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# CARLEMAN ESTIMATES FOR THE THREE-DIMENSIONAL NON-STATIONARY LAMÉ SYSTEM AND THE APPLICATION TO AN INVERSE PROBLEM

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In this paper, we establish Carleman estimates for the three-dimensional isotropic non-stationary Lamé system with the homogeneous Dirichlet boundary conditions. Using this estimate, we prove the uniqueness and the stability in determining spatially varying density and two Lamé coefficients by a single measurement of solution over $(0,T)\times\omega$, where $T>0$ is sufficiently large and a subdomain $\omega$ satisfies a geometric condition.

## §1. Introduction.

This paper is concerned with Carleman estimates for the three-dimensional non-stationary isotropic Lamé system with the homogeneous Dirichlet boundary condition and an application to an inverse problem of determining spatially varying density and the Lamé coefficients by a single interior measurement of the solution.

We consider the three-dimensional isotropic non-stationary Lamé system:

$$\begin{aligned}(P\mathbf{u})(x_0,x') \equiv \rho(x')\partial_{x_0}^2\mathbf{u}(x_0,x') - (L_{\lambda,\mu}\mathbf{u})(x_0,x') = \mathbf{f}(x_0,x'),\\ x\equiv(x_0,x')\in Q\equiv(0,T)\times\Omega,\end{aligned}\tag{1.1}$$

where

$$\begin{aligned}(L_{\lambda,\mu}\mathbf{v})(x') \equiv \mu(x')\Delta\mathbf{v}(x') + (\mu(x')+\lambda(x'))\nabla_{x'}\mathrm{div}\,\mathbf{v}(x')\\ +(\mathrm{div}\,\mathbf{v}(x'))\nabla_{x'}\lambda(x') + (\nabla_{x'}\mathbf{v}+(\nabla_{x'}\mathbf{v})^T)\nabla_{x'}\mu(x'), \qquad x'\in\Omega.\end{aligned}\tag{1.2}$$

Throughout this paper, $\Omega\subset\mathbb{R}^3$ is a bounded domain whose boundary $\partial\Omega$ is of class $C^3$, $x_0$ and $x'=(x_1,x_2,x_3)$ denote the time variable and the spatial variable

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respectively, and $\mathbf{u} = (u_1, u_2, u_3)^T$ is displacement at $(x_0, x')$ where $\cdot^T$ denotes the transpose of matrices, $E_k$ is the $k \times k$ unit matrix,

$$\partial_{x_j}\phi = \phi_{x_j} = \frac{\partial \phi}{\partial x_j}, \quad j = 0,1,2,3.$$

We set $\nabla_{x'}\mathbf{v} = (\partial_{x_k} v_j)_{1\le j,k\le 3}$ for a vector function $\mathbf{v} = (v_1, v_2, v_3)^T$ and $\nabla_{x'}\phi = (\partial_{x_1}\phi, \partial_{x_2}\phi, \partial_{x_3}\phi)^T$ for a scalar function $\phi$. Henceforth $\nabla$ means $\nabla_x = (\partial_{x_0}, \partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ if we do not specify. Moreover the coefficients $\rho$, $\lambda$, $\mu$ satisfy

$$\rho, \lambda, \mu \in C^2(\overline{\Omega}), \quad \rho(x') > 0,\ \mu(x') > 0,\ \lambda(x') + \mu(x') > 0 \quad \text{for } x' \in \overline{\Omega}. \tag{1.3}$$

The Carleman estimate is an $L^2$-inequality of solution to a partial differential equation and is involved with a large parameter and a special weight function. The Carleman estimate was introduced by Carleman [Ca] for proving the unique continuation for an elliptic equation and general theories have been developed for single partial differential equations (e.g., [Hö]). Moreover the Carleman estimates have been effectively applied to the following problems:

(1) **Energy estimate called "observability inequality"**: Cheng, Isakov, Yamamoto and Zhou [CIYZ], Kazemi and Klibanov [KK], Klibanov and Malinsky [KM], Lasiecka and Triggiani [LT], Lasiecka, Triggiani and Zhang [LTZ].
(2) **Exact controllability and related control problems:** Bellassoued [B1] - [B3], Imanuvilov [Im1], Imanuvilov and Yamamoto [IY5], [IY6].
(3) **Inverse problems of determining functions in partial differential equations by a finite number of overlateral boundary data:** See Bukhgeim and Klibanov [BuK] as a pioneering paper. There are extensive references and we will give them in Section 3.

Thus it is first important to establish a Carleman estimate, which depends on types of partial differential equations under consideration. Especially for a single partial differential equation, the general theory for Carleman estimates has been well developed (e.g., [Hö], [Is2], [Is3]). In particular, for a single hyperbolic equation, see Imanuvilov [Im2]. However, for systems of partial differential equations where the principal terms are coupled, the results are still restricted, because of the intrinsic difficulty. The most general result for such a system is the Carleman type estimate obtained in the proof of the Carderon uniqueness theorem (see e.g., [E], [Zui]).

The non-stationary isotropic Lamé system is basic in the theory of elasticity, and unfortunately it does not satisfy all the conditions of the Calderon uniqueness theorem. In the existing papers, Carleman estimates for the Lamé system have been proved mainly for functions with compact supports (e.g., Eller, Isakov, Nakamura and Tataru [EINT], Ikehata, Nakamura and Yamamoto [INY], Imanuvilov, Isakov and Yamamoto [IIY1], Isakov [Is1]). Because of the restriction that $\mathbf{u}$ under consideration should have compact support, for the observability inequalities and the inverse problems, we have to take Cauchy data $\mathbf{u}$ and $\nabla\mathbf{u}$ on the whole boundary $(0,T)\times\partial\Omega$ or $\mathbf{u}$ in a neighbourhood of $\partial\Omega$ over $(0,T)$. Since we need not take Cauchy data on $(0,T)\times\partial\Omega$ or in such a neighbourhood for the wave equation (e.g., Lions [Li] for the observability inequality, and Imanuvilov and Yamamoto [IY2], [IY4] for the inverse problem for a single hyperbolic equation), we can naturally expect similar results also for the non-stationary isotropic Lamé system.

In the two-dimensional case, we have recently established Carleman estimates for $\mathbf{u}$ without compact supports to apply them to an inverse problem of determining the density and two Lamé coefficients:

(1) Imanuvilov and Yamamoto [IY8] for the case of the Dirichler boundary condition
(2) Imanuvilov and Yamamoto [IY9] for the case of the stress boundary condition.

In this paper, we will prove Carleman estimates in the case where the spatial dimension is three and $\mathbf{u}$ satisfies the homogeneous Dirichlet boundary condition and apply them to an inverse problem of determining $\rho$, $\lambda$ and $\mu$ by an interior measurement after suitably choosing single initial data. The three-dimensional case is handled similarly to the two-dimensional case [IY8], but the treatment should be modified.

We refer to Imanuvilov and Yamamoto [IY7] concerning the stationary isotropic Lamé system, and Isakov, Nakamura and Wang [INW], Lin and Wang [LW] concerning the Lamé system with residual stress which causes anisotropicity.

This paper is composed of seven sections. In Section 2, we state Carleman estimates (Theorems 2.1 - 2.3) for functions which do not necessarily have compact supports but satisfy the homogeneous Dirichlet boundary condition on $(0,T)\times\partial\Omega$. Theorem 2.1 is a Carleman estimate whose right hand side is estimated in $H^1$-space. Theorems 2.2 and 2.3 are Carleman estimates respectively with right hand sides in $L^2$-space and in $H^{-1}$-space, and are proved from Theorem 2.1 by the same method in [IY8]. In Section 3, we will apply the $H^{-1}$-Carleman estimate (Theorem 2.3), and prove the uniqueness and the conditional stability in the inverse problem with a single interior measurement. In Sections 4-7, we prove Theorem 2.1.

**Notations.** $H^{1,s}(Q)$ is the Sobolev space of scalar-valued functions equipped with the norm

$$\|u\|_{H^{1,s}(Q)} = \sqrt{\int_Q (|\nabla u|^2 + s^2u^2)dx},$$

$\mathbf{H}^{1,s}(Q) = H^{1,s}(Q)\times\cdots\times H^{1,s}(Q)$ is the corresponding space of vector-valued functions. Henceforth we set

$$i=\sqrt{-1}, \qquad D_{x_j} = \frac{1}{i}\partial_{x_j}, \qquad j=0,1,2,3$$

and $\overline{c}$ denotes the complex conjugate of $c\in\mathbb{C}$. By $\mathcal{L}(X,Y)$ we denote the Banach space of all the linear bounded operators defined on a Banach space $X$ to another Banach space $Y$. We set

$$\xi=(\xi_0,\xi_1,\xi_2,\xi_3), \quad \xi'=(\xi_0,\xi_1,\xi_2), \quad \zeta=(s,\xi_0,\xi_1,\xi_2).$$

By $\mathcal{O}(\delta)$ we denote the conic neighbourhood of a point $\zeta^*$:

$$\mathcal{O}(\delta)=\left\{\zeta;\left|\frac{\zeta}{|\zeta|}-\zeta^*\right|\le\delta\right\}.$$

**§2. Carleman estimates for the three-dimensional non-stationary Lamé system.**

Let us consider the three-dimensional Lamé system

$$P\mathbf{u}(x_0,x') \equiv \rho(x')\partial_{x_0}^2\mathbf{u}(x_0,x') - (L_{\lambda,\mu}\mathbf{u})(x_0,x') = \mathbf{f}(x_0,x') \quad \text{in } Q, \tag{2.1}$$

$$\mathbf{u}|_{(0,T)\times\partial\Omega} = 0, \quad \mathbf{u}(T,x') = \partial_{x_0}\mathbf{u}(T,x') = \mathbf{u}(0,x') = \partial_{x_0}\mathbf{u}(0,x') = 0, \tag{2.2}$$

where $\mathbf{u} = (u_1,u_2,u_3)^T, \mathbf{f} = (f_1,f_2,f_3)^T$ are vector-valued functions, and the partial differential operator $L_{\lambda,\mu}$ is defined by (1.2). The coefficients $\rho$, $\lambda$, $\mu \in C^2(\overline{\Omega})$ are assumed to satisfy (1.3). Let $\omega \subset \Omega$ be an arbitrarily fixed subdomain (not necessarily connected). By $\vec{n}(x') = (n_1(x'),n_2(x'),n_3(x'))$ and $\vec{t}(x')$ respectively denote the outward unit normal vector and a unit tangential vector to $\partial\Omega$ at $x'$ and set $\frac{\partial v}{\partial\vec{n}} = \nabla_{x'}v\cdot\vec{n}$ and $\frac{\partial v}{\partial\vec{t}} = \nabla_{x'}v\cdot\vec{t}$. Set

$$Q_\omega = (0,T)\times\omega.$$

We set

$$\begin{cases} p_1(x,\xi) = \rho(x')\xi_0^2 - \mu(x')(\xi_1^2+\xi_2^2+\xi_3^2), \\ p_2(x,\xi) = \rho(x')\xi_0^2 - (\lambda(x')+2\mu(x'))(\xi_1^2+\xi_2^2+\xi_3^2) \end{cases} \tag{2.3}$$

for $\xi = (\xi_0,\xi_1,\xi_2,\xi_3)$, and $\nabla_\xi = (\partial_{\xi_0},\partial_{\xi_1},\partial_{\xi_2},\partial_{\xi_3})$. For arbitrary smooth functions $\varphi(x,\xi)$ and $\psi(x,\xi)$, we define the Poisson bracket by the formula

$$\{\varphi,\psi\} = \sum_{j=0}^{3}(\partial_{\xi_j}\varphi)(\partial_{x_j}\psi) - (\partial_{\xi_j}\psi)(\partial_{x_j}\varphi).$$

We set $< a,b >= \sum_{k=1}^3 a_k\overline{b}_k$ for $a = (a_1,a_2,a_3)$ and $b = (b_1,b_2,b_3) \in \mathbb{C}^3$.

We assume that the density $\rho$, the Lamé coefficients $\lambda,\mu$ and the domains $\Omega,\omega$ satisfy the following condition (cf. [Hö]).

**Condition 2.1.** *There exists a function $\psi \in C^3(\overline{Q})$ such that $|\nabla_x\psi| \ne 0$ on $\overline{Q\setminus Q_\omega}$, and*

$$\{p_k,\{p_k,\psi\}\}(x,\xi) > 0, \quad \forall k \in \{1,2\} \tag{2.4}$$

*if $(x,\xi) \in (\overline{Q\setminus Q_\omega})\times(\mathbb{R}^4\setminus\{0\})$ satisfies $p_k(x,\xi) =< \nabla_\xi p_k, \nabla\psi >= 0$ and*

$$\frac{1}{2is}\{p_k(x,\xi - is\nabla\psi(x)), p_k(x,\xi+is\nabla\psi(x))\} > 0, \quad \forall k \in \{1,2\} \tag{2.5}$$

*if $(x,\xi,s) \in (\overline{Q\setminus Q_\omega})\times(\mathbb{R}^4\setminus\{0\})\times(\mathbb{R}_+\setminus\{0\})$ satisfies*

$$p_k(x,\xi+is\nabla\psi(x)) =< \nabla_\xi p_k(x,\xi+is\nabla\psi(x)), \nabla\psi(x) >= 0.$$

*On the lateral boundary, we assume that*

$$\sqrt{\rho}|\psi_{x_0}| < \frac{\mu}{\sqrt{\lambda+2\mu}}\left|\frac{\partial\psi}{\partial\vec{t}}\right| + \frac{\sqrt{\mu}\sqrt{\lambda+\mu}}{\sqrt{\lambda+2\mu}}\left|\frac{\partial\psi}{\partial\vec{n}}\right| \text{ for any unit tangential vector } \vec{t}(x'),\ x' \in \overline{\partial\Omega\setminus\partial\omega}$$

$$p_1(x,\nabla\psi) < 0 \quad \forall x \in \overline{(0,T)\times(\partial\Omega\setminus\partial\omega)} \quad \text{and} \quad \left.\frac{\partial\psi}{\partial\vec{n}}\right|_{(0,T)\times(\overline{\partial\Omega\setminus\partial\omega})} < 0. \tag{2.6}$$

Let $\psi(x)$ be the weight function in Condition 2.1. Using this function, we introduce the function $\phi(x)$ by

$$\phi(x) = e^{\tau\psi(x)}, \quad \tau > 1, \tag{2.7}$$

where the parameter $\tau > 0$ will be fixed below. Denote

$$\|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q)} = \int_Q \left( \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 + s|\nabla \mathrm{rot}\, \mathbf{u}|^2 + s^3|\mathrm{rot}\, \mathbf{u}|^2 + s|\nabla \mathrm{div}\, \mathbf{u}|^2 + s^3|\mathrm{div}\, \mathbf{u}|^2 \right) e^{2s\phi} dx,$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $\alpha_j \in \mathbb{N}_+ \cup \{0\}$, $j \in \{0,1,2,3\}$, $\partial_x^\alpha = \partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$.

Now we state our Carleman estimates as main results.

**Theorem 2.1.** *Let $\mathbf{f} \in \mathbf{H}^1(Q)$ and let the function $\psi$ satisfy Condition 2.1 and (1.3) holds true. Then there exists $\widehat{\tau} > 0$ such that for any $\tau > \widehat{\tau}$, there exists $s_0 = s_0(\tau) > 0$ such that for any solution $\mathbf{u} \in \mathbf{H}^1(Q) \cap L^2(0,T;\mathbf{H}^2(\Omega))$ to problem (2.1) - (2.2), the following estimate holds true:*

$$\begin{aligned} \|\mathbf{u}\|^2_{Y(\phi,Q)} &\equiv \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q)} + s\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}} e^{s\phi}\right\|^2_{\mathbf{H}^1((0,T)\times\partial\Omega)} + s\left\|\frac{\partial^2 \mathbf{u}}{\partial \vec{n}^2} e^{s\phi}\right\|^2_{\mathbf{L}^2((0,T)\times\partial\Omega)} \\ &\le C(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + \|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}), \quad \forall s \ge s_0(\tau), \end{aligned} \tag{2.8}$$

*where the constant $C = C(\tau) > 0$ is independent of $s$.*

Next we formulate Carleman estimates where norms of the function $\mathbf{f}$ are taken respectively in $\mathbf{L}^2(Q)$ and $L^2(0,T;\mathbf{H}^{-1}(\Omega))$. In particular, the latter Carleman estimate is used in Section 3 for obtaining our stability result in the inverse problem.

In addition to Condition 2.1, we assume that

$$\partial_{x_0}\psi(T,x') < 0, \quad \partial_{x_0}\psi(0,x') > 0, \qquad \forall x' \in \overline{\Omega}. \tag{2.9}$$

We have

**Theorem 2.2.** *Let $\mathbf{f} \in \mathbf{L}^2(Q)$ and let us assume (1.3), (2.9) and Condition 2.1. Then there exists $\widehat{\tau} > 0$ such that for any $\tau > \widehat{\tau}$, there exists $s_0 = s_0(\tau) > 0$ such that for any solution $\mathbf{u} \in \mathbf{H}^1(Q)$ to problem (2.1) - (2.2), the following estimate holds true:*

$$\begin{aligned} &\int_Q (|\nabla \mathbf{u}|^2 + s^2|\mathbf{u}|^2)e^{2s\phi} dx \\ \le &C\left(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + \int_{Q_\omega} (|\nabla \mathbf{u}|^2 + s^2|\mathbf{u}|^2)e^{2s\phi} dx\right), \quad \forall s \ge s_0(\tau), \end{aligned} \tag{2.10}$$

*where the constant $C = C(\tau) > 0$ is independent of $s$.*

**Theorem 2.3.** *Let* $\mathbf{f} = \mathbf{f}_{-1} + \sum_{j=0}^3 \partial_{x_j}\mathbf{f}_j$ *with* $\mathbf{f}_{-1} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ *and* $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \in \mathbf{L}^2(Q)$*, and let us assume (1.3), (2.9) and Condition 2.1. Then there exists* $\widehat{\tau} > 0$ *such that for any* $\tau > \widehat{\tau}$*, there exists* $s_0 = s_0(\tau) > 0$ *such that for any solution* $\mathbf{u} \in \mathbf{L}^2(Q)$ *to problem (2.1) - (2.2), the following estimate holds true:*

$$\int_Q |\mathbf{u}|^2 e^{2s\phi} dx$$
$$\le C\left( \|\mathbf{f}_{-1}e^{s\phi}\|^2_{L^2(0,T;\mathbf{H}^{-1}(\Omega))} + \sum_{j=0}^3 \|\mathbf{f}_j e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + \int_{Q_\omega} |\mathbf{u}|^2 e^{2s\phi} dx \right), \quad \forall s \ge s_0(\tau), \tag{2.11}$$

*where the constant* $C = C(\tau) > 0$ *is independent of* $s$*.*

In Theorems 2.2 and 2.3, the solution $\mathbf{u}$ is defined by the transposition method (e.g., [Li]). On the basis of Theorem 2.1, we can prove Theorems 2.2 and 2.3 exactly in the same way as the corresponding theorems in [IY8], and it suffices to prove only Theorem 2.1.

## §3. Inverse problem of determining the density and the Lamé coefficients by a single measurement.

Recall that the differential operator $L_{\lambda,\mu}$ is defined by (1.2). We assume (1.3) for $\rho, \lambda, \mu$. By $\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta)(x)$, we denote the sufficiently smooth solution to

$$\rho(x')(\partial_{x_0}^2 \mathbf{u})(x) = (L_{\lambda,\mu}\mathbf{u})(x), \quad x \in Q, \tag{3.1}$$

$$\mathbf{u}(x) = \eta(x), \qquad x \in (0,T) \times \partial\Omega, \tag{3.2}$$

$$\mathbf{u}(T/2, x') = \mathbf{p}(x'), \quad (\partial_{x_0}\mathbf{u})(T/2, x') = \mathbf{q}(x'), \quad x' \in \Omega, \tag{3.3}$$

with given $\eta$, $\mathbf{p}$ and $\mathbf{q}$. Let $\omega \subset \Omega$ be a suitably given subdomain.

In this section, we discuss

**Inverse Problem.** *Let* $\mathbf{p}_j, \mathbf{q}_j, \eta_j$, $1 \le j \le \mathcal{N}$*, be appropriately given. Then determine* $\lambda(x')$, $\mu(x')$, $\rho(x')$, $x' \in \Omega$*, by*

$$\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}_j, \mathbf{q}_j, \eta_j)(x), \quad x \in Q_\omega \equiv (0,T) \times \omega. \tag{3.4}$$

*In particular, we are concerned with the stability of the mapping*

$$\{\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}_j, \mathbf{q}_j, \eta_j)|_{Q_\omega}\}_{1 \le j \le \mathcal{N}} \longrightarrow \{\lambda, \mu, \rho\}.$$

This formulation of inverse problem is based on finite measurements and the research originated with Bukhgeim and Klibanov [BuK] where a Carleman estimate and an integral inequality with the weight function are combined to solve the inverse problem. As detailed accounts for such methodology, see [Is2], [Is3], [Kl], [KT]. Moreover, according to equations, we refer to the following papers:

(1) Baudouin and Puel [BP], Bukhgeim [Bu] for an inverse problem of determining potentials in Schrödinger equations,

(2) Imanuvilov and Yamamoto [IY1], Isakov [Is2], [Is3], Klibanov [Kl] for the corresponding inverse problems for parabolic equations,
(3) Bellasssoued [B4], [B5], Bellassoued and Yamamoto [BY], Bukhgeim, Cheng, Isakov and Yamamoto [BCIY], Imanuvilov and Yamamoto [IY2] - [IY4] (especially for conditional stability), Isakov [Is1], [Is2], [Is3], Isakov and Yamamoto [IsY], Khaĭdarov [Kh1], [Kh2], Klibanov [Kl], Puel and Yamamoto [PY1], [PY2], Yamamoto [Ya] for inverse problems of determining coefficients in scalar hyperbolic equations.
(4) Amirov [A] for an inverse problem of ultrahyperbolic equation.

As for the inverse problem of determining some (or all) of $\lambda$, $\mu$ and $\rho$, we can refer to Isakov [Is1], Ikehata, Nakamura and Yamamoto [INY], Imanuvilov, Isakov and Yamamoto [IIY1], Imanuvilov and Yamamoto [IY8]:
[Is1] established the uniqueness in determining a single coefficient $\rho(x')$, using four measurements (i.e., $\mathcal{N} = 4$).
[INY] decreased the number $\mathcal{N}$ of measurements to three for determining $\rho$.
[IIY1] proved conditional stability and the uniqueness in the determination of the three functions $\lambda(x')$, $\mu(x')$, $\rho(x')$, $x' \in \Omega$, with two measurements (i.e., $\mathcal{N} = 2$).

In all the papers [Is1], [INY], [IIY1], the authors have to assume that $\partial\omega \supset \partial\Omega$ because the technique based on Carleman estimates required that $\mathbf{u}$ has a compact support in $Q$. In the two-dimensional case, [IY8] reduced $\mathcal{N} = 2$ to $\mathcal{N} = 1$ (i.e., a single measurement) in determining all of $\lambda, \mu, \rho$ with more general $\omega$, and established conditional stability. As for other inverse problems for the Lamé systems, see Yakhno [Yak].

In this section, we will prove the conditional stability which is a three-dimensional version of [IY8]. As for the two-dimensional Lamé system with stress boundary condition, in [IY9] a similar inverse problem is discussed by a single measurement.

In order to formulate our main result, we will introduce notations and an admissible set of unknown parameters $\lambda, \mu, \rho$. Similarly to inverse hyperbolic problems, we have to assume that the observation subdomain $\omega$ should satisfy a geometric condition and the observation time $T$ has to be sufficiently large, which is a natural consequence of the hyperbolicity of the governing partial differential equation. First we formulate the geometric condition. Henceforth we set $(x', y') = \sum_{j=1}^3 x_j y_j$ for $x' = (x_1, x_2, x_3)$ and $y' = (y_1, y_2, y_3)$. Let a subdomain $\omega \subset \Omega$ satisfy

$$\partial\omega \supset \{x' \in \partial\Omega; ((x' - y'), \vec{n}(x')) \geq 0\} \tag{3.5}$$

with some $y' \notin \overline{\Omega}$.

**Remark.** Condition (3.5) is the same condition which yields the observability inequality for the wave equation $\partial_{x_0}^2 - \Delta$ if the observation time $T$ is larger than $2\sup_{x'\in\Omega} |x' - y'|$ (e.g., Section 2 of Chapter 7 in [Li]). Moreover, if (3.5) holds and $T > 0$ is sufficiently large, then $\omega$ and $T$ satisfy the geometric optics condition in [BLR].

Denote

$$d = (\sup_{x'\in\Omega} |x' - y'|^2 - \inf_{x'\in\Omega} |x' - y'|^2)^{\frac{1}{2}}. \tag{3.6}$$

Next we define an admissible set of unknown coefficients $\lambda$, $\mu$, $\rho$. Let $M_0 > 0$, $0 < \theta_0 \leq 1$ and $\theta_1 > 0$ be arbitrarily fixed and let us introduce the conditions on a function $\beta$:

$$\begin{cases} \beta(x') \ge \theta_1 > 0, \quad x' \in \overline{\Omega}, \\ \|\beta\|_{C^3(\overline{\Omega})} \le M_0, \quad \dfrac{(\nabla_{x'}\beta(x'), (x'-y'))}{2\beta(x')} \le 1-\theta_0, \quad x' \in \overline{\Omega \setminus \omega}. \end{cases} \tag{3.7}$$

For fixed functions $a$, $b$, $\eta$ on $\partial\Omega$ and $\mathbf{p}$, $\mathbf{q}$ in $\Omega$ and a fixed constant $M_1 > 0$, we set

$$\mathcal{W} = \mathcal{W}_{M_0,M_1,\theta_0,\theta_1,a,b} = \Bigg\{ (\lambda,\mu,\rho) \in (C^3(\overline{\Omega}))^3; \lambda = a, \mu = b \quad \text{on } \partial\Omega,$$

$$\frac{\lambda+2\mu}{\rho}, \ \frac{\mu}{\rho} \text{ satisfy (3.7)}, \ \|\mathbf{u}(\lambda,\mu,\rho,\mathbf{p},\mathbf{q},\eta)\|_{W^{7,\infty}(Q)} \le M_1,$$

$$\frac{\min\{\mu^2(x'), \mu(x')(\lambda+\mu)(x')\}}{\rho(x')(\lambda+2\mu)(x')} \ge \theta_1 \quad \text{on } \overline{\Omega} \Bigg\}. \tag{3.8}$$

**Remark.** If $\lambda, \mu, \rho$ are sufficiently close to positive constant functions, then $(\lambda,\mu,\rho) \in \mathcal{W}$. This suggests that $\mathcal{W}$ contains sufficiently many $(\lambda,\mu,\rho)$.

It is rather restrictive that $\frac{\lambda+2\mu}{\rho}$ and $\frac{\mu}{\rho}$ should satisfy (3.7), which is one possible sufficient condition for the pseudoconvexity (i.e., Condition 2.1). We can relax condition (3.7) to a more generous condition which can be related with a necessary condition for a Carleman estimate, and we refer to Imanuvilov, Isakov and Yamamoto [IIY2], where a scalar hyperbolic equation is discussed but the modification to the Lamé system is straightforward. Such a relaxed condition guarantees that the geodesics which are generated by the hyperbolic equations with principal symbol (2.3), cannot remain on the level sets given by the weight function $\phi$. In particular, by [IIY2], we can replace condition (3.7) by one that the Hessians

$$\left( \partial_{x_j}\partial_{x_k} \left(\frac{\rho}{\mu}\right)^{\frac{1}{2}} \right)_{1\le j,k\le 2}, \quad \left( \partial_{x_j}\partial_{x_k} \left(\frac{\rho}{\lambda+2\mu}\right)^{\frac{1}{2}} \right)_{1\le j,k\le 2}$$

are non-negative and $\left|\nabla\left(\frac{\rho}{\mu}\right)\right| \ne 0$ and $\left|\nabla\left(\frac{\rho}{\lambda+2\mu}\right)\right| \ne 0$ on $\overline{\Omega}$.

We choose $\theta > 0$ such that

$$\theta + \frac{M_0 d}{\sqrt{\theta_1}}\sqrt{\theta} < \theta_0\theta_1, \quad \theta_1 \inf_{x'\in\Omega} |x'-y'|^2 - \theta d^2 > 0. \tag{3.9}$$

Here we note that since $y' \not\in \overline{\Omega}$, such $\theta > 0$ exists. Let $E_3$ be the $3\times 3$ identity matrix. We note that $(L_{\lambda,\mu}\mathbf{p})(x')$ is a 3-column vector for 3-column vector $\mathbf{p}$. Moreover by $\{\mathbf{a}\}_j$ we denote the matrix (or vector) obtained from $\mathbf{a}$ after deleting the $j$-th row, and $< A >_j$ is the matrix which is obtained from $A$ by deleting the $j$-th column of $A$. Furthermore we assume that

$$x_1 - y_1 \ne 0 \qquad \text{for any } (x_1,x_2,x_3) \in \overline{\Omega}. \tag{3.10}$$

Now we are ready to state

**Theorem 3.1.** *Let $(\lambda,\mu,\rho)$ be an arbitrary element of $\mathcal{W}$. For $\mathbf{p}=(p_1,p_2,p_3)^T$ and $\mathbf{q}=(q_1,q_2,q_3)^T$, we assume that there exist $j_1,j_2,j_3\in\{1,2,3,4,5,6\}$ such that*

$$det\left\{\begin{array}{lll}(L_{\lambda,\mu}\mathbf{p})(x') & (div\,\mathbf{p}(x'))E_3 & (\nabla_{x'}\mathbf{p}(x')+(\nabla_{x'}\mathbf{p}(x'))^T)(x'-y') \\ (L_{\lambda,\mu}\mathbf{q})(x') & (div\,\mathbf{q}(x'))E_3 & (\nabla_{x'}\mathbf{q}(x')+(\nabla_{x'}\mathbf{q}(x'))^T)(x'-y')\end{array}\right\}_{j_1}\neq 0,$$
$$\forall x'\in\overline{\Omega}, \tag{3.11}$$

$$det\left\{\begin{array}{lll}(L_{\lambda,\mu}\mathbf{p})(x') & \nabla_{x'}\mathbf{p}(x')+(\nabla_{x'}\mathbf{p}(x'))^T & (div\,\mathbf{p})(x'-y') \\ (L_{\lambda,\mu}\mathbf{q})(x') & \nabla_{x'}\mathbf{q}(x')+(\nabla_{x'}\mathbf{q}(x'))^T & (div\,\mathbf{q})(x'-y')\end{array}\right\}_{j_2}\neq 0,\ \forall x'\in\overline{\Omega}, \tag{3.12}$$

$$det\left\{\begin{array}{lll}(L_{\lambda,\mu}\mathbf{p})(x') & (div\,\mathbf{p}(x'))<E_3>_1 & <(\nabla_{x'}\mathbf{p}(x')+(\nabla_{x'}\mathbf{p}(x'))^T>_1 \\ (L_{\lambda,\mu}\mathbf{q})(x') & (div\,\mathbf{q}(x'))<E_3>_1 & <(\nabla_{x'}\mathbf{q}(x')+(\nabla_{x'}\mathbf{q}(x'))^T)>_1\end{array}\right\}_{j_3}\neq 0,$$
$$\forall x'\in\overline{\Omega}, \tag{3.13}$$

*and that*

$$T>\frac{2}{\sqrt{\theta}}d. \tag{3.14}$$

*Then there exist constants $\kappa=\kappa(\mathcal{W},\omega,\Omega,T,\lambda,\mu,\rho)\in(0,1)$ and $C_1=C_1(\mathcal{W},\omega,\Omega,T,\lambda,\mu,\rho)>0$ such that*

$$\begin{aligned}&\|\widetilde{\lambda}-\lambda\|_{L^2(\Omega)}+\|\widetilde{\mu}-\mu\|_{L^2(\Omega)}+\|\widetilde{\rho}-\rho\|_{H^{-1}(\Omega)}\\ \le& C_1\|\mathbf{u}(\lambda,\mu,\rho,\mathbf{p},\mathbf{q},\eta)-\mathbf{u}(\widetilde{\lambda},\widetilde{\mu},\widetilde{\rho},\mathbf{p},\mathbf{q},\eta)\|^{\kappa}_{H^4(0,T;\mathbf{L}^2(\omega))}\end{aligned}$$

*for any $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\rho})\in\mathcal{W}$.*

Our stability and uniqueness result requires only one measurement: $\mathcal{N}=1$. In the case where $x_k-y_k\neq 0$ for $k=2$ or $3$, the conclusion is true if we replace (3.13) by

$$\det\left\{\begin{array}{lll}(L_{\lambda,\mu}\mathbf{p})(x') & (\mathrm{div}\,\mathbf{p}(x'))<E_3>_k & <(\nabla_{x'}\mathbf{p}(x')+(\nabla_{x'}\mathbf{p}(x'))^T>_k \\ (L_{\lambda,\mu}\mathbf{q})(x') & (\mathrm{div}\,\mathbf{q}(x'))<E_3>_k & <(\nabla_{x'}\mathbf{q}(x')+(\nabla_{x'}\mathbf{q}(x'))^T)>_k\end{array}\right\}_{j_3}\neq 0,$$
$$\forall x'\in\overline{\Omega}.$$

For the determination of the three coefficients by a single measurement, we have to choose initial data which satisfy strong conditions (3.11) - (3.13) which do not generically hold, and we should satisfy them artificially and a posteriori. Moreover, as the following example shows, we can take such $\mathbf{p}$ and $\mathbf{q}$.

**Example of $\Omega$, p, q meeting (3.11) - (3.13).** For simplicity, we assume that $y'=(0,0,0)$, $\overline{\Omega}$ does not intersect any of the planes $\{x_1=0\}$, $\{x_2=0\}$, $\{x_3=0\}$ and $\{x_1+x_3=0\}$, and $\lambda$, $\mu$ are positive constants. Noting that the fifth columns of the matrices in (3.11) and (3.12) have $x'-y'$ as factors, we will take quadratic functions in $x'$. For example, we take

$$\mathbf{p}(x')=\begin{pmatrix}0\\ x_1x_2\\ 0\end{pmatrix},\quad \mathbf{q}(x')=\begin{pmatrix}x_2^2\\ 0\\ x_2^2\end{pmatrix}.$$

Then, choosing $j_1 = j_2 = j_3 = 6$, we can verify that (3.11) - (3.13) are all satisfied. We set

$$\psi(x) = |x' - y'|^2 - \theta \left( x_0 - \frac{T}{2} \right)^2, \quad \phi(x) = e^{\tau\psi(x)}, \quad x = (x_0, x') \in Q. \tag{3.15}$$

By $y' \notin \overline{\Omega}$, we note that $|\nabla_{x'}\psi| \ne 0$, $x \in \overline{Q}$.

First, in terms of (3.5), (3.8) and (3.9), we can prove the following lemma in the same way as in [IY8].

**Lemma 3.1.** *Let $(\lambda, \mu, \rho) \in \mathcal{W}$, and let us assume (3.9) and (3.14). Then, for sufficiently large $\tau > 0$, the function $\psi$ given by (3.15) satisfies Condition 2.1 and (2.9). Therefore the conclusion of Theorem 2.3 holds and the constants $C_1(\tau)$, $\widehat{\tau}$ and $s_0(\tau)$ in (2.11) can be taken independently of $(\lambda, \mu, \rho) \in \mathcal{W}$.*

Next we consider a first order partial differential operator

$$(P_0 g)(x') = \sum_{j=1}^{3} p_{0,j}(x')\partial_{x_j} g(x'),$$

where $p_{0,j} \in C^1(\overline{\Omega})$, $j = 1, 2, 3$. Then, by integration by parts, we can directly prove two Carleman estimates for $P_0$ (see [IY8] for the proof).

**Lemma 3.2.** *We assume*

$$\sum_{j=1}^{3} p_{0,j}(x')\partial_{x_j}\phi(T/2, x') > 0, \quad x' \in \overline{\Omega}. \tag{3.16}$$

*Then there exists a constant $\tau_0 > 0$ such that for all $\tau > \tau_0$, there exist $s_0 = s_0(\tau) > 0$ and $C_2 = C_2(s_0, \tau_0, \Omega, \omega) > 0$ such that*

$$\int_\Omega s^2 |g|^2 e^{2s\phi(T/2,x')} dx' \le C_2 \int_\Omega |P_0 g|^2 e^{2s\phi(T/2,x')} dx'$$

*for all $s > s_0$ and $g \in H^1(\Omega)$ satisfying*

$$g = 0 \quad on \left\{ x' \in \partial\Omega; \sum_{j=1}^{3} p_{0,j}(x')n_j(x') \ge 0 \right\}.$$

**Lemma 3.3.** *We assume*

$$\sum_{j=1}^{3} p_{0,j}(x')\partial_{x_j}\phi(T/2, x') \ne 0, \quad x' \in \overline{\Omega}. \tag{3.17}$$

*Then the conclusion of Lemma 3.2 is true for all $s > s_0$ and $g \in H^1_0(\Omega)$.*

Now we proceed to

**Proof of Theorem 3.1.** The proof is done by modifying the argument in Imanuvilov and Yamamoto [IY8]. We can separate $\partial\Omega$ into two relatively open subsets $\Gamma_1$ and $\Gamma_2$ such that

$$\begin{cases} \overline{\Gamma_1 \cup \Gamma_2} = \partial\Omega,\ n_1(x') \le 0 \text{ for } x' \in \overline{\Gamma_1},\ n_1(x') \ge 0 \text{ for } x' \in \overline{\Gamma_2}, \\ \text{and for any } x' = (x_1, x_2, x_3) \in \overline{\Omega}, \text{ there exists a unique point } \widetilde{x}' = (\widetilde{x_1}, x_2, x_3) \in \overline{\Gamma_1} \\ \text{such that the segment connecting } x' \text{ and } \widetilde{x}' \text{ is on } \overline{\Omega}. \end{cases} \tag{3.18}$$

In fact, we can choose straight lines parallel to the $x_1$-axis which divide $\Omega$ into parts $\Omega_1, ..., \Omega_m$ such that

$$\Omega_j = \{x';\ \gamma_{1j}(x_2, x_3) < x_1 < \gamma_{j2}(x_2, x_3),\ (x_2, x_3) \in \mathcal{D}_j\}$$

where $\mathcal{D}_j$ is a domain in $\mathbb{R}^2$ and $\gamma_{1j}, \gamma_{2j}$ are continuous functions on $\overline{\mathcal{D}_j}$. We set

$$\Gamma_1 = \bigcup_{j=1}^{m} \{x';\ x_1 = \gamma_{1j}(x_2, x_3), \quad (x_2, x_3) \in \mathcal{D}_j\}$$

and $\Gamma_2 = \partial\Omega \setminus \overline{\Gamma_1}$. Then we can easily see that condition (3.18) is satisfied.

By (3.18), for any $x' = (x_1, x_2, x_3) \in \overline{\Omega}$, we can prove that there exists a unique $(\gamma(x_2, x_3), x_2, x_3) \in \Gamma_1$. By (3.10), $x_1 - y_1 < 0$ for any $x' \in \overline{\Omega}$ or $x_1 - y_1 > 0$ for any $x' \in \overline{\Omega}$. First let $x_1 - y_1 < 0$. We set

$$F(x_1, x_2, x_3) = \int_{\gamma(x_2,x_3)}^{x_1} f(\xi, x_2, x_3) d\xi, \quad x' \in \overline{\Omega}. \tag{3.19}$$

Then

$$\frac{\partial F}{\partial x_1}(x') = f(x'), \qquad x' \in \overline{\Omega}. \tag{3.20}$$

On the other hand, if $x_1 - y_1 > 0$, then instead of $\Gamma_1$, we take $(\gamma(x_2, x_3), x_2, x_3) \in \Gamma_2$ in (3.19), and we can argue similarly to the case of $x_1 - y_1 < 0$. Therefore we will exclusively assume that $x_1 - y_1 < 0$.

Henceforth, for simplicity, we set

$$\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta), \quad \mathbf{v} = \mathbf{u}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}, \mathbf{p}, \mathbf{q}, \eta)$$

and

$$\mathbf{y} = \mathbf{u} - \mathbf{v}, \quad f = \rho - \widetilde{\rho}, \quad g = \lambda - \widetilde{\lambda}, \quad h = \mu - \widetilde{\mu}.$$

Then

$$\widetilde{\rho}\partial_{x_0}^2 \mathbf{y} = L_{\lambda,\mu}\mathbf{y} + \mathbb{G}\mathbf{u} \quad \text{in } Q, \tag{3.21}$$

$$\mathbf{y}\left(\frac{T}{2}, x'\right) = \partial_{x_0}\mathbf{y}\left(\frac{T}{2}, x'\right) = 0, \qquad x' \in \Omega \tag{3.22}$$

and

$$\mathbf{y} = 0 \qquad \text{in } (0, T) \times \partial\Omega. \tag{3.23}$$

Here we set

$$\mathbb{G}\mathbf{u}(x) = -\partial_{x_1}F(x')\partial_{x_0}^2\mathbf{u}(x) + (g+h)(x')\nabla_{x'}(\operatorname{div}\mathbf{u})(x) + h(x')\Delta\mathbf{u}(x)$$
$$+(\operatorname{div}\mathbf{u})(x)\nabla_{x'}g(x') + (\nabla_{x'}\mathbf{u}(x) + (\nabla_{x'}\mathbf{u}(x))^T)\nabla h(x'). \tag{3.24}$$

By (3.14), we have the inequality $\frac{\theta T^2}{4} > d^2$. Therefore, by (3.6) and definition (3.15) of the function $\phi$, we have

$$\phi(T/2, x') \ge d_1, \quad \phi(0, x') = \phi(T, x') < d_1, \qquad x' \in \overline{\Omega}$$

with

$$d_1 = \exp(\tau \inf_{x'\in\Omega} |x' - y'|^2). \tag{3.25}$$

Thus, for given $\varepsilon > 0$, we can choose a sufficiently small $\delta = \delta(\varepsilon) > 0$ such that

$$\phi(x) \ge d_1 - \varepsilon, \quad x \in \left[\frac{T}{2} - \delta, \frac{T}{2} + \delta\right] \times \overline{\Omega} \tag{3.26}$$

and

$$\phi(x) \le d_1 - 2\varepsilon, \quad x \in ([0, 2\delta] \cup [T - 2\delta, T]) \times \overline{\Omega}. \tag{3.27}$$

In order to apply Lemma 3.1, it is necessary to introduce a cut-off function $\chi$ satisfying $0 \le \chi \le 1$, $\chi \in C^\infty(\mathbb{R})$ and

$$\chi = \begin{cases} 0 & \text{on } [0, \delta] \cup [T - \delta, T], \\ 1 & \text{on } [2\delta, T - 2\delta]. \end{cases} \tag{3.28}$$

In the sequel, $C_j > 0$ denote generic constants depending on $s_0$, $\tau$, $M_0$, $M_1$, $\theta_0$, $\theta_1$, $\eta$, $\Omega$, $T$, $y'$, $\omega$, $\chi$ and $\mathbf{p}$, $\mathbf{q}$, $\varepsilon$, $\delta$, but independent of $s > s_0$. Setting $\mathbf{z}_1 = \chi\partial_{x_0}^2\mathbf{y}$, $\mathbf{z}_2 = \chi\partial_{x_0}^3\mathbf{y}$ and $\mathbf{z}_3 = \chi\partial_{x_0}^4\mathbf{y}$, we have

$$\begin{cases} \widetilde{\rho}\partial_{x_0}^2\mathbf{z}_1 = L_{\lambda,\mu}\mathbf{z}_1 + \chi\mathbb{G}(\partial_{x_0}^2\mathbf{u}) + 2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^3\mathbf{y} + \widetilde{\rho}(\partial_{x_0}^2\chi)\partial_{x_0}^2\mathbf{y}, \\ \widetilde{\rho}\partial_{x_0}^2\mathbf{z}_2 = L_{\lambda,\mu}\mathbf{z}_2 + \chi\mathbb{G}(\partial_{x_0}^3\mathbf{u}) + 2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^4\mathbf{y} + \widetilde{\rho}(\partial_{x_0}^2\chi)\partial_{x_0}^3\mathbf{y}, \\ \widetilde{\rho}\partial_{x_0}^2\mathbf{z}_3 = L_{\lambda,\mu}\mathbf{z}_3 + \chi\mathbb{G}(\partial_{x_0}^4\mathbf{u}) + 2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^5\mathbf{y} + \widetilde{\rho}(\partial_{x_0}^2\chi)\partial_{x_0}^4\mathbf{y} \quad \text{in } Q. \end{cases} \tag{3.29}$$

Henceforth we set

$$\mathcal{E} = \int_{Q_\omega} (|\partial_{x_0}^2\mathbf{y}|^2 + |\partial_{x_0}^3\mathbf{y}|^2 + |\partial_{x_0}^4\mathbf{y}|^2)e^{2s\phi}dx.$$

Noting that $\mathbf{u} \in W^{7,\infty}(Q)$, in view of (3.28) and Lemma 3.1, we can apply Theorem 2.3 to (3.29), so that

$$\sum_{j=2}^{4}\int_Q |\partial_{x_0}^j\mathbf{y}|^2\chi^2e^{2s\phi}dx \le C_3(\|Fe^{s\phi}\|^2_{L^2(Q)} + \|ge^{s\phi}\|^2_{L^2(Q)} + \|he^{s\phi}\|^2_{L^2(Q)})$$
$$+C_3\sum_{j=3}^{5}\|(\partial_{x_0}\chi)(\partial_{x_0}^j\mathbf{y})e^{s\phi}\|^2_{L^2(0,T;\mathbf{H}^{-1}(\Omega))}$$
$$+C_3\sum_{j=2}^{4}\|(\partial_{x_0}^2\chi)(\partial_{x_0}^j\mathbf{y})e^{s\phi}\|^2_{L^2(0,T;\mathbf{H}^{-1}(\Omega))} + C_3\mathcal{E}$$
$$\le C_4(\|Fe^{s\phi}\|^2_{L^2(Q)} + \|ge^{s\phi}\|^2_{L^2(Q)} + \|he^{s\phi}\|^2_{L^2(Q)}) + C_4e^{2s(d_1-2\varepsilon)} + C_4\mathcal{E} \tag{3.30}$$

for all large $s>0$. On the other hand,

$$\begin{aligned}&\int_\Omega |(\partial_{x_0}^2\mathbf{y})(T/2,x')|^2 e^{2s\phi(T/2,x')}dx'\\ =&\int_0^{T/2}\frac{\partial}{\partial x_0}\left(\int_\Omega |(\partial_{x_0}^2\mathbf{y})(x_0,x')|^2\chi^2(x_0)e^{2s\phi}dx'\right)dx_0\\ =&\int_0^{T/2}\int_\Omega 2(\partial_{x_0}^3\mathbf{y},\partial_{x_0}^2\mathbf{y})\chi^2 e^{2s\phi}dx\\ &+2s\int_0^{T/2}\int_\Omega |\partial_{x_0}^2\mathbf{y}|^2\chi^2(\partial_{x_0}\phi)e^{2s\phi}dx+\int_0^{T/2}\int_\Omega |\partial_{x_0}^2\mathbf{y}|^2(\partial_{x_0}(\chi^2))e^{2s\phi}dx\\ \le& C_5\int_Q s\chi^2(|\partial_{x_0}^3\mathbf{y}|^2+|\partial_{x_0}^2\mathbf{y}|^2)e^{2s\phi}dx+C_5e^{2s(d_1-2\varepsilon)}.\end{aligned}$$

Therefore (3.30) yields

$$\begin{aligned}&\int_\Omega |(\partial_{x_0}^2\mathbf{y})(T/2,x')|^2 e^{2s\phi(T/2,x')}dx'\\ \le& C_6 s\int_Q(|F|^2+|g|^2+|h|^2)e^{2s\phi}dx+C_6se^{2s(d_1-2\varepsilon)}+C_6s\mathcal{E}\end{aligned}$$

for all large $s>0$. Similarly we can estimate $\int_\Omega |(\partial_{x_0}^3\mathbf{y})(T/2,x')|^2e^{2s\phi(T/2,x')}dx'$ to obtain

$$\begin{aligned}&\int_\Omega(|(\partial_{x_0}^2\mathbf{y})(T/2,x')|^2+|(\partial_{x_0}^3\mathbf{y})(T/2,x')|^2)e^{2s\phi(T/2,x')}dx'\\ \le& C_6 s\int_Q(|F|^2+|g|^2+|h|^2)e^{2s\phi}dx+C_6se^{2s(d_1-2\varepsilon)}+C_6s\mathcal{E}\end{aligned}\tag{3.31}$$

for all large $s>0$.

Now we will consider first order partial differential equations satisfied by $h$, $g$ and $F$. That is, by (3.21), (3.22) and $\mathbf{u},\mathbf{v}\in W^{7,\infty}(Q)$, we have

$$\widetilde{\rho}\partial_{x_0}^2\mathbf{y}\left(\frac{T}{2},x'\right)=\mathbb{G}\mathbf{u}\left(\frac{T}{2},x'\right),\quad \widetilde{\rho}\partial_{x_0}^3\mathbf{y}\left(\frac{T}{2},x'\right)=\mathbb{G}\partial_{x_0}\mathbf{u}\left(\frac{T}{2},x'\right).\tag{3.32}$$

For simplicity, we set

$$\begin{cases}\mathbf{a}=\begin{pmatrix}-\frac{1}{\rho}L_{\lambda,\mu}\mathbf{p}\\ -\frac{1}{\rho}L_{\lambda,\mu}\mathbf{q}\end{pmatrix},\\ \mathbf{b}_1=\begin{pmatrix}\operatorname{div}\mathbf{p}\\0\\0\\\operatorname{div}\mathbf{q}\\0\\0\end{pmatrix},\quad \mathbf{b}_2=\begin{pmatrix}0\\\operatorname{div}\mathbf{p}\\0\\0\\\operatorname{div}\mathbf{q}\\0\end{pmatrix},\quad \mathbf{b}_3=\begin{pmatrix}0\\0\\\operatorname{div}\mathbf{p}\\0\\0\\\operatorname{div}\mathbf{q}\end{pmatrix},\\ (\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3)=\begin{pmatrix}\nabla\mathbf{p}+(\nabla\mathbf{p})^T\\ \nabla\mathbf{q}+(\nabla\mathbf{q})^T\end{pmatrix},\\ \mathbf{G}=\begin{pmatrix}\widetilde{\rho}\partial_{x_0}^2\mathbf{y}\left(\frac{T}{2},x'\right)-(g+h)\nabla_{x'}(\operatorname{div}\mathbf{p})-h\Delta\mathbf{p}\\ \widetilde{\rho}\partial_{x_0}^3\mathbf{y}\left(\frac{T}{2},x'\right)-(g+h)\nabla_{x'}(\operatorname{div}\mathbf{q})-h\Delta\mathbf{q}\end{pmatrix}\quad \text{on } \overline{\Omega}.\end{cases}\tag{3.33}$$

Then we can rewrite (3.32) as

$$\mathbf{a}\partial_{x_1}F + \mathbf{b}_1\partial_{x_1}g + \mathbf{b}_2\partial_{x_2}g + \mathbf{b}_3\partial_{x_3}g = \mathbf{G} - \mathbf{d}_1\partial_{x_1}h - \mathbf{d}_2\partial_{x_2}h - \mathbf{d}_3\partial_{x_3}h.$$

Therefore for $j_1 \in \{1,2,3,4,5,6\}$, we have

$$\begin{aligned}&\{\mathbf{a}\}_{j_1}\partial_{x_1}F + \{\mathbf{b}_1\}_{j_1}\partial_{x_1}g + \{\mathbf{b}_2\}_{j_1}\partial_{x_2}g + \{\mathbf{b}_3\}_{j_1}\partial_{x_3}g\\ =&\{\mathbf{G}\}_{j_1} - \{\mathbf{d}_1\}_{j_1}\partial_{x_1}h - \{\mathbf{d}_2\}_{j_1}\partial_{x_2}h - \{\mathbf{d}_3\}_{j_1}\partial_{x_3}h, \quad \text{on } \overline{\Omega}.\end{aligned} \tag{3.34}$$

Equality (3.34) is a system of five linear equations with respect to four unknowns $\partial_{x_1}F, \partial_{x_1}g, \partial_{x_2}g, \partial_{x_3}g$, and so for the existence of solutions, we need the consistency of the coefficients, that is,

$$\det\{\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{G} - \mathbf{d}_1\partial_{x_1}h - \mathbf{d}_2\partial_{x_2}h - \mathbf{d}_3\partial_{x_3}h\}_{j_1} = 0 \quad \text{on } \overline{\Omega},$$

that is,

$$\sum_{k=1}^{3}\det\{\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_k\}_{j_1}\partial_{x_k}h = \det\{\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{G}\}_{j_1} \quad \text{on } \overline{\Omega} \tag{3.35}$$

by the linearity of the determinant. In terms of condition (3.11) and $h \equiv \mu - \widetilde{\mu} = 0$ on $\partial\Omega$, considering (3.35) as a first order partial differential operator in $h$, we can apply Lemma 3.3, so that

$$\begin{aligned}&s^2\int_\Omega |h|^2 e^{2s\phi(T/2,x')}dx' \le C_7\|\mathrm{det}_{j_1}(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{G})e^{s\phi(T/2,\cdot)}\|^2_{L^2(\Omega)}\\ \le& C_8\int_\Omega\left(\left|\partial_{x_0}^2\mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 + \left|\partial_{x_0}^3\mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2\right)e^{2s\phi(T/2,x')}dx'\\ +&C_8\int_\Omega(|g|^2 + |h|^2)e^{2s\phi(T/2,x')}dx'\end{aligned} \tag{3.36}$$

in view of (3.33). Similarly to (3.34), we rewrite (3.32) and, by (3.12) we can similarly deduce

$$\begin{aligned}&s^2\int_\Omega |g|^2 e^{2s\phi(T/2,x')}dx' \le C_9\int_\Omega\left(\left|\partial_{x_0}^2\mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 + \left|\partial_{x_0}^3\mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2\right)e^{2s\phi(T/2,x')}dx'\\ +&C_9\int_\Omega(|g|^2 + |h|^2)e^{2s\phi(T/2,x')}dx'\end{aligned} \tag{3.37}$$

for all large $s > 0$. By (3.36) and (3.37), for sufficiently large $s > 0$, we have

$$\begin{aligned}&s^2\int_\Omega(|g|^2 + |h|^2)e^{2s\phi(T/2,x')}dx'\\ \le& C_{10}\int_\Omega\left(\left|\partial_{x_0}^2\mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 + \left|\partial_{x_0}^3\mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2\right)e^{2s\phi(T/2,x')}dx'.\end{aligned} \tag{3.38}$$

Finally, replacing $j_1$ by $j_3 \in \{1,2,3,4,5,6\}$, we consider (3.34) as a system of five linear equations with respect to four unknowns $\partial_{x_2} g$, $\partial_{x_3} g$, $\partial_{x_2} h$, $\partial_{x_3} h$. By the condition for the existence of solutions, we have

$$\det \{\mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_2, \mathbf{d}_3, \mathbf{G} - \mathbf{a}\partial_{x_1} F - \mathbf{b}_1 \partial_{x_1} g - \mathbf{d}_1 \partial_{x_1} h\}_{j_3} = 0$$

on $\overline{\Omega}$. Therefore

$$\begin{aligned} &- \partial_{x_1}(e_1 F + e_2 g + e_3 h) + (\partial_{x_1} e_1) F \\ =& -(\partial_{x_1} e_2) g - (\partial_{x_1} e_3) h - \det{}_{j_3} (\mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_2, \mathbf{d}_3, \mathbf{G}) \end{aligned}$$

on $\overline{\Omega}$. Here we set

$$e_1 = \det \{\mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_2, \mathbf{d}_3, \mathbf{a}\}_{j_3},$$

$$e_2 = \det \{\mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_2, \mathbf{d}_3, \mathbf{b}_1\}_{j_3}$$

and

$$e_3 = \det \{\mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_1\}_{j_3}.$$

In Lemma 3.2, we consider the case of $p_{0,1} = -1$ and $p_{0,2} = p_{0,3} = 0$. By (3.18), (3.19) and $g = h = 0$ on $\partial\Omega$, we see that if $-n_1(x') = \sum_{j=1}^{3} p_{0,j}(x') n_j(x') \ge 0$, then $(F + g + h)(x') = 0$. Moreover by $x_1 - y_1 < 0$ for $x' \in \overline{\Omega}$, condition (3.16) is satisfied. Consequently, choosing $s > 0$ sufficiently large and using (3.38), by Lemma 3.3 and (3.13), we obtain

$$\begin{aligned} & s^2 \int_\Omega |F|^2 e^{2s\phi(T/2,x')} dx' \\ \le & C_{11} \int_\Omega \left( \left| \partial_{x_0}^2 \mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 + \left| \partial_{x_0}^3 \mathbf{y}\left(\frac{T}{2}, x'\right)\right|^2 \right) e^{2s\phi(T/2,x')} dx' \end{aligned} \tag{3.39}$$

for all large $s > 0$. Consequently, substituting (3.38) and (3.39) into (3.31) and using $\phi(T/2, x') \ge \phi(x_0, x')$ for $(x_0, x') \in Q$, we obtain

$$\begin{aligned} & \int_\Omega (|F|^2 + |g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx' \\ \le & \frac{C_{12} T}{s} \int_\Omega (|F|^2 + |g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx' + \frac{C_{12}}{s} e^{2s(d_1 - 2\varepsilon)} + \frac{C_{12}}{s} \mathcal{E} \end{aligned}$$

for all large $s > 0$. Taking $s > 0$ sufficiently large and noting $e^{2s\phi(T/2,x')} \ge e^{2sd_1}$ for $x' \in \overline{\Omega}$, we obtain

$$\int_\Omega (|F|^2 + |g|^2 + |h|^2) dx' \le C_{13} e^{-4s\varepsilon} + C_{13} e^{2sC_{14}} \mathcal{E} \tag{3.40}$$

for all large $s > s_0$: a constant which is dependent on $\tau$, but independent of $s$. Next we take in (3.40) instead of the constant $C_{13}$ the constant $C_{13} e^{2s_0 C_{14}}$. Now this inequality holds true for all $s > 0$.

Now we choose $s > 0$ such that $e^{2sC_{14}}\mathcal{E} = e^{-4s\varepsilon}$, that is,

$$s = -\frac{1}{4\varepsilon + 2C_{14}} \ln \mathcal{E}.$$

Here we may assume that $\mathcal{E} < 1$ and so $s > 0$. Then it follows from (3.40) that

$$\int_\Omega (|F|^2 + |g|^2 + |h|^2)dx' \le 2C\mathcal{E}^{\frac{4\varepsilon}{4\varepsilon+2C}}.$$

By definition (3.19) of $F$, we have

$$\int_\Omega frdx' = \int_\Omega (\partial_{x_1}F)rdx' = \int_\Omega F(\partial_{x_1}r)dx'$$

for all $r \in H_0^1(\Omega)$ by integration by parts. Hence we can directly verify that $\|f\|_{H^{-1}(\Omega)} \le C\|F\|_{L^2(\Omega)}$, so that the proof of Theorem 3.1 is complete. ■

## §4. Proof of Theorem 2.1.

Without loss of generality, we may assume that $\rho \equiv 1$. Otherwise we introduce new coefficients $\mu_1 = \mu/\rho, \lambda_1 = \lambda/\rho$ to argue similarly. We can directly verify that the functions rot$\mathbf{u}$ and div$\mathbf{u}$ satisfy the equations

$$\partial_{x_0}^2 \mathrm{rot}\ \mathbf{u} - \mu\Delta \mathrm{rot}\mathbf{u} = m_1, \quad \partial_{x_0}^2 \mathrm{div}\mathbf{u} - (\lambda + 2\mu)\Delta \mathrm{div}\mathbf{u} = m_2 \qquad \text{in } Q, \tag{4.1}$$

where

$$m_1 = K_1\mathrm{rot}\mathbf{u} + K_2\mathrm{div}\mathbf{u} + \mathcal{K}_1\mathbf{u} + \mathrm{rot}\,\mathbf{f}, \quad m_2 = K_3\mathrm{rot}\mathbf{u} + K_4\mathrm{div}\mathbf{u} + \mathcal{K}_2\mathbf{u} + \mathrm{div}\,\mathbf{f}$$

and $K_j$, $\mathcal{K}_k$ are first order differential operators with $L^\infty$ coefficients. Thanks to Condition 2.1 on the weight function $\psi$, there exists $\widehat{\tau}$ such that for all $\tau > \widehat{\tau}$, we have (see [Ta]):

$$s\|(\mathrm{rot}\,\mathbf{u})e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + s\|(\mathrm{div}\,\mathbf{u})e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} \le C_1\Bigg(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + s\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{\mathbf{H}^{1,s}((0,T)\times\partial\Omega)}$$
$$+s\left\|\frac{\partial^2 \mathbf{u}}{\partial \vec{n}^2}e^{s\phi}\right\|^2_{\mathbf{L}^2((0,T)\times\partial\Omega)} + \|\mathbf{u}\|^2_{\mathcal{B}(Q_\omega)}\Bigg), \quad \forall s \ge s_0(\tau), \tag{4.2}$$

where the constant $C_1$ is independent of $s$. In order to estimate the $H^1(Q)$-norm of the function $\mathbf{u}$, we rewrite equations (1.1) in the form

$$\rho\partial_{x_0}^2\mathbf{u} - \mu\Delta\mathbf{u} = \mathbf{F}, \quad \mathbf{u}|_{\partial\Omega} = 0,$$

where

$$\mathbf{F} = \mathbf{f} + (\lambda(x') + \mu(x'))\nabla_{x'}\mathrm{div}\,\mathbf{u}(x) + (\mathrm{div}\,\mathbf{u}(x))\nabla_{x'}\lambda(x') + (\nabla_{x'}\mathbf{u} + (\nabla_{x'}\mathbf{u})^T)\nabla_{x'}\mu(x').$$

Thanks to Condition 2.1 we can apply the Carleman estimate in [Im2] to this equation

$$\begin{aligned}&s\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} \le C_2(\|\mathbf{F}e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + s\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q_\omega)})\\ \le & C_2(\|(\operatorname{div}\mathbf{u})e^{s\phi}\|^2_{L^2(Q)} + \|(\nabla_{x'}\operatorname{div}\mathbf{u})e^{s\varphi}\|^2_{\mathbf{L}^2(Q)} + \|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)}\\ &+\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + s\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q_\omega)}).\end{aligned}$$

This estimate and inequality (4.2) imply

$$\begin{aligned}s^2\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)}+s\|(\operatorname{rot}\mathbf{u})e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)}+s\|(\operatorname{div}\mathbf{u})e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} \le C_2\Bigg( \|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)}\\ + s\left\|\frac{\partial \mathbf{u}}{\partial \vec n}e^{s\phi}\right\|^2_{\mathbf{H}^{1,s}((0,T)\times\partial\Omega)} + s\left\|\frac{\partial^2 \mathbf{u}}{\partial \vec n^2}e^{s\phi}\right\|^2_{\mathbf{L}^2((0,T)\times\partial\Omega)} + \|\mathbf{u}\|^2_{\mathcal{B}(Q_\omega)}\Bigg), \quad \forall s\ge s_0.\end{aligned} \tag{4.3}$$

Next we estimate the second derivatives of the function $\mathbf{u}$. Denote $\operatorname{rot}\mathbf{u} = \mathbf{y}$. Using a well-known formula: $\operatorname{rot}\operatorname{rot} = -\Delta_{x'} + \nabla_{x'}\operatorname{div}$, we obtain

$$-\Delta_{x'}\mathbf{u} = -\operatorname{rot}\mathbf{y} - \nabla_{x'}\operatorname{div}\mathbf{u} \quad \text{in } \Omega, \quad \mathbf{u}|_{\partial\Omega} = 0.$$

Using the standard a priori estimate for the Laplace operator we have:

$$\sum_{j,k=1}^{3}\|(\partial_{x_j}\partial_{x_k}\mathbf{u})e^{s\phi}\|_{\mathbf{L}^2(Q)} \le C_2(s\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(Q)}+\|(\operatorname{div}\mathbf{u})e^{s\phi}\|_{\mathbf{H}^{1,s}(Q)}+\|(\operatorname{rot}\mathbf{u})e^{s\phi}\|_{\mathbf{H}^{1,s}(Q)}).$$

By (4.3) one can estimate the left hand side of this inequality by the right hand side of (4.2).

Next using this estimate and equation (1.1), we obtain the estimate for the norm $\|(\partial^2_{x_0}\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}$ via the right hand side of (4.2). Finally we obtain the estimate for $\|(\partial_{x_0}\partial_{x_j}\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}$ and $s^2\|(\partial_{x_0}\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}$ by an interpolation argument. Therefore, combining these estimates with (4.2) and (4.3), we have

$$\begin{aligned}\|\mathbf{u}\|^2_{Y(\phi,Q)} \le C_3\Bigg( \|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + s\left\|\frac{\partial \mathbf{u}}{\partial \vec n}e^{s\phi}\right\|^2_{\mathbf{H}^{1,s}((0,T)\times\partial\Omega)} + s\left\|\frac{\partial^2 \mathbf{u}}{\partial \vec n^2}e^{s\phi}\right\|^2_{\mathbf{L}^2((0,T)\times\partial\Omega)}\\ +\|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}\Bigg), \qquad \forall s\ge s_0(\tau),\end{aligned} \tag{4.4}$$

where the constant $C_3$ is independent of $s$.

Now we need to estimate the boundary integrals at the right hand side of (4.4). In order to do that, it is convenient to use another weight function $\varphi$ such that $\varphi|_{(0,T)\times\partial\Omega} = \phi|_{(0,T)\times\partial\Omega}$ and $\varphi(x) < \phi(x)$ for all $x$ in a small neighbourhood of $(0,T)\times\partial\Omega$. We construct such a function $\varphi$ locally near the boundary $\partial\Omega$:

$$\varphi(x) = e^{\tau\psi(x)}, \quad \widetilde\psi(x) = \psi(x) - \frac{1}{N^2}\ell_1(x') + N\ell_1^2(x'),$$

where $N>0$ is a large positive parameter, and $\ell_1 \in C^3(\overline{\Omega})$ is a function such that

$$\ell_1(x')>0, \quad \forall x'\in\Omega, \quad \ell_1|_{\partial\Omega}=0,\ \nabla_{x'}\ell_1|_{\partial\Omega}\ne 0.$$

Denote $\Omega_{\frac{1}{N^2}} = \{x'\in\Omega;\ \text{dist}\,(x',\partial\Omega)\le \frac{1}{N^2}\}$. Obviously there exists $N_0>0$ such that

$$\varphi(x)<\phi(x), \quad \forall x\in[0,T]\times\Omega_{\frac{1}{N^2}},\ N\in(N_0,\infty).$$

The following lemma plays a key role in our proof.

**Lemma 4.1.** *Under the conditions of Theorem 2.1, there exists $\widehat{\tau}>0$ such that for all $\tau>\widehat{\tau}$, there exists $s_0(\tau)>0$ such that*

$$\|\mathbf{u}\|^2_{Y(\varphi,Q)} + \sqrt{N}\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|(\partial_x^\alpha\mathbf{u})e^{s\varphi}\|^2_{\mathbf{L}^2(Q)} \le C_4(\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(Q)}$$
$$+\|\mathbf{u}\|^2_{\mathcal{B}(\varphi,Q_\omega)}), \quad \forall s\ge s_0(\tau,N),\ supp\,\mathbf{u}\subset[0,T]\times\overline{\Omega_{\frac{1}{N^2}}}, \tag{4.5}$$

*where the constant $C_4$ is independent of $s$ and $N$.*

We will postpone the proof of Lemma 4.1 and by means of the lemma, we continue the proof of Theorem 2.1. Let us fix the parameter $N$ such that (4.5) holds true. We take $\widetilde{\delta}\in\left(0,\frac{1}{N^2}\right)$ sufficiently small such that

$$\phi(x)>\varphi(x), \qquad \forall x\in[0,T]\times\overline{\Omega_\delta\setminus\Omega_{\delta/2}}. \tag{4.6}$$

We consider a cut-off function $\widetilde{\theta}\in C^3(\overline{\Omega}_\delta)$ such that $\widetilde{\theta}|_{\Omega_{\frac{\delta}{2}}}=1$ and $\widetilde{\theta}|_{\Omega_\delta\setminus\Omega_{\frac{3\delta}{4}}}=0$. The function $\widetilde{\theta}\mathbf{u}$ satisfies the equation

$$P(\widetilde{\theta}\mathbf{u})=\widetilde{\theta}\mathbf{f}+[P,\widetilde{\theta}]\mathbf{u}, \quad \mathbf{u}|_{(0,T)\times\partial\Omega}=0,$$
$$\mathbf{u}(0,\cdot)=\mathbf{u}_{x_0}(0,\cdot)=\mathbf{u}(T,\cdot)=\mathbf{u}_{x_0}(T,\cdot)=0. \tag{4.7}$$

Applying Carleman estimate (4.5) to (4.7), and using the fact that $(\varphi-\phi)|_{(0,T)\times\partial\Omega}=0$, we obtain

$$s\left\|\frac{\partial\mathbf{u}}{\partial\vec{n}}e^{s\phi}\right\|^2_{\mathbf{H}^{1,s}((0,T)\times\partial\Omega)} + s\left\|\frac{\partial^2\mathbf{u}}{\partial\vec{n}^2}e^{s\phi}\right\|^2_{\mathbf{L}^2((0,T)\times\partial\Omega)} \le C_5(\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(Q)}+\|[P,\widetilde{\theta}]\mathbf{u}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(Q)}$$
$$+\|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}), \quad \forall s\ge s_0(\tau). \tag{4.8}$$

Since the supports of the coefficients of the commutator $[P,\widetilde{\theta}]$ are in $[0,T]\times\overline{\Omega_\delta\setminus\Omega_{\delta/2}}$ by (4.6), we have

$$\|[P,\widetilde{\theta}]\mathbf{u}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(Q)} \le C_6\left(\sum_{|\alpha|=0}^{2} s^{3-2|\alpha|}\|(\partial_x^\alpha\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}+\|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}\right). \tag{4.9}$$

Combining (4.8) and (4.9), we obtain

$$s\left\|\frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{\mathbf{H}^{1,s}((0,T)\times\partial\Omega)}+s\left\|\frac{\partial^2 \mathbf{u}}{\partial \vec{n}^2}e^{s\phi}\right\|^2_{\mathbf{L}^2((0,T)\times\partial\Omega)}$$
$$\le C_7\left(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)}+\sum_{|\alpha|=0}^{2}s^{3-2|\alpha|}\|(\partial_x^\alpha \mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}+\|\mathbf{u}\|^2_{\mathcal{B}(\phi,Q_\omega)}\right),\qquad \forall s\ge s_0(\tau). \tag{4.10}$$

Finally we will estimate the surface integrals at the right hand side of (4.4) by the right hand side of (4.10). In the new inequality, the term

$$\sum_{|\alpha|=0}^{2}s^{3-2|\alpha|}\|(\partial_x^\alpha \mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}$$

which appears at the right hand side, can be absorbed by $\|\mathbf{u}\|^2_{Y(\phi,Q)}$. Thus the proof of Theorem 2.1 is complete. ■

The rest of the paper is devoted to the proof of the Lemma 4.1.

**Proof of Lemma 4.1.** First we note that, thanks to the large parameter $N$, it suffices to prove (4.5) only locally by assuming

$$\operatorname{supp}\mathbf{u}\subset B_\delta\cap([0,T]\times\overline{\Omega}_{\frac{1}{N^2}}),$$

where $B_\delta$ is the ball of the radius $\delta>0$ centered at some point $y^*\in[0,T]\times\partial\Omega$. In the case of $B_\delta\cap((0,T)\times\partial\Omega)=\emptyset$, we can prove the lemma in a usual way for a function with compact support (see e.g., [Hö]). Without loss of generality, we may assume that $y^*=(y_0^*,0,0,0)$. Moreover the parameter $\delta>0$ can be chosen arbitrarily small. Assume that near $(0,0,0)$, the boundary $\partial\Omega$ is locally given by the equation $x_3-\ell(x_1,x_2)=0$. Furthermore, since the function $\widetilde{\mathbf{u}}=\mathcal{O}\mathbf{u}(x_0,\mathcal{O}^{-1}x')$ satisfies system (2.1) and (2.2) with $\widetilde{\mathbf{f}}=\mathcal{O}\mathbf{f}(x_0,\mathcal{O}^{-1}x')$ for any orthogonal matrix $\mathcal{O}$, we may assume that

$$\left(\frac{\partial \ell}{\partial x_1}(0,0),\frac{\partial \ell}{\partial x_2}(0,0)\right)=0. \tag{4.11}$$

Next we make the change of variables $y_1=x_1,y_2=x_2$ and $y_3=x_3-\ell(x_1,x_2)$. We set $y_0=x_0$, $y=(y_0,y_1,y_2,y_3)$, $y'=(y_1,y_2,y_3)$. By $A(y,D)$ denote the Laplace operator after the change of variables. One can check that the principal symbol of this operator is equal to $a(y,\xi)=-\xi_1^2-\xi_2^2-|G|^2\xi_3^2+2(\nabla_{y'}\ell,\xi)\xi_3$, $|G|=\sqrt{1+|\nabla\ell|^2}$. In the new coordinates, the Lamé system has the form

$$\mathbb{P}(y,D)\mathbf{u}=D_{y_0}^2\mathbf{u}-\mu A(y,D)\mathbf{u}$$
$$-(\lambda+\mu)\left(\nabla_{y'}-\nabla_{y'}\ell\frac{\partial}{\partial y_3}\right)\left(\operatorname{div}\mathbf{u}-\left(\frac{\partial \mathbf{u}}{\partial y_3},\nabla_{y'}\ell\right)\right)$$
$$+\widetilde{K}_1\mathbf{u}=-\mathbf{f}, \tag{4.12}$$

where we use the same notations $\mathbf{u},\mathbf{f}$ after the change of variables and $\widetilde{K}_1$ is the partial differential operator of the first order. Denote by $(z_1,z_2,z_3)$ and $z_4$ the functions $\operatorname{rot}\mathbf{u}$ and $\operatorname{div}\mathbf{u}$ in the $y$ coordinates. These functions satisfy the equations:

$$P_\mu(y,D)z_j = D_0^2 z_j - \mu A(y,D)z_j = m_j \quad j\in\{1,2,3\}, \tag{4.13}$$

$$P_{\lambda+2\mu}(y,D)z_4 = D_0^2 z_4 - (\lambda+2\mu)A(y,D)z_4 = m_4. \tag{4.14}$$

Here we set $\mathbf{w}=(\mathbf{w}',w_4)$ where

$$\mathbf{w}' = (\operatorname{rot}\mathbf{u})e^{s\varphi}, \quad w_4 = (\operatorname{div}\mathbf{u})e^{s\varphi} \qquad \text{in the } y\text{-coordinate},$$

$$\mathbf{w}'_\nu = \chi_\nu(s,D')\mathbf{w}' \equiv \int_{\mathbb{R}^3}\chi_\nu(s,\xi')\widehat{\mathbf{w}'}(\xi_0,\xi_1,\xi_2,y_3)e^{i(y_0\xi_0+y_1\xi_1+y_2\xi_2)}d\xi_0 d\xi_1 d\xi_2,$$

where $\widehat{\mathbf{w}'}$ is the Fourier transform of $\mathbf{w}'$ with respect to the variables $(y_0,y_1,y_2)$.

We consider a finite covering of the unit sphere $S^3\equiv\{(s,\xi_0,\xi_1,\xi_2); s^2+\xi_0^2+\xi_1^2+\xi_2^2=1\}$. That is, $S^3\subset\cup_{\nu=1}^{K(\delta_1)}\{(s,\xi_0,\xi_1,\xi_2)\in S^3; |\zeta-\zeta_\nu^*|<\delta_1\}$ where $\zeta_\nu^*\in S^3$, and by $\{\chi_\nu(\zeta)\}_{1\le\nu\le K(\delta_1)}$ we denote the corresponding partition of unity: $\sum_{\nu=1}^{K(\delta_1)}\chi_\nu(\zeta)=1$ for any $\zeta\in S^3$ and $\operatorname{supp}\chi_\nu\subset\{\zeta\in S^3; |\zeta-\zeta_\nu^*|<\delta_1\}$. Henceforth we extend $\chi_\nu$ to the set $\{\zeta;|\zeta|>1\}$ as the homogeneous function of the order zero such that $\chi_\nu\in C^\infty(\mathbb{R}^3)$ and

$$\operatorname{supp}\chi_\nu\subset\mathcal{O}(\delta_1)\equiv\left\{\zeta;\ \left|\frac{\zeta}{|\zeta|}-\zeta_\nu^*\right|<\delta_1\right\}.$$

We set $\mathcal{G}=\mathbb{R}^3\times[0,\frac{1}{N^2})$. Let $\gamma=(y^*,\zeta^*)\equiv(y^*,s^*,\xi_0^*,\xi_1^*,\xi_2^*)\in\partial\mathcal{G}\times S^3$ be an arbitrary point. In order to finish the proof, we need the following lemma.

**Lemma 4.2.** *Let $\gamma=(y^*,\zeta^*)\in\partial\mathcal{G}\times S^3$ be an arbitrary point and $\operatorname{supp}\chi_\nu\subset\mathcal{O}(\delta_1)$. Then for all sufficiently small $\delta_1>0$, the following estimate holds true:*

$$\sqrt{|s|}\|\mathbf{w}_\nu\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\sqrt{|s|}\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_3},\mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\le C_8(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}). \tag{4.15}$$

Assume for the moment that Lemma 4.2 holds true. Using Carleman estimate (4.15) we have

$$\begin{aligned}&\sqrt{|s|}\|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\sqrt{|s|}\|(\frac{\partial\mathbf{w}}{\partial y_3},\mathbf{w})\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\\ \le&\sum_{\nu=1}^{K}\sqrt{|s|}\|\chi_\nu\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\sqrt{|s|}\|(\frac{\partial\mathbf{w}_\nu}{\partial y_3},\mathbf{w}_\nu)\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\\ \le&C_8(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})})\quad\forall|s|\ge s_0.\end{aligned} \tag{4.16}$$

By Proposition 5.1 and the argument similar to (5.10) and (5.11) in [IY8], we obtain

$$\sqrt{N}\left(\int_{\mathcal{G}}\sum_{|\alpha|=0}^{2}|s|^{4-2|\alpha|}|D_{y'}^\alpha\mathbf{u}e^{|s|\varphi}|^2dy\right)^{\frac12}\le C_8(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}),\quad\forall|s|\ge s_0. \tag{4.17}$$

Directly from equations (2.1) we can obtain the estimates for $(\partial^2_{y_0^2}\mathbf{u})e^{|s|\varphi}$ and $(\partial_{y_0}\partial_{y_1}\mathbf{u})e^{|s|\varphi}$ :

$$N^{\frac14}\left(\int_{\mathcal G}\sum_{|\alpha|=0}^{2}|s|^{4-2|\alpha|}|D^\alpha \mathbf{u}e^{|s|\varphi}|^2dy\right)^{\frac12}\le C_8(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal G)}+\|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal G)}),\quad \forall |s|\ge s_0. \tag{4.18}$$

Since the constant $C_8$ is independent of $N$, estimate (4.18) implies

$$N^{\frac14}\left(\int_{\mathcal G}\sum_{|\alpha|=0}^{2}|s|^{4-2|\alpha|}|D^\alpha \mathbf{u}e^{|s|\varphi}|^2dy\right)^{\frac12}+\sqrt{|s|}\left\|\left(\frac{\partial \mathbf{w}}{\partial y_3},\mathbf{w}\right)\right\|_{\mathbf{L}^2(\partial\mathcal G)\times \mathbf{H}^{1,s}(\partial\mathcal G)}$$
$$\le C_9\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal G)}\quad \forall |s|\ge s_0. \tag{4.19}$$

This estimate immediately implies (4.5). ■

Now it suffices to prove Lemma 4.2.

Before starting the proof of Lemma 4.2, we need to recall some facts from the theory of pseudodifferential operators and Carleman estimates.

We set

$$P_{\mu,s}(y,s,D)=P_\mu(y,\mathbf{D}),\quad P_{\lambda+2\mu,s}(y,s,D)=P_{\lambda+2\mu}(y,\mathbf{D}),\quad \mathbf{D}=D+i|s|\nabla\varphi.$$

Denote

$$p_\beta(y,s,\xi')=-(\xi_0+i|s|\varphi_{y_0})^2+\beta[(\xi_1+i|s|\varphi_{y_1})^2+(\xi_2+i|s|\varphi_{y_2})^2$$
$$-2\ell_{y_1}(\xi_1+i|s|\varphi_{y_1})(\xi_3+i|s|\varphi_{y_3})-2\ell_{y_2}(\xi_2+i|s|\varphi_{y_2})(\xi_3+i|s|\varphi_{y_3})+(\xi_3+i|s|\varphi_{y_3})^2|G|^2], \tag{4.20}$$

where $\beta\in\{\mu,\lambda+2\mu\}$ and $s\ne 0$ is a parameter. The roots $\Gamma^{\pm}_{\beta}(y,s,\xi')$ of the polynomial $p_\beta$ with respect to the variable $\xi_3$ are given by

$$\Gamma^{\pm}_{\beta}(y,s,\xi')=-i|s|\varphi_{y_3}+\alpha^{\pm}_{\beta}(y,s,\xi'), \tag{4.21}$$

where

$$\alpha^{\pm}_{\beta}(y,s,\xi')=\frac{(\xi_1+i|s|\varphi_{y_1})\ell_{y_1}+(\xi_2+i|s|\varphi_{y_2})\ell_{y_2}}{|G|^2}\pm\sqrt{r_\beta(y,s,\xi')}, \tag{4.22}$$

$$r_\beta(y,\zeta)=\frac{\{(\xi_0+i|s|\varphi_{y_0})^2-\beta((\xi_1+i|s|\varphi_{y_1})^2+(\xi_2+i|s|\varphi_{y_2})^2)\}|G|^2+\beta(\xi+i|s|\nabla\varphi,\nabla\ell)^2}{\beta|G|^4}. \tag{4.23}$$

In some situations we can factorize the operator $P_{\beta,s}$ as a product of two first order pseudodifferential operators.

**Proposition 4.1.** *Let $\beta\in\{\mu,\lambda+2\mu\}$ and $|r_\beta(y,\zeta)|\ge\widehat\delta|\zeta|^2>0$ for all $(y,\zeta)\in(B_\delta\cap\mathcal G)\times\mathcal O(2\delta_1)$. Then we can factorize the operator $P_{\beta,s}$ into the product of two first order pseudodifferential operators:*

$$P_{\beta,s}\chi_\nu(s,D')V=\beta|G|^2(D_{y_3}-\Gamma^-_\beta(y,s,D'))(D_{y_3}-\Gamma^+_\beta(y,s,D'))\chi_\nu(s,D')V+T_\beta V, \tag{4.24}$$

*where* $\mathrm{supp}\,V\subset B_\delta\cap\mathcal G$ *and $T_\beta$ is a continuous operator:*

$$T_\beta : L^2(0,1;H^{1,s}(\mathbb R^3))\to L^2(0,1;L^2(\mathbb R^3)).$$

Let us consider the equation

$$(D_{y_3}-\Gamma^-_\beta(y,s,D'))\chi_\nu(s,D')V=q,\quad V|_{y_3=\frac{1}{N^2}}=0,\quad \mathrm{supp}\ V\subset B_\delta\cap\mathcal G.$$

For solutions of this problem, similarly to Proposition 5.4 in [IY8], we can prove

**Proposition 4.2.** *Let $\beta\in\{\mu,\lambda+2\mu\}$ and $|r_\beta(y,\zeta)|\ge\widehat\delta|\zeta|^2>0$ for all $(y,\zeta)\in B_\delta\times\mathcal O(2\delta_1)$. Then there exists a constant $C_{10}>0$ independent of $N$ such that*

$$\|\sqrt{|s|}\chi_\nu(s,D')V|_{y_3=0}\|_{L^2(\mathbb R^3)}\le C_{10}\|q\|_{L^2(\mathcal G)}. \tag{4.25}$$

Let $\widetilde w(y)$ be a function which satisfies

$$P_{\beta,s}(y,s,D)\widetilde w=\widetilde q\quad\text{in } \mathcal G,\quad \frac{\partial\widetilde w}{\partial y_3}\Big|_{y_3=1/N^2}=\widetilde w|_{y_3=1/N^2}=0,\quad \mathrm{supp}\,\widetilde w\subset B_\delta\times[0,\frac{1}{N^2}).$$

Let $P^*_{\beta,s}$ be the formally adjoint operator to $P_{\beta,s}$, where $\beta\in\{\mu,\lambda+2\mu\}$. Set $L_{+,\beta}=\frac{P_{\beta,s}+P^*_{\beta,s}}{2}$ and $L_{-,\beta}=\frac{P_{\beta,s}-P^*_{\beta,s}}{2}$. Obviously $L_{+,\beta}\widetilde w+L_{-,\beta}\widetilde w=\widetilde q$. For almost all $s\in\mathbb R^1$, the following equality holds true:

$$\varXi_\beta+\|L_{-,\beta}\widetilde w\|^2_{L^2(\mathcal G)}+\|L_{+,\beta}\widetilde w\|^2_{L^2(\mathcal G)}+\mathrm{Re}\int_{\mathcal G}([L_{+,\beta},L_{-,\beta}]\widetilde w,\overline{\widetilde w})dy=\|\widetilde q\|^2_{L^2(\mathcal G)}, \tag{4.26}$$

where

$$\varXi_\beta=\int_{\partial\mathcal G}\widetilde p_\beta(y,\nabla\varphi,-\vec e_4)(|s|\widetilde p_\beta(y,\nabla\widetilde w,\overline{\nabla\widetilde w})-|s|^3p_\beta(y,\nabla\varphi)|\widetilde w|^2)dy_0dy_1dy_2$$
$$+\mathrm{Re}\int_{\partial\mathcal G}\widetilde p_\beta(y,\nabla\widetilde w,-\vec e_4)\overline{L_{-,\beta}\widetilde w}dy_0dy_1dy_2, \tag{4.27}$$

$\vec e_4=(0,0,0,1)$ and

$$\widetilde p_\beta(y,\xi,\widetilde\xi)=\xi_0\widetilde\xi_0-\beta(\xi_1\widetilde\xi_1+\xi_2\widetilde\xi_2-\ell_{y_1}(\xi_1\widetilde\xi_3+\xi_3\widetilde\xi_1)-\ell_{y_2}(\xi_2\widetilde\xi_3+\xi_3\widetilde\xi_2)+|G|^2\xi_3\widetilde\xi_3).$$

We note that $\phi_{y_k}|_{\partial\mathcal G}=\varphi_{y_k}|_{\partial\mathcal G}$ for $k\in\{0,1,2\}$. Therefore on $\partial\mathcal G$, the function $\nabla_{y'}\varphi$ is independent of $N$ and $|\nabla\phi(y')-\nabla\varphi(y')|\le C/N^2$ with the constant $C$ independent of $N$. It is convenient for us to rewrite (4.27) in the form

$$\varXi_\beta=\varXi^{(1)}_\beta+\varXi^{(2)}_\beta,$$

$$\Xi_\beta^{(1)} = \mathrm{Re}\int_{y_3=0} 2|s|\beta(y^*)\frac{\partial\widetilde{w}}{\partial y_3}\overline{\left(\beta(y^*)\frac{\partial\widetilde{w}}{\partial y_1}\varphi_{y_1}(y^*)+\beta(y^*)\frac{\partial\widetilde{w}}{\partial y_2}\varphi_{y_2}(y^*)\right.}$$
$$\overline{\left.+\beta(y^*)\frac{\partial\widetilde{w}}{\partial y_3}\varphi_{y_3}(y^*)-\frac{\partial\widetilde{w}}{\partial y_0}\varphi_{y_0}(y^*)\right)}dy_0dy_1dy_2$$
$$+\int_{y_3=0}|s|\beta(y^*)\varphi_{y_3}(y^*)\left\{\left|\frac{\partial\widetilde{w}}{\partial y_0}\right|^2-\beta(y^*)\left(\left|\frac{\partial\widetilde{w}}{\partial y_1}\right|^2+\left|\frac{\partial\widetilde{w}}{\partial y_2}\right|^2+\left|\frac{\partial\widetilde{w}}{\partial y_3}\right|^2\right)\right\}$$
$$-|s|^2(\varphi^2_{y_0}(y^*)-\beta(y^*)(\varphi^2_{y_1}(y^*)+\varphi^2_{y_2}(y^*)+\varphi^2_{y_3}(y^*)))|\widetilde{w}|^2)dy_0dy_1dy_2.$$

Then

$$|\Xi_\beta^{(2)}|\le\epsilon(\delta)|s|\left\|\left(\frac{\partial\widetilde{w}}{\partial y_3},\widetilde{w}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})},\tag{4.28}$$

where $\epsilon(\delta)\to 0$ as $\delta\to+0$. We can prove that there exists a parameter $\widehat{\tau}>1$ such that for any $\tau>\widehat{\tau}$ there exists $s_0(\tau)$ such that

$$\begin{aligned}&\frac{3}{4}\|L_{-,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})}+\frac{3}{4}\|L_{+,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})}+\mathrm{Re}\,([L_{+,\beta},L_{-,\beta}]\widetilde{w},\overline{\widetilde{w}})_{L^2(\mathcal{G})}\\&+C_{11}|s|\|\widetilde{w}\|_{L^2(\partial\mathcal{G})}\|\partial_{y_3}\widetilde{w}\|_{L^2(\partial\mathcal{G})}\\&\ge C_{12}|s|\|\widetilde{w}\|^2_{H^{1,s}(\mathcal{G})},\quad\forall|s|\ge s_0(\tau),\end{aligned}\tag{4.29}$$

where $C_{12}>0$ is independent of $s$. The proof of (4.29) is done exactly same as in Appendix II in [IY8]. Combining (4.26) and (4.29), we arrive at

$$\begin{aligned}&\frac{1}{4}\|L_{-,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})}+\frac{1}{4}\|L_{+,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})}+C_{12}|s|\|\widetilde{w}\|^2_{H^{1,s}(\mathcal{G})}+\Xi_\beta\\&\qquad\le C_{13}(\|q\|^2_{L^2(\mathcal{G})}+|s|\|\widetilde{w}\|_{L^2(\partial\mathcal{G})}\|\partial_{y_3}\widetilde{w}\|_{L^2(\partial\mathcal{G})}),\quad\forall|s|\ge s_0(\tau).\end{aligned}\tag{4.30}$$

By Rot, Div, Nab denote the operators obtained from rot, div, $\nabla_{y'}$ after the change of variables. In that case, on $\partial\mathcal{G}$ we can rewrite equation (4.12) and identity $\mathrm{div}\,\mathrm{rot}\,\mathbf{u}=0$ in the following way:

$$i\mu\mathrm{Rot}\,(y,\mathbf{D})\mathbf{w}'-i(\lambda+2\mu)\mathrm{Nab}\,(y,\mathbf{D})w_4=\mathbf{f}e^{|s|\varphi}+K(y,D,s)(\mathbf{u}e^{|s|\varphi}),\quad\mathrm{Div}\,(y,\mathbf{D})\mathbf{w}'=0,\tag{4.31}$$

where $K(y,D,s)$ is the first order differential operator. Applying the operator $\chi_\nu(s,D')$ to equation (4.31), we have

$$i\mu\mathrm{Rot}(y,\mathbf{D})\mathbf{w}'_\nu-i(\lambda+2\mu)\mathrm{Nab}(y,\mathbf{D})w_{4,\nu}=\mathbf{F}_1,\ \mathrm{Div}(y,\mathbf{D})\mathbf{w}'_\nu+[\chi_\nu,\mathrm{Div}]\mathbf{w}'=0\ \ y\in\partial\mathcal{G},\tag{4.32}$$

where

$$\mathbf{F}_1=\chi_\nu\mathbf{f}e^{|s|\varphi}-i[\chi_\nu,\mu\mathrm{Rot}(y,\mathbf{D})]\mathbf{w}'+i[\chi_\nu,(\lambda+2\mu)\mathrm{Nab}(y,\mathbf{D})]w_4+\chi_\nu K(y,D,s)(\mathbf{u}e^{|s|\varphi}).$$

We will prove Lemma 4.2 separately in the following three cases:

(1) $r_\mu(\gamma)=0$, $r_{\lambda+2\mu}(\gamma)\ne 0$ (Section 5)
(2) $r_\mu(\gamma)\ne 0$, $r_{\lambda+2\mu}(\gamma)=0$ (Section 6)
(3) $r_\mu(\gamma)\ne 0$, $r_{\lambda+2\mu}(\gamma)\ne 0$ or $r_\mu(\gamma)=r_{\lambda+2\mu}(\gamma)=0$ (Section 7).

**§5. The case : $r_\mu(\gamma)=0$ and $r_{\lambda+2\mu}(\gamma)\ne 0$.**

In this section, we treat the case where $r_\mu(\gamma)=0$ and $r_{\lambda+2\mu}(\gamma)\ne 0$. Taking the parameters $\delta$ and $\delta_1$ sufficiently small, we can assume that there exists a constant $\widehat{C}>0$ such that

$$|r_{\lambda+2\mu}(y,\zeta)|\ge \widehat{C}|\zeta|^2,\quad \forall (y,\zeta)\in B_\delta\times\mathcal{O}(\delta_1),\ |\zeta|\ge 1.$$

We note by (4.30) that there exist $C_1>0$ and $C_2>0$ such that

$$\begin{aligned}&C_1|s|\|w_{k,\nu}\|^2_{H^{1,s}(\mathcal{G})}+\Xi^{(1)}_{\mu,k}\\ &\le C_2(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})})+\epsilon(\delta)|s|\left\|\left(\frac{\partial \mathbf{w}'_\nu}{\partial y_2},\mathbf{w}'_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})},\quad k\in\{1,2,3\}\end{aligned}\tag{5.1}$$

and the parameter $\epsilon$ can be taken sufficiently small if we decrease $\delta$. Note that $\Xi^{(1)}_{k,\mu}$ can be written in the form:

$$\begin{aligned}\Xi^{(1)}_{k,\mu}&=\int_{\partial\mathcal{G}}\left(|s|(\mu^2\varphi_{y_3})(y^*)\left|\frac{\partial w_{k,\nu}}{\partial y_3}\right|^2+|s|^3(\mu^2\varphi^3_{y_3})(y^*)|w_{k,\nu}|^2\right)d\Sigma\\ &+\mathrm{Re}\int_{\partial\mathcal{G}}2|s|\mu(y^*)\frac{\partial w_{k,\nu}}{\partial y_3}\overline{\left((\mu\varphi_{y_1})(y^*)\frac{\partial w_{k,\nu}}{\partial y_1}+(\mu\varphi_{y_2})(y^*)\frac{\partial w_{k,\nu}}{\partial y_2}-\varphi_{y_0}(y^*)\frac{\partial w_{k,\nu}}{\partial y_0}\right)}d\Sigma\\ &+\int_{\partial\mathcal{G}}|s|(\mu\varphi_{y_3})(y^*)\{\xi_0^2-\mu(y^*)(\xi_1^2+\xi_2^2)-s^2\varphi^2_{y_0}(y^*)+s^2\mu(y^*)(\varphi^2_{y_1}(y^*)+\varphi^2_{y_2}(y^*))\}|\widehat{w}_{k,\nu}|^2d\Sigma\\ &\equiv J_1^{(k)}+J_2^{(k)}+J_3^{(k)}.\end{aligned}\tag{5.2}$$

By (4.21) - (4.23), there exists $C_3>0$ such that

$$\begin{aligned}&|\xi_0^2-s^2\varphi^2_{y_0}(y^*)-\mu(y^*)(\xi_1^2+\xi_2^2)+\mu(y^*)s^2(\varphi^2_{y_1}(y^*)+\varphi^2_{y_2}(y^*))|\\ &+|s||\xi_0\varphi_{y_0}(y^*)-\mu(y^*)\xi_1\varphi_{y_1}(y^*)-\mu(y^*)\xi_2\varphi_{y_2}(y^*)|\\ &\le\delta_1C_3(|\xi_0|^2+|\xi_1|^2+|\xi_2|^2+s^2),\quad\forall\zeta\in\mathcal{O}(\delta_1).\end{aligned}\tag{5.3}$$

Next we take the parameter $\delta_1$ sufficiently small such that

$$|\xi_0|^2\le C_4(\xi_1^2+\xi_2^2+s^2),\quad\forall\zeta\in\mathcal{O}(\delta_1),\tag{5.4}$$

where the constant $C_4>0$ is independent of $\zeta$. Then, by (5.3), we have

$$\sum_{k=1}^3|J_3^{(k)}|\le\delta_1(\mu\varphi_{y_3})(y^*)|s|\left\|\left(\frac{\partial\mathbf{w}_\nu}{\partial y_3},\mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.\tag{5.5}$$

Moreover we claim that there exists $\delta_0>0$ such that if $\delta_1\in(0,\delta_0)$, then there exists $C_5>0$ such that

$$|\xi_0|\le C_5(|\xi_1|+|\xi_2|+|s|),\quad\forall\zeta\in\mathcal{O}(\delta_1).\tag{5.6}$$

We set $V^+_{\lambda+2\mu} = (D_{y_3} - \Gamma^+_{\lambda+2\mu}(y,s,D'))w_{4,\nu}$. Then by Proposition 4.1

$$P_{\lambda+2\mu,s}w_{4,\nu} = (\lambda+2\mu)|G|^2(D_{y_3} - \Gamma^-_{\lambda+2\mu}(y,s,D'))V^+_{\lambda+2\mu} + T_{\lambda+2\mu}w_{4,\nu},$$

where $T_{\lambda+2\mu} \in \mathcal{L}(H^{1,s}(\mathcal{G}), L^2(\mathcal{G}))$. This decomposition and Proposition 4.2 immediately imply

$$\begin{aligned}&\|\sqrt{|s|}(D_{y_3} - \Gamma^+_{\lambda+2\mu}(y,s,D'))w_{4,\nu}|_{y_3=0}\|_{L^2(\partial\mathcal{G})}\\ \le & C_6(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}).\end{aligned} \tag{5.7}$$

Next we estimate the term $J^{(k)}_2$. First we note that thanks to the homogeneous Dirichlet boundary conditions, we have the a priori estimate

$$\sqrt{|s|}\|w_{3,\nu}\|_{H^{1,s}(\partial\mathcal{G})} \le \epsilon(\delta)\sqrt{|s|}\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}. \tag{5.8}$$

Using (5.8), from (4.30) with $w_{3,\nu}$ instead of $\widetilde{w}$, we have

$$\begin{aligned}&\sqrt{|s|}\left\|\left(\frac{\partial w_{3,\nu}}{\partial y_3}, w_{3,\nu}\right)\right\|_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})} \le C_7(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})})\\ &+\epsilon(\delta)\sqrt{|s|}\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.\end{aligned} \tag{5.9}$$

Now we consider the following two cases:

**Case 1.** Assume that $s^* \neq 0$. In this case by (5.3)

$$\sum_{k=1}^3 |J^{(k)}_2| + |J^{(k)}_3| \le \epsilon(\delta)|s|\left\|\left(\frac{\partial \mathbf{w}'_\nu}{\partial y_3}, \mathbf{w}'_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.$$

Therefore for some constant $C_8 > 0$

$$\sum_{k=1}^3 \Xi^{(1)}_{k,\mu} \ge |s|C_8\left\|\left(\frac{\partial \mathbf{w}'_\nu}{\partial y_3}, \mathbf{w}'_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}. \tag{5.10}$$

Combining (5.10) and (5.1), we have

$$\sqrt{|s|}\left\|\left(\frac{\partial \mathbf{w}'_\nu}{\partial y_3}, \mathbf{w}'_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_9(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{5.11}$$

Using (5.11) we obtain from equations (4.32)

$$\begin{aligned}&\sqrt{|s|}\|\mathrm{Nab}(y,\mathbf{D})w_{4,\nu}\|_{L^2(\partial\mathcal{G})}\\ \le & C_{10}(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}) + \epsilon(\delta)\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.\end{aligned} \tag{5.12}$$

On the other hand, thanks to (5.7) we have

$$\sqrt{|s|}\left\Vert\left(\frac{\partial w_{4,\nu}}{\partial y_3}, w_{4,\nu}\right)\right\Vert_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}$$
$$\le C_{11}\sqrt{|s|}\Vert \mathrm{Nab}(y,\mathbf{D})w_{4,\nu}\Vert_{L^2(\partial\mathcal{G})} + \epsilon(\delta)\left\Vert\left(\frac{\partial \mathbf{w}_{\nu}}{\partial y_3}, \mathbf{w}_{\nu}\right)\right\Vert_{\mathbf{L}^2(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})}.$$

Combining this estimate with (5.11) and (5.12), we obtain

$$\sqrt{|s|}\left\Vert\left(\frac{\partial \mathbf{w}_{\nu}}{\partial y_3}, \mathbf{w}_{\nu}\right)\right\Vert_{\mathbf{L}^2(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_{12}(\Vert \mathbf{f}e^{|s|\varphi}\Vert_{\mathbf{H}^{1,s}(\mathcal{G})} + \Vert\mathbf{w}\Vert_{\mathbf{H}^{1,s}(\mathcal{G})}). \quad (5.13)$$

Inequalities (5.13) and (5.1) imply (4.15).

**Case 2.** Assume that $s^* = 0$. By (4.32) the following equality is true:

$$R(y,s,D')(w_{1,\nu}, w_{2,\nu}) \equiv (\mathbf{D}_1 w_{1,\nu} + \mathbf{D}_2 w_{2,\nu}, -\mathbf{D}_2 w_{1,\nu} + \mathbf{D}_1 w_{2,\nu})$$
$$= \left(F_1, \frac{\lambda+2\mu}{\mu}\alpha^+_{\lambda+2\mu}(y,s,D')w_{4,\nu} + F_2\right),$$

where

$$\sqrt{|s|}\Vert(F_1,F_2)\Vert_{\mathbf{L}^2(\partial\mathcal{G})} \le \epsilon(\delta)\sqrt{|s|}\left\Vert\left(\frac{\partial \mathbf{w}_{\nu}}{\partial y_3}, \mathbf{w}_{\nu}\right)\right\Vert_{\mathbf{L}^2(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})} + C_{13}(\Vert \mathbf{f}e^{|s|\varphi}\Vert_{\mathbf{H}^{1,s}(\mathcal{G})} + \Vert\mathbf{w}\Vert_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

The principal symbol of the operator $R$ is

$$R(y^*, s, \xi_1, \xi_2) = \begin{pmatrix} \xi_1 + i|s|\varphi_{y_1}(y^*) & \xi_2 + i|s|\varphi_{y_2}(y^*) \\ -\xi_2 - i|s|\varphi_{y_2}(y^*) & \xi_1 + i|s|\varphi_{y_1}(y^*) \end{pmatrix}.$$

Since $\det R(y^*, s^*, \xi_1^*, \xi_2^*) \ne 0$, there exists a parametrix of the operator $R$ such that

$$(w_{1,\nu}, w_{2,\nu}) = R(y,s,D')^{-1}\left(0, \frac{\lambda+2\mu}{\mu}\alpha^+_{\lambda+2\mu}(y,s,D')w_{4,\nu}\right)$$
$$+ R(y,s,D')^{-1}(F_1,F_2) + T_{-1}(w_{1,\nu}, w_{2,\nu}). \quad (5.14)$$

By the first and second equations in (4.32), we have

$$(D_3 w_{1,\nu}, D_3 w_{2,\nu}) = \frac{\lambda+2\mu}{\mu}(D_2 w_{4,\nu}, -D_1 w_{4,\nu}) + (F_4, F_5), \quad (5.15)$$

where

$$\sqrt{|s|}\Vert(F_4,F_5)\Vert_{\mathbf{L}^2(\partial\mathcal{G})} \le \epsilon(\delta,\delta_1)\sqrt{|s|}\left\Vert\left(\frac{\partial \mathbf{w}_{\nu}}{\partial y_3}, \mathbf{w}_{\nu}\right)\right\Vert_{\mathbf{L}^2(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})}$$
$$+ C_{14}(\Vert \mathbf{f}e^{|s|\varphi}\Vert_{\mathbf{H}^{1,s}(\mathcal{G})} + \Vert\mathbf{w}\Vert_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

Using equalities (5.14) and (5.15), we can reduce $J_2^{(1)}, J_2^{(2)}$ to the form

$$J_2^{(1)} = \mathrm{Re}\int_{\partial\mathcal{G}} 2|s|(\lambda+2\mu)(y^*)\frac{\partial w_{4,\nu}}{\partial y_2}\overline{\{i(\mu\varphi_{y_1})(y^*)D_1 + i(\mu\varphi_{y_2})(y^*)D_2 - i\varphi_{y_0}(y^*)D_0\}}$$
$$\overline{(R(y,s,D)^{-1}(0,\alpha^+_{\lambda+2\mu}(y,s,D')w_{4,\nu})\cdot\vec{j}_1)}d\Sigma + I_1, \tag{5.16}$$

$$J_2^{(2)} = -\mathrm{Re}\int_{\partial\mathcal{G}} 2|s|(\lambda+2\mu)(y^*)\frac{\partial w_{4,\nu}}{\partial y_1}\overline{\{i(\mu\varphi_{y_1})(y^*)D_1 + i(\mu\varphi_{y_2})(y^*)D_2 - i\varphi_{y_0}(y^*)D_0\}}$$
$$\overline{(R(y,s,D)^{-1}(0,\alpha^+_{\lambda+2\mu}(y,s,D')w_{4,\nu})\cdot\vec{j}_2)}d\Sigma + I_2, \tag{5.17}$$

where $\vec{j}_1 = (1,0)$, $\vec{j}_2 = (0,1)$, $I_1$ and $I_2$ are terms which are estimated by

$$|I_1|+|I_2| \le \epsilon(\delta)|s|\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} + C_{15}(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

Since $\mathrm{Re}\,\alpha^+_\mu(\gamma) = 0$ and $\mathrm{Im}\,R(\gamma)^{-1} = 0$, by Gårding's inequality we obtain from (5.16) and (5.17) that

$$|J_2^{(1)}|+|J_2^{(2)}| \le \epsilon(\delta)\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} + C_{16}(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

This inequality and (5.5) imply for $k \in \{1,2,3\}$

$$\Xi^{(1)}_{k,\mu} \ge \int_{\partial\mathcal{G}}\left\{|s|(\mu^2\varphi_{y_3})(y^*)\left|\frac{\partial w_{k,\nu}}{\partial y_3}\right|^2 + |s|^3(\mu^2\varphi^3_{y_3})(y^*)|w_{k,\nu}|^2\right\}d\Sigma$$
$$-\epsilon|s|\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}. \tag{5.18}$$

In terms of (5.15) and (5.18), we have

$$|s|\|w_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} \le C_{17}\sum_{k=1}^3 \Xi^{(1)}_{k,\mu} + \epsilon|s|\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.$$

This inequality and (5.7) imply

$$|s|\left\|\left(\frac{\partial w_{4,\nu}}{\partial y_3}, w_{4,\nu}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})} \le C_{18}\left(\sum_{k=1}^3 \Xi^{(1)}_{k,\mu} + \|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right)$$
$$+\epsilon|s|\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} + C_{18}(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{5.19}$$

By (5.14), (5.18) and (5.19), we obtain

$$|s|\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_{19}\left(\sum_{k=1}^3 \Xi^{(1)}_{k,\mu} + \|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\right)$$
$$+\epsilon|s|\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.$$

This estimate and (4.30) imply (4.15). ■

**§6. The case: $r_{\lambda+2\mu}(\gamma)=0$ and $r_\mu(\gamma)\ne 0$.**

Let $\gamma=(y^*,\zeta^*)=(y^*,s^*,\xi_0^*,\xi_1^*,\xi_2^*)$ be a point on $\partial\mathcal{G}\times S^3$ such that $r_{\lambda+2\mu}(\gamma)=0, r_\mu(\gamma)\ne 0$ and $\mathrm{supp}\chi_\nu\subset\mathcal{O}(\delta_1)$. Taking the parameters $\delta$ and $\delta_1$ sufficiently small, we can assume that there exists a constant $\widehat{C}>0$ such that

$$|r_\mu(y,\zeta)|\ge \widehat{C}|\zeta|^2,\quad \forall(y,\zeta)\in B_\delta\times\mathcal{O}(\delta_1),\quad |\zeta|\ge 1.$$

By (4.21)-(4.23), there exist $\delta_0>0$ and $C_1>0$ such that for all $\delta_1\in(0,\delta_0)$ we have

$$\xi_0^2\le C_1(\xi_1^2+\xi_2^2+s^2),\quad \forall\zeta\in\mathcal{O}(\delta_1). \tag{6.1}$$

We consider the following three cases.

**Case A.** Assume that $s^*=0$ and $\varphi_{y_3}(y^*)>\frac{|\frac{1}{\mu(y^*)}\xi_0^*\varphi_{y_0}(y^*)-\xi_1^*\varphi_{y_1}(y^*)-\xi_2^*\varphi_{y_2}(y^*)|}{\frac{\lambda+\mu}{\mu}(y^*)|(\xi_1^*,\xi_2^*)|}$.

In that case, there exists a constant $C_2>0$ such that

$$-\mathrm{Im}\,\Gamma_\mu^\pm(y,\zeta)\ge C_2|s|,\quad \forall(y,\zeta)\in B_\delta\times\mathcal{O}(\delta_1),$$

provided that $|\delta|+|\delta_1|$ is sufficiently small. Since $s^*=0$, we may assume that

$$|\xi_0|^2+s^2\le C_3(\xi_1^2+\xi_2^2),\quad \forall\zeta\in\mathcal{O}(\delta_1) \tag{6.2}$$

for some constant $C_3>0$, taking a sufficiently small $\delta_1$. We set $V_\mu^\pm=(D_{y_3}-\Gamma_\mu^\pm(y,s,D'))\mathbf{w}'_\nu$. Then, by Proposition 4.1, we have

$$\begin{aligned}P_{\mu,s}(y,D)\mathbf{w}'_\nu&=|G|^2\mu(D_{y_3}-\Gamma_\mu^-(y,s,D'))V_\mu^++T_\mu^+\mathbf{w}'_\nu\\ &=|G|^2\mu(D_{y_3}-\Gamma_\mu^+(y,s,D'))V_\mu^-+T_\mu^-\mathbf{w}'_\nu,\end{aligned} \tag{6.3}$$

where $T_\mu^\pm\in\mathcal{L}(\mathbf{H}^{1,s}(\mathcal{G}),\mathbf{L}^2(\mathcal{G}))$. This decomposition and Proposition 4.2 imply

$$\|\sqrt{|s|}(D_{y_3}-\Gamma_\mu^\pm(y,s,D'))\mathbf{w}'_\nu|_{y_3=0}\|_{\mathbf{L}^2(\partial\mathcal{G})}\le C_4(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{w}'\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{6.4}$$

We have

$$(-V_\mu^++V_\mu^-)|_{y_3=0}=(\alpha_\mu^+(y,s,D')-\alpha_\mu^-(y,s,D'))\mathbf{w}'_\nu\quad\text{on }\partial\mathcal{G}. \tag{6.5}$$

Since $\alpha_\mu^+(y^*,\zeta^*)-\alpha_\mu^-(y^*,\zeta^*)=2\sqrt{r_\mu(y^*,\zeta^*)}\ne 0$, by (6.4), (6.5) and Gårding's inequality we have

$$\sqrt{|s|}\|\mathbf{w}'_\nu\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}\le C_5(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{w}'\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{6.6}$$

By (6.6) and (6.4), we obtain

$$\int_{\partial\mathcal{G}}|s|\left|\frac{\partial\mathbf{w}'_\nu}{\partial y_3}\right|^2d\Sigma\le C_6(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{6.7}$$

Finally, by (6.6) and (6.7) combined with (4.32), we obtain

$$|s|\left\|\left(\frac{\partial w_{4,\nu}}{\partial y_3}, w_{4,\nu}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})} \le C_7(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{6.8}$$

Inequalities (6.6) - (6.8) and (4.30) imply (4.15).

**Case B.** Assume that $s^* = 0$ and

$$\varphi_{y_3}(y^*) \le \frac{|\frac{1}{\mu(y^*)}\xi_0^*\varphi_{y_0}(y^*) - \xi_1^*\varphi_{y_1}(y^*) - \xi_2^*\varphi_{y_2}(y^*)|}{\sqrt{\frac{\lambda+\mu}{\mu}(y^*)}|(\xi_1^*, \xi_2^*)|}. \tag{6.9}$$

In that case $\lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(y^*, \zeta)/|s| \ne 0$. Since $s^* = 0$, we note that $\operatorname{Re} r_\mu(y^*, \zeta^*) > 0$. Set $I = \operatorname{sign} \lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(y^*, \zeta)/|s|$. Then we have

$$\Gamma^+_\mu(y^*, \zeta^*) = I\sqrt{\operatorname{Re} r_\mu(y^*, \zeta^*)}. \tag{6.10}$$

Therefore

$$-\operatorname{Re}\Gamma^+_\mu(y^*, \zeta^*)((\mu\varphi_{y_1})(y^*)\xi_1^* + (\mu\varphi_{y_2})(y^*)\xi_2^* - \varphi_{y_0}(y^*)\xi_0^*) > 0.$$

Taking the parameters $\delta > 0$ and $\delta_1 > 0$ sufficiently small, we obtain

$$-\operatorname{Re}\Gamma^+_\mu(y, \zeta)(\mu\varphi_{y_1}(y)\xi_1 + \mu\varphi_{y_2}(y)\xi_2 - \varphi_{y_0}(y)\xi_0) > 0, \quad \forall(y, \zeta) \in B_\delta\times\mathcal{O}(\delta_1). \tag{6.11}$$

Using the definition of $V^+_\mu$, we have

$$\begin{aligned}
J_2 =& \operatorname{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_3}\overline{\left(\mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_1}\varphi_{y_1}(y^*) + \mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_2}\varphi_{y_2}(y^*) - \frac{\partial \mathbf{w}'_\nu}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma\\
=&\operatorname{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)i\Gamma^+_\mu(y, s, D')\mathbf{w}'_\nu\overline{\left(\mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_1}\varphi_{y_1}(y^*) + \mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_2}\varphi_{y_2}(y^*) - \frac{\partial \mathbf{w}'_\nu}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma\\
&+\operatorname{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)iV^+_\mu(\cdot, 0)\overline{\left(\mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_1}\varphi_{y_1}(y^*) + \mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_2}\varphi_{y_2}(y^*) - \frac{\partial \mathbf{w}'_\nu}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma\\
=&\operatorname{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)(\mu(y^*)D_{y_1}\varphi_{y_1}(y^*) + \mu(y^*)D_{y_2}\varphi_{y_2}(y^*) - D_{y_0}\varphi_{y_0}(y^*))\Gamma^+_\mu(y, s, D')\mathbf{w}'_\nu\overline{\mathbf{w}'_\nu}d\Sigma\\
&+\operatorname{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)iV^+_\mu(\cdot, 0)\overline{\left(\mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_1}\varphi_{y_1}(y^*) + \mu(y^*)\frac{\partial \mathbf{w}'_\nu}{\partial y_2}\varphi_{y_2}(y^*) - \frac{\partial \mathbf{w}'_\nu}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma.
\end{aligned} \tag{6.12}$$

By (6.11) we obtain from Gårding's inequality that the first integral at the right hand side of (6.12) is negative. Consider two cases. First let

$$(\varphi_{y_1}(y^*)\xi_1^* + \varphi_{y_2}(y^*)\xi_2^*)\Gamma^+_\mu(y^*, \zeta^*) \ge 0.$$

This inequality and (6.11) yield that $|\xi_0^*\varphi_{y_0}(y^*)| > |\xi_1^*\varphi_{y_1}(y^*) + \xi_2^*\varphi_{y_2}(y^*)|$. If $\xi_0^*\varphi_{y_0}(y^*) > 0$, then $\Gamma_\mu^+(y^*,\zeta^*) = |\sqrt{r_\mu(\gamma)}|$ and $\xi_1^*\varphi_{y_1}(y^*) + \xi_2^*\varphi_{y_2}(y^*) \ge 0$. By the first condition in (2.6), we obtain

$$\frac{\mu}{\sqrt{\lambda+2\mu}}\frac{(\varphi_{y_1}(y^*)\xi_1^* + \varphi_{y_2}(y^*)\xi_2^*)}{|(\xi_1^*,\xi_2^*)|} + \frac{\sqrt{\mu}\sqrt{\lambda+\mu}}{\sqrt{\lambda+2\mu}}|\varphi_{y_3}(y^*)|$$
$$\ge |\varphi_{y_0}(y^*)|.$$

Again by the third condition in (2.6), we note that $|\varphi_{y_3}(y^*)| = \varphi_{y_3}(y^*)$. On the other hand, from $r_{\lambda+2\mu}(y^*,0,\xi_0^*,\xi_1^*,\xi_2^*) = 0$, we see that $|\xi_0^*| = \sqrt{(\lambda+2\mu)(y^*)}|(\xi_1^*,\xi_2^*)|$. By $\xi_0^*\varphi_{y_0}(y^*) > 0$, we obtain

$$\varphi_{y_3}(y^*) > \frac{-\varphi_{y_1}(y^*)\xi_1^* - \varphi_{y_2}(y^*)\xi_2^* + \frac{\varphi_{y_0}(y^*)}{\mu(y^*)}\xi_0^*}{\sqrt{\frac{\lambda+\mu}{\mu}(y^*)}|(\xi_1^*,\xi_2^*)|}.$$

This contradicts (6.9).

If $\xi_0^*\varphi_{y_0}(y^*) < 0$, then $\Gamma_\mu^+(y^*,\zeta^*) = -|\sqrt{r_\mu(\gamma)}|$ and $\xi_1^*\varphi_{y_1}(y^*) + \xi_2^*\varphi_{y_2}(y^*) < 0$. Therefore

$$\varphi_{y_3}(y^*) > \frac{|\varphi_{y_1}(y^*)\xi_1^* + \varphi_{y_2}(y^*)\xi_2^* - \frac{\varphi_{y_0}(y^*)\xi_0^*}{\mu(y^*)}|}{\sqrt{\frac{\lambda+\mu}{\mu}(y^*)}|(\xi_1^*,\xi_2^*)|}$$
$$= \frac{\varphi_{y_1}(y^*)\xi_1^* + \varphi_{y_2}(y^*)\xi_2^* - \frac{\varphi_{y_0}(y^*)}{\mu(y^*)}\xi_0^*}{\sqrt{\frac{\lambda+\mu}{\mu}(y^*)}|(\xi_1^*,\xi_2^*)|}.$$

By (2.6) this again contradicts (6.9).

As the second case, one has to consider $(\varphi_{y_1}(y^*)\xi_1^* + \varphi_{y_2}(y^*)\xi_2^*)\Gamma_\mu^+(y^*,\zeta^*) < 0$. By Gårding's inequality, for $k \in \{1,2\}$, we have

$$\mathrm{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)i\Gamma_\mu^+(y,D')w_{k,\nu}\overline{(\mu(y^*)\varphi_{y_1}(y^*)\frac{\partial w_{k,\nu}}{\partial y_1} + \mu(y^*)\varphi_{y_2}(y^*)\frac{\partial w_{k,\nu}}{\partial y_2})}d\Sigma < 0.$$

This inequality and the fact that $J_2$ is negative implies that

$$-\mathrm{Re}\int_{\partial\mathcal{G}} 2|s|\mu(y^*)i\Gamma_\mu^+(y,D')w_{k,\nu}\times$$
$$\overline{\left((\lambda+2\mu)(y^*)\frac{\partial w_{k,\nu}}{\partial y_1}\varphi_{y_1}(y^*) + (\lambda+2\mu)(y^*)\varphi_{y_2}(y^*)\frac{\partial w_{k,\nu}}{\partial y_2} - \frac{\partial w_{k,\nu}}{\partial y_0}\varphi_{y_0}(y^*)\right)}d\Sigma > 0. \tag{6.13}$$

Note that

$$\Xi^{(1)}_{\lambda+2\mu}=\int_{\partial\mathcal{G}}\left(|s|((\lambda+2\mu)^2\varphi_{y_3})(y^*)\left|\frac{\partial w_{4,\nu}}{\partial y_3}\right|^2+|s|^3((\lambda+2\mu)^2\varphi^3_{y_3})(y^*)|w_{4,\nu}|^2\right)d\Sigma$$

$$+\mathrm{Re}\int_{\partial\mathcal{G}}2|s|(\lambda+2\mu)(y^*)\frac{\partial w_{4,\nu}}{\partial y_3}$$

$$\overline{\times\left(((\lambda+2\mu)\varphi_{y_1})(y^*)\frac{\partial w_{4,\nu}}{\partial y_1}+((\lambda+2\mu)\varphi_{y_2})(y^*)\frac{\partial w_{4,\nu}}{\partial y_2}-\varphi_{y_0}(y^*)\frac{\partial w_{4,\nu}}{\partial y_0}\right)}d\Sigma$$

$$+\int_{\partial\mathcal{G}}|s|((\lambda+2\mu)\varphi_{y_3})(y^*)(\xi_0^2-(\lambda+2\mu)(y^*)(\xi_1^2+\xi_2^2)-s^2\varphi^2_{y_0}(y^*)$$

$$+s^2(\lambda+2\mu)(y^*)(\varphi^2_{y_1}(y^*)+\varphi^2_{y_2}(y^*)))|w_{4,\nu}|^2d\Sigma$$

$$\equiv\widetilde{J}_1+\widetilde{J}_2+\widetilde{J}_3.$$

Using equalities (4.32), we can transform $\widetilde{J}_2$ as

$$\widetilde{J}_2=-\mathrm{Re}\sum_{k=1}^{2}\int_{\partial\mathcal{G}}2|s|\frac{\mu^2}{\lambda+2\mu}\frac{\partial w_{k,\nu}}{\partial y_3}\times$$

$$\overline{\left((\lambda+2\mu)\varphi_{y_1}(y^*)\frac{\partial w_{k,\nu}}{\partial y_1}+(\lambda+2\mu)\varphi_{y_2}(y^*)\frac{\partial w_{k,\nu}}{\partial y_2}-\varphi_{y_0}(y^*)\frac{\partial w_{k,\nu}}{\partial y_0}\right)}d\Sigma+I,$$

where

$$|I|\le\epsilon(\delta)|s|\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3},\mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}+C_8(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

Then by (6.13)

$$\widetilde{J}_2>C_{10}|s|\left\|\left(\frac{\partial \mathbf{w}'_\nu}{\partial y_3},\mathbf{w}'_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}-C_9(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{6.14}$$

Since

$$|\widetilde{J_3}|\le C_{11}\delta_1|s|\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_3},w_{2,\nu}\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.$$

This inequality and (6.14) imply

$$\Xi^{(1)}_{\lambda+2\mu}\ge C_{12}\left\{\int_{\partial\mathcal{G}}\left(|s|\left|\frac{\partial w_{4,\nu}}{\partial y_3}\right|^2+|s|^3|w_{4,\nu}|^2\right)d\Sigma+|s|\left\|\left(\frac{\partial \mathbf{w}'_\nu}{\partial y_3},\mathbf{w}'_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\right\}$$
$$-\epsilon(\delta)|s|\left\|\left(\frac{\partial \mathbf{w}_\nu}{\partial y_3},\mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}-C_9(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{6.15}$$

Now we will estimate $\widetilde{J}_3$. By (4.21) and (4.23), there exists a constant $C_{13} > 0$ such that

$$\Big|\xi_0^2 - |s|^2\varphi_{y_0}^2(y^*) - (\lambda+2\mu)(y^*)\xi_1^2 + ((\lambda+2\mu)\varphi_{y_1}^2)(y^*)|s|^2 - (\lambda+2\mu)(y^*)\xi_2^2 + ((\lambda+2\mu)\varphi_{y_2}^2)(y^*)|s|^2\Big| \le C_{13}\delta_1(|\xi'|^2+s^2), \quad \forall\zeta\in\mathcal{O}(\delta_1). \tag{6.16}$$

Using (6.16), we obtain

$$\begin{aligned} &\xi_0^2 - \mu(y^*)\xi_1^2 - \mu(y^*)\xi_2^2 - s^2\varphi_{y_0}^2(y^*) + s^2(\mu\varphi_{y_1}^2)(y^*) + s^2(\mu\varphi_{y_2}^2)(y^*)\\ =&(\lambda+\mu)(y^*)\{\xi_1^2+\xi_2^2 - s^2\varphi_{y_1}^2(y^*) - s^2\varphi_{y_2}^2(y^*)\}\\ &+\xi_0^2 - (\lambda+2\mu)(y^*)\xi_1^2 - (\lambda+2\mu)(y^*)\xi_2^2 - s^2\varphi_{y_0}^2(y^*)\\ &+s^2((\lambda+2\mu)\varphi_{y_1}^2)(y^*) + s^2((\lambda+2\mu)\varphi_{y_2}^2)(y^*)\\ \ge&(\lambda+\mu)(y^*)\{\xi_1^2+\xi_2^2 - s^2\varphi_{y_1}^2(y^*) - s^2\varphi_{y_2}^2(y^*)\} - C_{14}\delta_1(|\xi'|^2+s^2). \end{aligned}$$

Therefore, for all sufficiently small $\delta_1$, there exists $C_{15} > 0$ such that for all $\zeta \in \mathcal{O}(\delta_1)$

$$\xi_0^2 - \mu(y^*)\xi_1^2 - \mu(y^*)\xi_2^2 - s^2\varphi_{y_0}^2(y^*) + s^2(\mu\varphi_{y_1}^2)(y^*) + s^2(\mu\varphi_{y_2}^2)(y^*) \ge C_{15}\delta_1(|\xi|^2+s^2). \tag{6.17}$$

By (6.17), we see that $\widetilde{J}_3 \ge 0$. Hence by (6.15) and (6.1), there exist constants $C_{16}, C_{17}, C_{18} > 0$ such that

$$\begin{aligned} &\sum_{k=1}^3 \Xi_{k,\mu}^{(1)} + C_{16}\Xi_{\lambda+2\mu}^{(1)} \ge C_{17}|s| \left\| \left(\frac{\partial \mathbf{w}'_\nu}{\partial y_3}, \mathbf{w}'_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\\ &-C_{18}(\delta,\delta_1)(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \end{aligned}$$

This inequality and (4.32) imply

$$\begin{aligned} &\sum_{k=1}^3 \Xi_{k,\mu}^{(1)} \ge C_{19}|s| \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\\ &-C_{18}(\delta,\delta_1)(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \end{aligned} \tag{6.18}$$

By (6.18), (5.1) and (4.30), we obtain (4.15).

**Case C.** Assume that $s^* \neq 0$. If $\delta_1 > 0$ is small enough, then there exists a constant $C_{20} > 0$ such that

$$|\xi_0\varphi_{y_0}(y^*) - (\lambda+2\mu)(y^*)\xi_1\varphi_{y_1}(y^*) - (\lambda+2\mu)(y^*)\xi_2\varphi_{y_2}(y^*)|^2 \le \delta_1^2 C_{20}(\xi_1^2+\xi_2^2+s^2). \tag{6.19}$$

By (4.30), there exists $C_{21} > 0$ such that

$$\begin{aligned} &\Xi_{\lambda+2\mu}^{(1)} + C_{21}\|\sqrt{|s|}w_{4,\nu}\|^2_{H^{1,s}(\mathcal{G})}\\ \le& C_{21}(\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) + \epsilon|s| \left\| \left(\frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu\right)\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}. \end{aligned} \tag{6.20}$$

By (6.16) and (6.19), we have

$$|\widetilde{J}_2 + \widetilde{J}_3| \le C_{22}\delta_1^2|s| \left\| \left( \frac{\partial w_{4,\nu}}{\partial y_3}, w_{4,\nu} \right) \right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}. \tag{6.21}$$

By (6.21) we see from (6.19) that there exists a constant $C_{23} > 0$ such that

$$\Xi^{(1)}_{\lambda+2\mu} \ge -\epsilon|s| \left\| \left( \frac{\partial w_{4,\nu}}{\partial y_3}, w_{4,\nu} \right) \right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}$$
$$+C_{23}\int_{\partial\mathcal{G}} \left( |s|((\lambda+2\mu)^2\varphi_{y_3})(y^*) \left| \frac{\partial w_{4,\nu}}{\partial y_3} \right|^2 + |s|^3((\lambda+2\mu)^2\varphi_{y_3}^3)(y^*)|w_{4,\nu}|^2 \right) d\Sigma. \tag{6.22}$$

Since $s^* \neq 0$, without loss of generality, taking $\delta_1$ sufficiently small, we can assume that

$$|\xi'| \le C_{24}|s|, \quad \forall \zeta \in \mathcal{O}(\delta_1). \tag{6.23}$$

By (6.22) and (6.23) for some constants $C_{25} > 0$ and $C_{26} > 0$, we have

$$\Xi^{(1)}_{\lambda+2\mu} \ge C_{25}|s| \left\| \left( \frac{\partial w_{4,\nu}}{\partial y_3}, w_{4,\nu} \right) \right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})} - C_{26}\|\mathbf{f}e^{|s|\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}. \tag{6.24}$$

By (4.32) we have

$$\mu\mathrm{Rot}(y,\mathbf{D})\mathbf{w}'_\nu = \mathbf{F}^* \quad \text{on } \partial\mathcal{G}, \tag{6.25}$$

where we set $\mathbf{F}^* = \frac{1}{i}\mathbf{F}_1 + (\lambda+2\mu)\mathrm{Nab}(y,\mathbf{D})w_{4,\nu}$ and $\|\mathbf{F}_1\|_{\mathbf{L}^2(\mathcal{G})} \le C_{27}\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{L}^2(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})})$. Next in the operator Rot, we put instead of $\mathbf{D}_3 w_{k,\nu}$ the function $\alpha^+(y,s,D')w_{k,\nu} + V^-_{k,\mu}$. We can represent

$$\mu\mathbb{G}(y,s,D')\mathbf{w}'_\nu = \mathbf{F}^* + \widetilde{R}(V^-_\mu). \tag{6.26}$$

By (6.4)

$$\sqrt{|s|}\|\widetilde{R}(V^-_\mu)\|_{\mathbf{L}^2(\partial\mathcal{G})} \le C_{28}(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}'\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{6.27}$$

The principal symbol of the operator $\mathbb{G}$ at the point $\gamma$ is given by the matrix

$$\mathbb{G}(\gamma) = \begin{pmatrix} 0 & \alpha^+_\mu(\gamma) & -\xi_2^* - i|s^*|\varphi_{y_2}(y^*) \\ -\alpha^+_\mu(\gamma) & 0 & \xi_1^* + i|s^*|\varphi_{y_1}(y^*) \\ -\xi_2^* - i|s^*|\varphi_{y_2}(y^*) & \xi_1^* + i|s^*|\varphi_{y_1}(y^*) & 0 \end{pmatrix}. \tag{6.28}$$

Thanks to the Dirichlet boundary condition, we note that

$$\|w_{3,\nu}\|_{H^{1,s}(\partial\mathcal{G})} \le \epsilon(\delta) \left\| \left( \frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu \right) \right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.$$

This inequality and (6.4) imply

$$|s|^{\frac12} \left\| \left( \frac{\partial w_{3,\nu}}{\partial y_3}, w_{3,\nu} \right) \right\|_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})} \le C_{29}(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}'\|_{\mathbf{H}^{1,s}(\mathcal{G})})$$
$$+\epsilon(\delta) \left\| \left( \frac{\partial \mathbf{w}_\nu}{\partial y_3}, \mathbf{w}_\nu \right) \right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}. \tag{6.29}$$

By the first two equations of (6.26) and $r_\mu(\gamma)\ne 0$, we obtain

$$|s|^{\frac12}\|w_{k,\nu}\|_{H^{1,s}(\partial\mathcal G)}\le C_{29}(\|\mathbf f e^{|s|\varphi}\|_{\mathbf H^{1,s}(\mathcal G)}+\|\mathbf w'\|_{\mathbf H^{1,s}(\mathcal G)}+|s|^{\frac12}\|\mathbf F^*\|_{\mathbf L^2(\partial\mathcal G)})$$
$$+\epsilon(\delta)\left\|\left(\frac{\partial \mathbf w_\nu}{\partial y_3},\mathbf w_\nu\right)\right\|_{\mathbf L^2(\partial\mathcal G)\times\mathbf H^{1,s}(\partial\mathcal G)}\qquad k=1,2. \tag{6.30}$$

By (6.4), (6.29) and (6.30)

$$|s|^{\frac12}\left\|\left(\frac{\partial \mathbf w_\nu}{\partial y_3}\mathbf w_\nu\right)\right\|_{\mathbf H^{1,s}(\partial\mathcal G)\times\mathbf L^2(\partial\mathcal G)}\le C_{29}(\|\mathbf f e^{|s|\varphi}\|_{\mathbf H^{1,s}(\mathcal G)}+\|\mathbf w'\|_{\mathbf H^{1,s}(\mathcal G)}+|s|^{\frac12}\|\mathbf F^*\|_{\mathbf L^2(\partial\mathcal G)})$$
$$+\epsilon(\delta)\left\|\left(\frac{\partial \mathbf w_\nu}{\partial y_3},\mathbf w_\nu\right)\right\|_{\mathbf L^2(\partial\mathcal G)\times\mathbf H^{1,s}(\partial\mathcal G)}. \tag{6.31}$$

By (6.21) and the definition of the function $\mathbf F^*$, we obtain

$$|s|\left\|\left(\frac{\partial \mathbf w_\nu}{\partial y_3}\mathbf w_\nu\right)\right\|^2_{\mathbf H^{1,s}(\partial\mathcal G)\times\mathbf L^2(\partial\mathcal G)}\le C_{30}(\|\mathbf f e^{|s|\varphi}\|^2_{\mathbf H^{1,s}(\mathcal G)}+\|\mathbf w'\|^2_{\mathbf H^{1,s}(\mathcal G)}+\Xi^{(1)}_{\lambda+2\mu}). \tag{6.32}$$

In view of (6.32) and (4.30), we obtain (4.15). ■

**§7. The case $r_\mu(\gamma)\ne 0$ and $r_{\lambda+2\mu}(\gamma)\ne 0$ or $r_\mu(\gamma)=r_{\lambda+2\mu}(\gamma)=0$.**
In order to treat this case, we use the Calderon method. First we introduce the new variables $U=(U_1,\dots,U_6)$, where

$$(U_1,U_2,U_3)=\Lambda(s,D')\mathbf u e^{|s|\varphi},\quad (U_4,U_5,U_6)=(D_3+i|s|\varphi_{y_3})\mathbf u e^{|s|\varphi},$$

and $\Lambda$ is the pseudodifferential operator with the symbol $(s^2+|\xi'|^2+1)^{\frac12}$. In the new notations, problem (4.12) can be written in the form

$$D_{y_3}U=M(y,s,D')U+\mathbf F\quad\text{in } \mathbb R^3\times[0,1],\quad (U_1,U_2,U_3)(y)|_{y_3=0}=0,\ U|_{y_3=\frac{1}{N^2}}=0, \tag{7.1}$$

where $\mathbf F=(0,\mathbf f e^{|s|\varphi})$. Here $M(y,s,D')$ is the matrix pseudodifferential operator with principal symbol $M_1(y,\zeta)$ given by

$$M_1(y,\zeta)=\begin{pmatrix}0 & \Lambda_1E_3\\ A^{-1}M_{21}\Lambda_1^{-1} & A^{-1}M_{22}\end{pmatrix}-i|s|\varphi_{y_3}E_6$$

(see [Y]). Here we set $\vec\theta=(\xi_1+i|s|\varphi_{y_1},\xi_2+i|s|\varphi_{y_2},0)$,
$G(y_1,y_2)=(-\partial\ell(y_1,y_2)/\partial y_1,-\partial\ell(y_1,y_2)/\partial y_2,1)$, $\Lambda_1=|\zeta|$, $M_{21}(y,\xi'+i|s|\nabla_{y'}\varphi(y))=((\xi_0+i|s|\varphi_{y_0}(y))^2-\mu((\xi_1+i|s|\varphi_{y_1}(y))^2+(\xi_2+i|s|\varphi_{y_2}(y))^2))E_3-(\lambda+\mu)(y)\vec\theta^T\vec\theta$,
$M_{22}(y,\xi')=-(\lambda+\mu)(y)(\vec\theta^TG+G^T\vec\theta)-2\mu\vec\theta G^TE_3$, $A=(\lambda+\mu)(y)G^TG+\mu(y)|G|^2E_3$.
Here $\vec\theta^T$ denotes the transpose of the row vector $\vec\theta$.

**Case A.** Suppose that $r_\mu(\gamma)=r_{\lambda+2\mu}(\gamma)=0$. Then $\operatorname{Im}\Gamma^\pm_\mu(\gamma)<0$ and $\operatorname{Im}\Gamma^\pm_{\lambda+2\mu}(\gamma)<0$, Therefore all the eigenvalues of the matrix $M_1(y,\zeta)$ have negative imaginary parts. There exists $C_1>0$ such that

$$\operatorname{Im}\Gamma^\pm_\mu(y,\zeta)<-C_1|\zeta|,\quad \operatorname{Im}\Gamma^\pm_{\lambda+2\mu}(y,\zeta)<-C_1|\zeta|,\quad \forall(y,\zeta)\in B_\delta\times\mathcal O(\delta_1).$$

Using the arguments in §4 of Chapter 7 in [Ku], we obtain

$$\|\chi_\nu U\|_{\mathbf{H}^{2,s}(\mathcal{G})} \le C_2(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{7.2}$$

This estimate implies (4.15).

**Case B.** Suppose that $r_\mu(\gamma) \ne r_{\lambda+2\mu}(\gamma), r_\mu(\gamma) \ne 0, r_{\lambda+2\mu}(\gamma) \ne 0$. In this case, the matrix $M_1$ has four smooth eigenvalues given by (4.21)-(4.23) and the corresponding six smooth eigenvectors $s_1^\pm$, $s_2^\pm$, $s_3^\pm$ given by the following formulae (e.g., [IY8], [Y]):

$$s_1^\pm = \left((\vec{\theta} + \alpha^\pm_{\lambda+2\mu} G)\Lambda_1^{-1},\ \alpha^\pm_{\lambda+2\mu}(\vec{\theta} + \alpha^\pm_{\lambda+2\mu} G)\Lambda_1^{-2}\right), s_2^\pm = (w_2^\pm, \alpha_\mu^\pm \Lambda_1^{-1} w_2^\pm),$$
$$s_3^\pm = (w_3^\pm, \alpha_\mu^\pm \Lambda_1^{-1} w_3^\pm),$$

where we set

$$w_2^\pm = \Lambda_1^{-1}(-\xi_2 - i|s|\varphi_{y_2} + \alpha_\mu^\pm \ell_{y_2}, \xi_1 + i|s|\varphi_{y_1} - \alpha_\mu^\pm \ell_{y_1}, 0), \tag{7.3}$$

$$w_3^\pm = \left(\alpha_\mu^\pm(\xi_1 + i|s|\varphi_{y_1} - \alpha_\mu^\pm \ell_{y_1}), \alpha_\mu^\pm(\xi_2 + i|s|\varphi_{y_2} - \alpha_\mu^\pm \ell_{y_2}), -\sum_{k=1}^{2}(\xi_k + i|s|\varphi_{y_k} - \alpha_\mu^\pm \ell_{y_k})^2\right)\Lambda_1^{-2}.$$

Now we describe the construction of the pseudodifferential operator $S$. We take the symbol $S$ in the form $S = (s_1^+, s_2^+, s_3^+, s_1^-, s_2^-, s_3^-)$. Denote

$$S(y,\zeta) = \begin{pmatrix} S_{11}(y,\zeta) & S_{12}(y,\zeta) \\ S_{21}(y,\zeta) & S_{22}(y,\zeta) \end{pmatrix},\ |\zeta|^2 = 1. \tag{7.4}$$

Let $S^{-1}(y,\zeta)$ be the inverse matrix to $S$. We extend the matrices $S$ and $S^{-1}$ within the $C^3$-class in $\zeta$ such that for $|\zeta| \ge 1$, the elements of these matrices are the homogeneous functions of order zero. Following [T] and using the change of variables $W = S^{-1}(y, s, D')U$ which is constructed above, we can reduce system (7.1) to the form

$$D_{y_3} W = \widetilde{M}(y, s, D')W + T(y, s, D')W + \widetilde{\mathbf{F}}, \tag{7.5}$$

where the matrix $\widetilde{M}$ is diagonal and $T \in L^\infty(0, \frac{1}{N^2}; \mathcal{L}(\mathbf{H}^{1,s}(\mathbb{R}^3), \mathbf{H}^{1,s}(\mathbb{R}^3)))$. Now using a standard argument (see [Ku], p.241), we can estimate the last three components of $W$ as follows:

$$\sqrt{s}\|(W_4, W_5, W_6)\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_3(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}), \tag{7.6}$$

where the constant $C_3$ is independent of $N$. Since the Lopatinskii determinant $\det S_{11}(\gamma)$ is not equal to zero, by (7.6) we have

$$\sqrt{|s|}\|(W_1, W_2, W_3)\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_4(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}). \tag{7.7}$$

By (7.6), (7.7) and (4.30) we have (4.15).

**Case C.** Suppose that $r_\mu(\gamma) = r_{\lambda+2\mu}(\gamma) \neq 0$. Obviously we may assume

$$\mathrm{Im}\,\frac{\Gamma^+_\mu}{|s|}(\gamma) \geq 0. \tag{7.8}$$

Otherwise (4.15) has been already obtained in Case A. The matrix $M_1(\gamma)$ has only two eigenvalues given by (4.21)-(4.23). Moreover it is known that the Jordan form of the matrix $M_1(\gamma)$ has two Jordan blocks of the form

$$M^\pm = \begin{pmatrix} \Gamma^\pm_\mu(\gamma) & 1 & 0 \\ 0 & \Gamma^\pm_\mu(\gamma) & 0 \\ 0 & 0 & \Gamma^\pm_\mu(\gamma) \end{pmatrix}.$$

Similarly to Case B, following [T] and using the change of variables $W = S^{-1}(y, s, D')U$ where $S^{-1}$ is constructed through $S$, we can reduce the system to (7.5) where the matrix $\widetilde{M}(y, \zeta)$ is represented by

$$\widetilde{M}(y,\zeta) = \begin{pmatrix} \widetilde{M}_+(y,\zeta) & 0 \\ 0 & \widetilde{M}_-(y,\zeta) \end{pmatrix}$$

with

$$\widetilde{M}_\pm(y,\zeta) = \begin{pmatrix} \Gamma^\pm_{\lambda+2\mu}(y,\zeta) & 0 & m^\pm_{13}(y,\zeta) \\ 0 & \Gamma^\pm_\mu(y,\zeta) & m^\pm_{23}(y,\zeta) \\ 0 & 0 & \Gamma^\pm_\mu(y,\zeta) \end{pmatrix},$$

and the operator $T$ is in $L^\infty\left(0, \frac{1}{N^2}; \mathcal{L}(\mathbf{H}^{1,s}(\mathbb{R}^3), \mathbf{H}^{1,s}(\mathbb{R}^3))\right)$, $m^\pm_{13}(y, s, D'), m^\pm_{23}(y, s, D')$ are first order operators and

$$\|\widetilde{\mathbf{F}}\|_{L^2(\mathbb{R}^1;\mathbf{H}^{1,s}(\mathbb{R}^3))} \leq C_5(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|U\|_{L^2(\mathbb{R}^1;\mathbf{H}^{1,s}(\mathbb{R}^3))}).$$

Now we describe the construction of the pseudodifferential operator $S$. We take the symbol $S$ in the form $S = (s^+_1, s^+_2, s^+_3, s^-_1, s^-_2, s^-_3)$. Here

$$s^\pm_1 = \left((\vec\theta + \alpha^\pm_{\lambda+2\mu}G)\Lambda_1^{-1},\, \alpha^\pm_{\lambda+2\mu}(\vec\theta + \alpha^\pm_{\lambda+2\mu}G)\Lambda_1^{-2}\right), \quad s^\pm_2 = (w^\pm_2, \alpha_\mu\Lambda_1^{-1}w^\pm_2)$$

are the eigenvectors of the matrix $M_1(y,\zeta)$ on the sphere $\zeta \in S^3$ which corresponds to the eigenvalue $\Gamma^\pm_{\lambda+2\mu}$ (with $w^\pm_2$ given by (7.3)) and the vector $s^\pm_3$ is given by the formula

$$s^\pm_3 = E_\pm s^\pm, \quad E_\pm = \frac{1}{2\pi i}\int_{C^\pm}(z - M_1(y,\zeta))^{-1}dz,$$

where $C^\pm$ are small circles, oriented counterclockwise, centered at $\Gamma^\pm_\mu(\gamma)$, and $s^\pm$ solves the equation $M_1(\gamma)s^\pm - \Gamma^\pm_\mu(\gamma)s^\pm = s^\pm_1(\gamma)$. For the explicit formula for the vector $s^\pm$ see [IY7]. By (7.8) the circles $C^\pm$ may be taken such that the disks bounded by these circles do not intersect, provided that $\delta_1, \delta$ are taken sufficiently small. Note that the vectors $s^\pm_j \in C^2(B_\delta \times \mathcal{O}_{\delta_1})$ are homogeneous functions of the order zero in $\zeta$. Now using a standard argument (see [Ku], p.241), we can estimate the last three components of $W$ as follows:

$$\|(W_4, W_5, W_6)\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} \le C_6(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}), \tag{7.9}$$

where the constant $C_6$ is independent of $N$. Now we need to estimate the first three components of the vector function $W$ on $\partial\mathcal{G}$. Thanks to the homogeneous boundary conditions for $U_4$, $U_5$, and $U_6$, we have

$$\begin{aligned} &S_{11}(y', 0, s, D')(W_1, W_2, W_3) \\ =& - S_{12}(y', 0, s, D')(W_4, W_5, W_6) + T_{-1}(y', 0, s, D')U, \end{aligned} \tag{7.10}$$

where $T_{-1} \in \mathcal{L}(\mathbf{H}^{1,s}(\mathbb{R}^3), \mathbf{H}^{2,s}(\mathbb{R}^3))$ and we set

$$S(y,\zeta) = \begin{pmatrix} S_{11}(y,\zeta) & S_{12}(y,\zeta) \\ S_{21}(y,\zeta) & S_{22}(y,\zeta) \end{pmatrix}.$$

The principal symbol of the pseudodifferential operator $S_{11}$ is a $3\times 3$ matrix such that the $j$-th column equals the last three coordinates of the vector $s_j^+$. Therefore $\det S_{11}(\gamma) \neq 0$. From (7.9), (7.10) and Gårding's inequality, we obtain

$$\left\| \left( \frac{\partial \mathbf{w}_\nu}{\partial y_2}, \mathbf{w}_\nu \right) \right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_7(\|\mathbf{f}e^{|s|\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\ \mathbf{u}e^{|s|\varphi}\|_{\mathbf{H}^2(\mathcal{G})}), \tag{7.11}$$

where the constant $C_7$ is independent of $N$. By (7.11) and (4.30), we obtain (4.15). ■

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