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New obstructions for the surjectivity of the Johnson homomorphism of the automorphism group of a free group

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NEW OBSTRUCTIONS FOR THE SURJECTIVITY OF THE JOHNSON HOMOMORPHISM OF THE AUTOMORPHISM GROUP OF A FREE GROUP

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ABSTRACT. In this paper we construct new obstructions for the surjectivity of the Johnson homomorphism of the automorphism group of a free group. We also determine the structure of the cokernel of the Johnson homomorphism for degree 2 or 3.

1. Introduction

Let F_n be a free group of rank $n \geq 2$ and $F_n = \Gamma_n(1), \Gamma_n(2), \ldots$ its lower central series. We denote by $\mathrm{Aut}\, F_n$ the group of automorphisms of F_n. For each $k \geq 0$, let $\mathcal{A}_n(k)$ be the group of automorphisms of F_n which induce the identity on the quotient group $F_n/\Gamma_n(k+1)$. Then we have a descending filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$. This filtration is introduced in 1963 with a remakable pioneer work by S. Andreadakis [1] who showed that $\mathcal{A}_n(1), \mathcal{A}_n(2), \ldots$ is a descending central series of $\mathcal{A}_n(1)$ and each graded quotient $\mathrm{gr}^k(\mathcal{A}_n) = \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank. He [1] also computed that $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^k(\mathcal{A}_2)$ for all $k \geq 1$ and $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^2(\mathcal{A}_3)$, and asserted $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^3(\mathcal{A}_3) = 44$. But in Section 5, we show that $\mathrm{gr}^3(\mathcal{A}_3) = 43$. Moreover, by a recent remakable work by A. Pettet [16] we have $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^2(\mathcal{A}_n) = \frac{1}{3}n^2(n^2-4) + \frac{1}{2}n(n-1)$ for all $n \geq 3$. However, it is difficult to compute the rank of $\mathrm{gr}^k(\mathcal{A}_n)$.

Let H be the abelianization of F_n and $H^* = \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ the dual group of H. Let $\mathcal{L}_n = \bigoplus_{k \geq 1} \mathcal{L}_n(k)$ be the free graded Lie algebra generated by H. Then for each $k \geq 1$, a $GL(n, \mathbf{Z})$-equivariant injective homomorphim

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

is defined. (For definition, see Section 2.) This is called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$. The theory of the Johnson homomorphism of a mapping class group of a compact Riemann surface began in 1980 by D. Johnson [7] and has been developed by many authors. There are many remarkable and variable results for the Johnson homomorphism of a mapping class group. (For example, see [6] and [14].) However, the properties of the Johnson homomorphism of $\mathrm{Aut}\, F_n$ are far from being well understood.

Our main interest of this paper is to determine the structure of the cokernel of the Johnson homomorphism τ_k as a $GL(n, \mathbf{Z})$-module. For $k = 1$, there is well known fact that the first Johnson homomorphism τ_1 is an isomorphism. (See [9].) For $k \geq 2$, the Johnson homomorphism τ_k is not surjective. In fact, A recent remarkable work by Shigeyuki Morita indicates that there is a symmetric product

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$S^kH_{\mathbf{Q}}$ of $H_{\mathbf{Q}} = H \otimes_{\mathbf{Z}} \mathbf{Q}$ in the cokernel of $\tau_{k,\mathbf{Q}} = \tau_k \otimes id_{\mathbf{Q}}$ for each $k \geq 2$. To show this, he introduced a homomorphism

$$\mathrm{Tr}_k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to S^k H,$$

called the trace map, and showed that Tr_k vanishes on the image of τ_k and is surjective after tensoring with $\mathbf{Q}$ for all $k \geq 2$.

The trace maps are introduced in the 1993 with almost simultaneous work by Morita [13] for a Johnson homomorphism of a mapping class group of a surface. He called these maps traces because they were defined using the trace of some matrix representation. Morita's traces are very important to study the Lie algebra structure of the target $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n = \mathrm{Der}(\mathcal{L}_n)$ of the Johnson homomorphisms. Here $\mathrm{Der}(\mathcal{L}_n)$ denotes the graded Lie algebra of derivations of $\mathcal{L}_n$. Morita conjectured that for any $n \geq 3$, the abelianization of the Lie algebra $\mathrm{Der}(\mathcal{L}_n)$ is given by

$$H_1(\mathrm{Der}(\mathcal{L}_n^{\mathbf{Q}})) \simeq (H_{\mathbf{Q}}^* \otimes_{\mathbf{Z}} \Lambda^2 H_{\mathbf{Q}}) \oplus (\bigoplus_{k\geq 2}^{\infty} S^k H_{\mathbf{Q}})$$

where $\mathcal{L}_n^{\mathbf{Q}} = \mathcal{L}_n \otimes_{\mathbf{Z}} \mathbf{Q}$ and the right hand side is understood to be an abelian Lie algebra. Recently, combining a work of Kassabov [8] with the concept of the traces, he [15] showed that the isomorphism above holds up to degree $n(n-1)$.

The subgroup $\mathcal{A}_n(1)$ is called the IA-automorphism group of F_n and denoted by IA_n. The group IA_n is the kernel of the natural map $\mathrm{Aut}\, F_n \to GL(n, \mathbf{Z})$ which is given by the action of $\mathrm{Aut}\, F_n$ on H. To study the structure of IA_n plays very important roles in the study of that of $\mathrm{Aut}\, F_n$. W. Magnus [11] showed that IA_n is finitely generated for all $n \geq 3$. However, it is not known whether IA_n is finitely presented or not for any $n \geq 4$. For $n = 3$, by a remakable work by S. Krstić and J. McCool [10], it is known that IA_3 is not finitely presented. On the other hand, the abelianization of IA_n is given by

$$IA_n^{\mathrm{ab}} \simeq H^* \otimes_{\mathbf{Z}} \Lambda^2 H$$

as a $GL(n, \mathbf{Z})$-module. (See [9].)

Now let $\mathcal{A}'_n(1), \mathcal{A}'_n(2), \ldots$ be the lower central series of $IA_n = \mathcal{A}_n(1)$ and $\mathrm{gr}^k(\mathcal{A}'_n)$ the graded quotient of it for each $k \geq 1$. In Section 2, we define a $GL(n, \mathbf{Z})$-equivariant homomorphism

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

which is also called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$. It is conjectured that $\mathrm{Coker}\, \tau'_k = \mathrm{Coker}\, \tau_k$ for $k \geq 1$. It is true for $1 \leq k \leq 3$. In fact, $\mathcal{A}_n(1) = \mathcal{A}'_n(1)$ by definition. We have $\mathcal{A}_n(2) = \mathcal{A}'_n(2)$ from the result stated above. (See [9].) Moreover, Pettet [16] showed $\mathcal{A}_n(3) = \mathcal{A}'_n(3)$. Hence, $\mathrm{Coker}\, \tau'_k = \mathrm{Coker}\, \tau_k$ for $1 \leq k \leq 3$.

In this paper, we construct new obstructions of the surjectivity of the Johnson homomorphism τ'_k. Let us denote the tensor products with $\mathbf{Q}$ of a $\mathbf{Z}$-module by attaching a subscript $\mathbf{Q}$ to the original one. For example, $H_{\mathbf{Q}} = H \otimes_{\mathbf{Z}} \mathbf{Q}$, $\mathcal{L}_n^{\mathbf{Q}}(k) = \mathcal{L}_n(k+1) \otimes_{\mathbf{Z}} \mathbf{Q}$. Similarly, for a $\mathbf{Z}$-linear map $f : A \to B$ we denote by $f_{\mathbf{Q}}$ the $\mathbf{Q}$-linear map $A_{\mathbf{Q}} \to B_{\mathbf{Q}}$ induced by f. Our main result is

Theorem 1.

(1) $\Lambda^k H_{\mathbf{Q}} \subset \mathrm{Coker}\, \tau'_{k,\mathbf{Q}}$ *for odd* k *and* $3 \leq k \leq n$.

(2) $H_{\mathbf{Q}}^{[2,1^{k-2}]} \subset \mathrm{Coker}\, \tau'_{k,\mathbf{Q}}$ *for even* k *and* $4 \leq k \leq n-1$.

Here $\Lambda^k H_{\mathbf{Q}}$ *denotes the* k*-th exterior product of* $H_{\mathbf{Q}}$*, and* $H_{\mathbf{Q}}^{[2,1^{k-2}]}$ *denotes the Schur-Weyl module of* $H_{\mathbf{Q}}$ *corresponding to the partition* $[2, 1^{k-2}]$.

In order to prove this, in Section 3, we introduce homomorphisms defined by

$$\mathrm{Tr}_{[1^k]} := f_{[1^k]} \circ \Phi_1^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to \Lambda^k H,$$

$$\mathrm{Tr}_{[2,1^{k-2}]} := (id_H \otimes f_{[1^{k-1}]}) \circ \Phi_2^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H \otimes_{\mathbf{Z}} \Lambda^{k-1} H$$

and show that these maps vanish on the image of the Johnson homomorphism τ_k'. Since these maps are constructed in a way similar to that of Morita's trace Tr_k, we also call these maps traces.

In Section 5, we determine the $GL(n, \mathbf{Z})$-module structure of the cokernel of the Johnson homomorphism τ_k for 2 and 3. Our result is

Theorem 2. *We have $GL(n, \mathbf{Z})$-equivariant exact sequences*

$$0 \to \mathrm{gr}^2(\mathcal{A}_n) \xrightarrow{\tau_2} H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3) \to S^2 H \to 0$$

and

$$0 \to \mathrm{gr}^3_{\mathbf{Q}}(\mathcal{A}_n) \xrightarrow{\tau_{3,\mathbf{Q}}} H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_n^{\mathbf{Q}}(4) \to S^3 H_{\mathbf{Q}} \oplus \Lambda^3 H_{\mathbf{Q}} \to 0$$

for $n \geq 3$.

Thus we have

Corollary 1. *For $n \geq 3$,*

$$\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^3(\mathcal{A}_n) = \frac{1}{12} n(3n^4 - 7n^2 - 8).$$

Contents

2. **Preliminaries**

In this section we review some basic facts. First, we note that the group $\mathrm{Aut}\, F_n$ acts on F_n on the right. For any $\sigma \in \mathrm{Aut}\, F_n$ and $x \in F_n$, the action of σ on x is denoted by x^σ.

2.1. **Commutators of higher weight.**

In this paper, we often use basic facts of commutator calculus. The reader is reffered to [12] and [3], for example. Let G be a group. For any elements x and y of G, the element

$$xyx^{-1}y^{-1}$$

is called the commutator of x and y, and denoted by $[x,y]$. In general, a commutator of higher weight is recursively defined as follows. First, a commutator of weight 1 is an element of G. For $k>1$, a commutator of weight k is an element of the type $C=[C_1,C_2]$ where C_j is a commutator of weight a_j $(j=1,2)$ such that $a_1+a_2=k$. The weight of the commutator C is denoted by $\mathrm{wt}\,(C)=k$. The commutator which has elements $g_1,\ldots,g_t\in G$ in the bracket components is called the commutator in the components $g_1,\ldots,g_t$. For elements $g_1,\ldots,g_t\in G$, a commutator of weight k in the components $g_1,\ldots,g_t$ of the type

$$[[\cdots[[g_{i_1},g_{i_2}],g_{i_3}],\cdots],g_{i_k}], \quad i_j\in\{1,\ldots,t\}$$

with all of its brackets to the left of all the elements occuring is called a simple k-fold commutator and is denoted by

$$[g_{i_1},g_{i_2},\cdots,g_{i_k}].$$

For each $k\geq 1$, the subgroups $\Gamma_G(k)$ of the lower central series of G are defined recursively by

$$\Gamma_G(1)=G, \quad \Gamma_G(k+1)=[\Gamma_G(k),G].$$

We use the following basic lemma in later sections.

Lemma 2.1. *If a group G is generated by $g_1,\ldots,g_t$, then each of the graded quotients $\Gamma_G(k)/\Gamma_G(k+1)$ for $k\geq 1$ is generated by the cosets of the simple k-fold commutators*

$$[g_{i_1},g_{i_2},\ldots,g_{i_k}], \quad i_j\in\{1,\ldots,t\}.$$

Now, for each $k\geq 1$, let $\Gamma_n(k)$ be the k-th subgroup $\Gamma_{F_n}(k)$ of the lower central series of a free group F_n of rank n and $\mathrm{gr}^k(\Gamma_n)$ its graded quotient $\Gamma_n(k)/\Gamma_n(k+1)$. We denote by $\mathrm{gr}\,(\Gamma_n)=\bigoplus_{k\geq 1}\mathrm{gr}^k(\Gamma_n)$ the associated graded sum. Then the set $\mathrm{gr}\,(\Gamma_n)$ naturally has a structure of a graded Lie algebra over $\mathbf{Z}$ induced from the commtator bracket on F_n. Let H be the abelianization of F_n and $\mathcal{L}_n=\bigoplus_{k\geq 1}\mathcal{L}_n(k)$ the free graded Lie algebla generated by H. It is well known that the Lie algebra $\mathrm{gr}\,(\Gamma_n)$ is isomorphic to $\mathcal{L}_n$ as a graded Lie algebra over $\mathbf{Z}$. Thus, in this paper, we identify $\mathrm{gr}\,(\Gamma_n)$ with $\mathcal{L}_n$. For any element $x\in\Gamma_n(k)$, we also denote by x the coset class of x in $\mathcal{L}_n(k)=\Gamma_n(k)/\Gamma_n(k+1)$. Let $T(H)$ be the tensor algebra of H over $\mathbf{Z}$. Then the algebra $T(H)$ is the universal envelopping algebra of the free Lie algebra $\mathcal{L}_n$ and the natural map $\mathcal{L}_n\to T(H)$ defined by

$$[X,Y]\mapsto X\otimes Y-Y\otimes X$$

for X, $Y\in\mathcal{L}_n$ is an injective Lie algebra homomorphism. Hence we also regard $\mathcal{L}_n(k)$ as a submodule of $H^{\otimes k}$ for each $k\geq 1$.

2.2. **IA-automorphism group.**

The kernel of the natural map $\mathrm{Aut}\,F_n\to GL(n,\mathbf{Z})$ which is given by the action of $\mathrm{Aut}\,F_n$ on H is called the IA-automorphism group of F_n and denoted by IA_n. Let $\{x_1,\ldots,x_n\}$ be a basis of a free group F_n. Magnus [11] showed that IA_n is finitely generated by automorphisms

$$K_{ab}:\begin{cases} x_a & \mapsto x_b^{-1}x_ax_b, \\ x_t & \mapsto x_t, \quad (t\neq a) \end{cases}$$

and

$$K_{abc} : \begin{cases} x_a & \mapsto x_a x_b x_c {x_b}^{-1} {x_c}^{-1}, \\ x_t & \mapsto x_t, \quad (t \neq a) \end{cases}$$

for any distinct a, b and $c \in \{1, 2, \ldots, n\}$. It is known that the abelianization IA_n^{ab} of the IA-automorphism group is free abelian group with generators K_{ab} for distinct a and b, and K_{abc} for distinct a, b, c and $b < c$. More precisely, if we denote by $H^* = \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ the dual group of H, we have a $GL(n, \mathbf{Z})$-module isomorphism $IA_n^{\mathrm{ab}} \simeq H^* \otimes_{\mathbf{Z}} \Lambda^2 H$. (For details, see [9].)

2.3. The associated graded Lie algebra.

Here we consider two descending filtrations of IA_n. The first one is $\{\mathcal{A}_n(k)\}_{k\geq 1}$ defined as above. Since the series $\mathcal{A}_n(1), \mathcal{A}_n(2), \ldots$ is central, the associated graded sum $\mathrm{gr}(\mathcal{A}_n) = \bigoplus_{k\geq 1} \mathrm{gr}^k(\mathcal{A}_n)$ naturally has a structure of a graded Lie algebla over $\mathbf{Z}$ induced from the commutator bracket on $\mathcal{A}_n(1)$. For each $k \geq 1$, the group $\mathcal{A}_n(0) = \mathrm{Aut}\, F_n$ naturally acts on $\mathcal{A}_n(k)$ by conjugation, hence on $\mathrm{gr}^k(\mathcal{A}_n)$. Since the group $\mathcal{A}_n(1) = IA_n$ trivially acts on $\mathrm{gr}^k(\mathcal{A}_n)$, we see that the group $GL(n, \mathbf{Z}) \simeq \mathcal{A}_n(0)/\mathcal{A}_n(1)$ naturally acts on $\mathrm{gr}^k(\mathcal{A}_n)$.

The other is a usual lower central series $\mathcal{A}'_n(1)$, $\mathcal{A}'_n(2)$, $\ldots$ of $IA_n(1)$. Let $\mathrm{gr}^k(\mathcal{A}'_n) = \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$ be the graded quotient for each $k \geq 1$. Similarly the associated graded sum $\mathrm{gr}(\mathcal{A}'_n) = \bigoplus_{k\geq 1} \mathrm{gr}^k(\mathcal{A}'_n)$ has a structure of a graded Lie algebra structure on $\mathbf{Z}$. Moreover, each graded quotient $\mathrm{gr}^k(\mathcal{A}'_n)$ is a $GL(n, \mathbf{Z})$-module. We remark that $\mathcal{A}_n(k) = \mathcal{A}'_n(k)$ for $1 \leq k \leq 3$ as mentioned in Section 1. From Lemma 2.1, for each $k \geq 1$, the graded quotient $\mathrm{gr}^k(\mathcal{A}'_n)$ is generated by (the cosets of) the simple k-fold commutators in the components K_{ab} and K_{abc}.

2.4. Johnson homomorphism.

Here we define the Johnson homomorphism of $\mathrm{Aut}\, F_n$. For each $k \geq 1$, let $\tau_k : \mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ be the map defined by

$$\sigma \mapsto (x \mapsto x^{-1} x^{\sigma}) \tag{1}$$

for $\sigma \in \mathcal{A}_n(k)$ and $x \in H$. Then the map τ_k is a $GL(n, \mathbf{Z})$-equivariant homomorphism and the kernel of τ_k is just $\mathcal{A}_n(k+1)$. Hence, identifying $\mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ with $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$, we obtain an injective homomorphism, also denoted by τ_k,

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

This homomorphism is called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$. Similarly, for each $k \geq 1$, we can define a $GL(n, \mathbf{Z})$-equivariant homomorphism $\tau'_k : \mathcal{A}'_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ as (1). Since $\mathcal{A}'_n(k+1)$ is contained in the kernel of τ'_k, we obtain a homomorphism, also denoted by τ'_k,

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

We also call the map τ'_k the Johnson homomorphism of $\mathrm{Aut}\, F_n$.

Let $\{x_1, \ldots, x_n\}$ be a basis of F_n. It defines a basis of H as a free abelian group, also denoted by $\{x_1, \ldots, x_n\}$. Let $\{x_1^*, \ldots, x_n^*\}$ be the dual basis of H^*. For any $\sigma \in \mathrm{A}'_n(k)$, if we set $s_i(\sigma) := x_i^{-1} x_i^{\sigma} \in \mathcal{L}_n(k+1)$ $(1 \leq i \leq n)$ then we have

$$\tau_k(\sigma) = \tau'_k(\sigma) = \sum_{i=1}^{n} x_i^* \otimes s_i(\sigma) \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

Let $\mathrm{Der}\,(\mathcal{L}_n)$ be the graded Lie algebra of derivations of $\mathcal{L}_n$. The degree k part of $\mathrm{Der}\,(\mathcal{L}_n)$ is expressed as $\mathrm{Der}\,(\mathcal{L}_n)(k) = H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k)$. Thus we sometimes identify $\mathrm{Der}\,(\mathcal{L}_n)$ with $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n$. Then the Johnson homomorphism $\tau = \bigoplus_{k\geq 1} \tau_k$ is a graded Lie algebra homomorphism. In fact, if we denote by $\partial\sigma$ the element of

Der $(\mathcal{L}_n)$ corresponding to an element $\sigma \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n$ and write the action of $\partial\sigma$ on $X \in \mathcal{L}_n$ as $X^{\partial\sigma}$ then we have

$$\tau'_{k+l}([\sigma, \tau]) = \sum_{i=1}^{n} x_i^* \otimes (s_i(\sigma)^{\partial\tau} - s_i(\tau)^{\partial\sigma}). \tag{2}$$

for any $\sigma \in \mathrm{A}'_n(k)$ and $\tau \in \mathrm{A}'_n(l)$.

3. **The contractions**

For $k \geq 1$ and $1 \leq l \leq k+1$, let $\varphi_l^k : H^* \otimes_{\mathbf{Z}} H^{\otimes(k+1)} \to H^{\otimes k}$ be the contraction map defined by

$$x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto x_i^*(x_{j_l}) \cdot x_{j_1} \otimes \cdots \otimes x_{j_{l-1}} \otimes x_{j_{l+1}} \otimes \cdots \otimes x_{j_{k+1}}.$$

For the natural embedding $\iota_n^{k+1} : \mathcal{L}_n(k+1) \to H^{\otimes(k+1)}$, we obtain a $GL(n, \mathbf{Z})$-equivariant homomorphism

$$\Phi_l^k = \varphi_l^k \circ (id_{H^*} \otimes \iota_n^{k+1}) : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^{\otimes k}.$$

We also call the map Φ_l^k contraction.

Here we introduce one of methods of the computation of $\Phi_l^k(x_i^* \otimes C)$ for a commutator $C \in \mathcal{L}_n(k+1)$ in the components $x_1, \ldots, x_n$. In this paper, whenever we compute $\Phi_l^k(x_i^* \otimes C)$, we use the following method. First, if x_i does not appear in the components of C, then $\Phi_l^k(x_i^* \otimes C) = 0$. On the other hand, if x_i appears in the components of C m times, then we distinguish them and write such x_i's as $x_{i_1}, \ldots, x_{i_m}$ in C. Then $\Phi_l^k(x_i^* \otimes C)$ is given by rewriting $x_{i_1}, \ldots, x_{i_m}$ as x_i in

$$\sum_{j=1}^{m} \Phi_l^k(x_{i_j}^* \otimes C).$$

Thus it suffices to compute $\Phi_l^k(x_i^* \otimes C)$ for a commutator C which has only one x_i in its components. Now, C is written as $[X, Y]$ for some commutators X and Y. Rewriting the commutator C as $-[Y, X]$ if x_i appears in Y, we may always consider $C = \pm[X, Y]$ such that x_i appears in the components of X. By a recursive argument, we have $C = \pm[x_i, C_1, \ldots, C_t]$ where each C_j $(1 \leq j \leq t)$ is a commutator of weight d_j and $d_1 + \cdots + d_t = k$.

Lemma 3.1. *For a commutator* $[x_i, C_1, \ldots, C_t] \in \mathcal{L}_n(k+1)$ *as above,*

$$\Phi_1^k(x_i^* \otimes [x_i, C_1, \ldots, C_t]) = C_1 \otimes \cdots \otimes C_t.$$

Proof Since C_t does not have x_i in the components, we have

$$\begin{aligned}\Phi_1^k(x_i^* \otimes [x_i, C_1, \ldots, C_t]) &= \Phi_1^k(x_i^* \otimes [x_i, C_1, \ldots, C_{t-1}] \otimes C_t) \\ &\qquad - \Phi_1^k(x_i^* \otimes C_t \otimes [x_i, C_1, \ldots, C_{t-1}]), \\ &= \Phi_1^k(x_i^* \otimes [x_i, C_1, \ldots, C_{t-1}] \otimes C_t).\end{aligned}$$

Thus by a recursive argument, we have

$$\Phi_1^k(x_i^* \otimes [x_i, C_1, \ldots, C_t]) = C_1 \otimes \cdots \otimes C_t. \quad \square$$

Lemma 3.2. *For a commutator* $[x_i, C_1, \ldots, C_t] \in \mathcal{L}_n(k+1)$ *as above,*

$$\begin{aligned}&\Phi_2^k(x_i^* \otimes [x_i, C_1, \ldots, C_t]) \\ &\qquad = -\sum_{\mathrm{wt}\,(C_j)=1} C_j \otimes C_1 \otimes \cdots \otimes C_{j-1} \otimes C_{j+1} \otimes \cdots \otimes C_t.\end{aligned}$$

Proof In

$$\Phi_2^k(x_i^* \otimes [x_i, C_1, \ldots, C_t]) = \Phi_2^k(x_i^* \otimes [x_i, C_1, \ldots, C_{t-1}] \otimes C_t) \\ - \Phi_2^k(x_i^* \otimes C_t \otimes [x_i, C_1, \ldots, C_{t-1}]),$$

if $\mathrm{wt}\,(C_t) \geq 2$, the last term of the right hand side is equal to zero. On the other hand, if $\mathrm{wt}\,(C_t) = 1$, it is equal to $-C_t \otimes C_1 \otimes \cdots \otimes C_{t-1}$ from Lemma 3.1. Thus, by a recursive argument, we have Lemma 3.2. □

Let $T(H) = \bigoplus_{k\geq 1} H^{\otimes k}$ and $S(H) = \bigoplus_{k\geq 1} S^k H$ be the tensor algebra and the symmetric algebra on H respectively. Then the kernel of a natural map $T(H) \to S(H)$ is a graded ideal of $T(H)$, and denoted by $I(H) = \bigoplus_{k\geq 1} I^k(H)$. For each $k \geq 2$, let $\mathcal{U}_n(k)$ be the $GL(n, \mathbf{Z})$-submodule of $H^{\otimes k}$ generated by elements type of

$$[A, B] := A \otimes B - B \otimes A$$

for $A \in H^{\otimes a}$, $B \in H^{\otimes b}$ and $a + b = k$. If we put $\mathcal{U}_n = \bigoplus_{k\geq 1} \mathcal{U}_n(k)$, then $\mathcal{U}_n$ is the kernel of the abelianizaton $T(H) \to T(H)^{\mathrm{ab}}$ as a Lie algebra. We have

$$\mathcal{L}_n(k) \subset \mathcal{U}_n(k) \subset I^k(H) \subset H^{\otimes k}.$$

3.1. The image of $\Phi_1^k \circ \tau_k'$.

Here we prove

Proposition 3.1. *For $n \geq 3$ and $k \geq 2$,* $\mathrm{Im}\,(\Phi_1^k \circ \tau_k') \subset \mathcal{U}_n(k)$.

It suffices to check that the image of any simple k-fold commutator σ in the components K_{ab} and K_{abc} is in $\mathcal{U}_n(k)$. We have

$$\tau_k'(\sigma) = \sum_{i=1}^{n} x_i^* \otimes s_i(\sigma).$$

In general, each $s_i(\sigma) \in \mathcal{L}_n(k+1)$ can not be uniquely written as a sum of commutators in the components $x_1, \ldots, x_n$. In this paper, each $s_i(\sigma)$ is recursively computed in the following way. First, for $\sigma = K_{abc}$, we can set

$$s_a(K_{abc}) = [x_b, x_c], \quad s_t(K_{abc}) = 0 \quad \text{if} \quad t \neq a.$$

For $\sigma = K_{ab}$, we see that

$$x_t^{-1} x_t^{\sigma} = \begin{cases} [x_a^{-1}, x_b^{-1}] & \text{if} \quad t = a, \\ 1 & \text{if} \quad t \neq a \end{cases}$$

in F_n. Since $[x_a^{-1}, x_b^{-1}] = [x_a, x_b]$ in $\mathcal{L}_n(2)$, so we can set

$$s_a(K_{ab}) = [x_a, x_b], \quad s_t(K_{ab}) = 0 \quad \text{if} \quad t \neq a.$$

Next, if $\sigma = [\tau, K_{ab}]$ for k-fold simple commutator τ, following from (2), we can set

$$s_i(\sigma) = s_i(\tau)^{\partial K_{ab}} - s_i(K_{ab})^{\partial \tau}$$

for each i. Furthermore, since a commutator bracket of weight l is considered as a l-fold multilinear map from the cartesian product of l copies of $\mathcal{L}_n(1)$ to $\mathcal{L}_n(l)$, we can also set

$$s_i(\sigma) = \sum_{p=1}^{\alpha(i)} (-1)^{e_{i,p}} C_{i,p}$$

where $e_{i,p} = 0$ or 1, and $C_{i,p}$ is a commutator of degree $k+1$ in the components $x_1, \ldots, x_n$. Similarly, we can set $s_i([\tau, K_{abc}])$ for $\sigma = [\tau, K_{abc}]$. Here we show the computation of $\tau_k'(\sigma)$ for some $\sigma \in \mathcal{A}_n'(k)/\mathcal{A}_n'(k+1)$ for example. For distinct a, b, c and d, we have

$$\begin{aligned}\tau_2'([K_{ab},K_{bac}]) &= x_a^* \otimes ([x_a,x_b])^{\partial K_{bac}} - x_b^* \otimes ([x_a,x_c])^{\partial K_{ab}},\\ &= x_a^* \otimes [x_a,[x_a,x_c]] - x_b^* \otimes [[x_a,x_b],x_c]\end{aligned}$$

and

$$\begin{aligned}\tau_3'([K_{ab},K_{bac},K_{ad}]) &= x_a^* \otimes ([x_a,[x_a,x_c]])^{\partial K_{ad}} - x_b^* \otimes ([[x_a,x_b],x_c])^{\partial K_{ad}}\\ &\quad - x_a^* \otimes ([x_a,x_d])^{\partial [K_{ab},K_{bac}]},\\ &= x_a^* \otimes [[x_a,x_d],[x_a,x_c]] + x_a^* \otimes [x_a,[[x_a,x_d],x_c]]\\ &\quad - x_b^* \otimes [[[x_a,x_d],x_b],x_c]\\ &\quad - x_a^* \otimes [[x_a,[x_a,x_c]],x_d].\end{aligned}$$

Now, for the convenience, for every $t \in \{1,\ldots,n\}$, if each $C_{i,p}$ has x_t in its components $\beta(i,p,t)$ times, we distinguish them and write such x_t's as $x_{t_1},\ldots,x_{t_{\beta(i,p,t)}}$ in $C_{i,p}$. We denote by $\bar{C}_{i,p}$ the element $C_{i,p}$ whose componets are distinguished as above. If we denote by $\Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural$ the element of $H^{\otimes k}$ which is given by rewriting $x_{t_1},\ldots,x_{t_{\beta(i,p,t)}}$ as x_t in $\Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p})$ for all t, then we have

$$\Phi_l^k \circ \tau_k'(\sigma) = \sum_{i=1}^{n}\sum_{p=1}^{\alpha(i)}(-1)^{e_{i,p}}\sum_{q=1}^{\beta(i,p,i)}\Phi_l^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural. \tag{3}$$

Then Proposition 3.1 follows from

Lemma 3.3. *Let k be an integer greater than 1. According to the notation as above, for each i, p and q, we have*

(i) $\Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural = 0$,
(ii) $\Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural = X$*; a commutator of weight k in $\mathcal{L}_n(k)$*
or
(iii) *There exist some j, p' and q' such that $(j,p',q') \neq (i,p,q)$,*

$$\begin{aligned}(-1)^{e_{i,p}}\Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural &= \pm A \otimes B,\\ (-1)^{e_{j,p'}}\Phi_1^k(x_{j_{q'}}^* \otimes \bar{C}_{j,p'})_\natural &= \mp B \otimes A\end{aligned}$$

where $A \in H^{\otimes \mu}$, $B \in H^{\otimes \nu}$ and $\mu + \nu = k$.

Proof We use induction on k. For $k=2$, the result follows. In fact, let us consider $\sigma = [K_{ab},K_{bac}]$ for example. Then we have

$$\begin{aligned}\Phi_1^2 \circ \tau_2'(\sigma) &= \Phi_1^2(x_a^* \otimes [x_a,[x_a,x_c]]) - \Phi_1^2(x_b^* \otimes [[x_a,x_b],x_c]),\\ &= \Phi_1^2(x_{a_1}^* \otimes [x_{a_1},[x_{a_2},x_c]])_\natural + \Phi_1^2(x_{a_2}^* \otimes [x_{a_1},[x_{a_2},x_c]])_\natural\\ &\quad - \Phi_1^2(x_b^* \otimes [[x_a,x_b],x_c])_\natural,\\ &= [x_a,x_c] - x_c \otimes x_a + x_a \otimes x_c.\end{aligned}$$

Hence we obtain the required result in this case. Similarly we can check for the other simple 2-fold commutators in the components K_{ab} and K_{abc}. The computations are left to the reader for exercises. Assume $k \geq 3$ and the result follows for $k-1$. Let σ be a simple $(k-1)$-fold commutator in the components K_{ab} and K_{abc}. First, for $\tau = K_{ab}$ we consider $[\sigma,\tau]$. Then set

$$\tau_{k-1}'(\sigma) = \sum_{i=1}^{n}\sum_{p=1}^{\alpha(i)} x_i^* \otimes (-1)^{e_{i,p}} C_{i,p}.$$

Here we also set $\tau_1'(\tau) = x_a^* \otimes [x_{a'}, x_{b'}]$ and distinguish a' and b' from any a and b which appear in $C_{i,p}$ for any i and p respectively. In general, for any $l \in \{1, \dots, n\}$, we have

$$
\begin{aligned}
\Phi_l^k \circ \tau_k'([\sigma,\tau]) = &\sum_{i=1}^{n}\sum_{p=1}^{\alpha(i)}(-1)^{e_{i,p}} \sum_{q=1}^{\beta(i,p,i)} \sum_{r=1}^{\beta(i,p,a)} \Phi_l^k(x_{i_q}^* \otimes \bar{C}_{i,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural \\
&+ \sum_{p=1}^{\alpha(a)}(-1)^{e_{a,p}} \sum_{r=1}^{\beta(a,p,a)} \Phi_l^k(x_{a'}^* \otimes \bar{C}_{a,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural \\
&+ \sum_{p=1}^{\alpha(b)}(-1)^{e_{b,p}} \sum_{r=1}^{\beta(b,p,a)} \Phi_l^k(x_{b'}^* \otimes \bar{C}_{b,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural \\
&- \sum_{p=1}^{\alpha(a)}(-1)^{e_{a,p}} \sum_{r=1}^{\beta(a,p,a)} \Phi_l^k(x_{a_r}^* \otimes [\bar{C}_{a,p}, x_{b'}])_\natural \\
&- \sum_{p=1}^{\alpha(b)}(-1)^{e_{b,p}} \sum_{r=1}^{\beta(b,p,a)} \Phi_l^k(x_{a_r}^* \otimes [x_{a'}, \bar{C}_{b,p}])_\natural \\
&- \sum_{p=1}^{\alpha(b)}(-1)^{e_{b,p}} \Phi_l^k(x_{a'}^* \otimes [x_{a'}, \bar{C}_{b,p}])_\natural
\end{aligned}
\tag{4}
$$

Here we consider the case where $l = 1$. First we consider each term of the last sum. Since

$$\Phi_1^k(x_{a'}^* \otimes [x_{a'}, \bar{C}_{b,p}])_\natural = C_{b,p} \in \mathcal{L}_n(k)$$

from Lemma 3.1, it satisfies (ii).

Next, we consider each term of the first sum. By the inductive hypothesis, we have $\Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural = 0$ if $\bar{C}_{i,p}$ does not have x_{a_r} in its components, $\Phi_1^{k-1}(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural = 0$ or $i_q = a_r$. Suppose $\bar{C}_{i,p}$ has x_{a_r} in its components. If $\Phi_1^{k-1}(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural$ is a commutator X of weight k and $i_q \neq a_r$, then we have $\Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural$ is a commutator $X^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])}$ of weight $k+1$. Suppose $(-1)^{e_{i,p}} \Phi_1^{k-1}(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural = \pm A \otimes B$ for some $A \in H^{\otimes \mu}$, $B \in H^{\otimes \nu}$ and $\mu + \nu = k$, and the element x_a which corresponds to x_{a_r} appears in A. If we consider the element $A' \in H^{\otimes \mu+1}$ given by A substituting $[x_a, x_b]$ into x_a which corresponds to x_{a_r}, then we have

$$(-1)^{e_{i,p}} \Phi_1^k(x_{i_q}^* \otimes \bar{C}_{i,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural = \pm A' \otimes B.$$

On the other hand, by the inductive hypothesis, we have

$$(-1)^{e_{j,p'}} \Phi_1^{k-1}(x_{j_{q'}}^* \otimes \bar{C}_{j,p'})_\natural = \mp B \otimes A$$

for some j, p' and q'. Hence there exists some r' corresponding to r and we have

$$(-1)^{e_{j,p'}} \Phi_1^k(x_{j_{q'}}^* \otimes \bar{C}_{j,p'}^{\partial(x_{a_{r'}}^* \otimes [x_{a'}, x_{b'}])})_\natural = \mp B \otimes A'.$$

Thus in this case, (iii) yields. Similarly we have the required result in the case where the element x_a corresponding to x_{a_r} appears in B.

Now, for any r $(1 \leq r \leq \beta(a,p,a))$, if we rewrite $\bar{C}_{a,p}$ as $\pm[a_r, X_1, \dots, X_t]$ stated as above, we have

$$
\begin{aligned}
\Phi_1^k(x_{a'}^* \otimes \bar{C}_{a,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural &= \pm \Phi_1^k(x_{a'}^* \otimes [[x_{a'}, x_{b'}], X_1, \dots, X_t])_\natural \\
&= \pm (x_{b'} \otimes X_1 \otimes \cdots \otimes X_t)_\natural
\end{aligned}
$$

and

$$\Phi_1^k(x_{a_r}^* \otimes [\bar{C}_{a,p}, b'])_\natural = \pm\Phi_1^k(x_{a_r}^* \otimes [[a_r, X_1, \dots, X_t], x_{b'}])_\natural$$
$$= \pm(X_1 \otimes \cdots \otimes X_t \otimes x_{b'})_\natural.$$

This shows

$$(-1)^{e_{i,p}}\Phi_1^k(x_{a'}^* \otimes \bar{C}_{a,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural \quad \text{and} \quad -(-1)^{e_{i,p}}\Phi_1^k(x_{a_r}^* \otimes [\bar{C}_{a,p}, x_{b'}])_\natural$$

satisfy (iii). Similarly we see

$$(-1)^{e_{b,p}}\Phi_1^k(x_{b'}^* \otimes \bar{C}_{b,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural \quad \text{and} \quad -(-1)^{e_{b,p}}\Phi_1^k(x_{a_r}^* \otimes [x_{a'}, \bar{C}_{b,p}])_\natural.$$

also satisfy (iii).

By an argument similar to above, we can also obtain the required result for $\tau = K_{abc}$. This completes the induction. □

3.2. **The image of $\Phi_2^k \circ \tau_k'$.**

Here we prove

Proposition 3.2. *For $n \geq 3$ and $k \geq 3$,* $\mathrm{Im}\,(\Phi_2^k \circ \tau_k') \subset H \otimes_{\mathbf{Z}} \mathcal{U}_n(k-1)$.

For each i, p and q in (3), if $\bar{C}_{i,p}$ has x_{i_q}, rewriting $\bar{C}_{i,p}$ as $\pm[x_{i_q}, D_{i,p}^1, \dots, D_{i,p}^{\gamma(i,p,q)}]$ we have,

$$\Phi_2^k \circ \tau_k'(x_{i_q}^* \otimes \bar{C}_{i,p}) = \sum_{\mathrm{wt}\,(D_{i,p}^t)=1} \mp(D_{i,p}^t \otimes D_{i,p}^1 \otimes \cdots \otimes D_{i,p}^{t-1} \otimes D_{i,p}^{t+1} \otimes \cdots \otimes D_{i,p}^{\gamma(i,p,q)})_\natural.$$

Set $T(\bar{C}_{i,p}) := \{t \,|\, \mathrm{wt}\,(D_{i,p}^t) = 1\}$. If $\bar{C}_{i,p}$ does not have x_{i_q} or $T(\bar{C}_{i,p}) = 0$ then $\Phi_2^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural = 0$. If $T(\bar{C}_{i,p}) = 1$ and $\gamma(i,p,q) = 2$, then

$$\Phi_2^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural = \pm x_s \otimes Z \in H \otimes_{\mathbf{Z}} \mathcal{L}_n(k-1)$$

for some commutator Z of weight $k-1$. Then Proposition 3.2 follows from

Lemma 3.4. *Let k be an integer greater than 2. According to the notation above, for each i, p and q, we have*

(i) *$\bar{C}_{i,p}$ does not have x_{i_q} or $T(\bar{C}_{i,p}) = 0$,*
(ii) *$T(\bar{C}_{i,p}) = 1$ and $\gamma(i,p,q) = 2$,*
or
(iii) *For each $t \in T(\bar{C}_{i,p})$, there exist some j, p', q' and t', $(j,p',q',t') \neq (i,p,q,t)$, such that if we set*

$$X := \mp(-1)^{e_{i,p}}(D_{i,p}^t \otimes D_{i,p}^1 \otimes \overset{\check{t}}{\cdots} \otimes D_{i,p}^{\gamma(i,p,q)})_\natural,$$
$$Y := \mp(-1)^{e_{j,p'}}(D_{j,p'}^{t'} \otimes D_{j,p'}^1 \otimes \overset{\check{t'}}{\cdots} \otimes D_{j,p'}^{\gamma(j,p',q')})_\natural$$

then $X + Y = 0$ or

$$X = \pm x_s \otimes A \otimes B, \quad Y = \mp x_s \otimes B \otimes A$$

where $A \in H^{\otimes\mu}$, $B \in H^{\otimes\nu}$ and $\mu + \nu = k-1$.

Proof We use induction on k. For $k = 3$, the result follows. The computations are reft to the reader for exercises. Assume $k \geq 4$ and the result follows for $k-1$. Let σ be a simple $(k-1)$-fold commutator in the components K_{ab} and K_{abc}. First, for $\tau = K_{ab}$ we consider $[\sigma, \tau]$. Then set

$$\tau_{k-1}'(\sigma) = \sum_{i=1}^{n}\sum_{p=1}^{\alpha(i)} x_i^* \otimes (-1)^{e_{i,p}} C_{i,p}.$$

Here we also set $\tau_1'(\tau) = x_a^* \otimes [x_{a'}, x_{b'}]$ and distinguish a' and b' from any a and b which appear in $C_{i,p}$ for any i and p respectively.

Now we consider (4) for $l = 2$. First, since $T([x_a', \bar{C}_{b,p}]) = 0$, each term of the last sum satisfies (i). For each term of the first sum, since $\bar{C}_{i,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])} = 0$ if $\bar{C}_{i,p}$ does not have x_{a_r}, so we may assume $\bar{C}_{i,p}$ has x_{a_r} in its components. If $i_q = a_r$ then the element $\bar{C}_{i,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])}$ also does not have x_{i_q}. Suppose $i_q \neq a_r$. If $T(\bar{C}_{i,p}) = 0$, then $T(\bar{C}_{i,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])}) = 0$. If $T(\bar{C}_{i,p}) = 1$, $\gamma(i,p,q) = 2$ and $\Phi_2^k(x_{i_q}^* \otimes \bar{C}_{i,p})_\natural = \pm x_s \otimes Z$ for some commutator Z of weight $k-1$ then we have

$$\Phi_2^k(x_{i_q}^* \otimes \bar{C}_{i,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural = \begin{cases} 0 & \text{if} \quad s = a_r, \\ \pm x_s \otimes Z^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])} & \text{if} \quad s \neq a_r. \end{cases}$$

So we see that each case above satisfies (i) or (ii). For $T(\bar{C}_{i,p}) = 1$ and $\gamma(i,p,q) \geq 3$, or $T(\bar{C}_{i,p}) \geq 2$, we have

$$\begin{aligned} &(-1)^{e_{i,p}} \Phi_2^k(x_{i_q}^* \otimes \bar{C}_{i,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural \\ &\quad = \sum_{\mathrm{wt}\,(D_{i,p}^t)=1,\, D_{i,p}^t \neq x_{a_r}} \mp(-1)^{e_{i,p}} \Big(D_{i,p}^t \otimes (D_{i,p}^1 \otimes \overset{\check{t}}{\cdots} \otimes D_{i,p}^{\gamma(i,p,q)})^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])} \Big)_\natural. \end{aligned}$$

Set

$$X' := \mp(-1)^{e_{i,p}} \Big(D_{i,p}^t \otimes (D_{i,p}^1 \otimes \overset{\check{t}}{\cdots} \otimes D_{i,p}^{\gamma(i,p,q)})^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])} \Big)_\natural.$$

By the inductive hypothesis, for $X = \mp(-1)^{e_{i,p}} (D_{i,p}^t \otimes D_{i,p}^1 \otimes \overset{\check{t}}{\cdots} \otimes D_{i,p}^{\gamma(i,p,q)})_\natural$, there exists $Y := \mp(-1)^{e_{j,p'}} (D_{j,p'}^{t'} \otimes D_{j,p'}^1 \otimes \overset{\check{t'}}{\cdots} \otimes D_{j,p'}^{\gamma(j,p',q')})_\natural$ for some j, p', q' and t' such that $X + Y = 0$, or $X = \pm x_s \otimes A \otimes B$ and $Y = \mp x_s \otimes B \otimes A$ for some $A \in H^{\otimes \mu}$, $B \in H^{\otimes \nu}$ and $\mu + \nu = k - 1$. By an argument similar to that in Lemma 3.3, we have

$$Y' := \mp(-1)^{e_{j,p'}} \Big(D_{j,p'}^{t'} \otimes (D_{j,p'}^1 \otimes \overset{\check{t'}}{\cdots} \otimes D_{j,p'}^{\gamma(j,p',q')})^{\partial(x_{a_{r'}}^* \otimes [x_{a'}, x_{b'}])} \Big)_\natural,$$

for some suitable r' corresponding to r such that $X' + Y' = 0$, or $X' = \pm x_s \otimes A' \otimes B'$ and $Y' = \mp x_s \otimes B' \otimes A'$. Here $A' \in H^{\otimes \mu'}$, $B' \in H^{\otimes \nu'}$ and $\mu' + \nu' = k$.

Finally, for any r ($1 \leq r \leq \beta(a,p,a)$), if we rewrite $\bar{C}_{a,p}$ as $\pm[a_r, X_1, \ldots, X_u]$ stated as above, we have

$$\begin{aligned} &\Phi_2^k(x_{a'}^* \otimes \bar{C}_{a,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural \\ &\qquad = \pm \Phi_2^k(x_{a'}^* \otimes [[x_{a'}, x_{b'}], X_1, \ldots, X_u])_\natural, \\ &\qquad = \mp \Big\{ \sum_{\mathrm{wt}\,(X_t)=1} (X_t \otimes x_{b'} \otimes X_1 \otimes \overset{\check{t}}{\cdots} \otimes X_u)_\natural \Big\} \mp (x_{b'} \otimes X_1 \otimes \cdots \otimes X_u)_\natural \end{aligned}$$

and

$$\begin{aligned} &-\Phi_2^k(x_{a_r}^* \otimes [\bar{C}_{a,p}, x_{b'}])_\natural \\ &\qquad = \mp \Phi_2^k(x_{a_r}^* \otimes [[x_{a_r}, X_1, \ldots, X_u], x_{b'}])_\natural \\ &\qquad = \pm \Big\{ \sum_{\mathrm{wt}\,(X_t)=1} (X_t \otimes X_1 \otimes \overset{\check{t}}{\cdots} \otimes X_u \otimes x_{b'})_\natural \Big\} \pm (x_{b'} \otimes X_1 \otimes \cdots \otimes X_u)_\natural. \end{aligned}$$

Hence we see that each term of the equations above satisfies condition (iii). Similarly we can show that each term of $\Phi_2^k(x_{b'}^* \otimes \bar{C}_{b,p}^{\partial(x_{a_r}^* \otimes [x_{a'}, x_{b'}])})_\natural$ and $\Phi_2^k(x_{a_r}^* \otimes [x_{a'}, \bar{C}_{b,p}])_\natural$ satisfies (iii).

By an argument similar to above, we can also obtain the required result for $\tau = K_{abc},$. This completes the induction. □

4. The trace maps

In this section, using the contractions defined in Section 3, we define a homomorphism called the trace map which vanishes on the image of the Johnson homomorphism. Here we use some basic facts of the representation theory of $GL(n, \mathbf{Z})$. The reader is referred to, for example, Fulton-Harris [5] and Fulton [4].

For any $k \geq 1$ and any partition λ of k, we denote by H^{λ} the Schur-Weyl module of H corresponding to the partition λ of k. Let $f_{\lambda} : H^{\otimes k} \to H^{\lambda}$ be a natural homomorphism. In this paper, we mainly consider the case for $\lambda = [k]$ or $[1^k]$. The modules $H^{[k]}$ and $H^{[1^k]}$ are the symmetric product $S^k H$ and the exterior product $\Lambda^k H$ respectively. Using the natural map $\iota_n^k : \mathcal{L}_n(k) \to H^{\otimes k}$, we denote $f_{[1^k]} \circ \iota_n^k(C)$ by $\widehat{C}$ for any $C \in \mathcal{L}_n(k)$.

Lemma 4.1. *For any commutator C of weight $k \geq 3$, $\widehat{C} = 0$ in $\Lambda^k H$*

Proof We use induction on k. For $k = 3$, the result is trivial. Assume $k \geq 4$ and $C = [C_1, C_2]$ for commutators C_1 and C_2. Then

$$\widehat{C} = \widehat{C_1} \wedge \widehat{C_2} - \widehat{C_2} \wedge \widehat{C_1}.$$

Set $\mathrm{wt}\,(C_1) = a$, $\mathrm{wt}\,(C_2) = b$. Then $a + b = k$. If either a or b is even, the result is trivial. If both a and b are odd, since $k \geq 3$, we have $3 \leq a < k$ or $3 \leq b < k$. By inductive hypothesis, we have $\widehat{C} = 0$. This completes the induction. □

Lemma 4.2. *For $1 \leq k \leq n$ and any commutator C of weight $k+1$ in the components $x_1, \ldots, x_n$ except for x_i, there exists an element $\sigma \in \mathcal{A}'_n(k)$ such that*

$$\tau'_k(\sigma) = x_i^* \otimes C \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

Proof We use induction on k. For $k = 1$, considering K_{abc}, the result holds. Assume $k \geq 2$ and $C = [C_1, C_2]$ for commutators C_1 and C_2. Moreover we may also assume $\mathrm{wt}\,(C_1) \geq \mathrm{wt}\,(C_2)$. Since $k \geq \mathrm{wt}\,(C_1) \geq \mathrm{wt}\,(C_2)$ and $\mathrm{wt}\,(C_1) + \mathrm{wt}\,(C_2) = k + 1 \geq 3$, we have $\mathrm{wt}\,(C_2) \leq k - 1$. Set $a = \mathrm{wt}\,(C_1)$ and $b = \mathrm{wt}\,(C_2)$. For any x_j which appears in C_1, by the inductive hypothesis, we have two elements $\sigma_1 \in \mathcal{A}'_n(b)$ and $\sigma_2 \in \mathcal{A}'_n(a)$ defined by

$$\tau'_b(\sigma_1) = x_i^* \otimes [x_j, C_2] \quad \text{and} \quad \tau'_{a-1}(\sigma_2) = x_j^* \otimes C_1.$$

Then, setting $\sigma = [\sigma_1, \sigma_2]$, we obtain $\tau'_k(\sigma) = x_i^* \otimes [C_1, C_2]$. This completes the induction. □

4.1. Morita's trace (Trace map for $S^k H$).

Here we consider the map

$$\mathrm{Tr}_{[k]} = f_{[k]} \circ \Phi_1^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to S^k H.$$

By definition, this map coincides with the Morita's trace Tr_k. For $n \geq 3$ and $k \geq 2$, Morita defined the trace map Tr_k using the Magnus representation of $\mathrm{Aut}\,F_n$ and showed that Tr_k vanishes on the image of τ_k. By a recent remakable work, he showed that $\mathrm{Tr}_k^{\mathbf{Q}}$ is surjective. Hence we have

Theorem 4.1. (Morita) *For $n \geq 3$ and $k \geq 2$,*

$$S^k H_{\mathbf{Q}} \subset \mathrm{Coker}\, \tau_{k,\mathbf{Q}}.$$

Corollary 4.1. *For $n \geq 3$ and $k \geq 2$,*

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{Coker}\,(\tau_k)) \geq \binom{n+k-1}{k}.$$

4.2. **Trace map for $\Lambda^k H$.**

Here we consider the map

$$\mathrm{Tr}_{[1^k]} := f_{[1^k]} \circ \Phi_1^k : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to \Lambda^k H.$$

Theorem 4.2.

(1) *For $3 \leq k \leq n$, $\mathrm{Tr}_{[1^k]}$ is surjective,*
(2) $\mathrm{Im}\,(\mathrm{Tr}_{[1^k]}(k) \circ \tau'_k) = 0$ *if k is odd and $3 \leq k \leq n$,*
(3) $\mathrm{Im}\,(\mathrm{Tr}_{[1^k]}(k) \circ \tau'_k) = 2(\Lambda^k H) \subset \Lambda^k H$ *if k is even and $4 \leq k \leq n-2$.*

Proof For $3 \leq k \leq n$, considering

$$x_i^* \otimes [x_i, x_{j_1}, x_i, x_{j_2} \ldots, x_{j_{k-1}}] \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

for distinct i, $j_1, \ldots, j_{k-1}$, we have

$$\mathrm{Tr}_{[1^k]}(x_i^* \otimes [x_i, x_{j_1}, x_i, x_{j_2} \ldots, x_{j_{k-1}}]) = -3\, x_i \wedge x_{j_1} \wedge \cdots \wedge x_{j_{k-1}}.$$

Similarly

$$\mathrm{Tr}_{[1^k]}(x_i^* \otimes [[x_i, x_{j_1}], [x_i, x_{j_2}], x_{j_3} \ldots, x_{j_{k-1}}]) = -4\, x_i \wedge x_{j_1} \wedge \cdots \wedge x_{j_{k-1}}.$$

Thus a generator $x_i \wedge x_{j_1} \wedge \cdots \wedge x_{j_{k-1}}$ of $\Lambda^k H$ is in the image of $\mathrm{Tr}_{[1^k]}$. This shows (1).

For an odd integer k, let us consider an element

$$[X, Y] = X \otimes Y - Y \otimes X \in \mathcal{U}_n(k)$$

for $X \in H^{\otimes a}$, $Y \in H^{\otimes b}$ and $a + b = k$. Since k is odd, either a or b is even. Hence

$$\begin{aligned} f_{[1^k]}([X, Y]) &= f_{[1^a]}(X) \wedge f_{[1^b]}(Y) - f_{[1^b]}(Y) \wedge f_{[1^a]}(X) \\ &= f_{[1^a]}(X) \wedge f_{[1^b]}(Y) - f_{[1^a]}(X) \wedge f_{[1^b]}(Y) \\ &= 0. \end{aligned}$$

Since $\mathcal{U}_n(k)$ is generated by the elements type of $[X, Y]$ as above, the map $f_{[1^k]}$ vanishes on $\mathcal{U}_n(k)$. Hence we obtain (2) from Proposition 3.1.

For an even integer k, $\mathrm{Im}\,(\mathrm{Tr}_{[1^k]}(k) \circ \tau'_k) \subset 2(\Lambda^k H)$ is shown by a similar argument as above. Thus it suffices to show $\mathrm{Im}\,(\mathrm{Tr}_{[1^k]}(k) \circ \tau'_k) \supset 2(\Lambda^k H)$. From Lemma 2.1, there are $\sigma_1 \in \mathcal{A}'_n(k-1)$ and $\sigma_2 \in \mathcal{A}'_n(1)$ such that

$$\tau'_{k-1}(\sigma_1) = x_{i_1}^* \otimes [x_{i_2}, x_{j_1}, \ldots, x_{j_{k-1}}] \quad \text{and} \quad \tau'_1(\sigma_2) = x_{i_2}^* \otimes [x_{i_1}, x_{j_k}]$$

for distinct $i_1, i_2, j_1, \ldots, j_k \in \{1, \ldots, n\}$. Then

$$\begin{aligned} \mathrm{Tr}_{[1^k]} \circ \tau'_k([\sigma_1, \sigma_2]) &= f_{[1^k]}(x_{j_k} \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k-1}} - x_{j_1} \otimes \cdots \otimes x_{j_{k-1}} \otimes x_{j_k}), \\ &= x_{j_k} \wedge x_{j_1} \wedge \cdots \wedge x_{j_{k-1}} - x_{j_1} \wedge \cdots \wedge x_{j_{k-1}} \wedge x_{j_k}, \\ &= -2\, x_{j_1} \wedge \cdots \wedge x_{j_k}. \end{aligned}$$

Since $2(\Lambda^k H)$ is generated by the elements $2\, x_{j_1} \wedge \cdots \wedge x_{j_k}$, we have $\mathrm{Im}\,(\mathrm{Tr}_{[1^k]}(k) \circ \tau'_k) \supset 2(\Lambda^k H)$. This completes the proof of (3). $\square$

Corollary 4.2. *For an odd k and $3 \leq k \leq n$,*

$$\Lambda^k H_{\mathbf{Q}} \subset \mathrm{Coker}\, \tau_{k,\mathbf{Q}}.$$

Corollary 4.3. *For an odd k and $3 \leq k \leq n$,*

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{Coker}\,(\tau'_k)) \geq \binom{n}{k}.$$

4.3. **Trace map for $H^{[2,1^{k-2}]}$.**
Here we consider the map

$$\mathrm{Tr}_{[2,1^{k-2}]} := (id_H \otimes f^{k-1}_{[1^{k-1}]}) \circ \Phi^k_2 : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H \otimes_{\mathbf{Z}} \Lambda^{k-1} H.$$

Let I be the $GL(n,\mathbf{Z})$-submodule of $H \otimes_{\mathbf{Z}} \Lambda^{k-1} H$ defined by

$$I = \langle x \otimes z_1 \wedge \cdots \wedge z_{k-2} \wedge y + y \otimes z_1 \wedge \cdots \wedge z_{k-2} \wedge x \mid x, y, z_t \in H \rangle.$$

Theorem 4.3. *For an even k and $4 \leq k \leq n-1$,*

(1) $\mathrm{Im}\,\mathrm{Tr}^{\mathbf{Q}}_{[2,1^{k-1}]}(k) = I_{\mathbf{Q}}$,
(2) $\mathrm{Im}\,(\mathrm{Tr}_{[2,1^{k-1}]}(k) \circ \tau'_k) = 0$.

Proof For any distinct $i, j_1, \ldots, j_k$, considering

$$x_i^* \otimes [x_i, x_{j_1}, x_{j_2}, [x_{j_3}, x_{j_4}], \ldots, [x_{j_{k-1}}, x_{j_k}]] \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1),$$

we have

$$\begin{aligned}
&\mathrm{Tr}_{[2,1^{k-2}]}(x_i^* \otimes [x_i, x_{j_1}, x_{j_2}, [x_{j_3}, x_{j_4}], \ldots, [x_{j_{k-1}}, x_{j_k}]]) \\
&\quad = -2^{\frac{k-2}{2}} (x_{j_1} \otimes x_{j_3} \wedge x_{j_4} \wedge \cdots \wedge x_{j_{k-1}} \wedge x_{j_k} \wedge x_{j_2} \\
&\qquad\qquad + x_{j_2} \otimes x_{j_3} \wedge x_{j_4} \wedge \cdots \wedge x_{j_{k-1}} \wedge x_{j_k} \wedge x_{j_1}).
\end{aligned}$$

Hence, $\mathrm{Im}\,\mathrm{Tr}^{\mathbf{Q}}_{[2,1^{k-2}]}(k) \supset I_{\mathbf{Q}}$. To prove $\mathrm{Im}\,\mathrm{Tr}^{\mathbf{Q}}_{[2,1^{k-2}]}(k) \subset I_{\mathbf{Q}}$, it suffices to show that the image of the element $x_i^* \otimes [x_i, C_1, \ldots, C_t] \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$, where x_i does not appear in the componets of each of C_j, is contained in $I_{\mathbf{Q}}$. From Lemma 3.2, we have

$$\begin{aligned}
&\mathrm{Tr}_{[2,1^{k-2}]}(x_i^* \otimes [x_i, C_1, \ldots, C_t]) \\
&\qquad = - \sum_{\mathrm{wt}\,(C_j)=1} C_j \otimes \widehat{C_1} \wedge \cdots \wedge \widehat{C_{j-1}} \wedge \widehat{C_{j+1}} \wedge \cdots \wedge \widehat{C_t}.
\end{aligned}$$

If $\mathrm{wt}\,(C_j) \geq 3$ for some j, the right hand side is equal to zero from Lemma 4.1. Hence we may assume $\mathrm{wt}\,(C_j) \leq 2$ for all j. Write the C_j's satisfying $\mathrm{wt}\,(C_j) = 1$ as $C_{j_1}, \ldots, C_{j_l}$. Then l is even and we have

$$\begin{aligned}
&\mathrm{Tr}_{[2,1^{k-2}]}(x_i^* \otimes [x_i, C_1, \ldots, C_t]) \\
&\qquad = -2^{\frac{k-2}{2}} \sum_{s=1}^{l/2} (C_{j_s} \otimes \widehat{C_1} \wedge \cdots \overset{\check{j_s}}{\cdots} \cdots \overset{\check{j_{s+1}}}{\cdots} \cdots \wedge \widehat{C_t} \wedge C_{j_{s+1}} \\
&\qquad\qquad + C_{j_{s+1}} \otimes \widehat{C_1} \wedge \cdots \overset{\check{j_s}}{\cdots} \cdots \overset{\check{j_{s+1}}}{\cdots} \cdots \wedge \widehat{C_t} \wedge C_{j_s}) \\
&\qquad \in I_{\mathbf{Q}}.
\end{aligned}$$

This shows (1).

Let us consider

$$x \otimes [X, Y] = x \otimes (X \otimes Y - Y \otimes X) \in H \otimes_{\mathbf{Z}} \mathcal{U}_n(k-1)$$

for $X \in H^{\otimes a}$, $Y \in H^{\otimes b}$ and $a + b = k-1$. Since $k-1$ is odd, either a or b is even. Thus

$$\begin{aligned}
(id_H \otimes f_{[1^{k-1}]})(x \otimes [X, Y]) &= x \otimes (f_{[1^a]}(X) \wedge f_{[1^b]}(Y) - f_{[1^b]}(Y) \wedge f_{[1^a]}(X)) \\
&= x \otimes (f_{[1^a]}(X) \wedge f_{[1^b]}(Y) - f_{[1^a]}(X) \wedge f_{[1^b]}(Y)) \\
&= 0.
\end{aligned}$$

Since $H \otimes_{\mathbf{Z}} \mathcal{U}_n(k-1)$ is generated by the elements above, the map $id_H \otimes f_{[1^{k-1}]}$ vanishes on $H \otimes_{\mathbf{Z}} \mathcal{U}_n(k-1)$. Hence we obtain (2) from Proposition 3.2. □

Now we have $H_{\mathbf{Q}} \otimes_{\mathbf{Z}} \Lambda^{k-1} H_{\mathbf{Q}} \simeq H_{\mathbf{Q}}^{[2,1^{k-2}]} \oplus \Lambda^k H_{\mathbf{Q}}$ from the representation theory of $GL(n, \mathbf{Z})$. For even k, since $I_{\mathbf{Q}}$ is contained in the kernel of a natural map $H_{\mathbf{Q}} \otimes_{\mathbf{Z}} \Lambda^{k-1} H_{\mathbf{Q}} \to \Lambda^k H_{\mathbf{Q}}$ defined by $x \otimes y_1 \wedge \cdots \wedge y_{k-1} \mapsto x \wedge y_1 \wedge \cdots \wedge y_{k-1}$, we have $I_{\mathbf{Q}} \simeq H_{\mathbf{Q}}^{[2,1^{k-2}]}$.

Corollary 4.4. *For an even k and $4 \le k \le n-1$,*

$$H_{\mathbf{Q}}^{[2,1^{k-2}]} \subset \mathrm{Coker}\, \tau'_{k,\mathbf{Q}}.$$

Corollary 4.5. *For an even k and $4 \le k \le n-1$,*

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{Coker}\,(\tau'_k)) \ge (k-1)\binom{n+1}{k}.$$

5. The cokernel of the Johnson homomorphism τ_k for $k=2$ and 3

5.1. the case for $k=2$.

In this subsection we consider the case where $n \ge 3$. From Theorem 4.1 and $\mathrm{rank}_{\mathbf{Z}}(\mathrm{Coker}\,(\tau'_2)) = \binom{n+1}{2}$ by Pettet [16], we have a $GL(n,\mathbf{Z})$-equivariant exact sequence

$$0 \to \mathrm{gr}^2_{\mathbf{Q}}(\mathcal{A}_n) \xrightarrow{\tau_{2,\mathbf{Q}}} H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_n^{\mathbf{Q}}(3) \to S^2 H_{\mathbf{Q}} \to 0.$$

In this subsection we show that the exact sequence above holds before tensoring with $\mathbf{Q}$. Here are some examples of commutators of degree 2 in the components K_{ab} and K_{abc} and their images by the Johnson homomorphism τ_2.

(C1): $[K_{ab}, K_{ac}]$, $x_a^* \otimes [[x_a, x_c], x_b] - x_a^* \otimes [[x_a, x_b], x_c]$,
(C2): $[K_{ab}, K_{acd}]$, $x_a^* \otimes [[x_c, x_d], x_b]$,
(C3): $[K_{ab}, K_{abc}]$, $x_a^* \otimes [[x_b, x_c], x_b]$,
(C4): $[K_{ab}, K_{bac}]$, $x_a^* \otimes [x_a, [x_a, x_c]] - x_b^* \otimes [[x_a, x_b], x_c]$,
(C5): $[K_{abc}, K_{bad}]$, $x_a^* \otimes [[x_a, x_d], x_c] - x_b^* \otimes [[x_b, x_c], x_d]$,
(C6): $[K_{abc}, K_{bac}]$, $x_a^* \otimes [[x_a, x_c], x_c] - x_b^* \otimes [[x_b, x_c], x_c]$.

Theorem 5.1. *For $n \ge 3$,*

$$0 \to \mathrm{gr}^2(\mathcal{A}_n) \xrightarrow{\tau_2} H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3) \to S^2 H \to 0$$

is a $GL(n,\mathbf{Z})$-equivariant exact sequence.

Proof First, we note that for any element $\delta \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3)$, we also denote by δ the coset class of it in $\mathrm{Coker}\, \tau_2$. For any $i, p, q, r \in \{1, \ldots, n\}$ and $p \neq q$, set

$$a_i(p,q,r) := x_i^* \otimes [[x_p, x_q], x_r] \in \mathrm{Coker}\, \tau_2.$$

From (C2) and (C3), we have $a_i(p,q,r) = 0$ for $p, q, r \neq i$. From Jacobi identity, we have

$$a_i(p,q,i) = -a_i(i,p,q) + a_i(i,q,p).$$

Hence, from (C1), $a_i(p,q,i) = 0$ for $p, q \neq i$. Since $a_i(p,q,r) = -a_i(q,p,r)$, from (C1) we can set

$$\alpha_i(q,r) := a_i(i,q,r) = a_i(i,r,q) = -a_i(q,i,r) = -a_i(r,i,q)$$

for $q, r \neq i$ and $q \neq r$. Moreover, from (C5) we can also set $\alpha(q,r) := \alpha_i(q,r)$ for $q \neq r$. Similarly, from (C4) and (C6), we can set $\alpha(p,p) := a_i(i,p,p) = -a_i(p,i,p)$ for $i \neq p$.

Let A be the free abelian group generated by the elements $\alpha(p,q)$ for $p \le q$. By the argument above, $\mathrm{Coker}\, \tau_2$ is isomorphic to a quotient group of A as an abelian group. On the other hand, since the rank of the free part of $\mathrm{Coker}\, \tau_2$ is $\frac{1}{2}n(n+1)$ from Corollary 4.1 and $\mathrm{rank}_{\mathbf{Z}}(A) = \frac{1}{2}n(n+1)$, we see that $\mathrm{Coker}\, \tau_2$ must be isomorphic to A. Considering the action of $GL(n,\mathbf{Z})$ on A, we verify $A \simeq S^2 H$. This completes the proof of Theorem 5.1. □

5.2. **the case for $k=3$.**

Next we compute the cokernel of the Johnson homomorphism $\tau_{3,\mathbf{Q}}$ for $n \geq 3$ using the fact that $\mathrm{Coker}\,\tau_3 = \mathrm{Coker}\,\tau_3'$. We use commutators of degree 3 in the components K_{ab} and K_{abc}:

(C1-1):	$[[K_{ab}, K_{ac}], K_{bd}]$,	(C1-2):	$[[K_{ab}, K_{ac}], K_{bc}]$,
(C1-3):	$[[K_{ab}, K_{ac}], K_{ba}]$,		
(C3-1):	$[[K_{ab}, K_{abc}], K_{cab}]$,	(C3-2):	$[[K_{ab}, K_{abc}], K_{ca}]$,
(C3-3):	$[[K_{ab}, K_{abc}], K_{bad}]$,		
(C4-1):	$[[K_{ab}, K_{bac}], K_{ac}]$,	(C4-2):	$[[K_{ab}, K_{bac}], K_{ba}]$,
(C4-3):	$[[K_{ab}, K_{bac}], K_{cd}]$,	(C4-4):	$[[K_{ab}, K_{bac}], K_{abc}]$,
(C4-5):	$[[K_{ab}, K_{bac}], K_{cab}]$,	(C4-6):	$[[K_{ab}, K_{bac}], K_{ca}]$,
(C4-7):	$[[K_{ab}, K_{bac}], K_{ab}]$,	(C4-8):	$[[K_{ab}, K_{bac}], K_{cb}]$,
(C4-9):	$[[K_{ab}, K_{bac}], K_{ad}]$.		

Here are a few examples of their images by τ_3:

(C1-1)': $x_a^* \otimes [[x_a, x_c], [x_b, x_d]] - x_a^* \otimes [[x_a, [x_b, x_d]], x_c]$,

(C3-1)': $x_a^* \otimes [[x_b, [x_a, x_b]], x_b] - x_c^* \otimes [[[x_b, x_c], x_b], x_b]$,

(C4-1)': $x_a^* \otimes [[x_c, [x_a, x_c]], x_a] + x_a^* \otimes [[x_c, x_a], [x_a, x_c]] + x_b^* \otimes [[x_b, [x_a, x_c]], x_c]$
$\qquad -x_a^* \otimes [[[x_c, x_a], x_a], x_c]$.

Theorem 5.2. *For $n \geq 3$,*

$$0 \to \mathrm{gr}^3_{\mathbf{Q}}(\mathcal{A}_n) \xrightarrow{\tau_{3,\mathbf{Q}}} H^*_{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathcal{L}_n^{\mathbf{Q}}(4) \to S^3 H_{\mathbf{Q}} \oplus \Lambda^3 H_{\mathbf{Q}} \to 0$$

is a $GL(n,\mathbf{Z})$-equivariant exact sequence.

Proof As before, for any element $\delta \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(4)$, we also denote by δ the coset class of it in $\mathrm{Coker}\,\tau_3$. For any $i,p,q,r,s \in \{1,\ldots,n\}$, set

$$a_i(p,q,r,s) := x_i^* \otimes [[[x_p, x_q], x_r], x_s] \quad \text{if} \;\; p \neq q,$$
$$b_i(p,q,r,s) := x_i^* \otimes [[x_p, x_q], [x_r, x_s]] \quad \text{if} \;\; p \neq q \text{ and } r \neq s$$

in $\mathrm{Coker}\,\tau_3$.

First, from Lemma 2.1, we have $a_i(p,q,r,s) = 0$ and $b_i(p,q,r,s) = 0$ for distinct i,p,q,r and s. Substituting $X = [x_b, x_d]$, $Y = x_c$ and $Z = x_a$ into Jacobi identity

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0, \tag{5}$$

we see (C1-1)' is equivalent to $x_a^* \otimes [[[x_b, x_d], x_c], x_a]$. Thus $a_i(p,q,r,i) = 0$ for i,p,q and r. Similarly $a_i(p,q,q,i) = 0$ for $p,q \neq i$ from (C1-2), and hence $a_i(p,q,p,i) = -a_i(q,p,p,i) = 0$. Substituting $X = [x_c, x_a]$, $Y = x_a$ and $Z = x_c$ into (5) we have $[[[x_c, x_a], x_c], x_a] = [[[x_c, x_a], x_a], x_c]$. Thus (C4-1)' is equivalent to $x_b^* \otimes [[[x_c, x_a], x_b], x_c]$. This shows $a_i(p,q,i,p) = 0$, and hence $a_i(p,q,i,q) = -a_i(q,p,i,q) = 0$ for $p,q \neq i$. Similarly, from (C4-2), we have $b_i(i,p,p,q) = 0$ for $p,q \neq i$.

Next, from (C3-1)', we have $a_i(i,p,p,p) = a_j(j,p,p,p)$ for distinct i,j and p. Thus we can set

$$\beta(p) := a_i(i,p,p,p) = -a_i(p,i,p,p).$$

If $n \geq 4$, from (C4-4) we have $a_i(i,p,q,q) + a_p(q,p,p,q) = 0$ and $a_j(j,p,q,q) + a_p(q,p,p,q) = 0$ for any distinct i,j,p and q. So $a_i(i,p,q,q) = a_j(j,p,q,q)$. Hence for $n \geq 3$ we can set

$$\beta(p,q) := a_i(i,p,q,q)$$

for distinct p and q. Then we can show that

$$a_i(i,p,i,p) = \beta(i,p),$$
$$a_i(i,p,p,i) = \beta(i,p),$$
$$a_i(i,p,i,i) = \beta(p,i),$$
$$a_i(i,p,p,q) = \beta(q,p),$$

from (C4-4), (C3-2), (C4-6) and (C4-5) respectively. Furthermore, considering the Jacobi identity obtained by substituting $X = [x_i, x_p]$, $Y = x_q$ and $Z = x_p$ into (5), we have

$$a_i(i,p,q,p) = \beta(q,p).$$

Thus we also have $a_i(p,i,i,p) = -\beta(i,p)$, $a_i(p,i,p,i) = -\beta(i,p)$ and so on.

Now set

$$\beta(p,q,i) := a_i(i,p,q,i) \quad \text{and} \quad \gamma(p,q,i) := a_i(p,q,i,i)$$

for distinct i, p and q. Clearly, $\gamma(p,q,r) = -\gamma(q,p,r)$. We have

$$a_i(i,p,i,q) = \beta(i,q,p) - \gamma(i,q,p)$$

from (C4-7) and considering the Jacobi identity obtained by substituting $X = [x_i, x_p]$, $Y = x_i$ and $Z = x_q$ into (5),

$$a_i(i,p,i,q) = \beta(p,i,q) - \gamma(p,i,q) - \beta(p,q,i). \tag{6}$$

On the other hand, we see $\beta(p,q,r) = \beta(r,p,q)$ from (C1-3), and considering $b_i(i,p,i,q) = -b_i(i,q,i,p)$ and (6), we have $\gamma(p,q,r) = \gamma(r,p,q)$. Then from (C4-8), we have $\beta(p,q,r) - \beta(p,r,q) = \gamma(r,q,p)$.

Finally, if $n \geq 4$, for distinct i, p, q and r, we obtain

$$a_i(i,p,q,r) = \beta(p,q,r),$$
$$a_i(p,q,i,r) = \gamma(p,q,r),$$
$$b_i(p,q,i,r) = \gamma(p,q,r)$$

from (C3-3), (C4-9) and (C4-3) respectively.

Let B the free abelian group generated by the elements

$$\begin{aligned} \beta(p,q,r) &\quad \text{for } p < q < r, \\ \beta(p,q) &\quad \text{for } p \neq q, \\ \beta(p) &\quad \text{for any } p \\ \gamma(p,q,r) &\quad \text{for } p < q < r. \end{aligned}$$

By the argument above, we see $\operatorname{Coker}\tau_3$ is isomorphic to a quotient group of B as an abelian group. On the other hand, from corollaries 4.1 and 4.3, and $\operatorname{rank}_{\mathbf{Z}} B = \binom{n+2}{3} + \binom{n}{3}$, we see that $\operatorname{Coker}\tau_3$ must be isomorphic to B. To consider the structure of $\operatorname{Coker}\tau_3$ as a $GL(n,\mathbf{Z})$-module, we define a $GL(n,\mathbf{Z})$-homomorphism $\Psi : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(4) \to S^3 H \oplus \Lambda^3 H$ by

$$w \mapsto (\,\mathrm{Tr}_{[3]}(w)\,,\ \mathrm{Tr}_{[1^3]}(w)\,).$$

Then from Theorem 4.2 and the argument above, we see $\mathrm{Im}(\tau_3) = \mathrm{Ker}(\Psi)$. On the other hand, since we have

$$\Psi(a_i(i,p,q,r)) = (x_p \cdot x_q \cdot x_r\,,\ x_p \wedge x_q \wedge x_r),$$
$$\Psi(a_i(p,q,i,r)) = (0, -2x_p \wedge x_q \wedge x_r),$$

$\Psi_{\mathbf{Q}}$ is surjective. This completes the proof of Theorem 5.2. □

Corollary 5.1. *For $n \geq 3$,*

$$\operatorname{rank}_{\mathbf{Z}} \mathrm{gr}^3(\mathcal{A}_n) = \frac{1}{12} n(3n^4 - 7n^2 - 8). \tag{7}$$

In particular, substituting $n = 3$ into (7), we have $\mathrm{rank}_{\mathbf{Z}}\, \mathrm{gr}^3(\mathcal{A}_3) = 43$.

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References

[1] S. Andreadakis; On the automorphisms of free groups and free nilpotent groups, Proc. London Math. Soc. (3) 15 (1965), 239-268.
[2] M. Asada; Two properties of the filtration of the outer automorphism groups of certain groups, Mathematische Zeitschrift.
[3] C. Reutenauer; Free Lie Algebras, London Mathematical Societymonographs, new series, no. 7, Oxford University Press (1993).
[4] W. Fulton; Young Tableaux, London Mathematical Society Student Texts 35, Cambridge University Press (1997).
[5] W. Fulton, J. Harris; Representation Theory, Graduate text in Mathematics 129, Springer-Verlag (1991).
[6] R. Hain; Infinitesimal presentations of the Torelli group, Journal of the American Mathematical Society 10 (1997), 597-651.
[7] D. Johnson; An abelian quotient of the mapping class group, Mathematische Annalen 249 (1980), 225-242.
[8] M. Kassabov; On the automorphism tower of free nilpotent groups, thesis, Yale University (2003).
[9] N. Kawazumi; Cohomological Aspects of Magnus Expansions, in preparation.
[10] S. Krstić, J. McCool; The non-finite presentability in $IA(F_3)$ and $GL_2(\mathbf{Z}[t,t^{-1}])$, Invent. Math. 129 (1997), 595-606.
[11] W. Magnus; Über n-dimensinale Gittertransformationen, Acta Math. 64 (1935), 353-367.
[12] W. Magnus, A. Karras, D. Solitar; Combinatorial group theory, Interscience Publ., New York (1966).
[13] S. Morita; Abelian quotients of subgroups of the mapping class group of surfaces, Duke Mathematical Journal 70 (1993), 699-726.
[14] S. Morita; Structure of the mapping class groups of surfaces: asurvey and a prospect, Geometry and Topology Monographs Vol. 2 (1999), 349-406.
[15] S. Morita; Cohomological structure of the mapping classs group and beyond, preprint.
[16] A. Pettet; The Johnson homomorphism and the second cohomology group of IA_n, preprint.

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