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Universal characters and q-Painlevé systems

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Abstract

We propose an integrable system of q-difference equations of which the universal characters satisfy and regard it as a q-analogue of the UC hierarchy; see [10]. Via a similarity reduction of this integrable system, rational solutions of the q-Painlevé systems are constructed in terms of the universal characters.

Introduction

We consider the q-Painlevé system of type $A^{(1)}_{N-1}$:

$$\begin{aligned}\overline{\varphi_n} &= \frac{\varphi_{n-1}}{a_n}\frac{G_{n-1}(\varphi)}{G_{n+1}(\varphi)},\\ G_n(\varphi) &= 1+\varphi_{n-1}+\varphi_{n-2}\varphi_{n-1}+\cdots+\varphi_{n-N+1}\cdots\varphi_{n-1},\end{aligned} \tag{0.1}$$

for $n \in \mathbb{Z}/N\mathbb{Z}$ ($N \geq 3$), which was introduced by Kajiwara *et al.* [2, 3]. Here $\varphi_n = \varphi_n(x)$ are the unknown variables and the symbol $\overline{\varphi_n}$ stands for $\varphi_n(qx)$; $a_n \in \mathbb{C}^\times$ are constant parameters such that $a_1a_2\cdots a_N = q^{-N}$. The system (0.1) in fact has symmetry under the affine Weyl group of type $A^{(1)}_{N-1}$ and goes to the (higher order) Painlevé equation of type $A^{(1)}_{N-1}$ through a certain limiting procedure as $q \to 1$; *cf.* [9]. We often denote the q-Painlevé system (0.1) by q-$P(A_{N-1})$. Note that q-$P(A_2)$ and q-$P(A_3)$ coincide with the q-Painlevé equations q-P_{IV} and q-P_{V} respectively; see [1, 7].

The q-Painlevé system arises, via a similarity reduction, from the q-KP hierarchy which is a q-analogue of the KP hierarchy; see [3]. As a consequence of this remarkable fact, (0.1) admits a class of rational solutions in terms of Schur polynomials; see [3] or Theorem 5.1 below. On the other hand, for the case $N = 4$ (q-P_{V}), Masuda [7] discovered another class of rational solutions which contains the former one in terms of the universal characters. Here the universal character is a generalization of Schur polynomial attached to a pair of partitions; see [5].

The aim of the present article is to give an answer to the question: why the universal character appears in the solutions of the q-Painlevé systems. First we propose an integrable system of q-difference equations of which the universal characters satisfy (see Definition 2.1 and Theorem 2.2). Since this integrable system is regarded as a q-difference analogue of the UC hierarchy (see [10]), we call it the *q-UC hierarchy*. Note that it contains the q-KP hierarchy as a special case (see Remark 2.4). Secondly a certain similarity reduction of the q-UC hierarchy is considered; and then turns out to be equivalent to the q-Painlevé system of type $A^{(1)}_{2g+1}$ ($g = 1, 2, \ldots$). This fact leads us to the

Theorem 0.1. *The q-Painlevé system of type $A^{(1)}_{2g+1}$ ($g = 1, 2, \ldots$) admits a class of rational solutions in terms of the universal characters attached to a pair of $(g+1)$-reduced partitions.*

(See Theorem 5.2.)

In Sect. 1, we recall the definition of the universal characters. Then we introduce the q-UC hierarchy in Sect. 2. In Sect. 3, we briefly review the derivation of q-Painlevé systems. Sect. 4 concerns a similarity reduction of q-UC hierarchy. Finally, in Sect. 5, we present an expression of the rational solutions of q-Painlevé systems in terms of universal characters. Sect. 6 is devoted to the proof of Theorem 2.2.

Note. Throughout the paper, we shall use the notations:

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-q^i a), \quad (a;q,p)_\infty = \prod_{i,j=0}^{\infty}(1-q^i p^j a);$$

use also $(a_1,\dots,a_r;q)_\infty = (a_1;q)_\infty \cdots (a_r;q)_\infty$ and $(a_1,\dots,a_r;q,p)_\infty = (a_1;q,p)_\infty \cdots (a_r;q,p)_\infty$. For a function $f = f(x)$, let

$$\overline{f} = f(qx), \quad \underline{f} = f(q^{-1}x).$$

1 Universal characters

1.1 Definition

We first recall the definition of the universal character. For a pair of sequences of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_{l'})$, the *universal character* $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ is a polynomial in $(\boldsymbol{x},\boldsymbol{y}) = (x_1, x_2, \dots, y_1, y_2, \dots)$ defined as follows (see [5, 10]):

$$S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = \det \begin{pmatrix} p_{\mu_{l'-i+1}+i-j}(\boldsymbol{y}), & 1 \le i \le l' \\ p_{\lambda_{i-l'}-i+j}(\boldsymbol{x}), & l'+1 \le i \le l+l' \end{pmatrix}_{1 \le i,j \le l+l'}, \tag{1.1}$$

where p_n are defined by the generating function:

$$\sum_{k \in \mathbb{Z}} p_k(\boldsymbol{x}) z^k = \exp\left(\sum_{n=1}^{\infty} x_n z^n\right). \tag{1.2}$$

Schur polynomial $S_\lambda(\boldsymbol{x})$ (see *e.g.* [6]) is regarded as a special case of the universal character:

$$S_\lambda(\boldsymbol{x}) = \det\big(p_{\lambda_i - i + j}(\boldsymbol{x})\big) = S_{[\lambda,\emptyset]}(\boldsymbol{x},\boldsymbol{y}). \tag{1.3}$$

If we count the degree of variables as

$$\deg x_n = n \quad \text{and} \quad \deg y_n = -n,$$

then the universal character $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ is a weighted homogeneous polynomial of degree

$$|\lambda| - |\mu|,$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_l$.

1.2 N-reduced partitions

A subset $M \subset \mathbb{Z}$ is said to be a *Maya diagram* if

$$m \in M \quad (m \ll 0) \quad \text{and} \quad m \notin M \quad (m \gg 0);$$

see [8]. Each Maya diagram $M = \{\dots, m_3, m_2, m_1\}$ corresponds to a unique partition $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $m_i - m_{i+1} = \lambda_i - \lambda_{i+1} + 1$. For each $\boldsymbol{n} = (n_1, n_2, \dots, n_N) \in \mathbb{Z}^N$, let us consider the Maya diagram:

$$M(\boldsymbol{n}) = \bigcup_{i=1}^{N} (N\mathbb{Z}_{<n_i} + i),$$

then denote by $\lambda(\boldsymbol{n})$ the corresponding partition. Notice that

$$\lambda(\boldsymbol{n}) = \lambda(\boldsymbol{n} + \mathbf{1}),$$

where $\mathbf{1} = (1, 1, \dots, 1)$. A partition of the form $\lambda(\boldsymbol{n})$ is said to be an N*-reduced partition*. We remark that a partition λ is N-reduced if and only if λ has no hook with length of a multiple of N. We prepare the notations:

$$\boldsymbol{e}_i = (0, \dots, 0, \overset{i}{\smile}{1}, 0, \dots, 0) \quad \text{and} \quad \boldsymbol{n}(k) = \boldsymbol{n} + \sum_{i=1}^{k} \boldsymbol{e}_i.$$

Lemma 1.1 (see [11, Lemma 2.2])**.** *For any $\boldsymbol{n} \in \mathbb{Z}^N$ and partition μ, we have*

$$S_{[(Nn_i - |\boldsymbol{n}|, \lambda(\boldsymbol{n}(i-1))), \mu]}(\boldsymbol{x}, \boldsymbol{y}) = \pm S_{[\lambda(\boldsymbol{n}(i)), \mu]}(\boldsymbol{x}, \boldsymbol{y}). \tag{1.4}$$

2 q-UC hierarchy

We introduce a q-difference analogue of the UC hierarchy (q-UC hierarchy); *cf.* [10].

Let $I \subset \mathbb{Z}_{>0}$ and $J \subset \mathbb{Z}_{<0}$. Let t_i ($i \in I \cup J$) be the independent variable and $T_i = T_{i;q}$ its q-shift operator defined as follows:

$$T_{i;q}(t_i) = \begin{cases} qt_i & (i \in I), \\ q^{-1}t_i & (i \in J), \end{cases} \tag{2.1}$$

and $T_{i;q}(t_j) = t_j$ ($i \neq j$). We use also the notation $T_{i_1} T_{i_2} \cdots T_{i_n} = T_{i_1 i_2 \dots i_n}$ for the sake of simplicity.

Definition 2.1. *The following system of q-difference equations is called the q-UC hierarchy:*

$$(t_i - t_j) T_{ij}(\tau_0) T_k(\tau_1) + (t_j - t_k) T_{jk}(\tau_0) T_i(\tau_1) + (t_k - t_i) T_{ik}(\tau_0) T_j(\tau_1) = 0, \tag{2.2}$$

where $i, j, k \in I \cup J$.

Let

$$x_n = \frac{\sum_{i\in I} t_i^n - q^n \sum_{j\in J} t_j^n}{n(1-q^n)}, \tag{2.3a}$$

$$y_n = \frac{\sum_{i\in I} t_i^{-n} - q^{-n} \sum_{j\in J} t_j^{-n}}{n(1-q^{-n})}, \tag{2.3b}$$

and define the function $s_{[\lambda,\mu]} = s_{[\lambda,\mu]}(\boldsymbol{t})$ in t_i $(i \in I \cup J)$ as

$$s_{[\lambda,\mu]}(\boldsymbol{t}) = S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}). \tag{2.4}$$

The universal characters solve the q-UC hierarchy in the sense of the

Theorem 2.2. *For any integer m and pair of sequences of integers $[\lambda, \mu]$,*

$$\tau_0 = s_{[\lambda,\mu]}(\boldsymbol{t}), \quad \tau_1 = s_{[(m,\lambda),\mu]}(\boldsymbol{t}), \tag{2.5}$$

satisfy the q-UC hierarchy (2.2).

The proof of the theorem is given in Sect. 6.

Remark 2.3. Let $h_n(\boldsymbol{t}) = p_n(\boldsymbol{x})$ and $H_n(\boldsymbol{t}) = p_n(\boldsymbol{y})$ under the change of variables (2.3). Notice the expression by the use of generating functions:

$$\sum_{k=0}^{\infty} h_k(\boldsymbol{t}) z^k = \prod_{i\in I, j\in J} \frac{(qt_j z; q)_\infty}{(t_i z; q)_\infty}, \tag{2.6a}$$

$$\sum_{k=0}^{\infty} H_k(\boldsymbol{t}) z^k = \prod_{i\in I, j\in J} \frac{(q^{-1} t_j^{-1} z; q^{-1})_\infty}{(t_i^{-1} z; q^{-1})_\infty}. \tag{2.6b}$$

Also function $s_{[\lambda,\mu]}(\boldsymbol{t})$ is defined as

$$s_{[\lambda,\mu]}(\boldsymbol{t}) = \det \begin{pmatrix} H_{\mu_{l'-i+1}+i-j}(\boldsymbol{t}), & 1 \le i \le l' \\ h_{\lambda_{i-l'}-i+j}(\boldsymbol{t}), & l'+1 \le i \le l+l' \end{pmatrix}_{1\le i,j\le l+l'}. \tag{2.7}$$

Remark 2.4. Let $J = \emptyset$ and put $t_k = 0$ in (2.2), we obtain the q-KP hierarchy (see [3]):

$$(t_i - t_j) T_{ij}(\tau_0)\tau_1 + t_j T_j(\tau_0) T_i(\tau_1) - t_i T_i(\tau_0) T_j(\tau_1) = 0, \quad i, j \in I. \tag{2.8}$$

If $\mu = \emptyset$, then it makes sense to substitute $t_k = 0$ in $s_{[\lambda,\mu]}(\boldsymbol{t})$. Hence we recover from Theorem 2.2 a solution of the q-KP hierarchy by means of Schur polynomials; *cf.* [3, Proposition 2.2].

3 q-Painlevé systems

We briefly review the derivation of the q-Painlevé system (0.1) from the q-KP hierarchy following [3].

Consider the case $I = \{1, 2\}$ in particular. Let us impose the N-periodic condition:

$$\tau_n = \tau_{n+N}, \tag{3.1}$$

and the similarity condition:

$$T_{12}(\tau_n) = \gamma_n \tau_n \quad (\gamma_n \in \mathbb{C}^\times : \text{constant}), \tag{3.2}$$

on the q-KP hierarchy:

$$(t_1 - t_2) T_{12}(\tau_{n-1}) \tau_n + t_2 T_2(\tau_{n-1}) T_1(\tau_n) - t_1 T_1(\tau_{n-1}) T_2(\tau_n) = 0. \tag{3.3}$$

Let $(t_1, t_2) = (x, 1)$, we have from (3.3) the following q-difference equation for $\sigma_n(x) = \tau_n(x, 1)$:

$$\underline{\sigma_{n-1}}\, \overline{\sigma_n} + (x-1)\sigma_{n-1}\, \sigma_n - \frac{\gamma_n}{\gamma_{n-1}} x\, \overline{\sigma_{n-1}}\, \underline{\sigma_n} = 0, \tag{3.4}$$

where $n \in \mathbb{Z}/N\mathbb{Z}$. This is the bilinear form of the q-Painlevé system; in fact, the functions

$$\varphi_n(x) = x \frac{\gamma_{n+1}}{\gamma_n} \frac{\sigma_{n+1}(q^{-1}x)\sigma_{n-1}(x)}{\sigma_{n+1}(x)\sigma_{n-1}(q^{-1}x)}, \tag{3.5a}$$

solve (0.1) with the parameters

$$a_n = \frac{\gamma_n^2}{\gamma_{n+1}\gamma_{n-1}} q^{-1}. \tag{3.5b}$$

4 Similarity reduction of q-UC hierarchy

Consider the q-UC hierarchy (2.2) in the case $I = \{1, 2\}$ and $J = \{-1, -2\}$; replace the base q with q^2. Then we have

$$\begin{aligned}(t_1 - t_2) T_{1,2;q^2}(\tau_0) T_{-1;q^2}(\tau_1) + (t_2 - t_{-1}) T_{-1,2;q^2}(\tau_0) T_{1;q^2}(\tau_1) \\ + (t_{-1} - t_1) T_{-1,1;q^2}(\tau_0) T_{2;q^2}(\tau_1) = 0.\end{aligned} \tag{4.1}$$

Notice that functions $\tau_i = \tau_i(t_{-2}, t_{-1}, t_1, t_2)$ $(i = 0, 1)$ can be regarded as functions in variables $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, x_2, \ldots, y_1, y_2, \ldots)$ via

$$\begin{aligned} x_n &= \frac{t_1^n + t_2^n - q^{2n}(t_{-1}^n + t_{-2}^n)}{n(1-q^{2n})}, \\ y_n &= \frac{t_1^{-n} + t_2^{-n} - q^{-2n}(t_{-1}^{-n} + t_{-2}^{-n})}{n(1-q^{-2n})}. \end{aligned}$$

We now suppose that τ_i satisfies the similarity condition:

$$\tau_i(pt_{-2}, pt_{-1}, pt_1, pt_2) = p^{d_i}\tau_i(t_{-2}, t_{-1}, t_1, t_2) \quad (d_i \in \mathbb{C} : \text{constant}), \tag{4.2}$$

for any $p \in \mathbb{C}^\times$; and let

$$t_1 = x, \quad t_2 = x^{-1}, \quad t_{-1} = -aq^{-2}, \quad t_{-2} = -aq^{-3}, \tag{4.3}$$

that is,

$$x_n = \frac{x^n + x^{-n} - (-a)^n(1 + q^{-n})}{n(1 - q^{2n})}, \tag{4.4a}$$

$$y_n = \frac{x^n + x^{-n} - (-a)^{-n}(1 + q^n)}{n(1 - q^{-2n})}. \tag{4.4b}$$

Under the specialization (4.3) (or (4.4)), let

$$f_i(x, a) = \tau_i(t_{-2}, t_{-1}, t_1, t_2).$$

Lemma 4.1. *The functions $f_i = f_i(x, a)$ satisfy the following equation:*

$$(x^{-1} + a)f_0(q^{-1}x, a)f_1(qx, aq) + q^{d_0 - d_1}(x - x^{-1})f_0(x, a)f_1(x, aq) \\ -(x + a)f_0(qx, a)f_1(q^{-1}x, aq) = 0. \tag{4.5}$$

Proof. We have

$$\begin{aligned} T_{-1,2;q^2}(x_n) &= \frac{t_1^n + (q^2t_2)^n - q^{2n}((q^{-2}t_{-1})^n + t_{-2}^n)}{n(1 - q^{2n})} \\ &= \frac{x^n + q^{2n}x^{-n} - (-a)^n(q^{-2n} + q^{-n})}{n(1 - q^{2n})} \\ &= q^n\left(\frac{q^{-n}x^n + q^nx^{-n} - (-aq^{-2})^n(q^{-n} + 1)}{n(1 - q^{2n})}\right), \end{aligned}$$

and similarly

$$T_{-1,2;q^2}(y_n) = q^{-n}\left(\frac{q^{-n}x^n + q^nx^{-n} - (-aq^{-2})^{-n}(q^n + 1)}{n(1 - q^{-2n})}\right).$$

Combine this with the similarity condition (4.2), we obtain

$$T_{-1,2;q^2}(\tau_0) = q^{d_0}f_0(q^{-1}x, aq^{-2}). \tag{4.6}$$

One can verify in the same way that $T_{1;q^2}(\tau_1) = q^{d_1}f_1(qx, aq^{-1})$; and also

$$\begin{aligned} &T_{1,2;q^2}(\tau_0) = q^{2d_0}f_0(x, aq^{-2}), \quad T_{-1;q^2}(\tau_1) = f_1(x, aq^{-1}), \\ &T_{-1,1;q^2}(\tau_0) = q^{d_0}f_0(qx, aq^{-2}), \quad T_{2;q^2}(\tau_1) = q^{d_1}f_1(q^{-1}x, aq^{-1}). \end{aligned} \tag{4.7}$$

Substitute (4.3) and the above formulae into (4.1); then replace a with aq^2, we get (4.5). ■

The universal character $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ is a homogeneous solution of the q-UC hierarchy whose degree equals $|\lambda|-|\mu|$; see Theorem 2.2. Hence we have in particular from Lemma 4.1 the

Proposition 4.2. *Let*

$$s_{[\lambda,\mu]}(x,a) = S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}), \tag{4.8}$$

with the specialization (4.4). *For any integer k and pair of sequences of integers $[\lambda,\mu]$, let*

$$f_0(x) = s_{[\lambda,\mu]}(x,a), \quad f_1(x) = s_{[(k,\lambda),\mu]}(x,aq). \tag{4.9}$$

Then we have

$$(x^{-1}+a)\underline{f_0}\,\overline{f_1} + q^{-k}(x-x^{-1})f_0\,f_1 - (x+a)\overline{f_0}\,\underline{f_1} = 0. \tag{4.10}$$

Remark 4.3. If we let $f_0 = 1$ and $f_1 = f(x)$, then equation (4.10) is reduced to the linear q-difference equation of the form:

$$(x^{-1}+a)f(qx) + q^{-k}(x-x^{-1})f(x) - (x+a)f(q^{-1}x) = 0. \tag{4.11}$$

Define $P_k = P_k(x,a)$ $(k \in \mathbb{Z}_{\geq 0})$ by the generating function:

$$\sum_{k=0}^{\infty} P_k(x,a)z^k = \frac{(-az,-aqz;q^2)_\infty}{(xz,x^{-1}z;q^2)_\infty}. \tag{4.12}$$

Then P_k is equivalent to the continuous q-Laguerre polynomial and solves the linear q-difference equation, (4.11), in fact; see *e.g.* [4].

Remark 4.4. Denote by λ^T the transpose of a partition λ; see [6]. It is easy to see the following properties:

$$S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = S_{[\mu,\lambda]}(\boldsymbol{y},\boldsymbol{x}) = \pm S_{[\lambda^T,\mu^T]}(-\boldsymbol{x},-\boldsymbol{y});$$

and for any $p \in \mathbb{C}^\times$

$$S_{[\lambda,\mu]}(px_1,p^2x_2,\ldots,p^{-1}y_1,p^{-2}y_2,\ldots) = p^{|\lambda|-|\mu|}S_{[\lambda,\mu]}(x_1,x_2,\ldots,y_1,y_2,\ldots).$$

Notice the formulae:

$$\begin{aligned} x_n &= \frac{x^n+x^{-n}-(-a)^n(1+q^{-n})}{n(1-q^{2n})} = -q^{-2n}\left(\frac{x^n+x^{-n}-(-a^{-1}q)^{-n}(1+q^n)}{n(1-q^{-2n})}\right), \\ y_n &= \frac{x^n+x^{-n}-(-a)^{-n}(1+q^n)}{n(1-q^{-2n})} = -q^{2n}\left(\frac{x^n+x^{-n}-(-a^{-1}q)^n(1+q^{-n})}{n(1-q^{2n})}\right). \end{aligned}$$

Then we see from Proposition 4.2 that the pair

$$F_0(x) = s_{[\lambda,\mu^T]}(x, aq), \quad F_1(x) = s_{[\lambda,(k,\mu)^T]}(x, a), \tag{4.13}$$

satisfies the q-difference equation:

$$(x^{-1} + a^{-1})\underline{F_0}\,\overline{F_1} + q^{-k}(x - x^{-1})F_0\,F_1 - (x + a^{-1})\overline{F_0}\,\underline{F_1} = 0, \tag{4.14}$$

which is the same equation as (4.10) except replacing a with a^{-1}.

5 Rational solutions of q-Painlevé systems

5.1 q-$P(A_{N-1})$ in terms of Schur polynomials

Since the q-Painlevé system is derived from the q-KP hierarchy via a similarity reduction, we obtain the following expression of rational solutions by means of N-reduced Schur polynomials.

Theorem 5.1 (see [3, Corollary 4.4]). *For any $\boldsymbol{n} \in \mathbb{Z}^N$, let*

$$\sigma_i(x) = S_{\lambda(\boldsymbol{n}(i))}(\boldsymbol{x}), \quad x_1 = x, \quad x_2 = 1, \quad x_l = 0 \quad (l \geq 3). \tag{5.1}$$

Then functions σ_i solve (3.4) *when*

$$\frac{\gamma_{i+1}}{\gamma_i} = q^{Nn_{i+1}-|\boldsymbol{n}|}.$$

Consequently

$$\varphi_i(x) = q^{Nn_{i+1}-|\boldsymbol{n}|}x\frac{\sigma_{i+1}(q^{-1}x)\sigma_{i-1}(x)}{\sigma_{i+1}(x)\sigma_{i-1}(q^{-1}x)}, \tag{5.2a}$$

gives a rational solution of q-$P(A_{N-1})$ with the parameters

$$a_i = q^{N(n_i-n_{i+1})-1}. \tag{5.2b}$$

5.2 q-$P(A_{2g+1})$ in terms of universal characters

From now on we deal with the q-Painlevé system, (0.1), in the case N is even. Let $N = 2g+2$ ($g = 1, 2, \ldots$). Consider the change of variables:

$$\sigma_{2j} = x^{d_{2j}}\frac{(-aqx, -a^{-1}q^2x; q, q^2)_\infty}{(-qx; q, q)_\infty}\rho_{2j}, \tag{5.3a}$$

$$\sigma_{2j+1} = x^{d_{2j+1}}\frac{(-a^{-1}qx, -aq^2x; q, q^2)_\infty}{(-qx; q, q)_\infty}\rho_{2j+1}. \tag{5.3b}$$

Here let $d_i \in \mathbb{C}$ be constant parameters such that

$$\frac{\gamma_{2j+1}}{\gamma_{2j}} = aq^{2k_{2j}}, \qquad \frac{\gamma_{2j}}{\gamma_{2j-1}} = a^{-1}q^{2k_{2j-1}}, \tag{5.4}$$

with $k_i = d_{i+1} - d_i$. Then the bilinear form of q-$P(A_{2g+1})$, (3.4) with $N = 2g+2$, is converted to the following system:

$$(x^{-1}+a)\underline{\rho_{2j}}\,\overline{\rho_{2j+1}} + q^{-k_{2j}}(x-x^{-1})\rho_{2j}\,\rho_{2j+1} - (x+a)\overline{\rho_{2j}}\,\underline{\rho_{2j+1}} = 0, \tag{5.5a}$$
$$(x^{-1}+a^{-1})\underline{\rho_{2j-1}}\,\overline{\rho_{2j}} + q^{-k_{2j-1}}(x-x^{-1})\rho_{2j-1}\,\rho_{2j} - (x+a^{-1})\overline{\rho_{2j-1}}\,\underline{\rho_{2j}} = 0, \tag{5.5b}$$

which coincides with the similarity reduction of the q-UC hierarchy; see Sect. 4.

By virtue of Proposition 4.2 (and also Remark 4.4), together with Lemma 1.1, we now arrive at the

Theorem 5.2. *For any $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^{g+1}$, the functions*

$$\begin{aligned} \rho_{2j}(x) &= s_{[\lambda(\boldsymbol{m}(j)),\lambda^T(\boldsymbol{n}(j))]}(x,a), \\ \rho_{2j+1}(x) &= s_{[\lambda(\boldsymbol{m}(j+1)),\lambda^T(\boldsymbol{n}(j))]}(x,aq), \end{aligned} \tag{5.6}$$

solve (5.5) *when*

$$k_{2j} = (g+1)m_{j+1} - |\boldsymbol{m}|, \quad k_{2j+1} = (g+1)n_{j+1} - |\boldsymbol{n}|.$$

Consequently

$$\varphi_i(x) = \frac{x}{(a_i q)^{\frac{1}{2}}} \frac{\rho_{i+1}(q^{-1}x)\rho_{i-1}(x)}{\rho_{i+1}(x)\rho_{i-1}(q^{-1}x)}, \tag{5.7a}$$

gives a rational solution of q-$P(A_{2g+1})$ with the parameters

$$\begin{aligned} a_{2j} &= a^{-2}q^{2(g+1)(n_j - m_{j+1}) + 2|\boldsymbol{m}| - 2|\boldsymbol{n}| - 1}, \\ a_{2j+1} &= a^{2}q^{2(g+1)(m_{j+1} - n_{j+1}) - 2|\boldsymbol{m}| + 2|\boldsymbol{n}| - 1}. \end{aligned} \tag{5.7b}$$

Remark 5.3. The rational solutions given in Theorems 5.1 and 5.2 are reduced to those of the Painlevé (differential) equation of type $A_{N-1}^{(1)}$, in parallel with the continuous limit from the q-Painlevé system to the equation; *cf.* [11].

Example 5.4. Consider the function:

$$\begin{aligned} &R_{[\lambda,\mu]}(x,a,q) \\ &\quad = a^{|\mu|}x^{[\lambda|+|\mu|}q^{l(\theta)-2|\theta|}\left(\prod_{(i,j)\in\lambda}\left(1-q^{2h(i,j)}\right)\right)\left(\prod_{(i,j)\in\mu}\left(q^{2h(i,j)}-1\right)\right)s_{[\lambda,\mu]}(x,aq), \end{aligned} \tag{5.8}$$

for a pair of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$. Here recall that $s_{[\lambda,\mu]}(x, a)$ is defined in (4.8) under the specialization (4.4); we denote by $h(i, j)$ the *hook-length*, *i.e.*, $h(i, j) = \lambda_i + \lambda_j^T - i - j + 1$; and let $\theta_i = \max(\mu_i^T - \lambda_i, 0)$, $|\theta| = \sum_i \theta_i$, $l(\theta) = \#\{i|\theta_i \neq 0\}$. Then it is observed that $R_{[\lambda,\mu]}$ seems to be a polynomial in x, a, q, whose coefficients are all positive integers. We give below some examples of the *special polynomials* $R_{[\lambda,\mu]}(x, a, q)$:

$$
\begin{aligned}
R_{[\emptyset,\emptyset]} &= 1, \\
R_{[\square,\emptyset]} &= 1 + x^2 + a(1+q)x, \\
R_{[\square\square,\emptyset]} &= 1 + x^4 + a(1+q)(1+q^2)(x+x^3) + (1+q^2)(1+a^2q(1+q))x^2, \\
R_{[\square\square\square,\emptyset]} &= 1 + x^6 + a(1+q)(1+q^2+q^4)(x+x^5) \\
&\quad +(1+q^2+q^4)(1+a^2q(1+q)(1+q^2))(x^2+x^4) \\
&\quad +a(1+q^2)(1+q^3)(1+q+q^2+a^2q^3(1+q))x^3, \\
R_{[\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\emptyset]} &= q^2(1+x^6) + a(1+q)(1+q^2+q^4)(x+x^5) + (1+q^2+q^4)(1+a^2(1+q)^2)(x^2+x^4) \\
&\quad +a(1+q^3)(2(1+q+q^2) + a^2q(1+q)^2)x^3, \\
R_{[\emptyset,\square]} &= aq(1+x^2) + (1+q)x, \\
R_{[\emptyset,\begin{smallmatrix}\square\\\square\end{smallmatrix}]} &= a^2q(1+x^4) + a(1+q)(1+q^2)(x+x^3) + (1+q^2)(1+q+a^2q)x^2, \\
R_{[\square,\square]} &= aq^2(1+x^4) + q(1+q)(1+a^2q)(x+x^3) + a(1+q^2)(1+q+q^2)x^2, \\
R_{[\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}]} &= a^2q^3(1+x^6) + aq^2(1+q)(1+q^2+a^2q)(x+x^5) \\
&\quad +q(q(1+q)(1+q^2) + a^2(1+q+3q^2+2q^3+2q^4+q^5+q^6))(x^2+x^4) \\
&\quad +a(1+q)(1+q^2)(1+q^2+q^3+q^4+a^2q^3)x^3.
\end{aligned}
$$

6 Verification of Theorem 2.2

We have in general the

Lemma 6.1. *Let $h_n = h_n(\boldsymbol{t})$ and $H_n = H_n(\boldsymbol{t})$ $(n \in \mathbb{Z})$ be functions such that*

$$
\begin{aligned}
T_i(h_n) &= h_n - t_i h_{n-1}, && (6.1a) \\
T_i(H_n) &= H_n - t_i^{-1} H_{n-1}, && (6.1b)
\end{aligned}
$$

for $i \in I \cup J$. Let

$$
\tau_0 = \det \begin{pmatrix} H_{n_i-j+1}, & 1 \leq i \leq r' \\ h_{n_i+j-1}, & r'+1 \leq i \leq r \end{pmatrix}_{1 \leq i,j \leq r}, \qquad (6.2)
$$

$$
\tau_1 = \det \begin{pmatrix} H_{n_i-j+1}, & 1 \leq i \leq r' \\ h_{n_i+j-2}, & r'+1 \leq i \leq r+1 \end{pmatrix}_{1 \leq i,j \leq r+1}. \qquad (6.3)
$$

Then the pair τ_0 and τ_1 solves the q-UC hierarchy (2.2).

Theorem 2.2 follows immediately from the lemma, since the functions defined by (2.6) in fact satisfy the relations (6.1).

Remark 6.2. Note that if we choose the functions h_n and H_n as

$$\begin{aligned} h_n &= \psi_{n,\alpha} = \alpha^{-n} \prod_{i\in I, j\in J} \frac{(\alpha q t_j; q)_\infty}{(\alpha t_i; q)_\infty}, \\ H_n &= \Psi_{n,\alpha} = \alpha^{n} \prod_{i\in I, j\in J} \frac{(\alpha^{-1} q^{-1} t_j^{-1}; q^{-1})_\infty}{(\alpha^{-1} t_i^{-1}; q^{-1})_\infty}, \end{aligned}$$

then obtain a q-analogue of the soliton solution; *cf.* [10] and Appendix below. It is easy to see that functions $\psi_{n,\alpha}$ and $\Psi_{n,\alpha}$ satisfy (6.1), in fact.

Proof of Lemma 6.1. We prove the lemma in three steps:

(i) Consider the row vector of size r:

$$(T_{ij}(h_n), T_{ij}(h_{n+1}), \ldots, T_{ij}(h_{n+r-1}))\,. \tag{6.4}$$

Add the l-th column multiplied by $(-t_k)$ to the $(l+1)$-th column for $1 \le l \le r-1$, we then obtain

$$(T_{ij}(h_n), T_{ijk}(h_{n+1}), \ldots, T_{ijk}(h_{n+r-1}))\,, \tag{6.5}$$

by using the relation (6.1a).

By the same procedure as above, the vector:

$$(T_{ij}(H_n), T_{ij}(H_{n-1}), \ldots, T_{ij}(H_{n-r+1}))\,, \tag{6.6}$$

is transformed into

$$-t_k \left(-t_k^{-1} T_{ij}(H_n), T_{ijk}(H_n), T_{ijk}(H_{n-1}), \ldots, T_{ijk}(H_{n-r+2})\right), \tag{6.7}$$

via (6.1b).

Summarizing above we thus have

$$T_{ij}(\tau_0) = (-t_k)^{r'} \left|\boldsymbol{u}_k, U\right|. \tag{6.8}$$

Here we let

$$\boldsymbol{u}_k = {}^T\left(-t_k^{-1} T_{ij}(H_{n_1}), \ldots, -t_k^{-1} T_{ij}(H_{n_{r'}}), T_{ij}(h_{n_{r'+1}}), \ldots, T_{ij}(h_{n_r})\right), \tag{6.9}$$

and $U = (U_{a,b})_{1\le a\le r, 1\le b\le r-1}$ denote the $r\times(r-1)$-matrix defined as

$$U_{a,b} = \begin{cases} T_{ijk}(H_{n_a-b+1}) & (1 \le a \le r'), \\ T_{ijk}(h_{n_a+b}) & (r'+1 \le a \le r). \end{cases} \tag{6.10}$$

(ii) Let us consider elementary transformations of the row vector of size $r+1$:

$$(T_k(h_{n-1}), T_k(h_n), \ldots, T_k(h_{n+r-1}))\,. \tag{6.11}$$

First we add the l-th column multiplied by $(-t_i)$ to the $(l+1)$-th column for $1 \le l \le r$, then we get

$$(T_k(h_{n-1}), T_{ik}(h_n), \ldots, T_{ik}(h_{n+r-1}))\,.$$

Secondly adding the l-th column multiplied by $(-t_j)$ to the $(l+1)$-th column for $2 \le l \le r$, we obtain

$$(T_k(h_{n-1}), T_{ik}(h_n), T_{ijk}(h_{n+1}), \ldots, T_{ijk}(h_{n+r-1}))\,.$$

Add the second column multiplied by $(t_i - t_j)^{-1}$ to the first column, we finally have the vector:

$$\big((t_i - t_j)^{-1} T_{jk}(h_n), T_{ik}(h_n), T_{ijk}(h_{n+1}), \ldots, T_{ijk}(h_{n+r-1})\big)\,. \tag{6.12}$$

Similarly, the vector:

$$(T_k(H_n), T_k(H_{n-1}), \ldots, T_k(H_{n-r}))\,, \tag{6.13}$$

is converted to

$$t_i t_j \left(-(t_i - t_j)^{-1} t_i^{-1} T_{jk}(H_n), -t_j^{-1} T_{ik}(H_n), T_{ijk}(H_n), \ldots, T_{ijk}(H_{n-r+2})\right), \tag{6.14}$$

by the same elementary transformations as above.

Hence we have the expression:

$$(t_i - t_j) T_k(\tau_1) = (t_i t_j)^{r'} \left|\boldsymbol{v}_i, \boldsymbol{v}_j, V\right|, \tag{6.15}$$

where we let

$$\boldsymbol{v}_i = {}^T\left(-t_i^{-1} T_{jk}(H_{n_1}), \ldots, -t_i^{-1} T_{jk}(H_{n_{r'}}), T_{jk}(h_{n_{r'+1}}), \ldots, T_{jk}(h_{n_{r+1}})\right), \tag{6.16}$$

and $V = (V_{a,b})_{1 \le a \le r+1, 1 \le b \le r-1}$ be the $(r+1) \times (r-1)$-matrix defined as

$$V_{a,b} = \begin{cases} T_{ijk}(H_{n_a - b + 1}) & (1 \le a \le r'), \\ T_{ijk}(h_{n_a + b}) & (r' + 1 \le a \le r+1). \end{cases} \tag{6.17}$$

(iii) Substitute (6.8) and (6.15) to the equation of q-UC hierarchy (2.2), we have

$$\begin{aligned} &(-t_i t_j t_k)^{-r'} \,(\text{LHS of (2.2)}) = \\ &\qquad |\boldsymbol{u}_k, U|\,|\boldsymbol{v}_i, \boldsymbol{v}_j, V| - |\boldsymbol{u}_j, U|\,|\boldsymbol{v}_i, \boldsymbol{v}_k, V| + |\boldsymbol{u}_i, U|\,|\boldsymbol{v}_j, \boldsymbol{v}_k, V|\,, \end{aligned} \tag{6.18}$$

which immediately turns out to be zero by the Plücker relation (a determinant identity). ■

A UC hierarchy and its discrete analogue

The UC hierarchy is an infinite-dimensional integrable system characterized by the universal characters and is an extension of the KP hierarchy; see [10]. In this appendix, we briefly summarize some results on the UC hierarchy and present its discrete analogue.

A.1 UC hierarchy and its solutions

Introduce the vertex operators:

$$X^{\pm}(z;\boldsymbol{x},\boldsymbol{y}) = \exp(\pm\xi(\boldsymbol{x}-\widetilde{\partial}_{\boldsymbol{y}},z))\exp(\mp\xi(\widetilde{\partial}_{\boldsymbol{x}},z^{-1})), \tag{A.1}$$

$$Y^{\pm}(z;\boldsymbol{x},\boldsymbol{y}) = \exp(\pm\xi(\boldsymbol{y}-\widetilde{\partial}_{\boldsymbol{x}},z^{-1}))\exp(\mp\xi(\widetilde{\partial}_{\boldsymbol{y}},z)), \tag{A.2}$$

where $\xi(\boldsymbol{x},z)=\sum_{n=1}^{\infty}x_nz^n$ and $\widetilde{\partial}_{\boldsymbol{x}}$ stands for $\left(\frac{\partial}{\partial x_1},\frac{1}{2}\frac{\partial}{\partial x_2},\frac{1}{3}\frac{\partial}{\partial x_3},\dots\right)$. The *UC hierarchy* is defined by the following bilinear equations for an unknown function $\tau=\tau(\boldsymbol{x},\boldsymbol{y})$ (see [10]):

$$\operatorname*{Res}_{z=0} X^{-}(z;\boldsymbol{x}',\boldsymbol{y}')\tau(\boldsymbol{x}',\boldsymbol{y}')X^{+}(z;\boldsymbol{x},\boldsymbol{y})\tau(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}z=0, \tag{A.3a}$$

$$\operatorname*{Res}_{z=\infty} Y^{-}(z;\boldsymbol{x}',\boldsymbol{y}')\tau(\boldsymbol{x}',\boldsymbol{y}')Y^{+}(z;\boldsymbol{x},\boldsymbol{y})\tau(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}z=0. \tag{A.3b}$$

Note that (A.3) is equivalently rewritten into a system of partial differential equations of infinite order.

Now we recall two classes of solutions of the UC hierarchy:
(i) All the universal characters $S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ are solutions of (A.3); see [10, Proposition 1.4].
(ii) $\tau=\tau_{m,n}(\boldsymbol{x},\boldsymbol{y};p_i,q_j,c_k)$ is a solution of (A.3), called the (m,n)-*soliton solution.* Here function $\tau_{m,n}$ has the following expression of 'twisted' Wronskian type:

$$\tau_{m,n}=\det\begin{pmatrix} \left(\frac{\partial}{\partial y_1}\right)^{j-1}\left(e^{\xi(\boldsymbol{y},p_i^{-1})}+c_ie^{\xi(\boldsymbol{y},q_i^{-1})}\right), & 1\le i\le n \\ \left(\frac{\partial}{\partial x_1}\right)^{m+n-j}\left(e^{\xi(\boldsymbol{x},p_i)}+c_ie^{\xi(\boldsymbol{x},q_i)}\right), & n+1\le i\le m+n \end{pmatrix}_{1\le i,j\le m+n}, \tag{A.4}$$

where p_i, q_i, c_i being constant parameters; see [10, Proposition 1.5].

Let $\tau_0(\boldsymbol{x},\boldsymbol{y})=\tau(\boldsymbol{x},\boldsymbol{y})$ be a solution of the UC hierarchy and let

$$\tau_1(\boldsymbol{x},\boldsymbol{y})=X^{+}(w;\boldsymbol{x},\boldsymbol{y})\tau(\boldsymbol{x},\boldsymbol{y}), \tag{A.5}$$

for an arbitrary parameter w. Then we have

$$\operatorname*{Res}_{z=0} zX^{-}(z;\boldsymbol{x}',\boldsymbol{y}')\tau_0(\boldsymbol{x}',\boldsymbol{y}')X^{+}(z;\boldsymbol{x},\boldsymbol{y})\tau_1(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}z=0, \tag{A.6a}$$

$$\operatorname*{Res}_{z=\infty} Y^{-}(z;\boldsymbol{x}',\boldsymbol{y}')\tau_0(\boldsymbol{x}',\boldsymbol{y}')Y^{+}(z;\boldsymbol{x},\boldsymbol{y})\tau_1(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}z=0, \tag{A.6b}$$

which is called the $(1,0)$-*modified* UC hierarchy. In particular we have the

Proposition A.1. *For any integer m and pair of sequences of integers $[\lambda,\mu]$,*

$$\tau_0=S_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}),\quad \tau_1=S_{[(m,\lambda),\mu]}(\boldsymbol{x},\boldsymbol{y}),$$

satisfy the system (A.6).

A.2 d-UC hierarchy

Consider the $(1,0)$-modified UC hierarchy (A.6). First we note that (A.6a) is equivalently rewritten into

$$\oint \frac{\mathrm{d}z}{2\pi\sqrt{-1}} z\tau_0(\boldsymbol{x}' + \boldsymbol{\epsilon}(z^{-1}), \boldsymbol{y}' + \boldsymbol{\epsilon}(z))\tau_1(\boldsymbol{x} - \boldsymbol{\epsilon}(z^{-1}), \boldsymbol{y} - \boldsymbol{\epsilon}(z))e^{\xi(\boldsymbol{x}-\boldsymbol{x}',z)} = 0, \tag{A.7}$$

with $\boldsymbol{\epsilon}(z) = (z, z^2/2, z^3/3, \ldots)$. Here the integration is taken along a small contour around $z = \infty$.

Let $I \subset \mathbb{Z}$ be a subset and fix $i, j, k \in I$. Let

$$\begin{aligned}\boldsymbol{x}' &= \boldsymbol{x} - \boldsymbol{\epsilon}(t_i) - \boldsymbol{\epsilon}(t_j) - \boldsymbol{\epsilon}(t_k),\\ \boldsymbol{y}' &= \boldsymbol{y} - \boldsymbol{\epsilon}(t_i^{-1}) - \boldsymbol{\epsilon}(t_j^{-1}) - \boldsymbol{\epsilon}(t_k^{-1}),\end{aligned}$$

where t_i's are arbitrary small parameters such that t_i^{-1}'s lie inside the contour. Then we have

$$e^{\xi(\boldsymbol{x}-\boldsymbol{x}',z)} = \frac{1}{(1-t_iz)(1-t_jz)(1-t_kz)}. \tag{A.8}$$

Equation (A.7) is rewritten into

$$\oint F(z)\frac{\mathrm{d}z}{2\pi\sqrt{-1}} = 0, \tag{A.9}$$

where

$$\begin{aligned}F(z) = &\frac{z}{(1-t_iz)(1-t_jz)(1-t_kz)}\\ &\times\tau_0(\boldsymbol{x} - \boldsymbol{\epsilon}(t_i) - \boldsymbol{\epsilon}(t_j) - \boldsymbol{\epsilon}(t_k) + \boldsymbol{\epsilon}(z^{-1}), \boldsymbol{y} - \boldsymbol{\epsilon}(t_i^{-1}) - \boldsymbol{\epsilon}(t_j^{-1}) - \boldsymbol{\epsilon}(t_k^{-1}) + \boldsymbol{\epsilon}(z))\\ &\times\tau_1(\boldsymbol{x} - \boldsymbol{\epsilon}(z^{-1}), \boldsymbol{y} - \boldsymbol{\epsilon}(z)).\end{aligned}$$

Now let us assume that: the sum of residues of $F(z)$ vanishes inside the contour except at $z = t_i^{-1}, t_j^{-1}, t_k^{-1}$. Then we have from (A.9)

$$\operatorname*{Res}_{z=t_i^{-1}} F(z)\,\mathrm{d}z + \operatorname*{Res}_{z=t_j^{-1}} F(z)\,\mathrm{d}z + \operatorname*{Res}_{z=t_k^{-1}} F(z)\,\mathrm{d}z = 0,$$

which is equivalent to the equation:

$$(t_i - t_j)\widetilde{T}_{ij}(\tau_0)\widetilde{T}_k(\tau_1) + (t_j - t_k)\widetilde{T}_{jk}(\tau_0)\widetilde{T}_i(\tau_1) + (t_k - t_i)\widetilde{T}_{ik}(\tau_0)\widetilde{T}_j(\tau_1) = 0, \tag{A.10}$$

via the change of variables:

$$x_n = \frac{\sum_{i\in I}\alpha_i t_i^n}{n} \quad \text{and} \quad y_n = \frac{\sum_{i\in I}\alpha_i t_i^{-n}}{n}. \tag{A.11}$$

Here $\widetilde{T}_i$ ($i \in I$) stands for the shift operator defined as

$$\widetilde{T}_i(\alpha_i) = \alpha_i - 1, \quad \widetilde{T}_i(\alpha_j) = \alpha_j \quad (i \neq j); \tag{A.12}$$

and let $\widetilde{T}_{ij} = \widetilde{T}_i\widetilde{T}_j$. In the same way, we obtain (A.10) also from (A.6b). Note that both classes of solutions of the (modified) UC hierarchy, universal characters and soliton solutions, satisfy the above assumption. We call (A.10) the *d-UC hierarchy*. The q-UC hierarchy, (2.2), is derived from (A.10) by replacing $\widetilde{T}_i$ with the q-shift operator T_i formally; *cf.* Sect. 2.

Remark A.2. The d-UC hierarchy can be recovered from the q-UC hierarchy through a certain limiting procedure as follows. Let us consider the substitution:

$$t_i = s_i e^{\varepsilon\alpha_i}, \quad q = e^{-\varepsilon}, \tag{A.13}$$

in (2.2). Noticing that

$$T_i(t_i) = qt_i = s_i e^{\varepsilon(\alpha_i - 1)},$$

if we take $\varepsilon \to 0$, then we immediately obtain from (2.2) the d-UC hierarchy, (A.10).

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