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Nonsynchronous covariation measurement for continuous semimartingales

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Summary: We present a methodology to compute covariance of two time series when the data are sampled from continuous semimartingales at general stopping times in a nonsynchronous manner. This generalizes the result recently obtained by Hayashi and Yoshida (2003).

Key words: Continuous semimartingales, high-frequency data, nonsynchronous trading, quadratic variation, realized volatility.

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1 Introduction

Consider the case when two semimartingales evolve continuously in time and state-space, but are observed only at discrete, random times in a *nonsynchronous* manner. We intend to answer to the question: How do we measure the *covariance* of the two processes well in such a situation?

This type of question arises frequently in *high-frequency finance*. A modern, popular approach for this is to compute

$$V_{\pi(m)} := \sum_{i=1}^{m} (P^1_{t_i} - P^1_{t_{i-1}})(P^2_{t_i} - P^2_{t_{i-1}}), \tag{1.1}$$

which is often called the *realized* covariance (estimator) in the literature. Here, P^1 and P^2 are log-prices, $0 = t_0 < t_1 < \cdots < t_m = T$ are grid points for measuring their respective changes. The popularity of the estimator comes from its *consistency*, i.e., as $\pi(m) := \max_{1\le i\le m} |t_i - t_{i-1}| \to 0$, one has $V_{\pi(m)} \to V$ in probability, not to mention from its ease of implementation. For practical convenience it is standard to take equal spacing, i.e., $t_i - t_{i-1} = T/m$ $(=: h)$, $i \geq 1$. See Andersen, Bollerslev, Diebold, and Labys (2001), for instance, who studied properties of realized covariances/correlations. (Note that the usual *sample covariance* is among alternative approaches. However, its usage may not be justified except for some special cases since it is not generally consistent in the sense just described. Also, it is not immune to the problem that the realized estimator has, as described in the next paragraph.)

Actual transaction data are recorded at random times, in a nonsynchronous, irregular manner. This fact requires one who adopts (1.1) to "synchronize" two time series a priori; choose a common interval length h first, then impute missing observations by either previous-tick interpolation or linear interpolation. Inevitably, the value of V_h depends heavily on the choice of h as well as the interpolation method. In fact, as h gets smaller compared to the (expected) observation intervals, realized covariance estimators (1.1) tend to be biased (Hayashi and Yoshida (2003)). Such a phenomenon has been reported in the empirical finance literature, often referred to as the Epps effect (Epps (1979)).

Hayashi and Yoshida (2003) have proposed a new class of estimators which are free of h and of interpolation scheme. In the case of diffusion-type processes with independent random observation times, they showed that their estimators are consistent for the underlying covariations as the size of observation intervals goes to zero, which is *not* in general possessed by realized estimators *once h is fixed* (as supposed to be so).

This paper provides a moment bound for the error of the estimators in a general situation–processes are *continuous semimartingales* and observation times are *stopping times*, which covers the case discussed in the aforementioned paper by Hayashi and Yoshida. As an immediate corollary its consistency, i.e., the convergence of the estimators to the true, underlying covariation processes as the size of observation intervals goes to zero, is established.

With a similar motivation in mind, Malliavin and Mancino (2002) have developed a Fourier transform based estimator for the covariation. However, their approach is completely different from ours.

2 Main results

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete, filtered probability space. Let $M^k, k = 1, 2$, be L^4 continuous martingales, τ_i^k, $k = 1, 2$, $i \geq 0$, be stopping times with $\tau_0^k = 0$, $k = 1, 2$, such that $\tau_i^k \uparrow \infty$ as $i \to \infty$ almost surely. For each $T \geq 0$, let us define the size of the random partition $\{\tau_i^k \wedge T, i \geq 0\}$ over $[0, T]$ by

$$\Delta_T := \max\{\tau_i^k \wedge T - \tau_{i-1}^k \wedge T;\, k = 1, 2,\, i \geq 1\}.$$

For any interval $I = [a, b)$, $0 \leq a < b$, we denote the increment of M^k over I by

$$\Delta M^k(I) := M^k(b) - M^k(a),\, k = 1, 2.$$

Also, the random intervals associated with $\{\tau_i^k, i \geq 1\}$, truncated at T, are denoted by

$$I_i^k(T) := [\tau_{i-1}^k \wedge T, \tau_i^k \wedge T), k = 1, 2, i \geq 1.$$

We define the "modulus of continuity" of a stochastic process X, by

$$\delta_0(\epsilon, T; X) := \sup\{|X(t) - X(s)|\,;\, s, t \in [0, T],\, |t - s| \leq \epsilon\},$$

for $\epsilon \geq 0$, $T \geq 0$. Also, we define

$$\delta_1(\epsilon, T; X) := \sup\{|\langle X \rangle (t) - \langle X \rangle (s)|\,; s, t \in [0, T], |t - s| \leq \epsilon\},$$

provided that the quadratic variation $\langle \cdot \rangle$ exists and is finite. We let $\delta(\epsilon, T; X) := \delta_0(\epsilon, T; X) + \delta_1(\epsilon, T; X)^{1/2}$ whenever the right-hand side is defined.

With the notation at hand, we are now ready to assert the first main result of the paper as follows.

Theorem 2.1 *There exists an absolute constant C such that*

$$E\left[\left\{\sum_{i,j=1}^{\infty} \Delta M^1(I_i^1(T))\Delta M^2(I_j^2(T))1_{\{I_i^1(T)\cap I_j^2(T)\neq\emptyset\}} - \langle M^1, M^2\rangle (T)\right\}^2\right]$$

$$\leq CE\left[\widetilde{\delta}(3\Delta_T, T; M^1, M^2)^4\right]^{1/2}\left(E\left[M^1(T)^4\right]^{1/2} + E\left[M^2(T)^4\right]^{1/2}\right)$$

for every $T > 0$, where $\widetilde{\delta}(\epsilon, T; M^1, M^2) := \delta(\epsilon, T; M^1) + \delta(\epsilon, T; M^2)$.

All the proofs are placed in section 4.

Notice that, for continuous and adapted f^1 and f^2 with suitable integrability conditions, the stochastic integrals $f^1 \cdot M^1$ and $f^2 \cdot M^2$, and their respective approximations, $\sum_i f^1(\tau_{i-1}^1)(M^1(\tau_i^1 \wedge \cdot) - M^1(\tau_{i-1}^1 \wedge \cdot))$ and $\sum_j f^2(\tau_{j-1}^2)(M^2(\tau_j^2 \wedge \cdot) - M^2(\tau_{j-1}^2 \wedge \cdot))$, may become all L^4 martingales. So, Theorem 2.1 can be modified in the way that could broaden applicability of our approach. In particular, we have an immediate corollary which can be of practical use in statistics.

Corollary 2.1 *Let M and N be L^8 continuous martingales, $\tau_i^{(n)}$ and $\sigma_i^{(n)}$ be stopping times with $\tau_0^{(n)} = \sigma_0^{(n)} = 0$ such that $\tau_i^{(n)} \uparrow \infty$ and $\sigma_i^{(n)} \uparrow \infty$ as $i \to \infty$ almost surely, for every $n \geq 1$. Let f and g be continuous, adapted processes such that $f_t^* \in L^8$ and $g_t^* \in L^8$, for each $t \geq 0$. Let*

$$\Delta_T^{(n)} := \sup\{|\tau_i^{(n)} \wedge T - \tau_{i-1}^{(n)} \wedge T|; i \geq 1\} \vee \sup\{|\sigma_i^{(n)} \wedge T - \sigma_{i-1}^{(n)} \wedge T|; i \geq 1\}.$$

For each $T \geq 0$ and $n \geq 1$, define

$$Y_T^{(n)} := \sum_{i,j=1}^{\infty} f(\tau_{i-1}^{(n)})g(\sigma_{j-1}^{(n)})(M(\tau_i^{(n)} \wedge T) - M(\tau_{i-1}^{(n)} \wedge T))(N(\sigma_j^{(n)} \wedge T) - N(\sigma_{j-1}^{(n)} \wedge T))$$

$$1_{\{[\tau_{i-1}^{(n)}\wedge T, \tau_i^{(n)}\wedge T)\cap[\sigma_{j-1}^{(n)}\wedge T, \sigma_j^{(n)}\wedge T)\neq\emptyset\}}.$$

Then,

$$E\left[\left\{Y_T^{(n)} - \int_0^T f(t)g(t)d\langle M,N\rangle_t\right\}^2\right] \to 0 \quad as\ n\to\infty$$

for every $T \geq 0$, provided that $\Delta_T^{(n)} \to 0$ in probability.

What one needs in practice is to find appropriate f and g that are observable (together with M and N) at $\{\tau_i^{(n)}, i \geq 0\}$, and $\{\sigma_i^{(n)}, i \geq 0\}$.

The results obtained so far can generalize to *continuous semimartingales*. For a stochastic process X, for each $t \geq 0$, we let $|X|(t)(\omega)$ denote the total variation of the path $s \mapsto X(s)(\omega)$ on the interval $[0,t]$, for each $\omega \in \Omega$. A process X whose total variation is finite on each interval $[0,t]$ shall be referred to as a process of finite variation.

Let $S^k = S^k(0) + A^k + M^k$ be a continuous semimartingale, where A^k is a process of finite variation such that $E\left[\left|A^k\right|(T)^4\right] < \infty$, for every $T \geq 0$, and M^k is an L^4 martingale, $k = 1, 2$.

Theorem 2.2 *There exists an absolute constant C such that*

$$\begin{aligned}
&E\left[\left\{\sum_{i,j=1}^{\infty} \Delta S^1(I_i^1(T))\Delta S^2(I_j^2(T))1_{\{I_i^1(T)\cap I_j^2(T)\neq\emptyset\}} - \left\langle S^1, S^2\right\rangle(T)\right\}^2\right] \\
&\leq CE\left[\tilde{\tilde{\delta}}(3\Delta_T, T; S^1, S^2)^4\right]^{1/2}\left(E\left[M^1(T)^4\right]^{1/2} + E\left[M^2(T)^4\right]^{1/2} + 8E\left[\left|A^1\right|(T)^4\right]^{1/2} + 8E\left[\left|A^2\right|(T)^4\right]^{1/2}\right)
\end{aligned} \tag{2.2}$$

for every $T > 0$, where $\tilde{\tilde{\delta}}(\epsilon, T; S^1, S^2) := \delta(\epsilon, T; M^1) + \delta(\epsilon, T; M^2) + \delta_0(\epsilon, T; A^1) + \delta_0(\epsilon, T; A^2)$.

Then, similarly to Corollary 2.1, the sequence $\{Y_T^{(n)}\}$, which bases on stopping times $\{\tau_i^{(n)}, i \geq 0\}$ and $\{\sigma_i^{(n)}, i \geq 0\}$, converges in L^2 to $\left\langle S^1, S^2\right\rangle(T)$ as $n \to \infty$, whenever the mesh size shrinks to zero.

Corollary 2.2 *Let $\tau_i^{(n)}$ and $\sigma_i^{(n)}$ be defined as in Corollary 2.1. Let f and g be defined as in Corollary 2.1. For each $T \geq 0$ and $n \geq 1$, define*

$$Y_T^{(n)} := \sum_{i,j=1}^{\infty} f(\tau_{i-1}^{(n)})g(\sigma_{j-1}^{(n)})(S^1(\tau_i^{(n)}\wedge T) - S^1(\tau_{i-1}^{(n)}\wedge T))(S^2(\sigma_j^{(n)}\wedge T) - S^2(\sigma_{j-1}^{(n)}\wedge T))$$

$$1_{\{[\tau_{i-1}^{(n)}\wedge T, \tau_i^{(n)}\wedge T)\cap[\sigma_{j-1}^{(n)}\wedge T, \sigma_j^{(n)}\wedge T)\neq\emptyset\}}.$$

Then,

$$E\left[\left\{Y_T^{(n)} - \int_0^T f(t)g(t)d\left\langle S^1, S^2\right\rangle_t\right\}^2\right] \to 0 \quad \text{as } n \to \infty$$

for every $T \geq 0$, provided that $\Delta_T^{(n)} \to 0$ in probability.

Finally, by relaxing the L^P assumptions made above we can derive the following weaker but broader result.

Corollary 2.3 *Let $S^k = S^k(0) + A^k + M^k$ be a continuous semimartingale, where A^k is a process of finite variation and M^k is a local martingale, $k = 1, 2$. Suppose f and g are continuous and adapted. Let $\tau_i^{(n)}$, $\sigma_i^{(n)}$, and $Y_T^{(n)}$ be defined as above. Then, as $n \to \infty$,*

$$Y_T^{(n)} \overset{P}{\to} \int_0^T f(t)g(t)d\left\langle S^1, S^2\right\rangle_t,$$

for every $T \geq 0$, provided that $\Delta_T^{(n)} \to 0$ in probability.

Remark 2.1. Estimators of the form,

$$Y_T^{(n)} := \sum_{i,j=1}^{\infty} (M(\tau_i^{(n)} \wedge T) - M(\tau_{i-1}^{(n)} \wedge T))(N(\sigma_j^{(n)} \wedge T) - N(\sigma_{j-1}^{(n)} \wedge T))1_{\{[\tau_{i-1}^{(n)} \wedge T, \tau_i^{(n)} \wedge T) \cap [\sigma_{j-1}^{(n)} \wedge T, \sigma_j^{(n)} \wedge T) \neq \emptyset\}},$$

have been proposed by Hayashi and Yoshida (2003). They have shown the convergence of $Y_T^{(n)}$ to $\langle M, N\rangle_T$ in probability as $n \to \infty$, for diffusion-type processes M and N, when $\{\tau_i^{(n)}, i \geq 0\}$ and $\{\sigma_i^{(n)}, i \geq 0\}$ are *independent* of M and N (and when the limit $\langle M, N\rangle_T$ is non-random). To be more precise, they have consider the following observation times: Let $\tau_i^{(n)}$ and $\sigma_i^{(n)}$ be independent of M and N, such that

(i) as $n \to \infty$, $E[\Delta_T^{(n)}] = o(1)$, or

(ii) $P\left[\Delta_T^{(n)} > n^{-q}\right] = o(1)$ for some $q \in (0, 1)$.

This sampling scheme covers, for instance, the *Poisson* random sampling discussed in the same paper; that is, for two independent Poisson processes N^1 and N^2, independent of M and N, with intensity $\lambda^k = np^k$, $p^k \in (0, 1)$, $k = 1, 2$, $\tau_i^{(n)}$ and $\sigma_j^{(n)}$ are, respectively, the i-th and j-th jump arrival times of N^1 and N^2, with $\tau_0^{(n)} = \sigma_0^{(n)} = 0$.

Apparently, the independence of observation times is restrictive in financial modeling. The key contribution of the paper is to relax the assumption and to allow observations to be made at *arbitrary* stopping times. $\square$

3 Application to finance

The theorems developed can apply to estimation problems in finance. Let $I_i^k(T) := [\tau_{i-1}^k \wedge T, \tau_i^k \wedge T)$ be the ith observation interval of the kth security, as defined previously, such that $\max_{i,k} |I_i^k(T)| \to 0$ in probability.

Example 3.1. (from Hayashi and Yoshida (2003):) *The Black-Scholes model*:

Consider a market with d securities, $P^1, ..., P^d$, where P_t^k is the price of the k-th stock at $t \in [0,T]$, $k = 1, ..., d$. We suppose each P^k follows a geometric Brownian motion,

$$dP_t^k = \mu^k(t) P_t^k dt + \sigma^k(t) P_t^k dW_t^k, \quad P_0^k = p^k,\ k = 1, ..., d,$$

where W^ks $(k = 1, ..., d)$ are Brownian motions with $d\langle W^k, W^l\rangle_t = \rho^{k,l}(t)dt$, $\rho^{k,l}(t) \in (-1,1)$, $\mu^k(t)$, and $\sigma^k(t) > 0$ are all (unknown) deterministic and bounded functions, $\int_0^T |\mu^k(t)|dt < \infty$, k (or l) $= 1, ..., d$.

Put $\mathbf{X} := (X^1, ..., X^d)'$, where $X_t^k := \ln P_t^k$. Then, the $d \times d$-matrix $\mathbf{U}_n := \left(U_n^{k,l}\right)_{1 \le k,l \le d}$, the (k,l)-th element of which is defined by

$$U_n^{k,l} := \sum_{i,j} \Delta X^k(I_i^k(T)) \Delta X^l(I_j^l(T)) 1_{\{I_i^k(T) \cap I_j^l(T) \neq \emptyset\}},$$

is a consistent estimator for the cumulative covariance matrix of returns $\langle \mathbf{X}, \mathbf{X}\rangle_T := \left(\langle X^k, X^l\rangle_T\right)_{1 \le k,l \le d}$ with $\langle X^k, X^l\rangle_T = \int_0^T \sigma^k(t)\sigma^l(t)\rho^{k,l}(t)dt$. □

Example 3.2. *Diffusion models*:

Consider a market with d securities, $P^1, ..., P^d$, where P_t^k is the price of the k-th stock at $t \in [0,\infty)$, $k = 1, ..., d$. We suppose each P^k follows the following stochastic differential equation

$$dP_t^k = \mu^k(t, P_t^k)dt + \sigma^k(t, P_t^k)dW_t^k, \quad P_0^k = p^k(> 0),\ k = 1, ..., d,$$

where $(W^1, ..., W^d)$ is a Gaussian process with stationary increments with $d\langle W^k, W^l\rangle_t = v^{k,l}dt$, $\mu^k, k = 1, ..., d$, are constants (possibly unknown), and $\left(v^{k,l}\right)_{1 \le k,l \le d}$ is a fixed yet unknown, nonnegative definite, symmetric matrix. The (known) coefficients $\mu^k(\cdot,\cdot)$ and $\sigma^k(\cdot,\cdot)(> 0)$ are assumed to satisfy certain regularity conditions.

Then, the $d \times d$-matrix $\mathbf{V}_n := \left(V_n^{k,l}\right)_{1 \le k,l \le d}$, the (k,l)-th element of which is defined by

$$V_n^{k,l} := \sum_{i,j} \frac{\Delta P^k(I_i^k(T)) \Delta P^l(I_j^l(T))}{\sigma^k(\tau_{i-1}^k, P_{\tau_{i-1}^k}^k)\sigma^l(\tau_{j-1}^l, P_{\tau_{j-1}^l}^l)} 1_{\{I_i^k(T) \cap I_j^l(T) \neq \emptyset\}},$$

is a consistent estimator for the covariance matrix $\left(v^{k,l}\right)_{1\le k,l\le d}$ as $n\to\infty$. □

Example 3.3. *Covariance for dynamic portfolios*:

Suppose there are two self-financing strategies θ^1 and θ^2 (the number of shares) for two stocks S^1 and S^2, yielding two portfolios V^1 and V^2, respectively, i.e.,

$$dV_t^k = dB_t^k + \theta^k(t)dS_t^k, \quad k=1,2.$$

Here B^k is a finite variation process representing the holding amount of the liquid riskfree bond, $k=1,2$. $\theta^k(t)$ is assumed to be sufficiently regular.

The price dynamics of S^1 and S^2 are known to be continuous semimartingales, but not necessarily specified. Due to the random nature of transaction times, V^1 and V^2 are not marked-to-market concurrently.

Then, both

$$\widetilde{Y}_T^{(n)} := \sum_{i,j=1}^{\infty} \Delta V^1(I_i^1(T))\Delta V^2(I_j^2(T))1_{\{I_i^1(T)\cap I_j^2(T)\neq\emptyset\}}$$

and

$$\overline{Y}_T^{(n)} := \sum_{i,j=1}^{\infty} \theta^1(\tau_i^1\wedge T)\theta^2(\tau_j^2\wedge T)\Delta S^1(I_i^1(T))\Delta S^2(I_j^2(T))1_{\{I_i^1(T)\cap I_j^2(T)\neq\emptyset\}}$$

are consistent estimators for the cumulative "covariance" of V^1 and V^2, $\left\langle V^1,V^2\right\rangle_T = \int_0^T \theta^1(t)\theta^2(t)d\left\langle S^1,S^2\right\rangle_t$.

□

4 Proofs

4.1 Preliminaries:

Let $\sigma_i^k := \tau_i^k\wedge T$, $k=1,2$, $J_{ij} := 1_{\{I_i^1(T)\cap I_j^2(T)\neq\emptyset\}} = 1_{\{\sigma_{i-1}^1\vee\sigma_{j-1}^2<\sigma_i^1\wedge\sigma_j^2\}}$, and

$$U_{ij} := \Delta M^1(I_i^1(T))\Delta M^2(I_j^2(T))J_{ij} = \left(M^1(\sigma_i^1)-M^1(\sigma_{i-1}^1)\right)\left(M^2(\sigma_j^2)-M^2(\sigma_{j-1}^2)\right)J_{ij},$$

$$A_{ij} := \left(\left\langle M^1,M^2\right\rangle(\sigma_i^1\wedge\sigma_j^2) - \left\langle M^1,M^2\right\rangle(\sigma_{i-1}^1\vee\sigma_{j-1}^2)\right)J_{ij}.$$

Lemma 4.1 *For any* $i,j\ge 1$,

$$E\left[U_{ij}-A_{ij}\middle|\mathcal{F}_{\sigma_{i-1}^1\vee\sigma_{j-1}^2}\right]=0.$$

Proof. Fix i_1 and i_2. Decompose

$$M^k(\sigma^k_{i_k}) - M^k(\sigma^k_{i_k-1}) = I_{k1} + I_{k2} + I_{k1},\ k = 1, 2,$$

where

$$I_{k1} := M^k(\sigma^k_{i_k}) - M^k(\sigma^1_{i_1} \wedge \sigma^2_{i_2});\ I_{k2} := M^k(\sigma^1_{i_1} \wedge \sigma^2_{i_2}) - M^k(\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1});$$

$$I_{k3} := M^k(\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1}) - M^k(\sigma^k_{i_k-1}).$$

[i] Since $\sigma^1_{i_1} \wedge \sigma^2_{i_2} = \sigma^1_{i_1}$ or $\sigma^2_{i_2}$, either $I_{11} = 0$ or $I_{21} = 0$; hence, $I_{11}I_{21} = 0$.

[ii] Since $\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1} = \sigma^1_{i_1-1}$ or $\sigma^2_{i_2-1}$, either $I_{13} = 0$ or $I_{23} = 0$; hence $I_{13}I_{23} = 0$.

[iii] Observing that M^k is a martingale, one has

$$E\left[I_{k1} \middle| \mathcal{F}_{\sigma^1_{i_1} \wedge \sigma^2_{i_2}}\right] = 0,\ k = 1, 2.$$

In the meantime, for $Y \in L^1$,

$$E\left[Y J_{i_1 i_2} \middle| \mathcal{F}_{\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1}}\right] = E\left[E\left[Y \middle| \mathcal{F}_{\sigma^1_{i_1} \wedge \sigma^2_{i_2}}\right] J_{i_1 i_2} \middle| \mathcal{F}_{\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1}}\right].$$

Therefore, for $Y = I_{k1}(I_{l2} + I_{l3})$,

$$E\left[I_{k1}(I_{l2} + I_{l3}) J_{i_1 i_2} \middle| \mathcal{F}_{\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1}}\right] = 0,\ k, l = 1, 2,$$

noting that $(I_{l2} + I_{l3}) J_{i_1 i_2}$ is $\mathcal{F}_{\sigma^1_{i_1} \wedge \sigma^2_{i_2}}$-measurable.

[iv] Observe that

$$E\left[I_{k2} J_{i_1 i_2} \middle| \mathcal{F}_{\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1}}\right] = E\left[J_{i_1 i_2}\ E\left[I_{k2} \middle| \mathcal{F}_{(\sigma^1_{i_1} \wedge \sigma^2_{i_2}) \wedge (\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1})}\right] \middle| \mathcal{F}_{\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1}}\right] = 0$$

because

$$\begin{aligned}
&E\left[I_{k2} \middle| \mathcal{F}_{(\sigma^1_{i_1} \wedge \sigma^2_{i_2}) \wedge (\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1})}\right] \\
&= E\left[M^k(\sigma^1_{i_1} \wedge \sigma^2_{i_2}) - M^k\left((\sigma^1_{i_1} \wedge \sigma^2_{i_2}) \wedge (\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1})\right) \middle| \mathcal{F}_{(\sigma^1_{i_1} \wedge \sigma^2_{i_2}) \wedge (\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1})}\right] \\
&+ E\left[M^k\left((\sigma^1_{i_1} \wedge \sigma^2_{i_2}) \wedge (\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1})\right) - M^k(\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1}) \middle| \mathcal{F}_{(\sigma^1_{i_1} \wedge \sigma^2_{i_2}) \wedge (\sigma^1_{i_1-1} \vee \sigma^2_{i_2-1})}\right] \\
&= 0.
\end{aligned}$$

Therefore,

$$E\left[I_{k2}I_{l3}J_{i_1 i_2}\,|\,\mathcal{F}_{\sigma^1_{i_1-1}\vee\sigma^2_{i_2-1}}\right]=I_{l3}E\left[I_{k2}J_{i_1 i_2}\,|\,\mathcal{F}_{\sigma^1_{i_1-1}\vee\sigma^2_{i_2-1}}\right]=0,\ k,l=1,2,$$

noting that I_{k3} is $\mathcal{F}_{\sigma^1_{i_1-1}\vee\sigma^2_{i_2-1}}$-measurable.

With the aid of [i]-[iv] together, one has

$$\begin{aligned}
&E\left[(I_{11}+I_{12}+I_{13})(I_{21}+I_{22}+I_{23})J_{i_1 i_2}\,|\,\mathcal{F}_{\sigma^1_{i_1-1}\vee\sigma^2_{i_2-1}}\right]\\
&=E\left[I_{12}I_{22}J_{i_1 i_2}\,|\,\mathcal{F}_{\sigma^1_{i_1-1}\vee\sigma^2_{i_2-1}}\right]\\
&=E\left[\left(\langle M^1,M^2\rangle(\sigma^1_{i_1}\wedge\sigma^2_{i_2})-\langle M^1,M^2\rangle(\sigma^1_{i_1-1}\vee\sigma^2_{i_2-1})\right)J_{i_1 i_2}\,\middle|\,\mathcal{F}_{\sigma^1_{i_1-1}\vee\sigma^2_{i_2-1}}\right]
\end{aligned}$$

as claimed. ■

Lemma 4.2 *For $i<i'$ and $j'<j$,*

$$J_{ij}J_{i'j'}=0.$$

Proof. Suppose $i<i'$ and $j'<j$.

If $\sigma^1_{i-1}\vee\sigma^2_{j-1}<\sigma^1_i\wedge\sigma^2_j$, then

$$\begin{aligned}
\sigma^1_{i'-1}\vee\sigma^2_{j'-1}\geq\sigma^1_i\vee\sigma^2_{j'-1}&\geq\sigma^1_i\wedge\sigma^2_j\\
&\geq\sigma^2_{j-1}\geq\sigma^2_{j'}\geq\sigma^2_{i'}\wedge\sigma^2_{j'}.
\end{aligned}$$

Hence, $J_{ij}=1$ implies $J_{i'j'}=0$. Therefore, $J_{ij}J_{i'j'}=0$. ■

Lemma 4.3 *Suppose $k\leq l$. The following inequalities are true:*

(1) $\left|\sum_{j=k}^{l}U_{ij}\right|\leq\left|M^1(\sigma^1_i)-M^1(\sigma^1_{i-1})\right|\delta_0(3\Delta_T,T;M^2)$;

(2) $\left|\sum_{i=k}^{l}U_{ij}\right|\leq\left|M^2(\sigma^2_j)-M^2(\sigma^2_{j-1})\right|\delta_0(3\Delta_T,T;M^1)$;

(3) $\left|\sum_{j=k}^{l}A_{ij}\right|\leq\left(\langle M^1\rangle(\sigma^1_i)-\langle M^1\rangle(\sigma^1_{i-1})\right)^{1/2}\delta_1(\Delta_T,T;M^2)^{1/2}$;

(4) $\left|\sum_{i=k}^{l}A_{ij}\right|\leq\left(\langle M^2\rangle(\sigma^2_j)-\langle M^2\rangle(\sigma^2_{j-1})\right)^{1/2}\delta_1(\Delta_T,T;M^1)^{1/2}$;

Proof. We are going to prove (1) and (3) only.

Let i be fixed. Let

$$j_0 := \min\{j : \ J_{ij} = 1\}; \quad j_1 := \max\{j : \ J_{ij} = 1\}.$$

Note that

$$J_{ij} = 1 \quad \Longleftrightarrow \quad j_0 \leq j \leq j_1.$$

So,

$$\sum_{j=k}^{l} U_{ij} = \sum_{j=j_0 \vee k}^{j_1 \wedge l} U_{ij} = \left(M^1(\sigma_i^1) - M^1(\sigma_{i-1}^1)\right) \left(M^2(\sigma_{j_1 \wedge l}^2) - M^2(\sigma_{j_0 \vee k-1}^2)\right).$$

In the meantime, note that

$$\sigma_{j_1-1}^2 \leq \sigma_i^1 \leq \sigma_{j_1}^2 \quad \text{and} \quad \sigma_{j_0-1}^2 \leq \sigma_{i-1}^1 \leq \sigma_{j_0}^2,$$

which implies

$$\begin{aligned} \sigma_{j_1 \wedge l}^2 - \sigma_{j_0 \vee k-1}^2 &\leq \sigma_{j_1}^2 - \sigma_{j_0-1}^2 \\ &\leq (\sigma_{j_1}^2 - \sigma_{j_1-1}^2) + (\sigma_i^1 - \sigma_{i-1}^1) + (\sigma_{j_0}^2 - \sigma_{j_0-1}^2) \leq 3\Delta_T. \end{aligned}$$

Therefore,

$$\left| \sum_{j=k}^{l} U_{ij} \right| \leq \left| M^1(\sigma_i^1) - M^1(\sigma_{i-1}^1) \right| \delta_0(3\Delta_T, T; M^2).$$

In the meantime,

$$\begin{aligned} \left| \sum_{j=k}^{l} A_{ij} \right| &= \left| \sum_{j=k}^{l} \int_0^T 1_{I_i^1(T)}(t) 1_{I_j^2(T)}(t) d\left\langle M^1, M^2 \right\rangle_t \right| \\ &\leq \sum_{j=k}^{l} \int_0^T 1_{I_i^1(T)}(t) 1_{I_j^2(T)}(t) \left| d\left\langle M^1, M^2 \right\rangle_t \right| \\ &\leq \int_{\sigma_{i-1}^1}^{\sigma_i^1} \left| d\left\langle M^1, M^2 \right\rangle_t \right| \\ &\leq \left(\left\langle M^1 \right\rangle (\sigma_i^1) - \left\langle M^1 \right\rangle (\sigma_{i-1}^1) \right)^{1/2} \left(\left\langle M^2 \right\rangle (\sigma_i^1) - \left\langle M^2 \right\rangle (\sigma_{i-1}^1) \right)^{1/2} \quad \text{(Kunita-Watanabe)} \\ &\leq \left(\left\langle M^1 \right\rangle (\sigma_i^1) - \left\langle M^1 \right\rangle (\sigma_{i-1}^1) \right)^{1/2} \delta_1(\Delta_T, T; M^2)^{1/2}, \end{aligned}$$

as asserted. ∎

4.2 Proof of Theorem 2.1

Observe that

$$\begin{aligned}
\left\{\sum_{i,j} U_{ij} - \left\langle M^1, M^2\right\rangle(T)\right\}^2 &= \left\{\sum_{i,j}(U_{ij} - A_{ij})\right\}^2 \\
&= \sum_{i,j}\sum_{i',j'}(U_{ij} - A_{ij})(U_{i'j'} - A_{i'j'}) \\
&= 2\sum_{i<i'}\sum_{j,j'}(U_{ij} - A_{ij})(U_{i'j'} - A_{i'j'}) + \sum_{i}\sum_{j,j'}(U_{ij} - A_{ij})(U_{ij'} - A_{ij'}) \\
&= 2\sum_{i<i'}\sum_{j<j'} + 2\sum_{i<i'}\sum_{j'<j} + 2\left(\sum_{j}\sum_{i,i'} - \sum_{i}\sum_{j}\right) + \sum_{i}\sum_{j,j'}.
\end{aligned}$$

However, since $(U_{ij} - A_{ij}) = (U_{ij} - A_{ij})J_{ij}$, the second term vanishes,

$$\sum_{i<i'}\sum_{j'<j}(U_{ij} - A_{ij})(U_{i'j'} - A_{i'j'}) = \sum_{i<i'}\sum_{j'<j} J_{ij}J_{i'j'}(U_{ij} - A_{ij})(U_{i'j'} - A_{i'j'}) = 0,$$

thanks to Lemma 4.2.

Since $U_{ij} - A_{ij}$ is $\mathcal{F}_{\sigma_i^1 \vee \sigma_j^2}$-measurable, Lemma 4.1 implies that

$$\sum_{i<i'}\sum_{j<j'} E\left[(U_{ij} - A_{ij})E\left[U_{i'j'} - A_{i'j'} \,|\, \mathcal{F}_{\sigma^1_{i'-1} \vee \sigma^2_{j'-1}}\right]\right] = 0,$$

thus

$$\begin{aligned}
&E\left[\left\{\sum_{i,j} U_{ij} - \left\langle M^1, M^2\right\rangle(T)\right\}^2\right] \\
&= \sum_{j} E\left[\left(\sum_{i} U_{ij} - A_{ij}\right)^2\right] - \sum_{i}\sum_{j} E\left[(U_{ij} - A_{ij})^2\right] + \sum_{i} E\left[\left(\sum_{j} U_{ij} - A_{ij}\right)^2\right] \\
&\le 2\sum_{j} E\left[\left(\sum_{i} U_{ij}\right)^2 + \left(\sum_{i} A_{ij}\right)^2\right] + 2\sum_{i} E\left[\left(\sum_{j} U_{ij}\right)^2 + \left(\sum_{j} A_{ij}\right)^2\right].
\end{aligned}$$

With the aid of Lemma 4.3,

$$\sum_j E\left[\left(\sum_i U_{ij}\right)^2\right] \le E\left[\sum_j \left(M^2(\sigma_j^2) - M^2(\sigma_{j-1}^2)\right)^2 \cdot \delta_0(3\Delta_T, T; M^1)^2\right]$$

$$\le E\left[\left\{\sum_j \left(M^2(\sigma_j^2) - M^2(\sigma_{j-1}^2)\right)^2\right\}^2\right]^{1/2} E\left[\delta_0(3\Delta_T, T; M^1)^4\right]^{1/2}$$

$$\le C_4 E\left[M^{2^*}(T)^4\right]^{1/2} E\left[\delta_0(3\Delta_T, T; M^1)^4\right]^{1/2} \quad \text{(discrete-time Burkholder's inequality)}$$

$$\le 9\left(\frac{4}{3}\right)^4 C_4 E\left[M^2(T)^4\right]^{1/2} E\left[\delta_0(3\Delta_T, T; M^1)^4\right]^{1/2} \quad \text{(Doob's inequality)},$$

where C_4 is a constant for Burkholder's inequality.

In the meantime,

$$\sum_j E\left[\left(\sum_i A_{ij}\right)^2\right] \le E\left[\sum_j \left(\langle M^2\rangle(\sigma_j^2) - \langle M^2\rangle(\sigma_{j-1}^2)\right) \cdot \delta_1(\Delta_T, T; M^1)\right]$$

$$\le E\left[\langle M^2\rangle(T)^2\right]^{1/2} E\left[\delta_1(\Delta_T, T; M^1)^2\right]^{1/2}$$

$$\le \left(\frac{4}{3}\right)^4 C_4 E\left[M^2(T)^4\right]^{1/2} E\left[\delta_1(\Delta_T, T; M^1)^2\right]^{1/2} \quad \text{(Burkholder and Doob)}.$$

The theorem has been proved. ■

4.3 Proof of Corollary 2.1

Preceding the proof of Corollary 2.1, we define auxiliary symbols as follows. It should be noted that in this proof the system of notation that has been utilized in the Theorem 2.1 will be adopted for ease of writing. It is not the same as the one used in the statement of the Corollary, which pursues readability. Nevertheless, this treatment shall be little fear of confusion.

Let $M^k, k = 1, 2$, be L^8 continuous martingales. Let f^k, $k = 1, 2$, be continuous, adapted processes such that $f_t^{k*} \in L^8$, for each $t \ge 0$. Now, for each $n \ge 1$, we define the predictable step process $\overline{f}_n^k$ of f^k, associated with an increasing sequence of stopping times $\{\tau_i^{k,(n)}, i \ge 0\}$, by

$$\overline{f}_n^k(t) := f^k(0)1_{\{0\}}(t) + \sum_{i=0}^{\infty} f^k(\tau_i^{k,(n)})1_{\left(\tau_i^{k,(n)}, \tau_{i+1}^{k,(n)}\right]}(t), \quad t \ge 0,\ k = 1, 2.$$

The stochastic integral of $\overline{f}_n^k$ with respect to M^k is denoted as

$$N_n^k := \overline{f}_n^k \cdot M^k, \quad k = 1, 2.$$

Similarly, $N^k := f^k \cdot M^k$.

Set $\overline{h}_n(t) := \overline{f}^1_n(t)\overline{f}^2_n(t)$, $t \geq 0$, which is the predictable step process of $h(t) := f^1(t) \cdot f^2(t)$. Note that

$$\langle N_n^1, N_n^2 \rangle (T) = \int_0^T \overline{h}_n(t) d \langle M^1, M^2 \rangle (t); \ \ \langle N^1, N^2 \rangle (T) = \int_0^T h(t) d \langle M^1, M^2 \rangle (t).$$

In order to prove the assertion, it suffices to show that both $E\left[\left\{Y_T^{(n)} - \langle N_n^1, N_n^2 \rangle (T)\right\}^2\right] =: A_n$ and $E\left[\left\{\langle N_n^1, N_n^2 \rangle (T) - \langle N^1, N^2 \rangle (T)\right\}^2\right] =: B_n$ go to zero as $n \to \infty$.

[1] Consider A_n first. Because N_n^k, $k = 1, 2$, are L^4 continuous martingales, Theorem 2.1 implies that A_n is bounded by some constant times

$$E\left[\widetilde{\delta}(3\Delta_T, T; N_n^1, N_n^2)^4\right]^{1/2} \left(E\left[N_n^1(T)^4\right]^{1/2} + E\left[N_n^2(T)^4\right]^{1/2}\right).$$

We desire to show that this quantity goes to zero as $n \to \infty$. To this end, first note that

$$E\left[N_n^k(T)^4\right] \leq CE\left[\langle N_n^k(T) \rangle (T)^2\right] \leq CE\left[\left(\int_0^T \overline{f}^k_n(t)^2 d \langle M^k \rangle (t)\right)^2\right] \leq CE\left[f^{k*}(T)^4 \langle M^k \rangle (T)^2\right],$$

which is finite under the given assumptions, $k = 1, 2$.

In the meantime, recall that $\widetilde{\delta}(\cdot, \cdot; N_n^1, N_n^2) \equiv \delta_0(\cdot, \cdot; N_n^1) + \delta_1(\cdot, \cdot; N_n^1)^{1/2} + \delta_0(\cdot, \cdot; N_n^2) + \delta_1(\cdot, \cdot; N_n^2)^{1/2}$. Because

$$\delta_0(\epsilon, T; N_n^k) \leq \delta_0(\epsilon, T; N^k) + \delta_0(\epsilon, T; N^k - N_n^k) \leq \delta_0(\epsilon, T; N^k) + 2 \sup_{0 \leq t \leq T} \left|N^k(t) - N_n^k(t)\right|,$$

we have

$$E\left[\delta_0(3\Delta_T^{(n)}, T; N_n^k)^4\right] \leq 8E\left[\delta_0(3\Delta_T^{(n)}, T; N^k)^4\right] + 128E\left[\sup_{0 \leq t \leq T} \left|N^k(t) - N_n^k(t)\right|^4\right]. \tag{4.3}$$

Due to Burkholder's inequality, the second term of the r.h.s. is bounded by a constant multiple of

$$E\left[\langle N^k - N_n^k \rangle (T)^2\right] \leq E\left[\sup_{0 \leq t \leq T} \left|f^k(t) - f_n^k(t)\right|^8\right]^{1/2} E\left[\langle M^k \rangle (T)^4\right]^{1/2}. \tag{4.4}$$

Since

$$\sup_{0 \leq t \leq T} \left|f^k(t) - f_n^k(t)\right| \leq 2f^{k*}(T),$$

Lebesgue's dominated convergence theorem implies that $E\left[\sup_{0\le t\le T}\left|f^k(t)-f_n^k(t)\right|^8\right]^{1/2}\to 0$ as $n\to\infty$, hence, so does the second term of the r.h.s of (4.3).

Regarding the first term of the r.h.s. of (4.3), since, for arbitrary $\epsilon>0$,

$$\begin{aligned}
E\left[\delta_0(3\Delta_T^{(n)},T;N^k)^4\right] &\le E\left[\delta_0(\epsilon,T;N^k)^4\right]+E\left[\delta_0(3\Delta_T^{(n)},T;N^k)^4;3\Delta_T^{(n)}>\epsilon\right]\\
&\le E\left[\delta_0(\epsilon,T;N^k)^4\right]+E\left[2\sup_{0\le t\le T}\left|N^k(t)\right|^4;\Delta_T^{(n)}>\epsilon/3\right]\\
&\le E\left[\delta_0(\epsilon,T;N^k)^4\right]+CE\left[\left\langle N^k\right\rangle(T)^4\right]^{1/2}P\left[\Delta_T^{(n)}>\epsilon/3\right]^{1/2} \qquad (4.5)
\end{aligned}$$

for some constant $C>0$, we can see that the l.h.s of (4.5) goes to zero as $n\to\infty$, hence, so does the l.h.s. of (4.3), $\lim_{n\to\infty}E\left[\delta_0(3\Delta_T^{(n)},T;N_n^k)^4\right]=0$.

Besides, a similar argument applied to $\delta_1(\cdot,\cdot,T;N_n^k)$ yields, for arbitrary $\epsilon>0$,

$$\begin{aligned}
\frac{1}{2}E\left[\delta_1(3\Delta_T^{(n)},T;N_n^k)^2\right] &\le E\left[\delta_1(3\Delta_T^{(n)},T;N^k)^2\right]+E\left[\delta_1(3\Delta_T^{(n)},T;N^k-N_n^k)^2\right]\\
&\le E\left[\delta_1(\epsilon,T;N^k)^2\right]+E\left[\left\langle N^k\right\rangle(T)^2;\Delta_T^{(n)}>\epsilon/3\right]+E\left[\left\langle N^k-N_n^k\right\rangle(T)^2\right].
\end{aligned}$$

The inequality (4.4) bounds the third term of the r.h.s. Hence, $\lim_{n\to\infty}E\left[\delta_1(3\Delta_T^{(n)},T;N_n^k)^2\right]=0$ is shown.

It follows that $A_n\to 0$ as $n\to\infty$.

[2] Consider B_n next. Since

$$\begin{aligned}
\left|\left\langle N_n^1,N_n^2\right\rangle(T)-\left\langle N^1,N^2\right\rangle(T)\right| &\le \int_0^T\left|\overline{h}_n(t)-h(t)\right|\left|d\left\langle M^1,M^2\right\rangle\right|(t)\\
&\le\left\{\int_0^T\left|\overline{h}_n(t)-h(t)\right|d\left\langle M^1\right\rangle(t)\right\}^{1/2}\left\{\int_0^T\left|\overline{h}_n(t)-h(t)\right|d\left\langle M^2\right\rangle(t)\right\}^{1/2}
\end{aligned}$$

thanks to Kunita-Watanabe's inequality, B_n is bounded by

$$E\left[\sup_{0\le t\le T}\left\{\left|\overline{h}_n(t)-h(t)\right|^4\right\}\right]^{1/2}E\left[\left\langle M^1\right\rangle(T)^4\right]^{1/4}E\left[\left\langle M^2\right\rangle(T)^4\right]^{1/4}.$$

Now, because

$$\sup_{0\le t\le T}\left|\overline{h}_n(t)-h(t)\right|\le 2f^{1*}(T)f^{2*}(T)$$

as well as $\sup_{0\le t\le T}\left|\overline{h}_n(t)-h(t)\right|\to 0$ in probability as $n\to\infty$, Lebesgue's dominated convergence theorem implies that $E\left[\sup_{0\le t\le T}\left\{\left|\overline{h}_n(t)-h(t)\right|^4\right\}\right]$ goes to zero, hence, so does B_n. This ends the proof. ■

4.4 Proof of Theorem 2.2

The same notation defined in the proof of Theorem 2.1 shall be used. Besides, recall that $S^k = S^k(0) + A^k + M^k$ be a continuous semimartingale, where A^k is a process of bounded variation such that $E\left[\left|A^k\right|(T)^4\right] < \infty$, $T \geq 0$, and M^k is an L^4 martingale, $k = 1, 2$.

Beforehand, since $\Delta S^k(I_i^k) = \Delta A^k(I_i^k) + \Delta M^k(I_i^k)$, one may decompose

$$Y_T = \sum_{i,j=1}^{\infty} U_{ij} + \sum_{i,j=1}^{\infty} R_{1ij} + \sum_{i,j=1}^{\infty} R_{2ij} + \sum_{i,j=1}^{\infty} R_{3ij},$$

where

$$\begin{aligned}
U_{ij} &:= \Delta M^1(I_i^1(T))\Delta M^2(I_j^2(T))1_{\{I_i^1(T)\cap I_j^2(T)\neq\emptyset\}} \\
R_{1ij} &:= \Delta M^1(I_i^1(T))\Delta A^2(I_j^2(T))1_{\{I_i^1(T)\cap I_j^2(T)\neq\emptyset\}}, \\
R_{2ij} &:= \Delta A^1(I_i^1(T))\Delta M^2(I_j^2(T))1_{\{I_i^1(T)\cap I_j^2(T)\neq\emptyset\}}, \\
R_{3ij} &:= \Delta A^1(I_i^1(T))\Delta A^2(I_j^2(T))1_{\{I_i^1(T)\cap I_j^2(T)\neq\emptyset\}}.
\end{aligned}$$

Because $\langle S^1, S^2\rangle = \langle M^1, M^2\rangle$, one has

$$\left\{Y_T - \langle S^1, S^2\rangle(T)\right\}^2 \leq 2\left\{\sum_{i,j}(U_{ij} - A_{ij})\right\}^2 + 8\left\{\left(\sum_{i,j=1}^{\infty} R_{1ij}\right)^2 + \left(\sum_{i,j=1}^{\infty} R_{2ij}\right)^2 + \left(\sum_{i,j=1}^{\infty} R_{3ij}\right)^2\right\}.$$

The moment bound for the first term has been obtained already in Theorem 2.1, hence, in order to prove (2.2) one only needs to show $E[(\sum_{i,j=1}^{\infty} R_{kij})^2]$ are bounded by small constants, $k = 1, 2, 3$. Specifically, one can prove

Lemma 4.4

$$E\left[\left(\sum_{i,j=1}^{\infty} R_{1ij}\right)^2\right] \leq E\left[\delta_0(3\Delta_T, T; M^1)^4\right]^{1/2} E\left[\left|A^2\right|(T)^4\right]^{1/2};$$

$$E\left[\left(\sum_{i,j=1}^{\infty} R_{2ij}\right)^2\right] \leq E\left[\delta_0(3\Delta_T, T; M^2)^4\right]^{1/2} E\left[\left|A^1\right|(T)^4\right]^{1/2};$$

$$E\left[\left(\sum_{i,j=1}^{\infty} R_{3ij}\right)^2\right] \leq \left\{E\left[\delta_0(3\Delta_T, T; A^1)^4\right]^{1/2} E\left[\left|A^2\right|(T)^4\right]^{1/2}\right\} \wedge \left\{E\left[\delta_0(3\Delta_T, T; A^2)^4\right]^{1/2} E\left[\left|A^1\right|(T)^4\right]^{1/2}\right\}.$$

Proof. Similar arguments to Lemma 4.3 can apply. For $k \le l$,

$$\begin{aligned}\left|\sum_{i=k}^{l} R_{1ij}\right| &= \left|\sum_{i=k}^{l} \Delta M^1(I_i^1(T))\Delta A^2(I_j^2(T))J_{ij}\right| \\ &= \left|\left(A^2(\sigma_j^2) - A^2(\sigma_{j-1}^2)\right)\left(M^1(\sigma_{i\wedge l}^1) - M^1(\sigma_{i\vee k-1}^1)\right)\right| \\ &\le \left|A^2(\sigma_j^2) - A^2(\sigma_{j-1}^2)\right| \delta_0(3\Delta_T, T; M^1).\end{aligned}$$

Similarly,

$$\left|\sum_{i=k}^{l} R_{2ij}\right| \le \left|A^1(\sigma_j^1) - A^1(\sigma_{j-1}^1)\right| \delta_0(3\Delta_T, T; M^2),$$

$$\left|\sum_{i=k}^{l} R_{3ij}\right| \le \left|A^2(\sigma_j^2) - A^2(\sigma_{j-1}^2)\right| \delta_0(3\Delta_T, T; A^1); \quad \left|\sum_{j=k}^{l} R_{3ij}\right| \le \left|A^1(\sigma_j^1) - A^1(\sigma_{j-1}^1)\right| \delta_0(3\Delta_T, T; A^2).$$

Therefore,

$$\begin{aligned}E\left[\left(\sum_{i,j=1}^{\infty} R_{1ij}\right)^2\right] &= E\left[\left\{\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} R_{1ij}\right)\right\}^2\right] \\ &\le E\left[\delta_0(3\Delta_T, T; M^1)^2 \left\{\sum_{j=1}^{\infty} \left|A^2(\sigma_j^2) - A^2(\sigma_{j-1}^2)\right|\right\}^2\right],\end{aligned}$$

but $\sum_{j=1}^{\infty} \left|A^2(\sigma_j^2) - A^2(\sigma_{j-1}^2)\right| \le \left|A^2\right|(T)$, which proves the first inequality. The others can be shown by the same way. ■

Therefore, Theorem 2.2 has been proved. ■

4.5 Proof of Corollary 2.2

This can be shown by the same argument as that of the proof for Corollary 2.1.

We maintain the same hypotheses as Theorem 2.2 except that M^k is an L^8 martingale, $k = 1, 2$. Let f^k, $k = 1, 2$, be continuous, adapted processes such that $f_t^{k*} \in L^8$, for each $t \ge 0$. The symbols defined there carry over here.

We let $P_n^k := \overline{f}_n^k \cdot S^k$ denotes the stochastic integral of $\overline{f}_n^k$ with respect to the continuous semimartingale S^k, $k = 1, 2$. Similarly, $P^k := f^k \cdot S^k$.

To prove the assertion, it suffices to show that both $E\left[\left\{Y_T^{(n)} - \left\langle P_n^1, P_n^2\right\rangle(T)\right\}^2\right] =: A_n$ and $E\left[\left\{\left\langle P_n^1, P_n^2\right\rangle(T) - \left\langle P^1, P^2\right\rangle(T)\right\}^2\right] =: B_n$ go to zero as $n \to \infty$. For the evaluation of A_n, Theorem 2.2 can apply. For B_n, the argument [2] in the proof of Corollary 2.1 can be used, because $\left\langle P_n^1, P_n^2\right\rangle = \left\langle N_n^1, N_n^2\right\rangle$ and $\left\langle P^1, P^2\right\rangle = \left\langle N^1, N^2\right\rangle$. ■

4.6 Proof of Corollary 2.3

Usual localization argument can apply. The symbols defined in the proof above for Corollary 2.2 carry over. For simplicity we consider the case for $f^1 \equiv f^2 \equiv 1$; the same argument can apply to the general case. We introduce, for each $K \geq 1$, the stopping time

$$T_K := \inf\left\{t \geq 0;\ |M^k(t)| \geq K \text{ or } |A^k|(t) \geq K \text{ or } \left\langle M^k\right\rangle(t) \geq K,\ k = 1, 2\right\}, \text{ if } |S_0| < K;$$

$T_K := 0$, if $|S_0| \geq K$; $T_K := \infty$, if $|S_0| < K$ and $\{\cdots\} = \emptyset$. The finiteness of S_t implies that $T_K \uparrow \infty$ a.s., as $K \to \infty$.

Fix arbitrary $K \geq 1$. For arbitrary $\eta > 0$,

$$\begin{aligned} P\left[\left|Y_T^{(n)} - \left\langle S^1, S^2\right\rangle(T)\right| \geq \eta\right] &\leq P\left[T_K \leq T\right] + P\left[\left|Y_T^{(n)} - \left\langle S^1, S^2\right\rangle(T)\right| \geq \eta, T_K > T\right] \\ &\leq P\left[T_K \leq T\right] + P\left[\left|Y_T^{(n),K} - \left\langle S^{1,K}, S^{2,K}\right\rangle(T)\right| \geq \eta\right] \end{aligned}$$

where $S_t^{k,K} := S_{t\wedge T_K}^k$, $t \geq 0$, is the stopped process, $k = 1, 2$, and $Y_T^{(n),K}$ is constructed based on them. On $\{T_K > T\}$, $S_t^{k,K} \equiv S_t^k$, $0 \leq t \leq T$, $k = 1, 2$, hence, $Y_T^{(n),K} = Y_T^{(n)}$. In particular, $|A|^k$ and M^k are bounded on the event.

Therefore, in light of Theorem 2.2, for every fixed K,

$$\limsup_{n\to\infty} P\left[\left|Y_T^{(n)} - \left\langle S^1, S^2\right\rangle(T)\right| \geq \eta\right] \leq P\left[T_K \leq T\right].$$

Letting $K \to \infty$ implies

$$\lim_{n\to\infty} P\left[\left|Y_T^{(n)} - \left\langle S^1, S^2\right\rangle(T)\right| \geq \eta\right] = 0.$$

■

5 Conclusion

Extending Hayashi and Yoshida (2003), the paper provides a methodology to construct an estimator for the covariation processes of continuous semimartingales based only on samples taken at general stopping times in a nonsynchronous manner. The estimator is consistent in the sense that it tends to the true covariation as the sampling interval goes to zero, which is not possessed by the popular approach, realized estimators, which require to pre-fix a common regular interval $h > 0$ for "synchronizing" original time series. Because the setup is general the obtained results can be applicable widely in finance.

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References

Andersen, T., T. Bollerslev, F. Diebold, and P. Labys (2001). The Distribution of Realized Exchange Rate Volatility, *J. Amer. Statist. Assoc.* 96, 42–55.

Epps, T. (1979). Comovements in Stock Prices in the Very Short Run, *J. Amer. Statist. Assoc.* 74, 291–298.

Hayashi, T., and N. Yoshida (2003). On Covariance Estimation of Nonsynchronously Observed Diffusion Processes, in revision.

Malliavin, P., and M. Mancino (2002). Fourier Series Method for Measurement of Multivariate Volatilities, *Finance Stoch.* 6, 49–61.

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