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On the Kernel of the Magnus representation of the Torelli group

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Abstract

From our previous paper, it is known that the Magnus representation of the Torelli group is not faithful. In this paper, we characterize the kernel of its representation for a certain kind of elements.

1 Introduction

The linearity of the mapping class group of a surface of genus $g \geq 2$ has been one of the well-known open problems. A group is called linear if it admits a finite dimensional faithful representation. Recently, Korkmaz [K], Bigelow and Budney [B-B] proved that the mapping class group of a closed surface of genus 2 is linear. However, it still remains open for higher genera. Then it is significant to discuss whether some representations of the mapping class groups are faithful and to determine the kernel.

Let $\Sigma_{g,1}$ be an oriented surface obtained from a closed surface of genus g by removing an open disk. We denote by $\mathcal{M}_{g,1}$ the mapping class group of $\Sigma_{g,1}$ relative to the boundary, that is the group of path components of the group of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ which restrict to the identity on the boundary. Let $\mathcal{I}_{g,1}$ be the Torelli group of $\Sigma_{g,1}$, namely the normal subgroup of $\mathcal{M}_{g,1}$ consisting of all the elements which act trivially on the first homology group of $\Sigma_{g,1}$.

The Magnus representations of various subgroups of the automorphism group of a free group are defined making use of the Fox derivation [F], see [Bir] for details. The Magnus representation for the Torelli group

$$r_1 : \mathcal{I}_{g,1} \to GL(2g; \mathbb{Z}[H])$$

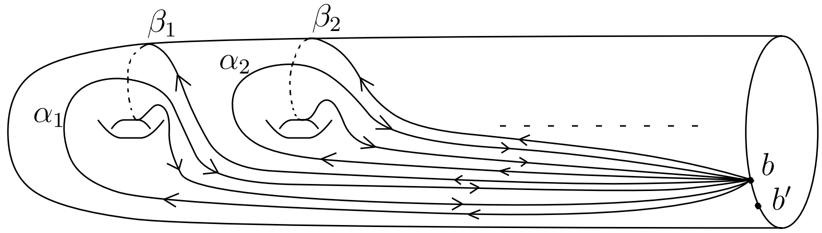

Figure 1: Generators of Γ_0 and base points b, b'

was introduced in [M1], where $H = H_1(\Sigma_{g,1};\mathbb{Z})$. From our previous paper [S1], the representation r_1 is not faithful for $g \geq 2$. Thus it makes sense to study the kernel of r_1. In this paper, we characterize the kernel of r_1 for the commutator of two BSCC maps, where the Dehn twist along a bounding simple closed curve is called BSCC map. The following is one of the main result of this paper.

Corollary 4.4 *The commutator of two BSCC maps φ_1, φ_2 belongs to the kernel of r_1 if and only if the characteristic polynomial of the Magnus matrix of the product $\varphi_1\varphi_2$ is trivial. Here the Magnus matrix means the image of r_1 for a mapping class.*

In Section 2, we will recall the definitions of the Magnus representation of the mapping class group and the Torelli group.

In Section 3, we will give a certain pairing for two curves on $\Sigma_{g,1}$ and show the relationship with the pairing and the kernel of r_1.

In Section 4, we will introduce another pairing for two curves on $\Sigma_{g,1}$ in order to obtain additional information of the kernel of r_1.

2 Definition of the Magnus representation of the Torelli group

In this section, we recall the definitions of the Magnus representation for the mapping class group and the Torelli group from [M1], [S1] and [S4].

Let $\mathbb{Z}[\Gamma_0]$ be the integral group ring of $\Gamma_0 = \pi_1(\Sigma_{g,1}, b)$. We fix a system of generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of the free group Γ_0 as shown in Figure 1. Let us simply write $\gamma_1, \ldots, \gamma_{2g}$ for them.

Definition 2.1 *We call the mapping*

$$\begin{array}{rccc} r: & \mathcal{M}_{g,1} & \longrightarrow & GL(2g;\mathbb{Z}[\Gamma_0]) \\ & \varphi & \longmapsto & \left(\overline{\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i}}\right)_{i,j} \end{array}$$

the Magnus representation for the mapping class group, where $\frac{\partial}{\partial \gamma_i}: \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ *is the Fox derivation and* $^- : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ *is the antiautomorphism induced by the mapping* $\gamma \mapsto \gamma^{-1}$.

This mapping is not a homomorphism but a crossed homomorphism.

Proposition 2.2 (Morita [M1]) *For any two elements* $\varphi, \psi \in \mathcal{M}_{g,1}$*, we have*

$$r(\varphi\psi) = r(\varphi) \cdot {}^{\varphi}r(\psi)$$

where ${}^{\varphi}r(\psi)$ *denotes the matrix obtained from* $r(\psi)$ *by applying the automorphism* $\varphi : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ *on each entry.*

It follows that if this mapping r is restricted to the Torelli group $\mathcal{I}_{g,1}$ and are reduced the coefficients to $\mathbb{Z}[H]$, then we obtain the following genuine representation:

$$r_1 : \mathcal{I}_{g,1} \longrightarrow GL(2g;\mathbb{Z}[H]).$$

Here the reduction is induced by the abelianization $\mathfrak{a} : \Gamma_0 \to H$ and r_1 denotes the composition $r^{\mathfrak{a}}$ of the mapping r by the abelianization $\mathfrak{a}$. We call r_1 the Magnus representation of the Torelli group.

We have another definition of this representation (see [S4]). Let $p : \widehat{\Sigma} \to \Sigma_{g,1}$ be the universal abelian covering, that is, the regular covering corresponding to the abelianization. An arbitrary element of the Torelli group induces an automorphism of $H_1(\widehat{\Sigma}, p^{-1}(b); \mathbb{Z})$ as a free $\mathbb{Z}[H]$-module of rank $2g$. Therefore we get the following representation:

$$r_1 : \mathcal{I}_{g,1} \longrightarrow GL(2g;\mathbb{Z}[H]).$$

3 A higher intersection number of two loops and the kernel of r_1

The non-triviality of the kernel of r_1 for $g \geq 2$ is proved in [S1]. Moreover, it is proved in [S2] that none of the terms of the lower central series of $\mathcal{I}_{g,1}$ is contained in the kernel. Then it is interesting to characterize and determine the kernel.

First, we define a pairing of two loops on $\Sigma_{g,1}$. This pairing is useful to give information about the kernel of r_1. Choose base points b and b' on $\partial\Sigma_{g,1}$ as depicted in Figure 1. Fix a point $\hat{b}$ which is a lift of b to the universal abelian covering $\widehat{\Sigma}$. The point $\hat{b}'$ is determined as follows, which is a lift of b'. We denote by bb' the path on $\partial\Sigma_{g,1}$ from b to b' with the orientation which is opposite to that of $\Sigma_{g,1}$. Let $\widehat{bb'}$ be the lift of bb' to $\widehat{\Sigma}$ starting at $\hat{b}$. Then we set $\hat{b}'$ for the endpoint of $\widehat{bb'}$.

Definition 3.1 *Let c_1, c_2 be two oriented loops on $\Sigma_{g,1}$ based at b, b' respectively. We define*

$$\langle c_1, c_2 \rangle_H = \sum_{h \in H} (h\hat{c}_1, \hat{c}_2)\, h.$$

Here $\hat{c}_1$ is the lift of c_1 to $\widehat{\Sigma}$ starting at $\hat{b}$, $\hat{c}_2$ is the lift of c_2 to $\widehat{\Sigma}$ starting at $\hat{b}'$ and $(\cdot,\cdot)$ denotes the algebraic intersection number of two arcs. We write $h\hat{c}_1$ for the curve which is acted on $\hat{c}_1$ by an element h of the covering transformation group H.

Suppose that c_1 and c_2 are bounding simple closed curves on $\Sigma_{g,1}$, where bounding means 0-homologous. If we regard c_1, c_2 as *oriented* loops *based at* b, b' respectively, then we can compute the pairing $\langle c_1, c_2 \rangle_H$ up to multiplication by ± 1 and by an element of H. That is to say, the pairing $\langle c_1, c_2 \rangle_H$ depends on how c_1, c_2 are represented as loops. However, whether $\langle c_1, c_2 \rangle_H$ is zero or not does not depend on the choices, and we will use this fact.

Proposition 3.2 *Suppose that c_1 and c_2 are two bounding simple closed curves on $\Sigma_{g,1}$, and φ_1 and φ_2 the Dehn twists along c_1 and c_2 respectively. If $\langle c_1, c_2 \rangle_H = 0$, then $[\varphi_1, \varphi_2] \in \ker r_1$.*

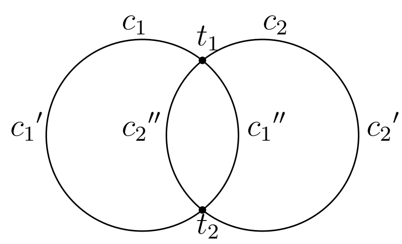

Figure 2: geometric intersection number 2

Proof. We denote by $\widehat{\varphi}_*$ the automorphism of the first homology group $H_1(\widehat{\Sigma}, p^{-1}(b); \mathbb{Z})$ induced by a diffeomorphism φ of $\Sigma_{g,1}$ representing an element of $\mathcal{M}_{g,1}$. Let $\hat{c}_1, \hat{c}_2$ be lifts of c_1, c_2 to $\widehat{\Sigma}$ respectively. Then $[\hat{c}_1], [\hat{c}_2]$ belong to $H_1(\widehat{\Sigma}, p^{-1}(b); \mathbb{Z})$. Since $\langle c_1, c_2 \rangle_H = 0$, the intersection number $(\hat{c}_1, \hat{c}_2)$ equals zero. For a loop c based at b, we denote by $\hat{c}$ a lift of c to $\widehat{\Sigma}$. Then we have an element $[\hat{c}]$ of $H_1(\widehat{\Sigma}, p^{-1}(b); \mathbb{Z})$ and

$$\widehat{\varphi}_{i*}([\hat{c}]) = [\hat{c}] + (\hat{c}_i, \hat{c})[\hat{c}_i] \qquad i = 1, 2.$$

Then we obtain

$$\begin{aligned} \widehat{\varphi_{1*}} \circ \widehat{\varphi_{2*}}([\hat{c}]) &= \widehat{\varphi_{1*}}([\hat{c}] + (\hat{c}_2, \hat{c})[\hat{c}_2]) \\ &= [\hat{c}] + (\hat{c}_1, \hat{c})[\hat{c}_1] + (\hat{c}_2, \hat{c})[\hat{c}_2] \\ &= \widehat{\varphi_{2*}} \circ \widehat{\varphi_{1*}}([\hat{c}]). \end{aligned}$$

It follows that $\widehat{\varphi_{1*}}$ commutes with $\widehat{\varphi_{2*}}$ and this completes the proof. ∎

Corollary 3.3 *Suppose that c_1 and c_2 are two bounding simple closed curves. If the geometric intersection number of c_1 and c_2 is two, then $[\varphi_1, \varphi_2] \in \ker r_1$.*

Proof. Let t_1, t_2 be the intersection points. Also, let $c_i{}'$ be the subarcs of c_i from t_1 to t_2, $c_i{}''$ from t_2 to t_1, see Figure 2. The number of the terms of $\langle c_1, c_2 \rangle_H$ is two. Each value of the terms is decided by the value at t_1 and t_2 respectively. We consider loops $c_1{}'c_2{}'', c_1{}'c_2{}'^{-1}, c_2{}'c_1{}''$ and $c_2{}''c_1{}''^{-1}$, where c^{-1} is the same arc as c with the opposite orientation. All of these are bounding simple closed curves. It follows that the value at t_1 is -1 times that of t_2. Then $\langle c_1, c_2 \rangle_H = 0$. By Proposition 3.2, this completes the proof. ∎

4 Another pairing of bounding simple closed curves and the kernel of r_1

We define another pairing for two bounding simple closed curves:

$$\langle\langle c_1, c_2 \rangle\rangle = -\langle c_1, c_2 \rangle_H \cdot \langle c_2, c_1 \rangle_H.$$

The pairing $\langle \cdot, \cdot \rangle_H$ depends on the way to assigning orientations and attaching basepoints to two bounding simple closed curves. However, the way does not have an effect on the pairing $\langle\langle \cdot, \cdot \rangle\rangle$. That is, we obtain the following lemma.

Lemma 4.1 *Let c_1, c_2 be two bounding simple closed curves on $\Sigma_{g,1}$. Then we have*

1. $\langle\langle c_1, c_2 \rangle\rangle = \langle\langle c_2, c_1 \rangle\rangle$
2. $\langle\langle \gamma c_1 \gamma^{-1}, c_2 \rangle\rangle = \langle\langle c_1, c_2 \rangle\rangle$
3. $\langle\langle {c_1}^{-1}, c_2 \rangle\rangle = \langle\langle c_1, c_2 \rangle\rangle$

where γ is a loop based at b and ${c_1}^{-1}$ is the same loop as c_1 with the opposite orientation.

We recall the following before proving Lemma 4.1.

Theorem 4.2 (Morita [M1]) *There exists a matrix $\widetilde{J}$ such that for any element $f \in \mathcal{M}_{g,1}$ the following equality holds:*

$$\overline{{}^t r(f)}\, \widetilde{J}\, r(f) = {}^f \widetilde{J}.$$

This means that the Magnus representation of the mapping class group is symplectic in a sense. The explicit expression of $\widetilde{J}$ can be found in [M1] and [S4] and is not included into this paper.

In this section, $\overrightarrow{c}$ denotes ${}^t\left(\mathfrak{a}\left(\frac{\partial c}{\partial \gamma_1}\right), \ldots, \mathfrak{a}\left(\frac{\partial c}{\partial \gamma_{2g}}\right)\right)$.

Proof.

1. It is obvious from the definition of the pairing $\langle\langle \cdot, \cdot \rangle\rangle$.

2. We can consider γ as an element of Γ_0 naturally. Because

$$\begin{aligned}\mathfrak{a}\left(\frac{\partial \gamma c_1 \gamma^{-1}}{\partial \gamma_i}\right) &= \mathfrak{a}\left(\frac{\partial \gamma}{\partial \gamma_i}\right) + \mathfrak{a}(\gamma)\mathfrak{a}\left(\frac{\partial c_1}{\partial \gamma_i}\right) + \mathfrak{a}(\gamma)\mathfrak{a}(c_1)\mathfrak{a}\left(\frac{\partial \gamma^{-1}}{\partial \gamma_i}\right) \\ &= \mathfrak{a}(\gamma)\mathfrak{a}\left(\frac{\partial c_1}{\partial \gamma_i}\right),\end{aligned}$$

then we get

$$\overrightarrow{\gamma c_1 \gamma^{-1}} = \mathfrak{a}(\gamma)\overrightarrow{c_1}.$$

By [S4, Lemma 4.4], we have $\langle c_1, c_2 \rangle_H = -{}^t\overrightarrow{c_2}\, J_1\, \overline{\overrightarrow{c_1}}$, where $\mathfrak{a}(\widetilde{J}) = J_1$, therefore

$$\begin{aligned}\langle\langle \gamma c_1 \gamma^{-1}, c_2 \rangle\rangle &= -{}^t\overrightarrow{c_2}\, J_1\, \overline{\mathfrak{a}(\gamma)}\ \overline{\overrightarrow{c_1}}{}^t\mathfrak{a}(\gamma)\overrightarrow{c_1}\, J_1\, \overline{\overrightarrow{c_2}} \\ &= -{}^t\overrightarrow{c_2}\, J_1\, \overline{\overrightarrow{c_1}}{}^t\overrightarrow{c_1}\, J_1\, \overline{\overrightarrow{c_2}} \\ &= \langle\langle c_1, c_2 \rangle\rangle.\end{aligned}$$

3. Since

$$\mathfrak{a}\left(\frac{\partial {c_1}^{-1}}{\partial \gamma_i}\right) = \mathfrak{a}({c_1}^{-1})\mathfrak{a}\left(\frac{\partial c_1}{\partial \gamma_i}\right) = -\mathfrak{a}\left(\frac{\partial c_1}{\partial \gamma_i}\right),$$

we deduce this lemma.

∎

The relation between the pairing $\langle\langle \cdot, \cdot \rangle\rangle$ and the Magnus representation r_1 of the Torelli group can be expressed as the following formula.

Theorem 4.3 *Suppose that c_1 and c_2 are two bounding simple closed curves on $\Sigma_{g,1}$. Then we obtain*

$$\langle\langle c_1, c_2 \rangle\rangle = \mathrm{tr}(I_{2g} - r_1(\varphi_1\varphi_2)) = 2g - \mathrm{tr}(r_1(\varphi_1\varphi_2))$$

where φ_1, φ_2 are the Dehn twists along c_1, c_2 respectively.

Proof. Any bounding simple closed curve can be written as $f(d_k)$ for a certain element $f \in \mathcal{M}_{g,1}$ and for a bounding simple closed curve d_k which is shown in Figure 3. First, we will prove the statement in the case $c_1 = f(d_i), c_2 = d_j$. That is, we will consider the case $\varphi_1 = f\psi_i f^{-1}, \varphi_2 = \psi_j$,

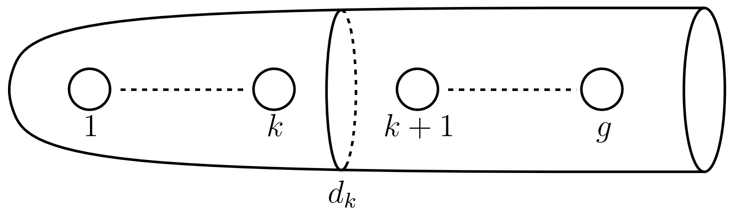

Figure 3: bounding simple closed curve

where ψ_k is the Dehn twist along d_k. By Lemma 4.1, we can assume that c_1 and c_2 have expressions as

$$c_1 = f([\beta_i, \alpha_i] \cdots [\beta_1, \alpha_1]), \quad c_2 = [\beta_j, \alpha_j] \cdots [\beta_1, \alpha_1].$$

We see from [S3] that

$$r_1(\psi_k) = I_{2g} + a_k b_k. \tag{4.1}$$

Here

$$\begin{aligned} a_k &= {}^t(\bar{y}_1 - 1 \ \cdots \ \bar{y}_k - 1 \quad \underbrace{0 \ \cdots \ 0}_{g-k \text{ times}} \quad 1 - \bar{x}_1 \ \cdots \ 1 - \bar{x}_k \quad \underbrace{0 \ \cdots \ 0}_{g-k \text{ times}} \ ) \\ b_k &= (1 - \bar{x}_1 \ \cdots \ 1 - \bar{x}_k \quad \underbrace{0 \ \cdots \ 0}_{g-k \text{ times}} \quad 1 - \bar{y}_1 \ \cdots \ 1 - \bar{y}_k \quad \underbrace{0 \ \cdots \ 0}_{g-k \text{ times}} \ ), \end{aligned}$$

and x_i, y_i are the homology classes of α_i, β_i respectively. Note that $\mathrm{tr}(a_k b_k) = b_k a_k = 0$. We denote by $r^{\mathfrak{a}}$ the composition of the mapping r by the abelianization $\mathfrak{a} : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[H]$. If we consider elements of the Torelli group, we write r_1 for $r^{\mathfrak{a}}$ as before. By the abelianization, Theorem 4.2 can be stated as

$$\overline{{}^t r^{\mathfrak{a}}(f)} \, J_1 \, r^{\mathfrak{a}}(f) = {}^f J_1. \tag{4.2}$$

The following equalities can be checked easily:

$$b_k J_1^{-1} = \overline{{}^t a_k}, \quad \overline{{}^t a_k} J_1 = b_k. \tag{4.3}$$

We will compute $\overline{\overrightarrow{c_1}}$ by an explicit calculation. Since

$$\begin{aligned}
&\mathfrak{a}\left(\frac{\partial c_1}{\partial \gamma_l}\right)\\
&= \sum_{k=1}^{i} \mathfrak{a}\left(\frac{\partial f([\beta_k,\alpha_k])}{\partial \gamma_l}\right)\\
&= \sum_{k=1}^{i} \Bigg\{ \mathfrak{a}\left(\frac{\partial f(\beta_k)}{\partial \gamma_l}\right) + \mathfrak{a}(f(\beta_k))\cdot \mathfrak{a}\left(\frac{\partial f(\alpha_k)}{\partial \gamma_l}\right)\\
&\qquad + \mathfrak{a}(f(\beta_k))\cdot \mathfrak{a}(f(\alpha_k))\cdot \mathfrak{a}\left(\frac{\partial f({\beta_k}^{-1})}{\partial \gamma_l}\right) + \mathfrak{a}(f(\alpha_k))\cdot \mathfrak{a}\left(\frac{\partial f({\alpha_k}^{-1})}{\partial \gamma_l}\right)\Bigg\}\\
&= \sum_{k=1}^{i} \Bigg\{ \mathfrak{a}\left(\frac{\partial f(\beta_k)}{\partial \gamma_l}\right) + f(y_k)\cdot \mathfrak{a}\left(\frac{\partial f(\alpha_k)}{\partial \gamma_l}\right)\\
&\qquad - f(x_k)\cdot \mathfrak{a}\left(\frac{\partial f(\beta_k)}{\partial \gamma_l}\right) - \mathfrak{a}\left(\frac{\partial f(\alpha_k)}{\partial \gamma_l}\right)\Bigg\}\\
&= \sum_{k=1}^{i} \left\{ (f(y_k)-1)\cdot \mathfrak{a}\left(\frac{\partial f(\alpha_k)}{\partial \gamma_l}\right) + (1-f(x_k))\cdot \mathfrak{a}\left(\frac{\partial f(\beta_k)}{\partial \gamma_l}\right)\right\},
\end{aligned}$$

we obtain

$$\overline{\overrightarrow{c_1}} = r^{\mathfrak{a}}(f)\cdot {}^{f}a_i. \tag{4.4}$$

Similarly, $\overline{\overrightarrow{c_2}} = \overline{a_j}$. Therefore

$$\begin{aligned}
&\mathrm{tr}(I_{2g} - r_1(\varphi_1\varphi_2))\\
&= \mathrm{tr}(I_{2g} - r^{\mathfrak{a}}(f)\cdot {}^{f}r_1(\psi_i)\cdot r^{\mathfrak{a}}(f)^{-1}\cdot r_1(\psi_j))\\
&= \mathrm{tr}(I_{2g} - r^{\mathfrak{a}}(f)\cdot (I_{2g} + {}^{f}a_i\,{}^{f}b_i)\cdot r^{\mathfrak{a}}(f)^{-1}\cdot (I_{2g}+a_jb_j)) \qquad \text{Because (4.1)}\\
&= \mathrm{tr}(-r^{\mathfrak{a}}(f)\cdot {}^{f}a_i\,{}^{f}b_i\cdot r^{\mathfrak{a}}(f)^{-1} - a_jb_j - r^{\mathfrak{a}}(f)\cdot {}^{f}a_i\,{}^{f}b_i\cdot r^{\mathfrak{a}}(f)^{-1}\cdot a_jb_j)\\
&= -\mathrm{tr}(r^{\mathfrak{a}}(f)\cdot {}^{f}a_i\,{}^{f}b_i\cdot r^{\mathfrak{a}}(f)^{-1}\cdot a_jb_j)\\
&= -\mathrm{tr}(r^{\mathfrak{a}}(f)\cdot {}^{f}a_i\,{}^{f}b_i\cdot {}^{f}{J_1}^{-1}\cdot \overline{{}^{t}r^{\mathfrak{a}}(f)}\cdot J_1\cdot a_jb_j\,) \qquad \text{Because (4.2)}\\
&= -\mathrm{tr}(r^{\mathfrak{a}}(f)\cdot {}^{f}a_i\,\overline{{}^{tf}a_i}\cdot \overline{{}^{t}r^{\mathfrak{a}}(f)}\cdot J_1\cdot a_j\,\overline{{}^{t}a_j}\cdot J_1) \qquad \text{Because (4.3)}\\
&= -\overline{{}^{tf}a_i}\cdot \overline{{}^{t}r^{\mathfrak{a}}(f)}\cdot J_1\cdot a_j\cdot \mathrm{tr}(r^{\mathfrak{a}}(f)\cdot {}^{f}a_i\,\overline{{}^{t}a_j}\cdot J_1)\\
&= -{}^{t}\overline{\overrightarrow{c_1}}\,J_1\,\overline{\overline{\overrightarrow{c_2}}}\cdot \overline{{}^{t}a_j}\,J_1\,r^{\mathfrak{a}}(f)\cdot {}^{f}a_i \qquad \text{Because (4.4)}\\
&= -{}^{t}\overline{\overrightarrow{c_1}}\,J_1\,\overline{\overline{\overrightarrow{c_2}}}\cdot {}^{t}\overline{\overrightarrow{c_2}}\,J_1\,\overline{\overline{\overrightarrow{c_1}}}\\
&= \langle\langle c_2, c_1\rangle\rangle = \langle\langle c_1, c_2\rangle\rangle.
\end{aligned}$$

Next, we consider the general case $\varphi_1\varphi_2 = g\,f\psi_i f^{-1}\,\psi_j\,g^{-1}$ for $g \in \mathcal{M}_{g,1}$. The pairing $\langle\langle \cdot, \cdot \rangle\rangle$ is $\mathcal{M}_{g,1}$-equivariant by [S4, Lemma 4.3], that is,

$$\langle\langle g(c_1), g(c_2) \rangle\rangle = g(\langle\langle c_1, c_2 \rangle\rangle).$$

Moreover, we see from [S3, Proposition 3.2] that

$$\mathrm{tr}(r_1(g\varphi_1\varphi_2 g^{-1})) = g(\mathrm{tr}(r_1(\varphi_1\varphi_2))).$$

This means that $\mathrm{tr}(r_1(\cdot))$ is also $\mathcal{M}_{g,1}$-equivariant. Therefore this completes the proof. ∎

The Dehn twist along a bounding simple closed curve is called a BSCC map. From our previous paper [S3], it is known that any BSCC map φ does not lie in the kernel of r_1, and the characteristic polynomial of the Magnus matrix of φ is trivial:

$$\det(\lambda I_{2g} - r_1(\varphi)) = (\lambda - 1)^{2g}.$$

It follows that $\mathcal{K}_{g,1}$ is not contained in the kernel of r_1, where $\mathcal{K}_{g,1}$ denotes the subgroup generated by the BSCC maps. We remark that the characteristic polynomial of the Magnus matrix on $\mathcal{K}_{g,1}$ is not always trivial (see [S4] for details).

Theorem 4.3 gives a characterization of the kernel of r_1 for the commutator of two BSCC maps.

Corollary 4.4 *The commutator of two BSCC maps φ_1, φ_2 belongs to the kernel of r_1 if and only if the characteristic polynomial of the Magnus matrix of the product $\varphi_1\varphi_2$ is trivial. Here the Magnus matrix means the image of r_1 for a mapping class.*

Proof. In general, if the characteristic polynomials of two matrices A, B are trivial and A commutes with B, then the characteristic polynomial of AB is also trivial.

Suppose that the commutator of two BSCC maps φ_1, φ_2 belongs to the kernel of r_1, that is, $r_1(\varphi_1)$ commutes with $r_1(\varphi_2)$. Because the characteristic polynomial of the Magnus matrix for any BSCC map is trivial, we get

$$\det(\lambda I_{2g} - r_1(\varphi_1\varphi_2)) = (\lambda - 1)^{2g}.$$

Conversely, suppose that the characteristic polynomial is trivial. Then we have

$$-\mathrm{tr}(r_1(\varphi_1\varphi_2)) = -2g.$$

By Theorem 4.3, we conclude that $\langle\langle c_1, c_2\rangle\rangle = 0$. This means $\langle c_1, c_2\rangle_H = 0$ or $\langle c_2, c_1\rangle_H = 0$, because $\mathbb{Z}[H]$ is an integral domain. In virtue of Proposition 3.2, $\langle c_1, c_2\rangle_H = 0$ gives $[\varphi_1, \varphi_2] \in \ker r_1$ and $\langle c_2, c_1\rangle_H = 0$ gives $[\varphi_2, \varphi_1] \in \ker r_1$. This completes the proof. ∎

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References

[Big] S. Bigelow, *The Burau representation is not faithful for* $n = 5$, Geom. Topol. Vol.3 (1999) 397-404

[B-B] S. Bigelow and R. Budney, *The mapping class group of a genus two surface is linear*, Alg. Geom. Topol. Vol.1 (2001) 699-708

[Bir] J. Birman, *Braids, Links and Mapping Class Groups*, Annals of Math. Studies No. 82, Princeton University Press, 1975

[F] R. H. Fox, *Free Differential Calculus I*, Annals of Math. 57 (1953), 547-560

[J] D. Johnson, *A survey of the Torelli group*, Contemporary Math. 20 (1983) 165-179

[K] M. Korkmaz, *On the linearity of certain mapping class groups*, Turkish J. Math. 24 (2000) 367-371

[M1] S. Morita, *Abelian quotients of subgroups of the mapping class group of surfaces*, Duke Math. J. 70 (1993), 699-726

[M2] S. Morita, *Structure of the mapping class groups of surfaces: a survey and a prospect*, Geom. Topol. Monogr. Vol.2: Proceedings of the Kirbyfest (1999) 349-406

[S1] M. Suzuki, *The Magnus representation of the Torelli group $\mathcal{I}_{g,1}$ is not faithful for $g \geq 2$*, Proc. Amer. Math. Soc. 130 (2002) 909-914

[S2] M. Suzuki, *Irreducible decomposition of the Magnus representation of the Torelli group*, Bull. Austral. Math. Soc. 67 (2003) 1-14

[S3] M. Suzuki, *A class function on the Torelli group*, Kodai Math. J. 26 (2003) 304-316

[S4] M. Suzuki, *A geometric interpretation of the Magnus representation of the mapping class group*, preprint

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