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ESTIMATION OF POINT SOURCES AND THE APPLICATIONS TO INVERSE PROBLEMS

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ABSTRACT. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We introduce a distance $\|A-B\|$ between two n-tuples of point sources $A=(a_1,...,a_n), B=(b_1,....,b_n) \in \Omega^n$ and we establish upper and lower estimates between $\|A-B\|$ and a norm of $\sum_{j=1}^n \delta_{a_j} - \sum_{j=1}^n \delta_{b_j}$. Next we will apply the estimates to the following two inverse problems and prove best possible conditional stability estimates:

(i) inverse problem of determining point heat sources by final overdetermining data.

(ii) inverse problem of determining point wave sources by lateral overdetermining data.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary $\partial\Omega$ is smooth. For a point $a=(a^1,...,a^N) \in \Omega$, by δ_a we denote the Dirac delta function:

$$\delta_a(\varphi) =< \delta_a, \varphi >= \varphi(a) \quad \text{for } \varphi \in C(\Omega).$$

In relation with inverse problems, the Dirac delta function describes a point source.

For example, let us consider

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \Delta u(x,t) + \sigma(t)\delta_A(x), \quad x \in \Omega,\ t>0$$

where $\sigma \not\equiv 0$ is a given function, and $A=(a_1,...,a_n) \in \Omega^n$, $\delta_A = \sum_{j=1}^n \delta_{a_j}$.

This is a very simplified model for an earthquake (e.g., Aki and Richards [2]), and in an inverse problem, we are required to estimate locations $a_1,...,a_n$ of point sources. By a general framework, we can estimate the locations in $H^{-s}(\Omega)$ with some $s>0$, which is the dual of $H_0^s(\Omega)$: the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$. However, it is not directly clear how much $\|\delta_A - \delta_B\|_{H^{-s}(\Omega)}$ characterizes geometric distances between $A=(a_1,...,a_n), B=(b_1,...,b_n) \in \Omega^n$. Therefore we need an independent research and we can refer to Bruckner and Yamamoto [4], El Badia and Ha-Duong [7], Komornik and Yamamoto [12], where the authors discuss determination of point sources in a one-dimensional wave equation. In particular, in [12] we introduced some quasi-distance $\|\cdot\|_*$ between two sequences $A=(a_1,...,a_n)$, $B=(b_1,...,b_n) \in (0,1)^n$, and established an upper and lower estimate between $\|\delta_A - \delta_B\|_{W^{-s,q}(0,1)}$ and $\|A-B\|_*$.

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The purpose of this paper is to extend the estimate given in [12] to the multidimensional domain Ω, and then to apply it to two inverse problems.

The paper is composed of four sections. In Section 2, we will introduce an optimal distance $\|\cdot\|$ between $A, B \in \Omega^n$, and show our main results on upper and lower estimates between Sobolev norms of $\delta_A - \delta_B$ and $\|A - B\|$. In Section 3, we apply the estimate to an inverse heat source problem with final overdetermination and in Section 4, we discuss an inverse wave source problem by lateral overdetermination.

2. Estimation of point sources

First we investigate the relation between the distance of two points $a, b \in \Omega$ and the distance of the corresponding Dirac masses δ_a, δ_b in suitable Sobolev spaces. Here we recall that $\delta_a(\varphi) = \varphi(a)$ for $\varphi \in C(\Omega)$.

Henceforth, for the usual Sobolev space $W^{s,p}(\Omega)$, by $\widetilde{W}^{s,p}(\Omega)$ we denote its factor space with respect to constants, and by $(\widetilde{W}^{s,p}(\Omega))'$ its dual space. We can identify $\widetilde{W}^{s,p}(\Omega)$ with

$$\Big\{u \in W^{s,p}(\Omega) : \quad \int_\Omega u(x)dx = 0\Big\}.$$

In order to obtain an upper and lower estimate of $|a - b|$ without extra constraints on $a, b \in \Omega$, we take the norm of the dual space of $\widetilde{W}^{s,p}(\Omega)$.

Proposition 2.1. *Let $s > 1$ and $p > 1$ be two real numbers.*

(a) If $\frac{N-1}{p} < s - 1 < \frac{N}{p}$, then there exist two constants $C_1, C_2 > 0$ such that

$$C_1|a - b|^{s-\frac{N}{p}} \le \|\delta_a - \delta_b\|_{(\widetilde{W}^{s,p}(\Omega))'} \le C_2|a - b|^{s-\frac{N}{p}} \tag{2.1}$$

for all $a, b \in \Omega$.

In case $\frac{N-1}{p} = s - 1$ the first inequality still holds; if, moreover, $1 < p \le 2$, then the second inequality holds, too.

(b) If $s - 1 > \frac{N}{p}$, then there exist two constants $C_1, C_2 > 0$ such that

$$C_1|a - b| \le \|\delta_a - \delta_b\|_{(\widetilde{W}^{s,p}(\Omega))'} \le C_2|a - b| \tag{2.2}$$

for all $a, b \in \Omega$.

Remark. The first inequality of (2.1) and its proof are valid under the weaker assumption $s > \frac{N}{p}$, which is in fact necessary in order to have $\delta_a - \delta_b \in (\widetilde{W}^{s,p}(\Omega))'$.

Proof of the second inequality of (2.1). It follows from our assumptions on s that the equality

$$\frac{1}{r} = \frac{N}{p} - \frac{s-1}{1} \tag{2.3}$$

defines a number r such that $p < r < \infty$. Given a and b, introduce a unit vector e satisfying $a - b = |a - b|e$ and set $f(t) := \varphi(b + te)$ for any $\varphi \in W^{s,p}(\Omega)$. Using the Hölder inequality, the Sobolev embedding and trace theorems (see e.g., Adams [1], Theorem 7.58 (i), p. 218), we have the following inequalities:

$$\begin{aligned}|(\delta_a - \delta_b)(\varphi)| &= |\varphi(a) - \varphi(b)| \\ &= \left| \int_0^{|a-b|} f'(t)\, dt \right| \\ &\le |a-b|^{1-\frac{1}{r}} \cdot \|f'\|_{L^r(0,|a-b|)} \\ &\le |a-b|^{1-\frac{1}{r}} \cdot \|\nabla\varphi\|_{L^r([a,b])} \\ &\le C|a-b|^{1-\frac{1}{r}} \cdot \|\nabla\varphi\|_{W^{s-1,p}(\Omega)} \\ &= C|a-b|^{1-\frac{1}{r}} \cdot \|\varphi\|_{\widetilde{W}^{s,p}(\Omega)}.\end{aligned}$$

Finally let $1 < p \le 2$ and let $\frac{N-1}{p} \le s-1$. Then we can take $p \le r < \infty$ satisfying (2.3). Therefore Theorem 7.58 (iii) in Adams [1], p. 218 yields

$$\|\nabla\varphi\|_{L^r([a,b])} \le C\|\nabla\varphi\|_{W^{s-1,p}(\Omega)}.$$

Taking (2.3) into account, the second estimate of (2.1) follows. □

Proof of the first inequality of (2.1) . Fix a function $\psi \in C^\infty(\mathbb{R}^N)$ satisfying the following conditions:

$$\left\{ \begin{array}{l} 0 \le \psi \le 1, \\ \psi(0) = 0, \\ \psi(x) = 1 \quad \text{if } |x| \ge 1. \end{array} \right. \tag{2.4}$$

Given two points $a, b \in \mathbb{R}^N$ with $a \ne b$, consider the function $\varphi \in \widetilde{W}^{s,p}(\mathbb{R}^N)$ given by the formula

$$\varphi(x) := \psi\Big(\frac{x-a}{r}\Big), \quad r := |a-b|.$$

We clearly have

$$(\delta_a - \delta_b)(\varphi) = \varphi(a) - \varphi(b) = -1.$$

Furthermore, for $s = 1, 2, \ldots$ we also have for every $0 \le r \le M$ the following estimates:

$$\begin{aligned}\|\varphi\|^p_{\widetilde{W}^{s,p}(\mathbb{R}^N)} &= \sum_{1\le|\alpha|\le s} \Big\| r^{-|\alpha|} D^\alpha \psi\Big(\frac{x-a}{r}\Big)\Big\|^p_{L^p(\mathbb{R}^N)} \\ &= \sum_{1\le|\alpha|\le s} \int_{\mathbb{R}^N} r^{-|\alpha|p} \Big| D^\alpha\psi\Big(\frac{x-a}{r}\Big)\Big|^p dx \\ &= \sum_{1\le|\alpha|\le s} \int_{\mathbb{R}^N} r^{N-|\alpha|p} \Big| D^\alpha\psi\Big(\frac{x-a}{r}\Big)\Big|^p d\left(\frac{x-a}{r}\right) \\ &= \sum_{1\le|\alpha|\le s} r^{N-|\alpha|p} \Big\| D^\alpha\psi\Big\|^p_{L^p(\mathbb{R}^N)} \\ &\le C_M^p r^{N-sp}.\end{aligned}$$

Here $C_M > 0$ is a constant depending on M. These estimates remain valid for all real numbers $s \ge 1$ by the interpolation argument.

Now choosing a number M which is larger than the diameter of Ω and using the obvious inequality

$$\|\varphi\|_{\widetilde{W}^{s,p}(\Omega)} \le \|\varphi\|_{\widetilde{W}^{s,p}(\mathbb{R}^N)},$$

it follows that

$$\begin{aligned}\|\delta_a-\delta_b\|_{(\widetilde{W}^{s,p}(\Omega))'} &\ge \frac{|(\delta_a-\delta_b)(\varphi)|}{\|\varphi\|_{\widetilde{W}^{s,p}(\Omega)}}\\ &\ge \frac{|(\delta_a-\delta_b)(\varphi)|}{\|\varphi\|_{\widetilde{W}^{s,p}(\mathbb{R}^N)}}\\ &\ge C_M^{-1}r^{s-\frac{N}{p}}\\ &= C_M^{-1}|a-b|^{s-\frac{N}{p}}.\end{aligned}$$

□

Proof of the second inequality of (2.2). We modify the proof of the corresponding inequality in (2.1) as follows. We may assume, by diminishing s if needed, that

$$\frac{N}{p} < s-1 < \frac{N+1}{p},$$

because $(\widetilde{W}^{s_1,p}(\Omega))' \subset (\widetilde{W}^{s_2,p}(\Omega))'$ for $s_1 < s_2$. Then we may choose $\varepsilon > 0$ such that

$$s-1 = \frac{N}{p} + \frac{1}{p} - \frac{1}{p+\varepsilon}.$$

Then, applying Theorem 7.58 in [1], p. 218, we obtain that

$$W^{s-1,p}(\Omega) \subset W^{1/p,p+\varepsilon}([a,b]).$$

Furthermore, since

$$\frac{p+\varepsilon}{p} > 1,$$

applying Theorem 7.57 in [1], p. 217 (the condition $p < n$ of that theorem is unnecessary, see a remark on p. 218), we also have

$$W^{1/p,p+\varepsilon}([a,b]) \subset L^\infty([a,b]).$$

It follows that

$$\begin{aligned}|(\delta_a-\delta_b)(\varphi)| &= |\varphi(a)-\varphi(b)|\\ &= \Big|\int_0^{|a-b|} f'(t)\ dt\Big|\\ &\le |a-b|\cdot\|f'\|_{L^\infty(0,|a-b|)}\\ &\le |a-b|\cdot\|\nabla\varphi\|_{L^\infty([a,b])}\\ &\le C|a-b|\cdot\|\nabla\varphi\|_{W^{s-1,p}(\Omega)}\\ &= C|a-b|\cdot\|\varphi\|_{\widetilde{W}^{s,p}(\Omega)}.\end{aligned}$$

□

Proof of the first inequality of (2.2). Choosing the test functions $\varphi(x^1,\ldots,x^N) := x^j$, $j=1,\ldots,N$, we have obviously

$$\|\varphi\|^p_{\widetilde{W}^{s,p}(\Omega)} = |\Omega|$$

(the volume of Ω) and

$$|(\delta_a - \delta_b)(\varphi)| = |a^j - b^j|,$$

so that

$$\|\delta_a - \delta_b\|_{(\widetilde{W}^{s,p}(\Omega))'} \geq \frac{|(\delta_a - \delta_b)(\varphi)|}{\|\varphi\|_{\widetilde{W}^{s,p}(\Omega)}} \geq |\Omega|^{-1}|a^j - b^j|.$$

It follows that

$$N\|\delta_a - \delta_b\|^2_{(\widetilde{W}^{s,p}(\Omega))'} \geq |\Omega|^{-2}\sum_{j=1}^{N}|a^j - b^j|^2 = |\Omega|^{-2}|a-b|^2,$$

so that

$$\|\delta_a - \delta_b\|_{(\widetilde{W}^{s,p}(\Omega))'} \geq \frac{1}{\sqrt{N}|\Omega|}|a-b|.$$

□

Next we will extend Proposition 2.1 to $A = (a_1, ..., a_n)$, $B = (b_1, ..., b_n) \in \Omega^n$ where a_j and b_j may be not distinct.

For the optimal distance $\|A - B\|$ with $A = (a_1, ..., a_n), B = (b_1, ..., b_n) \in \Omega^n$, we define

$$\|A - B\| = \min_{\pi} \max_{1\leq i\leq n} |a_i - b_{\pi(i)}| \tag{2.5}$$

where the minimum is taken over all the permutations of $1, ..., n$. Then $\|\cdot\|$ is a distance: $\|A - C\| \leq \|A - B\| + \|B - C\|$ for $A, B, C \in \Omega^n$ and $\|A - B\| = 0$ if and only if $A = B$ (after renumbering if necessary). We state the main result which assures that the distance $\|\cdot\|$ defined by (2.5) gives an optimal estimate for $\|\delta_A - \delta_B\|_{(\widetilde{W}^{s,p}(\Omega))'}$.

Theorem 2.2. *Let n be a positive integer and $s > 1$, $p > 1$ two real numbers.*

(a) If $\frac{N-1}{p} < s - 1 < \frac{N}{p}$, then there exist two constants $C_1, C_2 > 0$ such that

$$C_1\|A - B\|^{s-\frac{N}{p}} \leq \|\delta_A - \delta_B\|_{(\widetilde{W}^{s,p}(\Omega))'} \leq C_2\|A - B\|^{s-\frac{N}{p}} \tag{2.6}$$

for all $A, B \in \Omega^n$.

In case $\frac{N-1}{p} = s - 1$ the first inequality still holds; if, moreover, $1 < p \leq 2$, then the second inequality holds, too.

(b) If $s - 1 > \frac{N}{p}$, then there exist two constants $C_1, C_2 > 0$ such that

$$C_1\|A - B\|^{s-\frac{N}{p}} \leq \|\delta_A - \delta_B\|_{(\widetilde{W}^{s,p}(\Omega))'} \leq C_2\|A - B\| \tag{2.7}$$

for all $A, B \in \Omega^n$.

Estimates (2.6) are clearly optimal because the expressions on the left- and right-hand sides have the same size. Concerning the optimality of this theorem for $s - 1 > \frac{N}{p}$, we shall also prove the following proposition:

Proposition 2.3. *(a) Unlike in* (2.2) *for the case of $n = 1$, the first inequality of* (2.7) *cannot be improved as*

$$C_1\|A - B\| \leq \|\delta_A - \delta_B\|_{(\widetilde{W}^{s,p}(\Omega))'}$$

in general.

(b) There exist a positive constant C_2 and special configurations of points a_i and b_i with arbitrarily small norm $\|A-B\|$ such that

$$\|\delta_A - \delta_B\|_{(\widetilde{W}^{s,p}(\Omega))'} \le C_2 \|A-B\|^{s-\frac{N}{p}}$$

with a constant C_2 independent of $\|A-B\|$. That is, the second inequality in (2.7) is not the best possible.

For the proof of the first inequalities of (2.6) and (2.7), we will use a theorem by Hall and Rado [8], [17], [18] from the graph theory.

Theorem 2.4. *(Hall–Rado). Consider an even graph having $2n$ points $a_1, ..., a_n$ and $b_1, ..., b_n$ which may not necessarily distinct each other. We connect some pairs (a_i, b_j) such that the following two conditions hold:*

- *for every $k \in \{1, ..., n\}$ and every subsequence $A' = (a_{i_1}, ..., a_{i_k})$ of $A = (a_1, ..., a_n)$, at least k elements b_j of the sequence B are connected to one of them;*
- *for every $k \in \{1, ..., n\}$ and every subsequence $B' = (b_{i_1}, ..., b_{i_k})$ of $B = (b_1, ..., b_n)$, at least k elements a_j of the sequence A are connected to one of them.*

Then there exists a permutation π of the integers $1, \dots, n$ such that a_i is connected to $b_{\pi(i)}$ for every i.

Remark. Let us recall an interpretation of this result as a solution of a "marriage problem". Assume that each of a set of n boys is acquainted with some of a set of n girls. Under what conditions is it possible for each boy to marry one of his acquaintances? It is clearly necessary that every set of k boys be, collectively, acquainted with at least k girls. The theorem, first proved in [8], [17], [18], asserts that this condition is also sufficient. See [9] for a simple proof and for generalizations, and [3] for other applications.

Proof of Theorem 2.2. The second inequalities follow by combining Proposition 2.1 and the triangle inequality. Indeed, for every permutation π we have

$$\|\delta_A - \delta_B\|_{(\widetilde{W}^{s,p}(\Omega))'} \le \sum_{i=1}^{n} \|\delta_{a_i} - \delta_{b_{\pi(i)}}\|_{(\widetilde{W}^{s,p}(\Omega))'}$$

$$\le \sum_{i=1}^{n} C_2 |a_i - b_{\pi(i)}|^{s-\frac{N}{p}} \le nC_2 \max_{1\le i\le n} |a_i - b_{\pi(i)}|^{s-\frac{N}{p}}.$$

Taking the minimum of the right-hand side of this inequality, the second inequality of (2.6) follows. The second inequality of (2.7) follows similarly.

Turning to the proof of the first inequalities, we may assume that

$$\|A-B\| > 0.$$

Fix $0 < r < \|A-B\|$ arbitrarily. We will consider an even graph with points a_i, b_i, $i = 1, ..., n$, where a_i is connected to b_j if $|a_i - b_j| < r$. Then there exist an integer $1 \le k \le n$ and a subsequence $(a_{i_1}, ..., a_{i_k})$ of A such that the union of the k balls $B_r(a_{i_1})$,...., $B_r(a_{i_k})$ contains at most $(k-1)$ points b_j. Indeed, otherwise there would exist, by the Hall–Rado theorem, a permutation π such that $|a_i - b_{\pi(i)}| < r$ for $i = 1, \dots, n$, contradicting the choice of r.

Now introduce the product function

$$\varphi(x) := \psi\Big(\frac{x-a_{i_1}}{r}\Big)\dots\psi\Big(\frac{x-a_{i_k}}{r}\Big), \quad x\in\Omega,$$

where ψ is defined by (2.4).

Then φ belongs to $W^{s,p}(\Omega)$ and

$$\|\varphi\|_{\widetilde{W}^{s,p}(\Omega)} \le Cr^{\frac{N}{p}-s} \tag{2.8}$$

by a simple computation, using the Leibniz formula and the interpolation inequality (e.g., Lions and Magenes [14]). Furthermore, by the definition, we have

$$\begin{aligned}
&\varphi(a_{i_1}) = \dots = \varphi(a_{i_k}) = 0,\\
&\varphi(a_j) \le 1 \quad \text{for all other values of} \quad j,\\
&\varphi(b_j) = 1 \quad \text{for all but at most } k-1 \text{ indices } j,\\
&\varphi(b_j) \ge 0 \quad \text{for all other values of} \quad j.
\end{aligned}$$

It follows that

$$\delta_A(\varphi) \le n-k \quad \text{and} \quad \delta_B(\varphi) \ge n-k+1.$$

Therefore we have

$$\|\delta_A-\delta_B\|_{(\widetilde{W}^{s,p}(\Omega))'} \ge \frac{|\delta_B(\varphi)-\delta_A(\varphi)|}{\|\varphi\|_{\widetilde{W}^{s,p}(\Omega)}} \ge C^{-1}r^{s-\frac{N}{p}}.$$

Thus the proof of Theorem 2.2 is complete. □

Proof of Proposition 2.3. (a) The following counterexample shows that we may not be able to replace the exponent $s-\frac{N}{p}$ by 1 in the first inequality of (2.7) if the gravicentres of A and B are equal and

$$\max\{\max_{1\le i,j\le n}|a_i-a_j|,0\}, \quad \max\{\max_{1\le i,j\le n}|b_i-b_j|,0\}$$

are small. For example, let us take $N=1$, $n=2$, $\Omega=(-1,1)$ and $A_k=\big(-\frac{1}{k},\frac{1}{k}\big)$, $B=(0,0)\in\Omega^2$. In the case of (b), we have $s>1+\frac{1}{p}$, and $\varphi\in\widetilde{W}^{s,p}(-1,1)$ implies $\varphi\in C^{1+\theta}[-1,1]$ by the Sobolev embedding where $\theta>0$. Moreover, by the definition of $\widetilde{W}^{s,p}(-1,1)$, we have

$$\sup_{-1\le x,y\le 1}\left|\frac{d\varphi}{dx}(x)-\frac{d\varphi}{dx}(y)\right| \le C\|\varphi\|_{\widetilde{W}^{s,p}(-1,1)}|x-y|^\theta.$$

Let $\varphi\in\widetilde{W}^{s,p}(-1,1)$ be arbitrary. Then the mean value theorem yields

$$\begin{aligned}
&\delta_{A_n}(\varphi)-\delta_B(\varphi) = \varphi\left(\frac{1}{k}\right)+\varphi\left(-\frac{1}{k}\right)-2\varphi(0)\\
&=\left(\frac{d\varphi}{dx}(\xi_1)-\frac{d\varphi}{dx}(\xi_2)\right)\frac{1}{k}
\end{aligned}$$

with some $0<\xi_1<\frac{1}{k}$ and $-\frac{1}{k}<\xi_2<0$. By $\frac{d\varphi}{dx}\in C^\theta[-1,1]$, we have

$$\begin{aligned}
&|\delta_{A_k}(\varphi)-\delta_B(\varphi)| \le C\|\varphi\|_{\widetilde{W}^{s,p}(-1,1)}|\xi_1-\xi_2|^\theta\frac{1}{k}\\
&\le 2^\theta C\|\varphi\|_{\widetilde{W}^{s,p}(-1,1)}\frac{1}{k^{1+\theta}},
\end{aligned}$$

that is,

$$\frac{\|\delta_{A_k}-\delta_B\|_{(\widetilde{W}^{s,p}(-1,1))'}}{\|A_k-B\|}\le\frac{2^\theta C}{k^\theta}.$$

Therefore

$$\lim_{k\to\infty}\frac{\|\delta_{A_k}-\delta_B\|_{(\widetilde{W}^{s,p}(-1,1))'}}{\|A_k-B\|}=0,$$

so that we cannot replace the exponent $s-\frac{N}{p}$ by 1 in the first inequality of (2.7).

(b) Write $s-\frac{N}{p}=m+\theta$ with an integer m and with $0<\theta<1$, set $n=2^m$ and choose the points a_i and b_i as follows. Set

$$s_{2j}:=\begin{pmatrix} m+1\\ 0\end{pmatrix}+\begin{pmatrix} m+1\\ 2\end{pmatrix}+\cdots+\begin{pmatrix} m+1\\ 2j\end{pmatrix}$$

$$s_{2j+1}:=\begin{pmatrix} m+1\\ 1\end{pmatrix}+\begin{pmatrix} m+1\\ 3\end{pmatrix}+\cdots+\begin{pmatrix} m+1\\ 2j+1\end{pmatrix}$$

for $0\le j\le[(m+1)/2]$. Fix $a_1\in\Omega$ arbitrarily and let h be a small nonzero vector. Then let

$$a_i=a_1+2jh\quad\text{if } s_{2j}\le i<s_{2j+2};$$

$$b_i=a_1+(2j+1)h\quad\text{if } s_{2j+1}-s_1\le i<s_{2j+3}-s_1$$

for $0\le j\le[(m+1)/2]$. Then the points a_i and b_i lie on a straight line at equal distances. In order to simplify the notation, we may thus assume that they are real numbers and that h is a small positive number. Then using Newton's divided differences and Sobolev's imbedding theorem (see [1], p. 98, equation (9)), we have (here δ denotes the difference operator)

$$\begin{aligned}|(\delta_A-\delta_B)(\varphi)|&=|h^{m+1}(\Delta^{m+1}\varphi(a_1))|\\&=|h^m(\Delta^m\varphi(a_1+h)-\Delta^m\varphi(a_1))|\\&=|h^m(\varphi^{(m)}(c_1)-\varphi^{(m)}(c_2))|\\&\le ch^m|c_1-c_2|^\theta\\&\le ch^{s-\frac{N}{p}},\end{aligned}$$

because c_1 and c_2 belong to the interval $(a_1,a_1+(m+1)h)$ by the generalized Rolle theorem. See, e.g., [6], pp. 50 and 65 (Corollary 3.4.2). Thus the proof of (b) is complete. □

Theorem 2.2 gives an unconditional upper and lower estimate between $\|A-B\|$ and $\delta_A-\delta_B$, but the choice of the space $\widetilde{W}^{s,p}(\Omega)$ makes the direct application to our inverse source problems difficult, and a relevant norm may be not $\|\delta_A-\delta_B\|_{(\widetilde{W}^{s,p}(\Omega))'}$. By $W_0^{s,p}(\Omega)$, we denote the closure of $C_0^\infty(\Omega)$ by the norm in $W^{s,p}(\Omega)$ and we set $W^{-s,q}(\Omega)=(W_0^{s,p}(\Omega))'$: the dual where $s>0$, $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then we will modify Theorem 2.2:

Theorem 2.5. *Let $n\in\mathbb{N}$ and $s>1$, $p>1$ two real numbers. We assume that $A=(a_1,...,a_n)$, $B=(b_1,....,b_n)\in\Omega^n$ satisfy*

(2.9) $$\text{dist }(a,\partial\Omega)>\varepsilon\quad\textit{for any } a\in A\cup B$$

for some $\varepsilon>0$.

(a) If $\frac{N-1}{p} < s-1 < \frac{N}{p}$, then there exist two constants $C_3(\varepsilon) > 0$ and $C_4 > 0$ such that

$$C_3\|A-B\|^{s-\frac{N}{p}} \le \|\delta_A - \delta_B\|_{W^{-s,q}(\Omega)} \le C_4\|A-B\|^{s-\frac{N}{p}}. \tag{2.10}$$

In case $\frac{N-1}{p} = s-1$ the first inequality still holds; if, moreover, $1 < p \le 2$, then the second inequality holds, too.

(b) If $s-1 > \frac{N}{p}$, then there exist two constants $C_3 = C_3(\varepsilon) > 0$ and $C_4 > 0$ such that

$$C_3\|A-B\|^{s-\frac{N}{p}} \le \|\delta_A - \delta_B\|_{W^{-s,q}(\Omega)} \le C_4\|A-B\|. \tag{2.11}$$

Proof. The second inequalities in (2.10) and (2.11) are proved in the same way as the ones in (2.6) and (2.7) of Theorem 2.2. Thus it is sufficient to prove the first inequalities. The proof is similar to Theorem 2.2 with a different choice of a test function.

We set

$$R = \max\{\sup_{x,x'\in\Omega} |x-x'|, \varepsilon\}. \tag{2.12}$$

Then $\frac{\varepsilon}{R} \le 1$. Fix a function $\psi_1 \in C^\infty(\mathbb{R}^N)$ satifying

$$0 \le \psi_1 \le 1, \quad \psi_1(0) = 0, \quad \psi_1(x) = 1 \quad \text{if } |x| \ge \frac{\varepsilon}{R}. \tag{2.13}$$

We may assume that $\|A-B\| > 0$. Fix $0 < r < \|A-B\|$ arbitrarily. Then, in view of the Hall–Rado theorem, there exists an integer $1 \le k \le n$ and a subsequence $(a_{i_1}, \dots, a_{i_k})$ of A such that

$$\bigcup_{m=1}^{k} \{x; |x-a_{i_m}| < r\} \tag{2.14}$$

contains at most $(k-1)$ points b_j. Now introduce a test function

$$\varphi_1(x) = 1 - \psi_1\Big(\frac{x-a_{i_1}}{r}\Big)\dots\psi_1\Big(\frac{x-a_{i_k}}{r}\Big), \quad x \in \Omega.$$

Then $\varphi_1 \in W^{s,p}(\Omega)$ and $\|\varphi_1\|_{W^{s,p}(\Omega)} \le Cr^{\frac{N}{p}-s}$ by the same way as (2.8). Since

$$\left|\frac{x-a_{i_m}}{r}\right| > \frac{\varepsilon}{R}$$

for any $x \in \partial\Omega$ by (2.9), we see from (2.13) that $\varphi_1 \in W_0^{s,p}(\Omega)$. Furthermore it follows from (2.12), (2.13) and (2.14) that

$$\begin{aligned}
&\varphi_1(a_{i_1}) = \dots = \varphi(a_{i_k}) = 1,\\
&\varphi_1(a_j) \ge 0 \quad \text{for all other values of} \quad j,\\
&\varphi_1(b_j) = 0 \quad \text{for all but at most } k-1 \text{ indices } j,\\
&0 \le \varphi_1(b_j) \le 1 \quad \text{for all other values of} \quad j.
\end{aligned}$$

Therefore

$$\delta_A(\varphi_1) \ge k \quad \text{and} \quad \delta_B(\varphi_1) \le k-1,$$

so that

$$\|\delta_A - \delta_B\|_{(W_0^{s,p}(\Omega))'} \ge \frac{|\delta_B(\varphi_1) - \delta_A(\varphi_1)|}{\|\varphi_1\|_{W^{s,p}(\Omega)}} \ge C^{-1}r^{s-\frac{N}{p}}.$$

Thus the proof of Theorem 2.5 is complete. □

3. Application to an inverse heat source problem with final overdetermination

In this section, we will consider

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + h(t)\delta_A(x), \quad x \in \Omega,\ t > 0, \tag{3.1}$$

$$u(x,0) = 0, \qquad x \in \Omega, \tag{3.2}$$

and

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0. \tag{3.3}$$

Here we set $A = (a_1, ..., a_n) \in \Omega^n$, $\delta_A = \sum_{j=1}^n \delta_{a_j}$, and $h = h(t), \not\equiv 0, \in C^1[0,\infty)$ is a given function. Let $T > 0$ be given.

Then we will consider an *inverse heat source problem* of determining A by the final overdetermining data $u(x,T)$, $x \in \Omega$.

This inverse problem is determination of hot spots in a heat process, and there are several papers in the case where δ_A in (3.1) is replaced by L^2-functions. For example, Choulli and Yamamoto [5], Isakov [10], Prilepko, Orlovsky and Vasin [16], Rundell [19], and the references therein.

For the statement of our result, we introduce operators and function spaces. Let us define an operator $-L$ in $L^2(\Omega)$ by

$$(Lu)(x) = -\Delta u(x),\ x \in \Omega, \quad \mathcal{D}(L) = H^2(\Omega) \cap H_0^1(\Omega). \tag{3.4}$$

Then L^{-1} is bounded from $L^2(\Omega)$ to $L^2(\Omega)$ and $-L$ generates an analytic semigroup in $L^2(\Omega)$ (e.g., Pazy [15]). Moreover we can choose $\lambda_j \in \mathbb{R}$ and $\varphi_j \in L^2(\Omega)$, $j \in \mathbb{N}$ such that $L\varphi_j = \lambda_j \varphi_j$, $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty$ and $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$. For $\alpha \in \mathbb{R}$, we can define the fractional power L^α (e.g., [15]), and by the interpolation theory (e.g., Lions and Magenes [14]), we see:

$$\begin{cases} \mathcal{D}(L^{\frac{s}{2}}) \subset H^s(\Omega), \quad s \ge 0, \quad H_0^s(\Omega) = \mathcal{D}(L^{\frac{s}{2}}) \quad \text{if } \frac{1}{2} < s \le 2, \\ \text{there exists a constant } C > 0 \text{ such that} \\ C\|y\|_{H^s(\Omega)} \le \left\|L^{\frac{s}{2}} y\right\|_{L^2(\Omega)} \le C^{-1}\|y\|_{H^s(\Omega)}, \quad y \in \mathcal{D}(L^{\frac{s}{2}}),\ s \ge 0. \end{cases} \tag{3.5}$$

For $s \ge 0$, we define a Hilbert space $X(s)$ by $\mathcal{D}(L^{\frac{s}{2}})$ with the norm $\|y\|_{X(s)} = \left\|L^{\frac{s}{2}} y\right\|_{L^2(\Omega)}$. For $s > 0$, we introduce the triple:

$$X(s) \subset X_0 \equiv L^2(\Omega) \subset X(-s). \tag{3.6}$$

By extending $L^{-\frac{s}{2}}$ to $X(-s)$, henceforth we can write $\|y\|_{X(-s)} = \|L^{-\frac{s}{2}} y\|_{L^2(\Omega)}$ for any $y \in X(-s)$. Moreover there exists a constant $C > 0$ such that

$$\|y\|_{H^{-s}(\Omega)} \le C\|L^{-\frac{s}{2}} y\|_{L^2(\Omega)} \quad \text{for } y \in X(-s). \tag{3.7}$$

In particular, by the Sobolev embedding and the duality, we see that $\delta_A \in X(-s)$ if $s > \frac{N}{2}$. Moreover, as is proved later, there exists a unique solution $u = u_A \in C^1([0,\infty); X(-s))$ with $s > \frac{N}{2}$ to (3.1)–(3.3).

We are ready to state the conditional stability in our inverse heat source problem.

Theorem 3.1. *We assume that $h \in C^1[0,\infty)$ satisfies*

$$h(T) \ne 0, \quad \int_0^T e^{\lambda_j \eta} h(\eta) d\eta \ne 0 \quad \textit{for any } j \in \mathbb{N}. \tag{3.8}$$

Let $s > 1$. Then

(a) If $\frac{N-1}{2} \le s-1$*, then there exist two positive constants* $C_1 = C_1(\varepsilon)$ *and* C_2 *such that*

$$C_1\|A-B\|^{s-\frac{N}{2}} \le \|(u_A-u_B)(\cdot,T)\|_{X\left(1-\frac{s}{2}\right)} \le C_2\|A-B\|^{s-\frac{N}{2}} \tag{3.9}$$

for any $A, B \in \Omega^n$ *satisfying* (2.9) *with* $\varepsilon > 0$.

(b) If $s-1 > \frac{N}{2}$*, then there exist two positive constants* $C_1 = C_1(\varepsilon)$ *and* C_2 *such that*

$$C_1\|A-B\|^{s-\frac{N}{2}} \le \|(u_A-u_B)(\cdot,T)\|_{X\left(1-\frac{s}{2}\right)} \le C_2\|A-B\| \tag{3.10}$$

for any $A, B \in \Omega^n$ *satisfying* (2.9) *with* $\varepsilon > 0$.

The first inequalities in (3.9) and (3.10) are concerned with the stability for the inverse problem. In the case of $1-\frac{s}{2} \ge 0$, we can replace $\|\cdot\|_{X\left(1-\frac{s}{2}\right)}$ by $\|\cdot\|_{H^{2-s}(\Omega)}$:

$$\|A-B\|^{s-\frac{N}{2}} \le \|(u_A-u_B)(\cdot,T)\|_{H^{2-s}(\Omega)}.$$

Remark. Since $-L$ generates an analytic semigroup in $L^p(\Omega)$ with $p>1$ (e.g., [15]), we can discuss the inverse problem in general $W^{s,p}(\Omega)$, but the argument is more complicated, and here we restrict ourselves to the L^2-framework.

The choice $s = \frac{N}{2}+\frac{1}{2}$ in case (a) gives the optimal stability rate:

$$C_1\|A-B\| \le \|(u_A-u_B)(\cdot,T)\|^2_{X\left(-\frac{N}{4}+\frac{6}{4}\right)} \le C_2\|A-B\|$$

for all $A, B \in \Omega^n$ satisfying (2.9).

Proof. First step. We will prove a variant of Theorem 2.5.

Proposition 3.2. *Let* $s>1$ *and let us assume that* $A, B \in \Omega^n$ *satisfy* (2.9)*. Then (a) If* $\frac{N-1}{2} \le s-1$*, then there exist two positive constants* $C_3 = C_3(\varepsilon)$ *and* C_4 *such that*

$$C_3\|A-B\|^{s-\frac{N}{2}} \le \|\delta_A-\delta_B\|_{X(-s)} \le C_4\|A-B\|^{s-\frac{N}{2}}. \tag{3.11}$$

(b) If $s-1 > \frac{N}{2}$*, then there exist two positive constants* $C_3 = C_3(\varepsilon)$ *and* C_4 *such that*

$$C_3\|A-B\|^{s-\frac{N}{2}} \le \|\delta_A-\delta_B\|_{X(-s)} \le C_4\|A-B\|. \tag{3.12}$$

Proof. The first inequalities in (3.11) and (3.12) follow directly from (3.7) and the first inequalities in (2.10) and (2.11). As for the second ineqiualities, choosing $\varphi \in H_0^s(\Omega)$ in the proofs of the second inequalities in (2.1) and (2.2), we follows the arguments there. Then, similarly to Theorem 2.2, the triangle inequality completes the proofs. □

Second step. Let $s > \frac{N}{2}$. Then we will prove that there exist two positive constants C_5 and C_6 such that

$$C_5\left\|L^{-\frac{s}{2}}(\delta_A-\delta_B)\right\|_{L^2(\Omega)} \le \left\|L^{1-\frac{s}{2}}(u_A-u_B)(\cdot,T)\right\|_{L^2(\Omega)}$$

$$\le C_6\left\|L^{-\frac{s}{2}}(\delta_A-\delta_B)\right\|_{L^2(\Omega)} \tag{3.13}$$

for all $A, B \in \Omega^n$.

Proof of (3.13). We rewrite (3.1)–(3.3) in terms of semigroup (e.g., [15]):

$$\begin{cases} \frac{du_A}{dt}(t) = -Lu_A(t) + h(t)\delta_A & \text{in } X(-s),\ t > 0, \\ u_A(0) = 0. \end{cases}$$

Setting $v = u_A - u_B$, we have

$$\begin{cases} \frac{dv}{dt}(t) = -Lv(t) + h(t)(\delta_A - \delta_B) & \text{in } X(-s),\ t > 0, \\ v = 0. \end{cases}$$

Then there exists a unique solution $v \in C^1([0,\infty); X(-s))$ such that

$$Lv \in C([0,\infty); X(-s)).$$

(See, e.g., Theorem 3.5, p. 114, in [15].) We further set $w(t) = v'(t) = \frac{dv}{dt}(t)$, $t > 0$ and $d = (u_A - u_B)(T)$. Then

$$\begin{cases} \frac{dw}{dt}(t) = -Lw(t) + h'(t)(\delta_A - \delta_B) & \text{in } X(-s),\ t > 0, \\ w(0) = h(0)(\delta_A - \delta_B), \\ w(T) = -Ld + h(T)(\delta_A - \delta_B). \end{cases}$$

Therefore, since $h(T) \neq 0$ and

$$w(t) = e^{-tL}w(0) + \int_0^t e^{-(t-\eta)L}h'(\eta)(\delta_A - \delta_B)d\eta, \quad t > 0,$$

we have

$$\delta_A - \delta_B = \frac{Ld}{h(T)} + \frac{1}{h(T)}e^{-TL}h(0)(\delta_A - \delta_B) + \frac{1}{h(T)}\int_0^T e^{-(T-\eta)L}h'(\eta)(\delta_A - \delta_B)d\eta.$$

For simplicity, we set $g = L^{-\frac{s}{2}}(\delta_A - \delta_B) \in L^2(\Omega)$. Then

$$g = \frac{L^{1-\frac{s}{2}}d}{h(T)} + \frac{h(0)}{h(T)}e^{-TL}g + \frac{1}{h(T)}\int_0^T e^{-(T-\eta)L}h'(\eta)gd\eta \equiv \frac{L^{1-\frac{s}{2}}d}{h(T)} + Kg. \tag{3.14}$$

First we will verify that the operator $K : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact.

In fact, we see that for $\alpha > 0$, there exists $C_7 > 0$ such that $\|L^\alpha e^{-tL}\| \leq C_7 t^{-\alpha}$ for $0 < t \leq T$ (e.g., [15]). Take $\alpha \in (0,1)$. Hence for any $y \in L^2(\Omega)$, we have

$$\begin{aligned} \|L^\alpha Ky\|_{L^2(\Omega)} &\leq \left|\frac{h(0)}{h(T)}\right| \|L^\alpha e^{-TL}y\|_{L^2(\Omega)} \\ &\quad + \left|\frac{1}{h(T)}\right| \int_0^T \|L^\alpha e^{-(T-\eta)L}\| \cdot \|h'\|_{L^\infty(0,T)}\|y\|_{L^2(\Omega)}d\eta \\ &\leq C_8\left(T^{-\alpha} + \int_0^T (T-\eta)^{-\alpha}d\eta\right)\|y\|_{L^2(\Omega)} \\ &\leq C_9\|y\|_{L^2(\Omega)} \end{aligned}$$

by $0 < \alpha < 1$.

By (3.5), we see that $\|Ky\|_{H^{2\alpha}(\Omega)} \leq C_{10}\|y\|_{L^2(\Omega)}$. Since the embedding

$$H^{2\alpha}(\Omega) \longrightarrow L^2(\Omega)$$

is compact by the Rellich theorem, we see that $K : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact.

Next we can verify that $(I-K)y=0$ implies $y=0$.

In fact, let $(I-K)y=0$, that is,

$$L^{-1}h(T)y = L^{-1}h(0)e^{-TL}y + \int_0^T L^{-1}e^{-(T-\eta)L}h'(\eta)yd\eta.$$

Integration by parts yields

$$\int_0^T e^{-(T-\eta)L}h(\eta)yd\eta = 0. \tag{3.15}$$

Since the system $\{\varphi_j\}_{j\in\mathbb{N}}$ of the eigenfunctions forms an orthonormal basis in $L^2(\Omega)$, we have

$$e^{-tL}z = \sum_{j=1}^{\infty} e^{-\lambda_j t}(z,\varphi_j)\varphi_j$$

for any $z\in L^2(\Omega)$, where the series converges in $L^2(\Omega)$ and $(\cdot,\cdot)$ denotes the L^2-scalar product. Therefore (3.15) implies

$$\sum_{j=1}^{\infty} e^{-\lambda_j T}\left(\int_0^T e^{\lambda_j\eta}h(\eta)d\eta\right)(y,\varphi_j)\varphi_j = 0,$$

and

$$\left(\int_0^T e^{\lambda_j\eta}h(\eta)d\eta\right)(y,\varphi_j) = 0, \quad j\in\mathbb{N}.$$

Hence assumption (3.8) yields $(y,\varphi_j)=0$, $j\in\mathbb{N}$, that is, $y=0$. Thus $I-K$ is injective.

Therefore from (3.14), the Fredholm alternative implies

$$C_5'\|g\|_{L^2(\Omega)} \le \|L^{1-\frac{s}{2}}d\|_{L^2(\Omega)} \le C_6'\|g\|_{L^2(\Omega)}.$$

Thus the proof of (3.13) is complete. □

Hence by combining (3.13) with (3.11) and (3.12), the proof of Theorem 3.1 is complete. □

4. Application to an inverse problem of determining impulsive point wave sources

Consider the system

$$\begin{cases} \frac{\partial^2 w}{\partial t^2}(x,t) = \Delta w(x,t) + \sigma(t)\delta_A(x), & x\in\Omega,\ t>0, \\ w(x,0) = \frac{\partial w}{\partial t}(x,0) = 0, & x\in\Omega, \\ w(x,t) = 0, & x\in\partial\Omega,\ t>0, \end{cases} \tag{4.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain whose boundary $\partial\Omega$ is of C^2-class. Throughout this section, we assume

$$\sigma\in C^1[0,\infty), \quad \sigma(0)\ne 0.$$

In (4.1), the vibration is assumed to be caused by multiple impulses at n-points $A=(a_1,...,a_n)$, and we are required to determine A by boundary measurements. We will study the stability in determining A.

For a suitable subboundary $\Gamma \subset \partial\Omega$, we introduce Hilbert spaces $H = V_0 \equiv L^2(0,T;L^2(\Gamma))$ and $V_s = H_0^s(0,T;L^2(\Gamma))$ with scalar product $(\cdot,\cdot)_{V_s}$. For $s = m \in \mathbb{N}$, we have

$$V_m = \left\{ g \in H^m(0,T;L^2(\Gamma));\ \frac{\partial^j g}{\partial t^j}(\cdot,0) = \frac{\partial^j g}{\partial t^j}(\cdot,T) = 0, \quad j = 0,1,...,m-1 \right\}$$

and

$$(g,h)_{V_m} = \left(\frac{d^m g}{dt^m}, \frac{d^m h}{dt^m} \right)_H .$$

Identifying H with its dual, we have the dense and continuous imbeddings:

$$V_s \subset H = H' \subset V_s'.$$

We write $V_s = V_{-s}$.

Here we recall that the operator L and the spaces $X(s)$, $X(-s)$ are defined by (3.4) and (3.6). Then, for $s > \frac{N}{2}$, we have $X(s) \subset C(\overline{\Omega})$ by Sobolev imbedding. Furthermore (see, e.g., Komornik [11]) we see that there exists a unique solution $w_A \in C([0,\infty);X(-s+2)) \cap C^2([0,\infty);X(-s+1))$ to (4.1).

Let $x_0 \in \mathbb{R}^N$ be arbitrarily fixed and let

$$\Gamma = \{x \in \partial\Omega;\ (x - x_0)\cdot \nu(x) \ge 0\},$$

where $\nu = \nu(x)$ denotes the unit outward normal vector to $\partial\Omega$ at x. In terms of Theorem 2.5, we will show the theorem concerning estimation of implusive wave sources.

Theorem 4.1. *Let $s > 1$ and*

$$T > \sup_{x\in\Omega} |x - x_0|.$$

(a) If $\frac{N-1}{2} \le s - 1$, then there exist two constants $C_1(\varepsilon), C_2 > 0$ such that

$$C_1\|A - B\|^{s-\frac{N}{2}} \le \|\partial_\nu w_A - \partial_\nu w_B\|_{V_{-s+1}} \le C_2\|A - B\|^{s-\frac{N}{2}}$$

for all $A, B \in \Omega^n$ satisfying (2.9) *with $\varepsilon > 0$.*

(b) If $s - 1 > \frac{N}{2}$, then there exist two constants $C_1(\varepsilon), C_2 > 0$ such that

$$C_1\|A - B\|^{s-\frac{N}{2}} \le \|\partial_\nu w_A - \partial_\nu w_B\|_{V_{-s+1}} \le C_2\|A - B\|$$

for all $A, B \in \Omega^n$ satisfying (2.9) *with $\varepsilon > 0$.*

Similarly to Theorem 3.1, the choice $s = \frac{N}{2} + \frac{1}{2}$ in case (a) gives the optimal stability rate:

$$C_1\|A - B\| \le \|\partial_\nu w_A - \partial_\nu w_B\|^2_{V_{(1-N)/2}} \le C_2\|A - B\|$$

for all $A, B \in \Omega^n$ satisfying (2.9).

Proof of Theorem 4.1.

First Step.

We will prove:

Proposition 4.2. *Let $s \ge 1$. For $f \in X(-s)$, let*

$$v_f \in C([0,\infty);X(-s+1)) \cap C^1([0,\infty);X(-s))$$

be the solution to

$$\begin{cases} \frac{\partial^2 v}{\partial t^2}(x,t) = \Delta v(x,t), & x \in \Omega,\, t > 0, \\ v(x,0) = 0, \quad \frac{\partial v}{\partial t}(x,0) = f(x), & x \in \Omega, \\ v(x,t) = 0, & x \in \partial\Omega,\, t > 0. \end{cases} \tag{4.2}$$

We assume that $T > \sup_{x\in\Omega} |x - x_0|$. Then there exist positive constants C_3, C_4 such that

$$C_3 \|f\|_{X(-s)} \le \|\partial_\nu v_f\|_{V_{-s}} \le C_4 \|f\|_{X(-s)} \tag{4.3}$$

for any $f \in X(-s)$.

Remark. In the case of $s = 0$, estimate (4.3) corresponds to a usual observability inequality and a direct inequality (e.g., Lions [13], or Komornik [11]). In our case, we assume that $v(\cdot, 0) = 0$, which is related with the notion "contrôlabilité exacte élargie" where we can reduce the critical time $\sup_{x\in\Omega} |x - x_0|$ (Section 9 of Chapter 1 in Lions [13]). As for the case $s = 1$, we refer to Komornik and Yamamoto [12] (the one dimensional case) and, Yamamoto and Zhang [20] for example.

Here and henceforth we write $p \asymp q$ if there exist positive constants C and C', which are independent of p, q, such that $Cp \le q \le C'p$.

Proof of Proposition 4.2. By the interpolation argument (e.g., Lions and Magenes [14]), it suffices to prove (4.3) for $s = m \in \mathbb{N}$.

We recall that φ_j, λ_j are the eigenvectors and the eigenvalues of the operator L. By $(\cdot, \cdot)$ we denote the scalar product in $L^2(\Omega)$. Then, by the "contrôlabilité exacte élargie" (Section 9 of Chapter 1 in [13]), we have

$$\|f\|_{L^2(\Omega)} \asymp \|\partial_\nu v_f\|_H$$

for all $f \in L^2(\Omega)$, provided that $T > \sup_{x\in\Omega} |x - x_0|$. Hence, noting that

$$v_f(x,t) = \sum_{k=1}^{\infty} (f, \varphi_k) \frac{\sin \sqrt{\lambda_k} t}{\sqrt{\lambda_k}} \varphi_k(x)$$

in $C^1([0,T]; L^2(\Omega))$ and

$$(\partial_\nu v_f)(x,t) = \sum_{k=1}^{\infty} (f, \varphi_k) \frac{\sin \sqrt{\lambda_k t}}{\sqrt{\lambda_k}} \partial_\nu \varphi_k(x)$$

in H, by a usual density argument we see that

$$\left(\sum_{k=1}^{\infty} \alpha_k^2 \right)^{\frac{1}{2}} \asymp \left\| \sum_{k=1}^{\infty} \alpha_k \frac{\sin \sqrt{\lambda_k t}}{\sqrt{\lambda_k}} \partial_\nu \varphi_k \right\|_H \tag{4.4}$$

for $\alpha_k \in \mathbb{R}$. Similarly for

$$\begin{cases} \frac{\partial^2 V}{\partial t^2}(x,t) = \Delta V(x,t), & x \in \Omega,\, t > 0, \\ V(x,0) = f(x), \quad \frac{\partial V}{\partial t}(x,0) = 0, & x \in \Omega, \\ V(x,t) = 0, & x \in \partial\Omega,\, t > 0, \end{cases}$$

in view of (3.5), we can see that

$$\|f\|_{H_0^1(\Omega)} \asymp \|\partial_\nu V\|_H$$

for $f \in H_0^1(\Omega) = \mathcal{D}(L^{\frac{1}{2}})$. This means

$$\left(\sum_{k=1}^{\infty} \lambda_k \alpha_k^2\right)^{\frac{1}{2}} \asymp \left\| \sum_{k=1}^{\infty} \alpha_k \cos \sqrt{\lambda_k t} \partial_\nu \varphi_k \right\|_H . \tag{4.5}$$

By a usual density argument, it suffices to prove the proposition in the case where $v_f(x,t) = \sum_{k=1}^{\infty} (f, \varphi_k) \frac{\sin \sqrt{\lambda_k t}}{\sqrt{\lambda_k}} \varphi_k(x)$ is a finite sum. For any $\psi \in H_0^m(0,T; L^2(\Gamma))$, by integration by parts we see that

$$\begin{aligned}
(\partial_\nu v_f, \psi)_{V_{-m}, V_m} &= \int_0^T \int_\Gamma (\partial_\nu v_f) \psi dS dt \\
&= \sum_k \int_\Gamma \int_0^T (f, \varphi_k) \frac{\sin \sqrt{\lambda_k t}}{\sqrt{\lambda_k}} \partial_\nu \varphi_k(x) \psi(x,y) dt dS \\
&= \sum_k \int_\Gamma \int_0^T (f, \varphi_k) \frac{\cos \sqrt{\lambda_k t}}{\lambda_k} \partial_\nu \varphi_k(x) \frac{\partial \psi}{\partial t}(x,t) dt dS \\
&= -\sum_k \int_\Gamma \int_0^T (f, \varphi_k) \frac{\sin \sqrt{\lambda_k t}}{\lambda_k^{\frac{3}{2}}} \partial_\nu \varphi_k(x) \frac{\partial^2 \psi}{\partial t^2}(x,t) dt dS \\
&= \cdots\cdots\cdots \\
&= \sum_k \int_\Gamma \int_0^T (f, \varphi_k) \frac{\Phi(\sqrt{\lambda_k} t)}{\lambda_k^{\frac{1+m}{2}}} \partial_\nu \varphi_k(x) \frac{\partial^m \psi}{\partial t^m}(x,t) dt dS.
\end{aligned}$$

Here we set $\Phi(t) = \cos t$ or $-\cos t$ or $\sin t$ or $-\sin t$. Therefore

$$\begin{aligned}
\|\partial_\nu v_f\|_{V_{-m}} &= \sup_{\|\psi\|_{V_m}=1} |(\partial_\nu v_f, \psi)_{V_{-m}, V_m}| \\
&= \sup_{\|\partial_t^m \psi\|_H = 1} \left| \left(\sum_k (f, \varphi_k) \frac{\Phi(\sqrt{\lambda_k} t)}{\lambda_k^{\frac{1+m}{2}}} \partial_\nu \varphi_k, \partial_t^m \psi \right)_H \right| = \left\| \sum_k (f, \varphi_k) \frac{\Phi(\sqrt{\lambda_k} t)}{\lambda_k^{\frac{1+m}{2}}} \partial_\nu \varphi_k \right\|_H .
\end{aligned}$$

Hence, setting $\alpha_k = (f, \varphi_k) \lambda_k^{-\frac{m}{2}}$ or $\alpha_k = (f, \varphi_k) \lambda_k^{-\frac{1+m}{2}}$ in (4.4) and (4.5), we have

$$\|\partial_\nu v_f\|_{V_{-m}} \asymp \left(\sum_k (f, \varphi_k)^2 \lambda_k^{-m} \right)^{\frac{1}{2}} = \|L^{-\frac{m}{2}} f\|_{L^2(\Omega)}.$$

The last equality is seen because

$$L^{-\frac{m}{2}} f = \sum_k \frac{(f, \varphi_k)}{\lambda_k^{\frac{m}{2}}} \varphi_k$$

and $\{\varphi_k\}_{k \in \mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$. Thus the proof of Proposition 4.2 is complete. □

Second Step. We will prove

Proposition 4.3. *Let $s \geq 1$ and let $T > \sup_{x \in \Omega} |x - x_0|$. Then there exist positive constants C_5, C_6 such that*

$$C_5 \|f\|_{X(-s)} \leq \|\partial_\nu w_f\|_{V_{-s+1}} \leq C_6 \|f\|_{X(-s)} \tag{4.6}$$

for any $f \in X(-s)$.

Once Proposition 4.3 will be proved, Proposition 3.2 completes the proof of Theorem 4.1.

Proof of Proposition 4.3. By the interpolation argument, it is sufficient to prove (4.6) for $s = m \in \mathbb{N}$. First

$$\left\| \int_0^t \frac{(t-\eta)^{m-1}}{(m-1)!} h(\cdot,\eta)d\eta \right\|_H = \|h\|_{V_{-m}}. \tag{4.7}$$

Indeed, by the density argument, it suffices to prove (4.7) for $h \in H = L^2(0,T;L^2(\Gamma))$. By the definition of the dual norm and integration by parts, noting that $\frac{\partial^j \psi}{\partial t^j}(\cdot,0) = \frac{\partial^j \psi}{\partial t^j}(\cdot,T) = 0$, $j = 0,1,...,m-1$ if $\psi \in V_m$, we have

$$\begin{aligned}
\|h\|_{V_{-m}} &= \sup_{\|\psi\|_{V_m}=1} |(h,\psi)_H| \\
&= \sup_{\|\psi\|_{V_m}=1} \left| \int_\Gamma \int_0^T h(x,\eta)\psi(x,\eta)d\eta dS \right| \\
&= \sup_{\|\psi\|_{V_m}=1} \left| \int_\Gamma \int_0^T \left(\int_0^\eta h(x,\theta)d\theta \right) \frac{\partial \psi}{\partial \eta}(x,\eta)d\eta dS \right| \\
&= \cdots \\
&= \sup_{\|\psi\|_{V_m}=1} \left| \int_\Gamma \int_0^T \left(\int_0^\eta \frac{(\eta-\theta)^{m-1}}{(m-1)!} h(x,\theta)d\theta \right) \frac{\partial^m \psi}{\partial \eta^m}(x,\eta)d\eta dS \right| \\
&= \sup_{\|\Psi\|_H=1} \left| \left(\int_0^\eta \frac{(\eta-\theta)^{m-1}}{(m-1)!} h(x,\theta)d\theta, \Psi \right)_H \right| \\
&= \left\| \int_0^\eta \frac{(\eta-\theta)^{m-1}}{(m-1)!} h(x,\theta)d\theta \right\|_H,
\end{aligned}$$

which is (4.7).

Next we will verify that

$$\begin{aligned}
&\int_0^t \frac{(t-\eta)^{m-2}}{(m-2)!} (\partial_\nu w_f)(x,\eta)d\eta \\
&= \sigma(0) \int_0^t \frac{(t-\eta)^{m-1}}{(m-1)!} (\partial_\nu v_f)(x,\eta)d\eta + \int_0^t \sigma'(\theta) \left(\int_0^{t-\theta} \frac{(t-\theta-\eta)^{m-1}}{(m-1)!} (\partial_\nu v_f)(x,\eta)d\eta \right) d\theta
\end{aligned} \tag{4.8}$$

for all $f \in C_0^\infty(\Omega)$.

Proof of (4.8). By the Duhamel principle (e.g., [12]), we have

$$w_f(x,\eta) = \int_0^\eta \sigma(\eta-\theta)v_f(x,\theta)d\theta, \quad x \in \Omega,\ t > 0.$$

Hence, by integration by parts, we have

$$\begin{aligned}
&\int_0^t (\partial_\nu w_f)(x,\eta)\frac{(t-\eta)^{m-2}}{(m-2)!}d\eta \\
&= \int_0^t \frac{(t-\eta)^{m-2}}{(m-2)!}\left(\int_0^\eta \sigma(\eta-\theta)\partial_\nu v_f(x,\theta)d\theta\right)d\eta \\
&= \int_0^t \sigma(0)(\partial_\nu v_f)(x,\eta)\frac{(t-\eta)^{m-1}}{(m-1)!}d\eta + \int_0^t \frac{(t-\eta)^{m-1}}{(m-1)!}\left(\int_0^\eta \sigma'(\eta-\theta)\partial_\nu v_f(x,\theta)d\theta\right)d\eta \\
&= \int_0^t \sigma(0)(\partial_\nu v_f)(x,\eta)\frac{(t-\eta)^{m-1}}{(m-1)!}d\eta + \int_0^t \frac{(t-\eta)^{m-1}}{(m-1)!}\left(\int_0^\eta \sigma'(\theta)\partial_\nu v_f(x,\eta-\theta)d\theta\right)d\eta.
\end{aligned}$$

On the other hand, we change orders of integrals and the variables $\eta \longrightarrow \eta - \theta$ in the resulting integral, so that

$$\begin{aligned}
&\int_0^t \frac{(t-\eta)^{m-1}}{(m-1)!}\left(\int_0^\eta \sigma'(\theta)\partial_\nu v_f(x,\eta-\theta)d\theta\right)d\eta \\
&\qquad = \int_0^t \sigma'(\theta)\left(\int_\theta^t \frac{(t-\eta)^{m-1}}{(m-1)!}(\partial_\nu v_f)(x,\eta-\theta)d\eta\right)d\theta \\
&\qquad = \int_0^t \sigma'(\theta)\left(\int_0^{t-\theta} \frac{(t-\theta-\eta)^{m-1}}{(m-1)!}(\partial_\nu v_f)(x,\eta)d\eta\right)d\theta.
\end{aligned}$$

Thus the proof of (4.8) is complete.

Let $f \in C_0^\infty(\Omega)$. Setting

$$W(x,t) = \int_0^t \frac{(t-\eta)^{m-1}}{(m-1)!}(\partial_\nu v_f)(x,\eta)d\eta,$$

we can rewrite (4.8) as

$$\begin{aligned}
&\sigma(0)W(x,t) + \int_0^t \sigma'(\theta)W(x,t-\theta)d\theta \\
&\qquad = \sigma(0)W(x,t) + \int_0^t \sigma'(t-\theta)W(x,\theta)d\theta \\
&\qquad = \int_0^t \frac{(t-\eta)^{m-2}}{(m-2)!}(\partial_\nu w_f)(x,\eta)d\eta, \quad x \in \Gamma,\ 0 < t < T,
\end{aligned}$$

which is a Volterra equation of the second kind by $\sigma(0) \neq 0$. Consequently we have

$$\left\| \int_0^t \frac{(t-\eta)^{m-1}}{(m-1)!}(\partial_\nu v_f)(\cdot,\eta)d\eta \right\|_H \asymp \left\| \int_0^t \frac{(t-\eta)^{m-2}}{(m-2)!}(\partial_\nu w_f)(\cdot,\eta)d\eta \right\|_H.$$

By (4.7), we obtain

$$\|\partial_\nu v_f\|_{V_{-m}} \asymp \|\partial_\nu w_f\|_{V_{-m+1}}.$$

Hence Proposition 4.2 implies that $\|\partial_\nu w_f\|_{V_{-m+1}} \asymp \|f\|_{X(-m))}$ for $f \in C_0^\infty(\Omega)$. The density argument yields (4.6) for any $f \in X(-m)$. Thus the proof of Proposition 4.3 is complete.

Hence, in view of Propositions 3.2 and 4.3, we can complete the proof of Theorem 4.1.

References

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York–London, 1975.

[2] K. Aki and P. G. Richards, *Quantitative Seismology Theory and Methods*, Vol. I, Freeman, New York, 1980.

[3] E. Bertram and P. Horák, *Some applications of graph theory to other parts of mathematics*, The Math. Intelligencer 21 (1999), 3, 6–11.

[4] G. Bruckner and M. Yamamoto, *Determination of point wave sources by pointwise observations: stability and reconstruction*, Inverse Problems 16 (2000), 723-748.

[5] M. Choulli and M. Yamamoto, *An inverse parabolic problem with non-zero initial condition*, Inverse Problems 13 (1997), 19–27.

[6] P. J. Davies, *Interpolation and Approximation*, Dover, New York, 1975.

[7] A. El Badia and T. Ha-Duong, *Determination of point wave sources by boundary measurements*, Inverse Problems 17 (2001), 1127–1139.

[8] P. Hall, *On representatives of subsets*, J. London Math. Soc. 10 (1935), 26-30.

[9] P. R. Halmos and H. E. Vaughan, *The marriage problem*, Amer. Math. Monthly 72 (1950), 214-215.

[10] V. Isakov, *Inverse parabolic problems with the final overdetermination*, Comm. Pure Appl. Math. 44 (1991), 185–209.

[11] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*, Masson–John Wiley, Paris–Chicester, 1994.

[12] V. Komornik and M. Yamamoto, *On the determination of point sources*, Inverse Problems 18 (2002), 319–329.

[13] J.-L. Lions, *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués*, Vol. 1-2, Masson, Paris, 1988.

[14] J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications I-III*, Springer-Verlag, New York-Heidelberg, 1972–73.

[15] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[16] A.I. Prilepko, D.G. Orlovsky and I.A.Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*, Marcel Dekker, New York, 2000.

[17] R. Rado, *Bemerkungen zur Kombinatorik im Anschluss an Untersuchungen von Herrn D. König*, Berliner Sitzungsberichte 32 (1933), 61-68.

[18] R. Rado, *Factorisation of even graphs*, Quart. J. Math. Oxford Ser. (2) 20 (1949), 95-104.

[19] W. Rundell, *Determination of an unknown nonhomogeneous term in a linear partial differential equation from overspecified boundary data*, Appl. Anal. 10 (1980), 231–242.

[20] M. Yamamoto and X.Zhang, *Global uniqueness and stability for an inverse wave source problem for less regular data*, J. Math. Anal. Appl. 263 (2001), 479–500.

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