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by

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A STABILITY RESULT VIA CARLEMAN ESTIMATES FOR AN INVERSE PROBLEM RELATED TO A HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATION

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Abstract. First we prove a Carleman estimate for a hyperbolic integro-differential equation. Next we apply such a result to identify a spatially dependent function in a source term by a single measurement of boundary data.

§1. Introduction and the main results.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and let $\nu = \nu(x)$ be the outward unit normal vector to $\partial\Omega$ at x, $\partial_\nu u = \nabla u \cdot \nu$. We consider a hyperbolic integro-differential equation:

$$\partial_t^2 u(x,t) = \mathrm{div}\,(p(x)\nabla u(x,t)) + \mathrm{div}\left(\int_0^t K(x,t,\eta)\nabla u(x,\eta)d\eta\right) + F(x,t), \tag{1.1}$$
$$x \in \Omega,\ t > 0.$$

Here $p \in C^2(\overline{\Omega})$, > 0 on $\overline{\Omega}$ and $K \in C^2(\overline{\Omega} \times [0,\infty)^2)$. We set $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1, 2, ..., n$, $\nabla_{x,t} = (\nabla, \partial_t) = (\partial_1, ..., \partial_n, \partial_t)$, $\Delta = \sum_{j=1}^n \partial_j^2$. Equation (1.1) appears in various cases such as viscoelasticity.

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One of the fundamental question for (1.1) is the unique continuation: if u satisfies (1.1) with $F \equiv 0$ and $u = \partial_\nu u = 0$ on $\Gamma \times (0, T)$ where $\Gamma \subset \partial\Omega$, then can we conclude that $u = 0$ in $U \times (T', T'')$ where U is a neighbourhood of Γ and $0 < T' < T'' < T$?

To prove the unique continuation and other application to inverse problems, a Carleman estimate is a main tool. In this paper, we will establish a Carleman estimate for (1.1), and apply it to determine an unknown source term. We stress that our result is the first step to determine $p(x)$ or an x-dependent function in $K(x, t, \eta)$. In a forthcoming paper, we discuss details for the unique continuation.

In addition to the assumption that $p \in C^2(\overline{\Omega})$, $p(x) > 0$ on $\overline{\Omega}$, throughout this paper, we suppose that there exists $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ such that

$$\frac{1}{2}p(x)^2 - (\nabla p(x) \cdot (x - x_0)) \geq 0, \qquad x \in \overline{\Omega}. \tag{1.2}$$

We set

$$\varphi(x, t) = |x - x_0|^2 - \beta t^2, \tag{1.3}$$

where $\beta > 0$ is a sufficiently small positive constant depending on Ω, p, x_0. Furthermore for a fixed $R > 0$, let

$$Q(\varepsilon) = \{(x, t) \in \Omega \times (0, T);\ \varphi(x, t) > R^2 + \varepsilon, t > 0\}$$

for $\varepsilon \geq 0$. Then we can show

Theorem 1 (Carleman estimate). *Let $p \in C^2(\overline{\Omega})$, $K \in C^2(\overline{\Omega} \times [0, T]^2)$ and let $u = u(x, t) \in H^2(Q(\varepsilon))$ satisfy*

$$\begin{aligned} Pu(x,t) &\equiv \partial_t^2 u(x,t) - div\,(p(x)\nabla u(x,t)) - div\left(\int_0^t K(x,t,\eta)\nabla u(x,\eta)d\eta\right) \\ &= F(x,t), \qquad x \in \Omega,\ t > 0 \end{aligned} \tag{1.4}$$

and

$$u(x,0) = 0 \quad or \quad \partial_t u(x,0) = K(x,0,0) = 0, \quad x \in Q(0) \cap \{t = 0\}. \tag{1.5}$$

Then there exists $s_0 > 0$ such that we can choose a constant $C = C(s_0) > 0$ which is independent of u and s, such that

$$\begin{aligned} &\int_{Q(\varepsilon)} (s|\nabla_{x,t}u|^2 + s^3u^2)e^{2s\varphi}dxdt \\ \le &C\int_{Q(\varepsilon)} |F|^2 e^{2s\varphi}dxdt + Ce^{Cs}\int_{\partial Q(\varepsilon)\setminus(Q(\varepsilon)\cap\{t=0\})} (|\nabla_{x,t}u|^2 + u^2)dS \end{aligned} \tag{1.6}$$

for any $s \ge s_0$.

Remark. In the weight function φ, we have to choose $\beta = \beta(\Omega, p, x_0) > 0$ sufficiently small. In paricular, if $p \equiv 1$, then we can choose any $\beta \in (0, 1)$.

Inequality (1.6) is called a Carleman estimate. Carleman estimates have been well known for elliptic, parabolic and hyperbolic operators (e.g., Hörmander [8], Isakov [12], [13], Klibanov and Timonov [20], Lavrent'ev, Romanov and Shishat·skiĭ[23]). However our system is involved with the integral term

$$\operatorname{div}\left(\int_0^t K(x,t,\eta)\nabla u(x,\eta)d\eta\right), \tag{1.7}$$

so that a Carleman estimate is not directly proved for (1.4) in the existing papers. In Yong and Zhang [31], one can find such an argument for the exact controllability but the result is not related with inverse problems. To treat the integral term (1.7), we have to assume the extra information (1.5). In other words, a usual Carleman estimate is proved for the extended domain

$$\{(x,t);\, x \in \Omega,\, \varphi(x,t) > R^2 + \varepsilon\},$$

but not for

$$\{(x,t);\, x \in \Omega,\, \varphi(x,t) > R^2 + \varepsilon\} \cap \{t > 0\},$$

so that for applying a usual Carleman estimate to an inverse problem in $t > 0$, we have to extend the solution u to $t < 0$. Such an extension requires an extra argument by term (1.7). On the contrary, for an inverse problem over a time interval $(0,T)$ under (1.5), we need not extend u to $(-T,0)$ and directly apply our Carleman estimate (1.6). This kind of Carleman estimates in $t > 0$ are derived by a pointwise inequality in Klibanov and Timonov [20], Lavrent'ev, Romanov and Shishat·skiĭ[23], and is quite different from the Carleman estimates in Hörmander [8], Isakov [12], [13], etc.

Next we will consider

The Inverse Source Problem. Let $\varepsilon > 0$ be arbitrarily fixed. Ler $R = R(x,t) \in W^{1,\infty}(0,T;L^\infty(\Omega))$ and let us consider

$$(Pu)(x,t) = R(x,t)f(x), \qquad x \in \Omega, \quad 0 < t < T, \tag{1.8}$$

$$u(x,0) = \partial_t u(x,0) = 0, \qquad x \in \Omega. \tag{1.9}$$

Then determine $f = f(x)$ in $\Omega(\varepsilon)$ from the knowledge of

$$u|_{\Gamma\times(0,T)}, \quad \partial_\nu|_{\Gamma\times(0,T)}.$$

Here Γ be some sub-boundary of $\partial\Omega$.

The problem to be solved is actually a sort of "double" Cauchy problem, since we are given Cauchy conditions on both $t = 0$ and Γ. Note that we are given only incomplete boundary conditions, since no conditions on u or $\partial_\nu u$ are prescribed on the whole of $\partial\Omega$.

We set

$$\Omega(\varepsilon) = Q(\varepsilon) \cap \{t = 0\}. \tag{1.10}$$

Let us assume

$$\overline{\Omega(0)} \subset \Omega \cup \Gamma. \tag{1.11}$$

Condition (1.11) follows if Ω is convex near Γ, that is, for any $x \in \Gamma$, there exists a neighbourhood U of x such that $U \cap \Omega$ is convex.

We are ready to state the stability result for our inverse source problem.

Theorem 2. *Let $u \in C^3([0,T]; L^2(\Omega)) \cap C^2([0,T]; H^1(\Omega)) \cap C^1([0,T]; H^2(\Omega))$ satisfy (1.8) and (1.9). We assume*

$$|R(x,0)| > 0, \qquad x \in \overline{\Omega} \tag{1.12}$$

and

$$T > \frac{\sup_{x\in\Omega(0)}(|x-x_0|^2 - R^2)^{\frac{1}{2}}}{\sqrt{\beta}}. \tag{1.13}$$

Then for any $\delta > 0$, there exist contants $C = C(\Omega, T, p, x_0, \beta, \delta, R) > 0$ and $\kappa = \kappa(\Omega, T, p, x_0, \beta, \delta, R) \in (0,1)$ such that

$$\begin{aligned} &\|f\|_{L^2(\Omega(\delta))} \le C(\|u\|_{H^1(Q(0))} + \|\partial_t u\|_{H^1(Q(0))} + \|f\|_{L^2(\Omega(0))})^{1-\kappa} \\ &(\|u\|_{H^1(\Gamma\times(0,T))} + \|\partial_t u\|_{H^1(\Gamma\times(0,T))})^{\kappa}. \end{aligned} \tag{1.14}$$

The factor $(\|u\|_{H^1(\Gamma\times(0,T))} + \|\partial_t u\|_{H^1(\Gamma\times(0,T))})$ is observation data and (1.14) shows the stability of Hölder type which is conditional under an a priori boundedness of $\|u\|_{H^1(Q(0))} + \|\partial_t u\|_{H^1(Q(0))} + \|f\|_{L^2(\Omega(0))}$.

Theorem 2 is derived from Theorem 1 by means of the method created by Bukhgeim and Klibanov [3]. As related works on inverse problems by Carleman estimates, see Bellassoued [1], Bukhgeim [2], Imanuvilov and Yamamoto [9] - [11], Isakov [12] - [14], Khaĭdarov [18], Klibanov [19], Klibanov and Timonov [20], Klibanov and Yamamoto [21], Kubo [22], Yamamoto [30] and the references therein. As for other inverse problems of determining time dependent factor in the kernel $K(x,t,\eta)$ and related inverse problems, see Cavaterra [4], Cavaterra and Grasselli [5], Cavaterra and Lorenzi [6], Janno and Lorenzi [15], Janno and von Wolfersdorf [16], Kabanikhin and Lorenzi [17], Lorenzi [24], Lorenzi and Messina [25], [26], Lorenzi and Romanov [27], Lorenzi and Yahkno [28], von Wolfersdorf [29] and the references therein.

The rest of this paper is composed of two sections: In Section 2, we will prove Theorem 1, while Section 3 is devoted to the proof of Theorem 2.

§2. Proof of Theorem 1.

Henceforth $C > 0$, $C_j > 0$ denote generic constants which are independent of $s > 0$, and may vary from line to line. Set

$$v(x,t) = p(x)u(x,t) + \int_0^t K(x,t,\eta)u(x,\eta)d\eta, \quad x \in \Omega,\ t > 0. \tag{2.1}$$

Then direct calculations yield

$$\begin{aligned} &\partial_t^2 v(x,t) = p(x)\Delta v(x,t) - p\nabla p \cdot \nabla u \\ &-p(x)\int_0^t \nabla K(x,t,\eta)\cdot\nabla u(x,\eta)d\eta + \left(\frac{\partial(K(x,t,t))}{\partial t} + (\partial_t K)(x,t,t) - p\Delta p\right)u \\ &+K(x,t,t)\partial_t u + \int_0^t (\partial_t^2 K(x,t,\eta) - p\Delta K)u(x,\eta)d\eta + pF, \quad x \in \Omega,\ t > 0 \end{aligned} \tag{2.2}$$

and

$$v(x,0)=0 \quad \text{or} \quad \partial_t v(x,0)=0, \qquad x\in\Omega(0). \tag{2.3}$$

Noting (2.3), to (2.2) we apply a pointwise Carleman estimate for a hyperbolic operator (Theorem 2.2.4 in Klibanov and Timonov [20, pp.45–46], or Lemma 2 in [23, p.128]) for the case of $p\equiv 1$, see also Cheng, Isakov, Yamamoto and Zhou [7]), so that for some postive constant s_0, we obtain

$$\begin{aligned}
&\int_{Q(\varepsilon)} (s|\nabla_{x,t}v|^2+s^3v^2)e^{2s\varphi}dxdt\\
\le& C\int_{Q(\varepsilon)}\left\{|p\nabla p\cdot\nabla u|^2+\left|\left(\frac{\partial(K(x,t,t))}{\partial t}+(\partial_t K)(x,t,t)-p\Delta p\right)u\right|^2\right.\\
&\left.+|K(x,t,t)\partial_t u|^2\right\}e^{2s\varphi}dxdt+C\int_{Q(\varepsilon)}|pF|^2e^{2s\varphi}dxdt\\
&+C\int_{Q(\varepsilon)}\left\{\left|p\int_0^t\nabla K(x,t,\eta)\cdot\nabla u(x,\eta)d\eta\right|^2\right.\\
&\left.+\left|\int_0^t((\partial_t^2K)(x,t,\eta)-p\Delta K)u(x,\eta)d\eta\right|^2\right\}e^{2s\varphi}dxdt\\
&+Ce^{Cs}\int_{\partial Q(\varepsilon)\setminus\Omega(\varepsilon)}(|\nabla_{x,t}v|^2+|v|^2)dS,
\end{aligned} \tag{2.4}$$

if $s\ge s_0$. Since $p\in C^2(\overline{\Omega})$ and $K\in C^2(\overline{\Omega}\times[0,T]^2)$, we have

$$\begin{aligned}
&\int_{Q(\varepsilon)}(s|\nabla_{x,t}v|^2+s^3v^2)e^{2s\varphi}dxdt\\
\le& C\int_{Q(\varepsilon)}(|\nabla_{x,t}u|^2+u^2)e^{2s\varphi}dxdt+C\int_{Q(\varepsilon)}|F|^2e^{2s\varphi}dxdt\\
&+C\int_{Q(\varepsilon)}\left(\int_0^t(|\nabla u(x,\eta)|^2+u(x,\eta)^2)d\eta\right)e^{2s\varphi}dxdt\\
&+Ce^{Cs}\int_{\partial Q(\varepsilon)\setminus\Omega(\varepsilon)}(|\nabla_{x,t}v|^2+s^3v^2)dS, \quad s\ge s_0.
\end{aligned} \tag{2.5}$$

We show

Lemma 1.

$$\int_{Q(\varepsilon)} \left(\int_0^t |w(x,\xi)|d\xi\right)^2 e^{2s\varphi}dxdt \le \frac{C}{s}\int_{Q(\varepsilon)} |w(x,t)|^2 e^{2s\varphi}dxdt$$

for $w \in L^2(Q(\varepsilon))$.

Lemma 1 is fundamental to derive a Carleman estimate for our inverse problem. We note that it was proved in Bukhgeim and Klibanov [3], Klibanov [19], but with a factor not containing $\frac{1}{s}$ at the right hand side. On the contrary, for our proof, the factor $\frac{1}{s}$ is essential. As for the proof of Lemma 1, see Lemma 3.1.1 (pp.77-78) in [20]. However, for completeness, we will give the proof of it in Appendix.

By (2.1) and $p > 0$ on $\overline{\Omega}$,

$$u(x,t) = \frac{1}{p(x)}v(x,t) - \int_0^t \frac{K(x,t,\eta)}{p(x)}u(x,\eta)d\eta. \tag{2.6}$$

Hence, in terms of Lemma 1, we have

$$\int_{Q(\varepsilon)} u^2 e^{2s\varphi}dxdt \le C\int_{Q(\varepsilon)} v^2 e^{2s\varphi}dxdt + \frac{C}{s}\int_{Q(\varepsilon)} u^2 e^{2s\varphi}dxdt.$$

Taking $s > 0$ sufficiently large, we can absorb the second term at the right hand side into the left hand side, and we have

$$\int_{Q(\varepsilon)} u^2 e^{2s\varphi}dxdt \le C\int_{Q(\varepsilon)} v^2 e^{2s\varphi}dxdt. \tag{2.7}$$

Similarly from (2.6), we obtain

$$\int_{Q(\varepsilon)} |\nabla_{x,t}u|^2 e^{2s\varphi}dxdt \le C\int_{Q(\varepsilon)} (|\nabla_{x,t}v| + v^2)e^{2s\varphi}dxdt, \quad s \ge s_0. \tag{2.8}$$

Hence, substituing (2.7) and (2.8) into the left hand side of (2.5) and applying

Lemma 1 to the third term at the right hand side of (2.5), we have

$$\begin{aligned}
&\int_{Q(\varepsilon)} (s|\nabla_{x,t}u|^2 + s^3u^2)e^{2s\varphi}dxdt \\
\le& C\int_{Q(\varepsilon)} (|\nabla_{x,t}u|^2 + u^2)e^{2s\varphi}dxdt + C\int_{Q(\varepsilon)} F^2e^{2s\varphi}dxdt \\
+& Ce^{Cs}\int_{\partial Q(\varepsilon)\setminus\Omega(\varepsilon)} (|\nabla_{x,t}v|^2 + v^2)dS.
\end{aligned} \tag{2.9}$$

By (2.1), we have

$$\begin{aligned}
&\|v\|_{L^2(\partial Q(\varepsilon))} \le C\|u\|_{L^2(\partial Q(\varepsilon))}, \\
&\|\nabla_{x,t}v\|_{L^2(\partial Q(\varepsilon))} \le C\|u\|_{H^1(\partial Q(\varepsilon))}.
\end{aligned} \tag{2.10}$$

Again taking $s > 0$ sufficiently large, we absorb the first term at the right hand side into the left hand side in (2.9), and we apply (2.10) to the third term at the right hand side of (2.9). Thus the proof of Theorem 1 is complete.

§3. Proof of Theorem 2.

The proof is based on the modification by Imanuvilov and Yamamoto [10] of the original method by Bukhgeim and Klibanov [3]. First we modify Theorem 1 as follows.

Corollary. *Let $u = u(x,t) \in H^2(Q(\varepsilon))$ satisfy (1.4) and $u(x,0) = 0$ for $x \in \Omega(0)$. Then there exist $s_0 > 0$ and a constant $C = C(s_0) > 0$ independent of u and s, such that*

$$\begin{aligned}
&\int_{Q(\varepsilon)} (s|\nabla_{x,t}u|^2 + s^3u^2)e^{2s\varphi}dxdt \le C\int_{Q(\varepsilon)} |F|^2e^{2s\varphi}dxdt \\
&+Ce^{Cs}\int_{\partial Q(\varepsilon)\cap(\Gamma\times(0,\infty))} (|\nabla_{x,t}u|^2 + u^2)dS + Cs^3e^{2s(R^2+3\varepsilon)}\|u\|^2_{H^1(Q(\varepsilon))}
\end{aligned} \tag{3.1}$$

for any $s \ge s_0$.

Proof of Corollay 1. Let $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ satisfy $0 \le \chi \le 1$ in $\mathbb{R}^{n+1}$ and

$$\chi(x,t) = \begin{cases} 1, & (x,t) \in Q(3\varepsilon), \\ 0, & (x,t) \in Q(\varepsilon) \setminus Q(2\varepsilon). \end{cases} \tag{3.2}$$

We set $v = \chi u$. Then $|v| = |\nabla_{x,t} v| = 0$ on $\partial Q(\varepsilon) \setminus \{(\Gamma \times (0,\infty)) \cup \Omega(\varepsilon)\}$ and $v = 0$ on $\Omega(\varepsilon)$. Therefore Theorem 1 yields

$$\begin{aligned} &\int_{Q(3\varepsilon)} (s|\nabla_{x,t} u|^2 + s^3|u|^2) e^{2s\varphi} dxdt \le \int_{Q(\varepsilon)} (s|\nabla_{x,t}(\chi u)|^2 + s^3|\chi u|^2) e^{2s\varphi} dxdt \\ \le& C \int_{Q(\varepsilon)} |F|^2 e^{2s\varphi} dxdt + Ce^{Cs} \int_{\partial Q(\varepsilon)\cap(\Gamma\times(0,\infty))} (|\nabla_{x,t}(\chi u)|^2 + |\chi u|^2) dS, \end{aligned} \tag{3.3}$$

for any $s \ge s_0$. Since

$$\begin{aligned} &\int_{Q(\varepsilon)} (s|\nabla_{x,t} u|^2 + s^3 u^2) e^{2s\varphi} dxdt \\ =& \left(\int_{Q(3\varepsilon)} + \int_{Q(\varepsilon)\setminus Q(3\varepsilon)} \right) (s|\nabla_{x,t} u|^2 + s^3 u^2) e^{2s\varphi} dxdt \end{aligned}$$

and $\chi = 1$ in $Q(3\varepsilon)$, $\varphi(x,t) \le R^2 + 3\varepsilon$ for $(x,t) \in Q(\varepsilon) \setminus Q(3\varepsilon)$, we have

$$\begin{aligned} &\int_{Q(\varepsilon)} (s|\nabla_{x,t} u|^2 + s^3 u^2) e^{2s\varphi} dxdt \\ \le& \int_{Q(3\varepsilon)} (s|\nabla_{x,t}(\chi u)|^2 + s^3|\chi u|^2) e^{2s\varphi} dxdt + Cs^3 e^{2s(R^2+3\varepsilon)} \|u\|^2_{H^1(Q(\varepsilon))}. \end{aligned}$$

Substituting this into (3.3), we complete the proof of Corollary.

Now we proceed to the proof of Theorem 2. By (1.13), there exists $\varepsilon > 0$ such that $|x - x_0|^2 - \beta T^2 < R^2 + \varepsilon$ for all $x \in \Omega(0)$. Hence $(x,t) \in \overline{Q(\varepsilon)}$ implies that $0 \le t \le T$. In particular,

$$(\Gamma \times (0,\infty)) \cap \partial Q(\varepsilon) \subset \Gamma \times (0,T).$$

For simplicity, we set

$$\begin{cases} D = \|u\|^2_{H^1(\Gamma\times(0,T))} + \|\partial_t u\|^2_{H^1(\Gamma\times(0,T))}, \\ M = \|u\|^2_{H^1(Q(0))} + \|\partial_t u\|^2_{H^1(Q(0))} + \|f\|^2_{L^2(\Omega(0))}. \end{cases} \tag{3.4}$$

Applying Corollary to (1.8), we have

$$\begin{aligned}&\int_{Q(\varepsilon)} (s|\nabla_{x,t}u|^2 + s^3u^2)e^{2s\varphi}dxdt \\ \le &C\int_{Q(\varepsilon)} |f|^2e^{2s\varphi}dxdt + Ce^{Cs}D + Cs^3e^{2s(R^2+3\varepsilon)}M, \quad s \ge s_0. \end{aligned} \tag{3.5}$$

On the other hand, (1.8) yields

$$\begin{aligned}\Delta u = &-\int_0^t \frac{K(x,t,\eta)}{p(x)}\Delta u(x,\eta)d\eta + \frac{1}{p}\partial_t^2 u(x,t) - \frac{\nabla p}{p}\cdot\nabla u(x,t) \\ &-\frac{1}{p}\int_0^t \nabla K(x,t,\eta)\cdot\nabla u(x,\eta)d\eta - \frac{1}{p}R(x,t)f(x), \quad (x,t)\in Q(\varepsilon).\end{aligned}$$

Therefore Lemma 1 implies

$$\begin{aligned}&\int_{Q(\varepsilon)} |\Delta u|^2e^{2s\varphi}dxdt \le \frac{C}{s}\int_{Q(\varepsilon)} |\Delta u|^2e^{2s\varphi}dxdt \\ &+C\int_{Q(\varepsilon)} |\partial_t^2 u|^2e^{2s\varphi}dxdt + C\int_{Q(\varepsilon)} |\nabla u|^2e^{2s\varphi}dxdt + C\int_{Q(\varepsilon)} f^2e^{2s\varphi}dxdt,\end{aligned}$$

that is,

$$\begin{aligned}&\int_{Q(\varepsilon)} |\Delta u|^2e^{2s\varphi}dxdt \le C\int_{Q(\varepsilon)} |\partial_t^2 u|^2e^{2s\varphi}dxdt \\ &+C\int_{Q(\varepsilon)} |\nabla u|^2e^{2s\varphi}dxdt + C\int_{Q(\varepsilon)} f^2e^{2s\varphi}dxdt, \quad s\ge s_0,\end{aligned} \tag{3.6}$$

by taking $s>0$ sufficiently large. Setting $v=\partial_t u$, we have

$$\begin{aligned}&\partial_t^2 v = \mathrm{div}\,(p(x)\nabla v) + \mathrm{div}\,(K(x,t,t)\nabla u(x,t)) \\ &+\mathrm{div}\left(\int_0^t \partial_t K(x,t,\eta)\nabla u(x,\eta)d\eta\right) + (\partial_t R)(x,t)f(x), \qquad (x,t)\in Q(\varepsilon)\end{aligned}$$

and $v(x,0) = 0$ for $x\in\Omega(\varepsilon)$. Hence we apply Corollary to the operator $\partial_t^2 -$

$\mathrm{div}\,(p(x)\nabla)$ to have

$$
\begin{aligned}
&\int_{Q(\varepsilon)} (s|\nabla_{x,t}\partial_t u|^2 + s^3|\partial_t u|^2)e^{2s\varphi}dxdt \\
\le& C\int_{Q(\varepsilon)} |f|^2 e^{2s\varphi}dxdt + C\int_{Q(\varepsilon)} |\mathrm{div}\,(K(x,t,t)\nabla u(x)|^2 e^{2s\varphi}dxdt \\
+&C\int_{Q(\varepsilon)} \left|\int_0^t \mathrm{div}\,(\partial_t K(x,t,\eta)\nabla u(x,\eta))d\eta\right|^2 e^{2s\varphi}dxdt \\
+&Ce^{Cs}D + Cs^3 e^{2s(R^2+3\varepsilon)}M \\
\le& C\int_{Q(\varepsilon)} |f|^2 e^{2s\varphi}dxdt + C\int_{Q(\varepsilon)} (|\nabla u|^2 + |\Delta u|^2)e^{2s\varphi}dxdt \\
+&Ce^{Cs}D + Cs^3 e^{2s(R^2+3\varepsilon)}M, \quad s \ge s_0,
\end{aligned}
\tag{3.7}
$$

where in the last inequality, we have used Lemma 1 again. Combining (3.5) - (3.7) and taking $s > 0$ sufficiently large, we obtain

$$
\begin{aligned}
&\int_{Q(\varepsilon)} (|\Delta u|^2 + s|\nabla_{x,t}u|^2 + s|\nabla_{x,t}\partial_t u|^2 + s^3|\partial_t u|^2 + s^3u^2)e^{2s\varphi}dxdt \\
\le& C\int_{Q(\varepsilon)} |f|^2 e^{2s\varphi}dxdt + Ce^{Cs}D + Cs^3 e^{2s(R^2+3\varepsilon)}M, \quad s \ge s_0.
\end{aligned}
\tag{3.8}
$$

We now set $z = \chi(\partial_t u)e^{s\varphi}$. Then by direct calculations, we can see that z satisfies the equation

$$
\begin{aligned}
&\partial_t^2 z - \mathrm{div}\,(p\nabla z) = \chi e^{s\varphi}\mathrm{div}\,(K(x,t,t)\nabla u) + \chi e^{s\varphi}\int_0^t \mathrm{div}\,(\partial_t K(x,t,\eta)\nabla u(x,\eta))d\eta \\
&+\chi e^{s\varphi}(\partial_t R)f + 2s\chi(\partial_t\varphi)(\partial_t^2 u)e^{s\varphi} - 2p\chi s(\nabla\varphi\cdot\nabla(\partial_t u))e^{s\varphi} \\
&+[s^2\{(\partial_t\varphi)^2 - p|\nabla\varphi|^2\} + s\{(\partial_t^2\varphi) - p\Delta\varphi - (\nabla\varphi\cdot\nabla p)\}]z \\
&+2(\partial_t^2 u)(\partial_t\chi)e^{s\varphi} + (\partial_t u)\{\partial_t^2\chi + 2s(\partial_t\chi)\partial_t\varphi\}e^{s\varphi} \\
&-2pe^{s\varphi}\nabla(\partial_t u)\cdot\nabla\chi - e^{s\varphi}\{p\Delta\chi + 2sp(\nabla\chi\cdot\nabla\varphi) + \nabla\chi\cdot\nabla p\}\partial_t u \equiv J.
\end{aligned}
$$

Then we have

$$
(3.9)\quad \begin{aligned}
&|J(x,t)| \le Ce^{s\varphi}(|\nabla u(x,t)| + |\Delta u(x,t)| + s|\nabla_{x,t}(\partial_t u)(x,t)| + s^2|\partial_t u|)\\
&+Ce^{s\varphi}|f(x)| + Ce^{s\varphi}\int_0^t (|\nabla u(x,\eta)| + |\Delta u(x,\eta)|)d\eta, \quad (x,t)\in Q(\varepsilon).
\end{aligned}
$$

Multiply $-\partial_t^2 z + \mathrm{div}\,(p\nabla z) = -J$ by $2\partial_t z$ and integrate over $Q(\varepsilon)$ to obtain

$$
(3.10)\quad \begin{aligned}
&-\int_{Q(\varepsilon)} 2(\partial_t^2 z)\partial_t z dxdt + \int_{Q(\varepsilon)} 2(\partial_t z)\mathrm{div}\,(p\nabla z)dxdt\\
=&-2\int_{Q(\varepsilon)} J(\partial_t z)dzdt.
\end{aligned}
$$

Henceforth let $(\nu, \nu_{n+1}) = (\nu_1, ..., \nu_n, \nu_{n+1})$ denote the unit outward normal vector to $\partial Q(\varepsilon)$. Hence, in terms of (1.9) and (3.2), we obtain $z = |\nabla_{x,t} z| = 0$ on $\partial Q(\varepsilon)\setminus((\Gamma\times(0,T))\cup\Omega(\varepsilon))$, $\nabla z = 0$ on $\Omega(\varepsilon)$, $\nu_{n+1} = 0$ on $\partial Q(\varepsilon)\cap(\Gamma\times(0,T))$ and $\nu = (0, ..., 0, -1)$ on $\Omega(\varepsilon)$. An integration by parts gives

$$
(3.11)\quad \begin{aligned}
&-\int_{Q(\varepsilon)} 2(\partial_t^2 z)\partial_t z dxdt + \int_{Q(\varepsilon)} 2(\partial_t z)\mathrm{div}\,(p\nabla z)dxdt\\
=&-\int_{Q(\varepsilon)} \partial_t(|\partial_t z|^2)dxdt - \int_{Q(\varepsilon)} p\partial_t(|\nabla z|^2)dxdt + \int_{\partial Q(\varepsilon)} 2(\partial_t z)p\nabla z\cdot\nu dS\\
=&-\int_{\Omega(\varepsilon)} |\partial_t z|^2\nu_{n+1} dS + 2\int_{\partial Q(\varepsilon)\cap(\Gamma\times(0,T))} p(\partial_t z)\nabla z\cdot\nu dS.
\end{aligned}
$$

On the other hand, we see that

$$
|\partial_t z(x,t)| \le Cs|\partial_t u(x,t)|e^{s\varphi} + C|\partial_t^2 u(x,t)|e^{s\varphi}, \quad (x,t)\in Q(\varepsilon).
$$

Therefore, by (3.9), we have

$$
\begin{aligned}
&\left|-2\int_{Q(\varepsilon)} J(\partial_t z)dxdt\right|\\
\le& C\int_{Q(\varepsilon)} (|\nabla u| + |\Delta u| + s|\nabla_{x,t}\partial_t u| + s^2|\partial_t u|)(|\partial_t^2 u| + s|\partial_t u|)e^{2s\varphi}dxdt\\
&+C\int_{Q(\varepsilon)} |f|(|\partial_t^2 u| + s|\partial_t u|)e^{2s\varphi}dxdt\\
&+C\int_{Q(\varepsilon)} e^{2s\varphi}(|\partial_t^2 u| + s|\partial_t u|)\left(\int_0^t (|\nabla u(x,\eta)| + |\Delta u(x,\eta)|)d\eta\right)dxdt.
\end{aligned}
$$

We apply the Cauchy-Schwarz inequality to obtain

$$s^2|\nabla_{x,t}\partial_t u||\partial_t u| \le s|\nabla_{x,t}\partial_t u|^2 + s^3|\partial_t u|^2,$$

$$|f|(|\partial_t^2 u| + s|\partial_t u|) \le 2|f|^2 + |\partial_t^2 u|^2 + s^2|\partial_t u|^2.$$

Hence we have

$$\begin{aligned}&\left|-2\int_{Q(\varepsilon)} J(\partial_t z)dxdt\right|\\ \le& C\int_{Q(\varepsilon)} (|\Delta u|^2 + s|\nabla_{x,t}u|^2 + s|\nabla_{x,t}\partial_t u|^2 + s^3|\partial_t u|^2)e^{2s\varphi}dxdt\\ +&C\int_{Q(\varepsilon)} |f|^2 e^{2s\varphi}dxdt.\end{aligned}$$

Hence inequality (3.8) yields

$$\begin{aligned}&\left|-2\int_{Q(\varepsilon)} J(\partial_t z)dxdt\right|\\ \le& C\int_{Q(\varepsilon)} |f|^2 e^{2s\varphi}dxdt + Ce^{Cs}D + Cs^3 e^{2s(3\varepsilon+R^2)}M, \quad s \ge s_0.\end{aligned} \tag{3.12}$$

Consequently we derive from (3.10) - (3.12) that

$$\begin{aligned}&\int_{\Omega(\varepsilon)} |\partial_t z(x,0)|^2 dx \le C\int_{\Gamma\times(0,T)} (|\partial_t z|^2 + |\nabla z|^2)dS\\ +&C\int_{Q(\varepsilon)} |f|^2 e^{2s\varphi}dxdt + Ce^{Cs}D + Cs^3 e^{2s(3\varepsilon+R^2)}M\\ \le& C\int_{Q(\varepsilon)} |f|^2 e^{2s\varphi}dxdt + Ce^{Cs}D + Cs^3 e^{2s(3\varepsilon+R^2)}M, \quad s \ge s_0.\end{aligned} \tag{3.13}$$

By (1.8) and (1.9), we have

$$(\partial_t z)(x,0) = \chi(x,0)(\partial_t^2 u)(x,0)e^{s\varphi(x,0)} = \chi(x,0)R(x,0)f(x)e^{s\varphi(x,0)}$$

for $x \in \Omega(\varepsilon)$. Hence (1.12), (3.2) and (3.13) imply

$$\begin{aligned}&\int_{\Omega(3\varepsilon)} |f|^2 e^{2s\varphi(x,0)}dx \le C\int_{\Omega(\varepsilon)} |\partial_t z(x,0)|^2 dx\\ \le& C\int_{Q(\varepsilon)} |f|^2 e^{2s\varphi}dxdt + Ce^{Cs}D + Cs^3 e^{2s(3\varepsilon+R^2)}M, \quad s \ge s_0.\end{aligned}$$

We have

$$C\int_{Q(\varepsilon)} f^2 e^{2s\varphi} dxdt = \int_{\Omega(\varepsilon)} |f(x)|^2 e^{2s\varphi(x,0)} \left(\int_0^{(|x-x_0|^2-R^2-\varepsilon)^{\frac{1}{2}}\beta^{-\frac{1}{2}}} e^{2s(\varphi(x,t)-\varphi(x,0))} dt \right) dx$$

$$\le \int_{\Omega(\varepsilon)} |f(x)|^2 e^{2s\varphi(x,0)} \left(\int_0^{\infty} e^{-2s\beta t^2} dt \right) dx = \frac{\sqrt{\pi}}{2\sqrt{2\beta}} \frac{1}{\sqrt{s}} \int_{\Omega(\varepsilon)} |f(x)|^2 e^{2s\varphi(x,0)} dx$$

$$\le \frac{C}{\sqrt{s}} \int_{\Omega(3\varepsilon)} |f(x)|^2 e^{2s\varphi(x,0)} dx + CMe^{2s(3\varepsilon+R^2)},$$

because

$$\int_{\Omega(\varepsilon)} |f(x)|^2 e^{2s\varphi(x,0)} dx = \left(\int_{\Omega(3\varepsilon)} + \int_{\Omega(\varepsilon)\setminus\Omega(3\varepsilon)} \right) |f(x)|^2 e^{2s\varphi(x,0)} dx$$

$$\le \int_{\Omega(3\varepsilon)} |f(x)|^2 e^{2s\varphi(x,0)} dx + \|f\|^2_{L^2(\Omega(\varepsilon))} e^{2s(3\varepsilon+R^2)}.$$

Therefore

$$\int_{\Omega(3\varepsilon)} |f|^2 e^{2s\varphi(x,0)} dx \le \frac{C}{\sqrt{s}} \int_{\Omega(3\varepsilon)} |f|^2 e^{2s\varphi(x,0)} dx + Ce^{Cs}D + Cs^3 e^{2s(3\varepsilon+R^2)}M.$$

Hence for sufficiently large $s > 0$, we obtain

$$\int_{\Omega(3\varepsilon)} |f|^2 e^{2s\varphi(x,0)} dx \le Ce^{Cs}D + Cs^3 e^{2s(3\varepsilon+R^2)}M, \quad s \ge s_0.$$

Consequently

$$e^{2s(4\varepsilon+R^2)} \|f\|^2_{L^2(\Omega(4\varepsilon))} \le \int_{\Omega(3\varepsilon)} |f(x)|^2 e^{2s\varphi(x,0)} dx \le Ce^{Cs}D + Cs^3 e^{2s(3\varepsilon+R^2)}M, \quad s \ge s_0,$$

that is,

$$\|f\|^2_{L^2(\Omega(4\varepsilon))} \le Ce^{Cs}D + Cs^3 e^{-2\varepsilon s}M \le Ce^{Cs}D + Ce^{-\varepsilon s}M, \quad s \ge s_0, \tag{3.14}$$

for a suitable $C > 0$. Then we replace $C > 0$ with Ce^{Cs_0} so that (3.14) holds for all $s > 0$. Without loss of generality, we can assume $M > D$. Finally, choosing $s = \frac{1}{C+\varepsilon} \log \frac{M}{D} > 0$, from (3.14), we obtain

$$\|f\|^2_{L^2(\Omega(4\varepsilon))} \le 2CM^{\frac{C}{C+\varepsilon}} D^{\frac{\varepsilon}{C+\varepsilon}}.$$

Choosing $\delta = 4\varepsilon$ concludes the proof of Theorem 2.

Appendix. Proof of Lemma 1.

First we have

$$te^{2s\varphi(x,t)} = -\frac{1}{4\beta s}\partial_t(e^{2s\varphi}).$$

Therefore, by the Cauchy-Schewarz inequality, we obtain

$$\begin{aligned}
&\int_{Q(\varepsilon)} \left(\int_0^t |w(x,\xi)|d\xi\right)^2 e^{2s\varphi} dxdt \\
\le &\int_{Q(\varepsilon)} t \left(\int_0^t |w(x,\xi)|^2 d\xi\right) e^{2s\varphi} dxdt \\
\le &\int_{|x-x_0|>\sqrt{R^2+\varepsilon}} \left\{\int_0^{\ell(x)} -\frac{1}{4\beta s}\partial_t(e^{2s\varphi}) \left(\int_0^t |w(x,\xi)|^2 d\xi\right) dt\right\} dx.
\end{aligned}$$

Here we set $\ell(x) = \left(\frac{|x-x_0|^2-R^2-\varepsilon}{\beta}\right)^{\frac{1}{2}}$. An integration by parts yields

$$\begin{aligned}
&\int_{Q(\varepsilon)} \left(\int_0^t |w(x,\xi)|d\xi\right)^2 e^{2s\varphi} dxdt \\
\le &\frac{1}{4\beta s}\left\{-e^{2s(R^2+\varepsilon)} \int_{|x-x_0|>\sqrt{R^2+\varepsilon}} \left(\int_0^{\ell(x)} |w(x,\xi)|^2 d\xi\right) dx + \int_{Q(\varepsilon)} |w(x,t)|^2 e^{2s\varphi} dxdt\right\} \\
\le &\frac{1}{4\beta s}\int_{Q(\varepsilon)} |w(x,t)|^2 e^{2s\varphi} dxdt.
\end{aligned}$$

The proof of Lemma 1 is complete.

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