to be its polynomial solutions, it is an interesting problem to construct a certain reduction procedure from the hierarchy to the Darboux chain.

2 Painlevé equations and universal characters

2.1 N-Periodic chain and Painlevé equation of type $A^{(1)}_{N-1}$

From now on, we consider the N-periodic chain, namely, equations (1.1) with the N-periodic condition:

$$v_{n+N} = v_n, \quad \alpha_{n+N} = \alpha_n. \tag{2.1}$$

We normalize

$$\sum_{n=1}^{N} \alpha_n = 1, \tag{2.2}$$

without loss of generality. As shown in [18] (and see also [1, 24]), through the change of variables:

$$f_n = v_n + v_{n-1}, \tag{2.3}$$

the N-periodic chain is converted to the Painlevé equation of type $A^{(1)}_{N-1}$ due to Noumi–Yamada [15], denoted by $P(A_{N-1})$; the system of differential equations for f_n is given as follows:

(i) if $N = 2g+1$ $(g = 1, 2, \ldots)$,

$$P(A_{2g}): \qquad \frac{\mathrm{d}f_n}{\mathrm{d}x} = f_n \left(\sum_{j=1}^{g} f_{n+2j-1} - \sum_{j=1}^{g} f_{n+2j} \right) + \alpha_n;$$

(ii) if $N = 2g+2$ $(g = 1, 2, \ldots)$,

$$\begin{aligned} P(A_{2g+1}): \qquad \frac{x}{2}\frac{\mathrm{d}f_n}{\mathrm{d}x} &= f_n \left(\sum_{1\le j\le k\le g} f_{n+2j-1} f_{n+2k} - \sum_{1\le j\le k\le g} f_{n+2j} f_{n+2k+1} \right) \\ &\quad + \left(\frac{1}{2} - \sum_{k=1}^{g} \alpha_{n+2k} \right) f_n + \alpha_n \sum_{j=1}^{g} f_{n+2j}. \end{aligned}$$

Note that $P(A_2)$ and $P(A_3)$ are equivalent to the Painlevé equations P_{IV} and P_{V} respectively.

Remark 2.1. The Painlevé equation $P(A_l)$ $(l \ge 2)$ has symmetry under the (extended) affine Weyl group of type $A^{(1)}_l$; that is, $P(A_l)$ is invariant under the action of the transformations s_i $(i = 0, 1, \ldots, l)$ and π given as follows (see [15]):

$$\begin{aligned} &s_i(\alpha_i) = -\alpha_i, \quad s_i(\alpha_j) = \alpha_j + \alpha_i \quad (j = i \pm 1), \quad s_i(\alpha_j) = \alpha_j \quad (j \ne i, i \pm 1), \\ &s_i(f_i) = f_i, \qquad s_i(f_j) = f_j \pm \frac{\alpha_i}{f_i} \quad (j = i \pm 1), \quad s_i(f_j) = f_j \quad (j \ne i, i \pm 1), \end{aligned} \tag{2.4}$$

and $\pi(\alpha_i) = \alpha_{i+1}$, $\pi(f_i) = f_{i+1}$; these define a representation of the extended affine Weyl group $\widetilde{W} = \langle s_0, \ldots, s_l, \pi \rangle$ of type $A^{(1)}_l$.

Now we shall consider the bilinear form of $P(A_{N-1})$. The periodic condition (2.1) requires that $c_{n+N} = c_n - 1$; see (1.3) and (2.2). Hence we have that, if

$$\epsilon = \frac{1}{N}, \tag{2.5}$$

then we can set

$$k_{n+N} = k_n, \quad \tau_{n+N} = \tau_n,$$

in the bilinear form of the chain (1.8). Thus we obtain the system of bilinear equations:

$$\left(D_x^2 + \frac{x}{N} D_x + \frac{k_n}{N} \right) \tau_n \cdot \tau_{n+1} = 0, \tag{2.6}$$

with $\tau_{n+N} = \tau_n$ and $k_{n+N} = k_n$, which is equivalent to Painlevé equation $P(A_{N-1})$ via

$$f_n = \frac{x}{N} - \frac{\mathrm{d}}{\mathrm{d}x} \log \frac{\tau_{n+1}}{\tau_{n-1}}, \quad \alpha_n = \frac{k_n - k_{n-1} + 1}{N}; \tag{2.7}$$

see (1.10) and (2.3).

2.2 N-reduced partitions and chain of vertex operators

We first recall the definition of N-reduced partition following [14]. A subset $M \subset \mathbb{Z}$ is said to be a *Maya diagram* if

$$m \in M \quad (m \ll 0) \quad \text{and} \quad m \notin M \quad (m \gg 0).$$

Each Maya diagram $M = \{\dots, m_3, m_2, m_1\}$ corresponds to a unique partition $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $m_i - m_{i+1} = \lambda_i - \lambda_{i+1} + 1$. For each $\boldsymbol{n} = (n_1, n_2, \dots, n_N) \in \mathbb{Z}^N$, let us consider the Maya diagram:

$$M(\boldsymbol{n}) = (N\mathbb{Z}_{<n_1} + 1) \cup (N\mathbb{Z}_{<n_2} + 2) \cup \cdots \cup (N\mathbb{Z}_{<n_N} + N);$$

then denote by $\lambda(\boldsymbol{n})$ the corresponding partition. Notice that

$$\lambda(\boldsymbol{n}) = \lambda(\boldsymbol{n} + \mathbf{1}),$$

where $\mathbf{1} = (1, 1, \dots, 1)$. A partition of the form $\lambda(\boldsymbol{n})$ is said to be an *N-reduced partition.* We remark that a partition λ is N-reduced if and only if λ has no hook with length of a multiple of N.

We have a cycle, connected with the action of the vertex operators (1.16), among the universal characters attached to N-reduced partitions:

Lemma 2.2. *The following formulae hold.*

$$\begin{aligned}
X_{Nn_1-|\boldsymbol{n}|}S_{[\lambda(\boldsymbol{n}),\mu]}(\boldsymbol{x},\boldsymbol{y}) &= \pm S_{[\lambda(\boldsymbol{n}+\boldsymbol{e}_1),\mu]}(\boldsymbol{x},\boldsymbol{y}),\\
X_{Nn_2-|\boldsymbol{n}|}S_{[\lambda(\boldsymbol{n}+\boldsymbol{e}_1),\mu]}(\boldsymbol{x},\boldsymbol{y}) &= \pm S_{[\lambda(\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2),\mu]}(\boldsymbol{x},\boldsymbol{y}),\\
&\ \vdots \\
X_{Nn_{N-1}-|\boldsymbol{n}|}S_{[\lambda(\boldsymbol{n}+\boldsymbol{e}_1+\cdots+\boldsymbol{e}_{N-2}),\mu]}(\boldsymbol{x},\boldsymbol{y}) &= \pm S_{[\lambda(\boldsymbol{n}+\boldsymbol{e}_1+\cdots+\boldsymbol{e}_{N-2}+\boldsymbol{e}_{N-1}),\mu]}(\boldsymbol{x},\boldsymbol{y}),\\
X_{Nn_N-|\boldsymbol{n}|}S_{[\lambda(\boldsymbol{n}+\boldsymbol{e}_1+\cdots+\boldsymbol{e}_{N-2}+\boldsymbol{e}_{N-1}),\mu]}(\boldsymbol{x},\boldsymbol{y}) &= \pm S_{[\lambda(\boldsymbol{n}),\mu]}(\boldsymbol{x},\boldsymbol{y}).
\end{aligned} \tag{2.8}$$

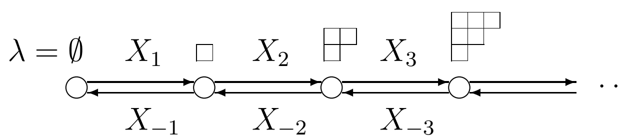
Here $\boldsymbol{n}=(n_1,n_2,\dots,n_N)\in\mathbb{Z}^N$*,* $|\boldsymbol{n}|=n_1+\cdots+n_N$*,* $\boldsymbol{e}_i=(0,\dots,0,\overset{\overset{i}{\smile}}{1},0,\dots,0)$*, and* μ *being an arbitrary sequence of integers.*

We can easily verify the above lemma by Proposition 1.2 together with the commutation relations among the vertex operators (see [19]):

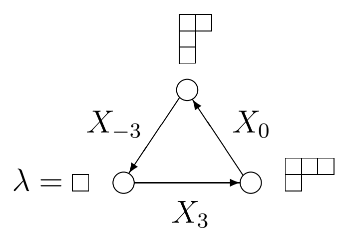
$$X_kX_l=-X_{l-1}X_{k+1} \quad \text{and} \quad [X_k,Y_l]=0.$$

Note that similar formulae as (2.8) hold also for the operators Y_n.

Example 2.3. (1) Consider the case $N=2$; we have a chain of universal characters $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ in which λ being a two-reduced partition, connected with vertex operators X_n ($n\in\mathbb{Z}$).



(2) Consider the case $N=3$; we have a cycle of a triple of universal characters corresponding to three-reduced partitions. We give below an example for $(n_1,n_2,n_3)=(2,1,0)$ in Lemma 2.2.



2.3 Rational solutions of $P(A_{N-1})$ in terms of Schur polynomials

Viewing Lemma 2.2 with $\mu=\emptyset$, we obtain from Theorem 1.4 a class of solutions of Painlevé equation $P(A_{N-1})$ expressed in terms of N-reduced Schur polynomials:

Theorem 2.4. *For any $\boldsymbol{n} \in \mathbb{Z}^N$, let*

$$\tau_i = S_{\lambda(\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2+\cdots+\boldsymbol{e}_i)}(\boldsymbol{x}), \tag{2.9}$$

where

$$x_1 = x, \quad x_2 = -\frac{N}{2}, \quad x_l = 0 \quad (l \geq 3).$$

(1) *The N-tuple of polynomials $\{\tau_i = \tau_i(x);\, i \in \mathbb{Z}/N\mathbb{Z}\}$ solves* (2.6)*, the bilinear form of $P(A_{N-1})$, when*

$$k_i = Nn_{i+1} - |\boldsymbol{n}|.$$

(2) *Consequently*

$$f_i = \frac{x}{N} - \frac{\mathrm{d}}{\mathrm{d}x} \log \frac{\tau_{i+1}}{\tau_{i-1}},$$

gives a rational solution of $P(A_{N-1})$ with the parameters:

$$\alpha_i = n_{i+1} - n_i + \frac{1}{N}. \tag{2.10}$$

Remark 2.5. The rational solutions given in Theorem 2.4 have been considered in the case when $N = 3$, that is, for the fourth Painlevé equation P_{IV}; *cf.* [7, 14].

2.4 Rational solutions of $P(A_{2g+1})$ in terms of universal characters

In what follows we consider only the case when N is even. Let $N = 2g + 2$, the N-periodic chain is equivalent to Painlevé equation $P(A_{2g+1})$. Equation $P(A_{2g+1})$ has another type of rational solutions than that given in the previous section, which is expressed by means of the $(g + 1)$-reduced universal characters.

Through the change of the variables:

$$t = -\frac{x^2}{4g+4}, \tag{2.11}$$

and

$$\tilde{\tau}_{2j} = t^{-\frac{1}{4}\left(a-\frac{1}{2}\right)\left(a+\frac{1}{2}\right)}\tau_{2j}, \quad \tilde{\tau}_{2j+1} = t^{-\frac{1}{4}\left(a+\frac{1}{2}\right)\left(a+\frac{3}{2}\right)}\tau_{2j+1}, \tag{2.12}$$

the bilinear form of $P(A_{2g+1})$, (2.6) with $N = 2g + 2$, is converted to the system:

$$\left(tD_t^2 - (t+a)D_t - \frac{1}{2}\left(k_{2j} - a - \frac{1}{2}\right)\right)\tilde{\tau}_{2j} \cdot \tilde{\tau}_{2j+1} + \tilde{\tau}_{2j}\frac{\mathrm{d}\tilde{\tau}_{2j+1}}{\mathrm{d}t} = 0, \tag{2.13a}$$

$$\left(tD_t^2 - (t-a-1)D_t - \frac{1}{2}\left(k_{2j-1} + a + \frac{1}{2}\right)\right)\tilde{\tau}_{2j-1} \cdot \tilde{\tau}_{2j} + \tilde{\tau}_{2j-1}\frac{\mathrm{d}\tilde{\tau}_{2j}}{\mathrm{d}t} = 0. \tag{2.13b}$$

Combine Lemma 2.2 with Theorem 1.5 (see also Remark 1.6), we thus have the following.

Theorem 2.6. *For any $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^{g+1}$, let*

$$\begin{aligned} \tilde{\tau}_{2j} &= S_{[\lambda(\boldsymbol{m}+\boldsymbol{e}_1+\cdots+\boldsymbol{e}_j),\lambda^T(\boldsymbol{n}+\boldsymbol{e}_1+\cdots+\boldsymbol{e}_j)]}(\boldsymbol{x},\boldsymbol{y}), \\ \tilde{\tau}_{2j+1} &= S_{[\lambda(\boldsymbol{m}+\boldsymbol{e}_1+\cdots+\boldsymbol{e}_j+\boldsymbol{e}_{j+1}),\lambda^T(\boldsymbol{n}+\boldsymbol{e}_1+\cdots+\boldsymbol{e}_j)]}(\boldsymbol{x}',\boldsymbol{y}'), \end{aligned} \tag{2.14}$$

with

$$x_n = t + \frac{a}{n}, \quad y_n = -t + \frac{a}{n}, \quad x'_n = t + \frac{a+1}{n}, \quad y'_n = -t + \frac{a+1}{n},$$

where

$$t = -\frac{x^2}{4(g+1)},$$

and $a \in \mathbb{C}$ being an arbitrary constant.

(1) *The $(2g+2)$-tuple of polynomials $\{\tilde{\tau}_i = \tilde{\tau}_i(t);\ i \in \mathbb{Z}/(2g+2)\mathbb{Z}\}$ solves* (2.13) *when*

$$\begin{aligned} k_{2j} &= 2(g+1)m_{j+1} - 2|\boldsymbol{m}| + a + \frac{1}{2}, \\ k_{2j+1} &= 2(g+1)n_{j+1} - 2|\boldsymbol{n}| - a - \frac{1}{2}. \end{aligned}$$

(2) *Consequently*

$$f_i = \frac{x}{2(g+1)} - \frac{\mathrm{d}}{\mathrm{d}x}\log\frac{\tilde{\tau}_{i+1}}{\tilde{\tau}_{i-1}},$$

gives a rational solution of $P(A_{2g+1})$ with the parameters:

$$\begin{aligned} \alpha_{2j} &= m_{j+1} - n_j + \frac{|\boldsymbol{n}| - |\boldsymbol{m}| + a + 1}{g+1}, \\ \alpha_{2j+1} &= n_{j+1} - m_{j+1} + \frac{|\boldsymbol{m}| - |\boldsymbol{n}| - a}{g+1}. \end{aligned} \tag{2.15}$$

Example 2.7. Consider the case when $g = 1$, namely, $N = 4$. Let $m_1 - m_2 = m$ and $n_1 - n_2 = n$. We have, from Lemma 2.2, a cycle of a quartet of universal characters $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ connected with the alternate action of vertex operators X_i and Y_i $(i \in \mathbb{Z})$:

$$\begin{tikzcd} {[(m-1)!, n!]} \arrow[d, "Y_{-n}"'] & {[m!, n!]} \arrow[l, "X_{-m}"'] \\ {[\lambda,\mu] = [(m-1)!, (n-1)!]} \arrow[r, "X_m"'] & {[m!, (n-1)!]} \arrow[u, "Y_n"'] \end{tikzcd}$$

Here we write the two-reduced partition as $m! = (m, m-1, \ldots, 1)$; and notice that $(m!)^T = m!$. Then the quartet

$$\begin{aligned} &(\tilde{\tau}_0, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3) = \\ &\quad \left(S_{[(m-1)!,(n-1)!]}(\boldsymbol{x},\boldsymbol{y}), S_{[m!,(n-1)!]}(\boldsymbol{x}',\boldsymbol{y}'), S_{[m!,n!]}(\boldsymbol{x},\boldsymbol{y}), S_{[(m-1)!,n!]}(\boldsymbol{x}',\boldsymbol{y}')\right), \end{aligned}$$

solves (2.13) with

$$(k_0, k_1, k_2, k_3) = \left(2m + a + \frac{1}{2}, 2n - a - \frac{1}{2}, -2m + a + \frac{1}{2}, -2n - a - \frac{1}{2}\right).$$

Remark 2.8. The expression of rational solutions for $g = 1, 2$ in Theorem 2.6 has been established in a different manner by Masuda *et al.* [11, 12]. We refer also to the results [5, 6, 10, 13, 20, 21, 22, 23, 26] where a class of rational (or algebraic) solutions is expressed in terms of Schur polynomials, or universal characters, for each of other Painlevé equations and Garnier's generalizations.

3 Proof of Theorem 1.5

Consider the change of variables:

$$x_n = t + \frac{a}{n}, \quad y_n = -t + \frac{a}{n}, \tag{3.1}$$

then let $S^{(a)}_{[\lambda,\mu]}(t)$ be a function of t equipped with a constant parameter $a \in \mathbb{C}$ defined by

$$S^{(a)}_{[\lambda,\mu]}(t) = S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}). \tag{3.2}$$

Similarly let

$$p^{(a)}_n(t) = p_n(\boldsymbol{x}) \quad \text{and} \quad q^{(a)}_n(t) = p^{(a)}_n(-t) = q_n(\boldsymbol{y}).$$

We remark that $p^{(a)}_n$ is essentially equivalent to the Laguerre polynomial. In fact we have the generating function:

$$\frac{e^{tk/(1-k)}}{(1-k)^a} = \sum_{n\in\mathbb{Z}} p^{(a)}_n(t) k^n, \tag{3.3}$$

and $p^{(a)}_n$ satisfies the linear differential equation of the form:

$$L_a p^{(a)}_n(t) = n p^{(a)}_n(t), \quad L_a = t\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (t+a)\frac{\mathrm{d}}{\mathrm{d}t}. \tag{3.4}$$

Lemma 3.1. *For any λ and μ, the function $S^{(a)}_{[\lambda,\mu]}(t)$ has the following expression of 'Wronskian' type:*

$$S^{(a)}_{[\lambda,\mu]}(t) = \det \begin{bmatrix} \left(\dfrac{\mathrm{d}}{\mathrm{d}t} - 1\right)^{l+l'-j} (-1)^{l+l'-1} q^{(\tilde{a})}_{\mu_{l'-i+1}+i-1}(t), & 1 \le i \le l' \\ \left(\dfrac{\mathrm{d}}{\mathrm{d}t}\right)^{l+l'-j} p^{(\tilde{a})}_{\lambda_{i-l'}-i+l+l'}(t), & l'+1 \le i \le l+l' \end{bmatrix}_{1\le i,j\le l+l'}, \tag{3.5}$$

where

$$\tilde{a} = a - l - l' + 1.$$

Proof. Polynomials $p_n^{(a)}(t)$ and $q_n^{(a)}(t)$ satisfy the formulae:

$$\begin{aligned} p_n^{(a-1)} &= p_n^{(a)} - p_{n-1}^{(a)}, && \text{(3.6a)} \\ \frac{\mathrm{d}}{\mathrm{d}t} p_n^{(a)} &= p_{n-1}^{(a+1)}, && \text{(3.6b)} \\ q_n^{(a-1)} &= q_n^{(a)} - q_{n-1}^{(a)}, && \text{(3.6c)} \\ \left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right) q_n^{(a)} &= -q_n^{(a+1)}, && \text{(3.6d)} \end{aligned}$$

which are essentially the contiguity relations for the Laguerre polynomial; *cf.* [2].

Consider the matrices

$$R_k = I + \sum_{k \le i \le l+l'-1} E_{i,i+1} \quad (1 \le k \le l + l' - 1),$$

where I denotes an identity; and $E_{i,i+1}$ a matrix element, that is, a matrix in which the $(i, i+1)$-th element is unity and all the others are zero. Applying the matrices $R_1, R_2, \ldots, R_{l+l'-1}$ successively from the right hand to the row vector:

$$\left(p_n^{(a)}, p_{n+1}^{(a)}, \ldots, p_{n+l+l'-1}^{(a)}\right),$$

then we obtain

$$\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{l+l'-1} p_{n+l+l'-1}^{(\tilde{a})}, \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{l+l'-2} p_{n+l+l'-1}^{(\tilde{a})}, \ldots, p_{n+l+l'-1}^{(\tilde{a})}\right),$$

by virtue of (3.6a) and (3.6b). By the same procedure as above, the vector:

$$\left(q_n^{(a)}, q_{n-1}^{(a)}, \ldots, q_{n-l-l'+1}^{(a)}\right),$$

is transformed into

$$\left(\left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)^{l+l'-1} (-1)^{l+l'-1} q_n^{(\tilde{a})}, \left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)^{l+l'-2} (-1)^{l+l'-1} q_n^{(\tilde{a})}, \ldots, (-1)^{l+l'-1} q_n^{(\tilde{a})}\right),$$

via (3.6c) and (3.6d). Since $\det R_k = 1$, the proof is now complete. ∎

One can easily verify the

Lemma 3.2. *For an $(l + l') \times (l + l')$-matrix $M = (M_{i,j}(t))_{1 \le i,j \le l+l'}$, we have the following formula of Leibniz type:*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - l'\right) \det M = \sum_{1 \le j \le l+l'} \det \begin{array}{c} \begin{array}{ccccc} \phantom{M_{1,1}} & & \overset{j}{\smile} & & \phantom{M_{1,l+l'}} \end{array} \\ \left[\begin{array}{ccccc} M_{1,1} & \cdots & \left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right) M_{1,j} & \cdots & M_{1,l+l'} \\ \vdots & & \vdots & & \vdots \\ M_{l',1} & \cdots & \left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right) M_{l',j} & \cdots & M_{l',l+l'} \\ \hline M_{l'+1,1} & \cdots & \frac{\mathrm{d}}{\mathrm{d}t} M_{l'+1,j} & \cdots & M_{l'+1,l+l'} \\ \vdots & & \vdots & & \vdots \\ M_{l+l',1} & \cdots & \frac{\mathrm{d}}{\mathrm{d}t} M_{l+l',j} & \cdots & M_{l+l',l+l'} \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} l' \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} l \end{array}.$$

Proof of Theorem 1.5. For the sake of simplicity, we prepare some notations; let

$$L_a = t\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (t+a)\frac{\mathrm{d}}{\mathrm{d}t}, \quad \widetilde{L}_a = t\frac{\mathrm{d}^2}{\mathrm{d}t^2} - (t-a)\frac{\mathrm{d}}{\mathrm{d}t} \quad \text{and} \quad \delta = \frac{\mathrm{d}}{\mathrm{d}t} - l'.$$

Then the left hand side of equation (1.31) is rewritten as follows:

$$\begin{aligned} &\left(tD_t^2 - (t+a)D_t - k\right) f\cdot g + f\frac{\mathrm{d}g}{\mathrm{d}t} \\ &\qquad = f(L_{a+1}g) - (L_a f)g - kfg \\ &\qquad\quad +2t\left\{(\delta^2 f)g + l'(\delta f)g - l' f(\delta g) - (\delta f)(\delta g)\right\}. \end{aligned} \tag{3.7}$$

Now we shall substitute (1.29) with (1.30) into the right hand side of the above formula. Let $f = \det(f_{i,j}(t))_{1\le i,j\le l+l'}$ and $g = \det(g_{i,j}(t))_{1\le i,j\le l+l'+1}$; namely, let

$$f_{i,j}(t) = \begin{cases} q^{(a)}_{\mu_{l'-i+1}+i-j}(t) & (1 \le i \le l'), \\ p^{(a)}_{\lambda_{i-l'}-i+j}(t) & (l'+1 \le i \le l+l'), \end{cases}$$

and so on. We have

$$\begin{aligned} L_a f \;=\; &\sum_j \det \begin{array}{c} \hphantom{f_{1,1}\cdots}\overbrace{\hphantom{j}}^{j}\hphantom{\cdots f_{1,l+l'}} \\ \left[\begin{array}{ccccc} f_{1,1} & \cdots & \widetilde{L}_a f_{1,j} & \cdots & f_{1,l+l'} \\ \vdots & & \vdots & & \vdots \\ f_{l',1} & \cdots & \widetilde{L}_a f_{l',j} & \cdots & f_{l',l+l'} \\ \hline f_{l'+1,1} & \cdots & L_a f_{l'+1,j} & \cdots & f_{l'+1,l+l'} \\ \vdots & & \vdots & & \vdots \\ f_{l+l',1} & \cdots & L_a f_{l+l',j} & \cdots & f_{l+l',l+l'} \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c} f \\ \vdots \\ f \end{array}}\right\} l' \\ \left.\vphantom{\begin{array}{c} f \\ \vdots \\ f \end{array}}\right\} l \end{array} \\ &+2t\sum_j \det \begin{array}{c} \hphantom{f_{1,1}\cdots}\overbrace{\hphantom{j}}^{j}\hphantom{\cdots f_{1,l+l'}} \\ \left[\begin{array}{ccccc} f_{1,1} & \cdots & \frac{\mathrm{d}}{\mathrm{d}t} f_{1,j} & \cdots & f_{1,l+l'} \\ \vdots & & \vdots & & \vdots \\ f_{l',1} & \cdots & \frac{\mathrm{d}}{\mathrm{d}t} f_{l',j} & \cdots & f_{l',l+l'} \\ \hline f_{l'+1,1} & \cdots & 0 & \cdots & f_{l'+1,l+l'} \\ \vdots & & \vdots & & \vdots \\ f_{l+l',1} & \cdots & 0 & \cdots & f_{l+l',l+l'} \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c} f \\ \vdots \\ f \end{array}}\right\} l' \\ \left.\vphantom{\begin{array}{c} f \\ \vdots \\ f \end{array}}\right\} l \end{array} \\ &+2t\sum_{j<k} \det \begin{array}{c} \overbrace{\hphantom{j}}^{j}\qquad\quad\overbrace{\hphantom{k}}^{k} \\ \left[\begin{array}{cc} \frac{\mathrm{d}}{\mathrm{d}t} f_{1,j} & \frac{\mathrm{d}}{\mathrm{d}t} f_{1,k} \\ \vdots & \vdots \\ \frac{\mathrm{d}}{\mathrm{d}t} f_{l+l',j} & \frac{\mathrm{d}}{\mathrm{d}t} f_{l+l',k} \end{array}\right] \end{array}. \end{aligned} \tag{3.8}$$

Noticing the relations:

$$L_a p_n^{(a)}(t) = np_n^{(a)}(t) \quad \text{and} \quad \widetilde{L}_a q_n^{(a)}(t) = -nq_n^{(a)}(t),$$

we see that the first term of right hand side of (3.8) is equal to $(|\lambda|-|\mu|)f$. The sum of the other terms coincides with

$$2t\sum_{j<k}\det\begin{array}{c} \begin{array}{cc} \qquad\qquad j \qquad\qquad & \qquad\qquad k \qquad\qquad \end{array} \\ \left[\begin{array}{cc} \left(\frac{\mathrm{d}}{\mathrm{d}t}-1\right)f_{1,j} & \left(\frac{\mathrm{d}}{\mathrm{d}t}-1\right)f_{1,k} \\ \vdots & \vdots \\ \left(\frac{\mathrm{d}}{\mathrm{d}t}-1\right)f_{l',j} & \left(\frac{\mathrm{d}}{\mathrm{d}t}-1\right)f_{l',k} \\ \hline \frac{\mathrm{d}}{\mathrm{d}t}f_{l'+1,j} & \frac{\mathrm{d}}{\mathrm{d}t}f_{l'+1,k} \\ \vdots & \vdots \\ \frac{\mathrm{d}}{\mathrm{d}t}f_{l+l',j} & \frac{\mathrm{d}}{\mathrm{d}t}f_{l+l',k} \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c} A\\ \vdots \\ A\end{array}}\right\} l' \\ \left.\vphantom{\begin{array}{c} A\\ \vdots \\ A\end{array}}\right\} l \end{array} + 2t\left(l'(\delta f)+\frac{l'(l'+1)}{2}f\right).$$

We can compute also $L_{a+1}g$ in the same way; then we obtain

$$\begin{aligned} \frac{1}{2t}\{\text{RHS of (3.7)}\} &= (\delta^2 f)g-(\delta f)(\delta g) \\ &\quad + f\times\sum_{j<k}\det\begin{array}{c} \begin{array}{cc} \qquad\qquad j \qquad\qquad & \qquad\qquad k \qquad\qquad \end{array} \\ \left[\begin{array}{cc} \left(\frac{\mathrm{d}}{\mathrm{d}t}-1\right)g_{1,j} & \left(\frac{\mathrm{d}}{\mathrm{d}t}-1\right)g_{1,k} \\ \vdots & \vdots \\ \left(\frac{\mathrm{d}}{\mathrm{d}t}-1\right)g_{l',j} & \left(\frac{\mathrm{d}}{\mathrm{d}t}-1\right)g_{l',k} \\ \hline \frac{\mathrm{d}}{\mathrm{d}t}g_{l'+1,j} & \frac{\mathrm{d}}{\mathrm{d}t}g_{l'+1,k} \\ \vdots & \vdots \\ \frac{\mathrm{d}}{\mathrm{d}t}g_{l+l'+1,j} & \frac{\mathrm{d}}{\mathrm{d}t}g_{l+l'+1,k} \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c} A\\ \vdots \\ A\end{array}}\right\} l' \\ \left.\vphantom{\begin{array}{c} A\\ \vdots \\ A\end{array}}\right\} l+1 \end{array} \\ &\quad - g\times\sum_{j<k}\det\begin{array}{c} \begin{array}{cc} \qquad\qquad j \qquad\qquad & \qquad\qquad k \qquad\qquad \end{array} \\ \left[\begin{array}{cc} \left(\frac{\mathrm{d}}{\mathrm{d}t}-1\right)f_{1,j} & \left(\frac{\mathrm{d}}{\mathrm{d}t}-1\right)f_{1,k} \\ \vdots & \vdots \\ \left(\frac{\mathrm{d}}{\mathrm{d}t}-1\right)f_{l',j} & \left(\frac{\mathrm{d}}{\mathrm{d}t}-1\right)f_{l',k} \\ \hline \frac{\mathrm{d}}{\mathrm{d}t}f_{l'+1,j} & \frac{\mathrm{d}}{\mathrm{d}t}f_{l'+1,k} \\ \vdots & \vdots \\ \frac{\mathrm{d}}{\mathrm{d}t}f_{l+l',j} & \frac{\mathrm{d}}{\mathrm{d}t}f_{l+l',k} \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c} A\\ \vdots \\ A\end{array}}\right\} l' \\ \left.\vphantom{\begin{array}{c} A\\ \vdots \\ A\end{array}}\right\} l \end{array} . \end{aligned} \tag{3.9}$$

From Lemma 3.1 we have the expression of $f=f(t)$ and $g=g(t)$ of 'Wronskian' type:

$$\begin{aligned} f &= \det\left[\boldsymbol{\xi}^{(m-1)},\boldsymbol{\xi}^{(m-2)},\ldots,\boldsymbol{\xi}^{(0)}\right], \\ g &= \det\left[\boldsymbol{\eta}^{(m)},\boldsymbol{\eta}^{(m-1)},\ldots,\boldsymbol{\eta}^{(0)}\right], \end{aligned}$$

with $m=l+l'$. Here we let $\boldsymbol{\xi}^{(j)}$ and $\boldsymbol{\eta}^{(j)}$ $(j\geq 0)$ be the column vectors of size m and $m+1$ respectively defined as follows:

$$\begin{aligned} \boldsymbol{\xi}^{(j)} &= D(l',l)^j\times{}^T\left((-1)^{m-1}q^{(\tilde{a})}_{\mu_{l'}},\ldots,(-1)^{m-1}q^{(\tilde{a})}_{\mu_1},p^{(\tilde{a})}_{\lambda_1+l-1},\ldots,p^{(\tilde{a})}_{\lambda_l}\right), \\ \boldsymbol{\eta}^{(j)} &= D(l',l+1)^j\times{}^T\left((-1)^{m-1}q^{(\tilde{a})}_{\mu_{l'}},\ldots,(-1)^{m-1}q^{(\tilde{a})}_{\mu_1},p^{(\tilde{a})}_{k+l},p^{(\tilde{a})}_{\lambda_1+l-1},\ldots,p^{(\tilde{a})}_{\lambda_l}\right), \end{aligned}$$

with

$$D(l',l) = \left[\begin{array}{cccccc} \frac{\mathrm{d}}{\mathrm{d}t}-1 & & & & & \\ & \ddots & & & & \\ & & \frac{\mathrm{d}}{\mathrm{d}t}-1 & & & \\ \hline & & & \frac{\mathrm{d}}{\mathrm{d}t} & & \\ & & & & \ddots & \\ & & & & & \frac{\mathrm{d}}{\mathrm{d}t} \end{array}\right]\begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} l' \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} l \end{array}.$$

Using the formula in Lemma 3.2, we have from (3.9) that

$$\begin{aligned} &\frac{1}{2t}\{\text{RHS of (3.7)}\} = \\ &\qquad \left|\boldsymbol{\xi}^{(m+1)},\boldsymbol{\xi}^{(m-2)},\ldots,\boldsymbol{\xi}^{(0)}\right| \times \left|\boldsymbol{\eta}^{(m)},\boldsymbol{\eta}^{(m-1)},\ldots,\boldsymbol{\eta}^{(0)}\right| \\ &\qquad -\left|\boldsymbol{\xi}^{(m)},\boldsymbol{\xi}^{(m-2)},\ldots,\boldsymbol{\xi}^{(0)}\right| \times \left|\boldsymbol{\eta}^{(m+1)},\boldsymbol{\eta}^{(m-1)},\ldots,\boldsymbol{\eta}^{(0)}\right| \\ &\qquad +\left|\boldsymbol{\xi}^{(m-1)},\boldsymbol{\xi}^{(m-2)},\ldots,\boldsymbol{\xi}^{(0)}\right| \times \left|\boldsymbol{\eta}^{(m+1)},\boldsymbol{\eta}^{(m)},\boldsymbol{\eta}^{(m-2)},\ldots,\boldsymbol{\eta}^{(0)}\right|, \end{aligned} \tag{3.10}$$

which immediately turns out to be zero by the Plücker relation (a determinant identity). ∎

Acknowledgements. The author wishes to thank Professors Tetsu Masuda, Kazuo Okamoto and Hidetaka Sakai for valuable discussions and useful comments.

References

[1] V.E. Adler, *Nonlinear chains and Painlevé equations*, Phys. D **73** (1994), 335–51.

[2] A. Erdélyi, *et al.*, *Higher transcendental functions Vol. II*, McGraw-Hill Book Company, Inc., New York–Toronto–London, 1953.

[3] M. Jimbo and T. Miwa, *Solitons and infinite dimensional Lie algebras*, Publ. Res. Inst. Math. Sci. **18** (1983), 943–1001.

[4] V.G. Kac, *Infinite dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990.

[5] K. Kajiwara and T. Masuda, *On the Umemura polynomials for the Painlevé III equation*, Phys. Lett. A **260** (1999), 462–7.

[6] K. Kajiwara and Y. Ohta, *Determinant structure of the rational solutions for the Painlevé II equation*, J. Math. Phys. **37** (1996), 4693–4704.

[7] ——, *Determinant structure of the rational solutions for the Painlevé IV equation*, J. Phys. A **31** (1998), 2431–46.

[8] K. Koike, *On the decomposition of tensor products of the representations of the classical groups: By means of the universal characters*, Adv. Math. **74** (1989), 57–86.

[9] I.G. Macdonald, *Symmetric Functions and Hall Polynomials* (Second Edition), Oxford Mathematical Monographs, Oxford University Press Inc., New York, 1995.

[10] T. Masuda, *On a class of algebraic solutions to the Painlevé VI equation, its determinant formula and coalescence cascade*, Funkcial. Ekvac. **46** (2003), 121–71.

[11] ——, *Special polynomials associated with the Noumi–Yamada system of type* $A_5^{(1)}$, submitted to Funkcial. Ekvac.

[12] T. Masuda, Y. Ohta and K. Kajiwara, *A determinant formula for a class of rational solutions of Painlevé V equation*, Nagoya Math. J. **168** (2002), 1–25.

[13] M. Noumi, S. Okada, K. Okamoto and H. Umemura, *Special polynomials associated with the Painlevé equations II*, Integrable Systems and Algebraic Geometry, eds. M.-H. Saito, Y. Shimizu and K. Ueno, (World Scientific, Singapore, 1998), pp. 349–72.

[14] M. Noumi and Y. Yamada, *Symmetries in the fourth Painlevé equation and Okamoto polynomials*, Nagoya Math. J. **153** (1999), 53–86.

[15] ——, *Higher order Painlevé equations of type* $A_l^{(1)}$, Funkcial. Ekvac. **41** (1998), 483–503.

[16] A.B. Shabat, *The infinite-dimensional dressing dynamical system*, Inverse Problems **8** (1992), 303–8.

[17] A.B. Shabat and R.I. Yamilov, *Symmetries of nonlinear lattices*, Leningrad Math. J. **2** (1991), 377–400.

[18] K. Takasaki, *Spectral curve, Darboux coordinates and Hamiltonian structure of periodic dressing chains*, Comm. Math. Phys. **241** (2003), 111–42.

[19] T. Tsuda, *Universal characters and an extension of the KP hierarchy*, Comm. Math. Phys., in press.

[20] ——, *Toda equation and special polynomials associated with the Garnier system*, submitted to Math. Ann.

[21] ——, *Rational solutions of the Garnier system in terms of Schur polynomials*, Internat. Math. Res. Notices **2003** (2003), no. 43, 2341–58.

[22] ——, *Tau functions of the fourth Painlevé equation in two variables*, Funkcial. Ekvac., in press.

[23] ——, *Universal characters and integrable systems*, Ph.D. thesis, The University of Tokyo, 2003.

[24] A.P. Veselov and A.B. Shabat, *A dressing chain and the spectral theory of the Schrödinger operator*, Funct. Anal. Appl. **27** (1993), 81–96.

[25] R. Willox and J. Hietarinta, *Painlevé equations from Darboux chains. I.* P_{III}–P_{V}, J. Phys. A **36** (2003), 10615–35.

[26] Y. Yamada, *Determinant formulas for the* τ*-functions of the Painlevé equations of type* A, Nagoya Math. J. **156** (1999), 123–34.

UTMS

2004–4 SAKAI Hidetaka: *A q-analog of the Garnier system.*

2004–5 Takuya Sakasai: *The Magnus representation for the group of homology cylinders.*

2004–6 Johannes Elschner and Masahiro Yamamoto: *Uniqueness in determining polygonal sound-hard obstacles.*

2004–7 Masaaki Suzuki: *Twisted Alexander polynomial for the Lawrence-Krammer representation.*

2004–8 Masaaki Suzuki: *On the Kernel of the Magnus representation of the Torelli group.*

2004–9 Hiroshi Kawabi: *Functional inequalities and an application for parabolic stochastic partial differential equations containing rotation.*

2004–10 Takashi Taniguchi: *On the zeta functions of prehomogeneous vector spaces for pair of simple algebras.*

2004–11 Harutaka Koseki and Takayuki Oda : *Matrix coefficients of representations of* $SU(2,2)$*: — the case of* P_J*-principal series —.*

2004–12 Takeo Satoh: *Twisted first homology groups of the automorphism group of a free group.*

2004–13 M. K. Klibanov and M. Yamamoto: *Lipschitz stability of an inverse problem for an acoustic equation.*

2004–14 Teruhisa Tsuda: *Universal characters, integrable chains and the Painlevé equations.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012