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A nontrivial algebraic cycle in the Jacobian variety of the Klein quartic

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A NONTRIVIAL ALGEBRAIC CYCLE IN THE JACOBIAN VARIETY OF THE KLEIN QUARTIC

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ABSTRACT. We prove some value of the harmonic volume for the Klein quartic C is nonzero modulo $\frac{1}{2}\mathbb{Z}$, using special values of the generalized hypergeometric function ${}_3F_2$. This result tells us the algebraic cycle $C - C^-$ is not algebraically equivalent to zero in the Jacobian variety $J(C)$.

1. INTRODUCTION

Let X be a compact Riemann surface of genus $g \geq 2$ and $J(X)$ its Jacobian variety. By the Abel-Jacobi map $X \to J(X)$, X is embedded in $J(X)$. The algebraic 1-cycle $X - X^-$ in $J(X)$ is homologous to zero. Here we denote by X^- the image of X under the multiplication map by -1. If X is hyperelliptic, $X = X^-$ in $J(X)$. For the rest of this paper, suppose $g \geq 3$. B. Harris [5] studied the problem whether the cycle $X - X^-$ in $J(X)$ is algebraically equivalent to zero or not. The harmonic volume I for X was introduced by Harris [4], using Chen's iterated integrals [2]. Let H denote the first integral homology group $H_1(X;\mathbb{Z})$ of X. The harmonic volume I is defined to be a homomorphism $(H^{\otimes 3})' \to \mathbb{R}/\mathbb{Z}$. Here $(H^{\otimes 3})'$ is a certain subgroup of $H^{\otimes 3}$. See Section 2 for the definition. Let ω be a third tensor product of holomorphic 1-forms on X. Suppose that $\omega + \overline{\omega}$ and $(\omega - \overline{\omega})/\sqrt{-1}$ belong to $(H^{\otimes 3})'$. If the cycle $X - X^-$ is algebraically equivalent to zero, then twice the values at both $\omega + \overline{\omega}$ and $(\omega - \overline{\omega})/\sqrt{-1}$ of the harmonic volume are zero modulo $\mathbb{Z}$. Harris proved twice the value at $\omega + \overline{\omega}$ of the harmonic volume for the Fermat quartic $F(4)$ are nonzero modulo $\mathbb{Z}$. This implies $F(4) - F(4)^-$ is not algebraically equivalent to zero in $J(F(4))$ ([5], [6]). Ceresa [1] showed that $X - X^-$ is not algebraically equivalent to zero for a generic X. We know few explicit nontrivial examples except for $F(4)$. Let C denote the Klein quartic. See Section 4.1 for the definition. The aim of this paper is to show

Theorem 4.14. *The algebraic cycle $C - C^-$ is not algebraically equivalent to zero in the Jacobian variety $J(C)$.*

Since Harris used the special feature of $F(4)$ that its normalized period matrix has entries in $\mathbb{Z}[\sqrt{-1}]$, it is not difficult to find some ω so that $\omega + \overline{\omega}$ and $(\omega - \overline{\omega})/\sqrt{-1}$ belong to $(H^{\otimes 3})'$ for $F(4)$. But, in general, it is not easy to find such an ω. For the Klein quartic C, we prove $(D + \overline{D})/7$ and $(D - \overline{D})/\sqrt{-7}$ belong to $(H^{\otimes 3})'$ (Proposition 4.7). See Section 4.3 for the definitions of them. In Theorem 4.9 we compute the value at $(D - \overline{D})/\sqrt{-7} \in (H^{\otimes 3})'$ of the harmonic volume for C

$$I((D - \overline{D})/\sqrt{-7}) = \frac{28}{\sqrt{-7}}\left(\frac{\zeta_7^2 - \zeta_7^6}{\zeta_7 + 1}x_{1,2} + \frac{\zeta_7^4 - \zeta_7^5}{\zeta_7^2 + 1}x_{2,3} + \frac{\zeta_7 - \zeta_7^3}{\zeta_7^4 + 1}x_{3,1}\right) \bmod \mathbb{Z}.$$

Here, $\zeta_7 = \exp(2\pi\sqrt{-1}/7)$ and $x_{i,j}$'s are real constants obtained from some special values of the generalized hypergeometric function ${}_3F_2$ (Lemma 4.13). By numerical computation using MATHEMATICA, we obtain Theorem 4.14. We give a calculation program in Appendix.

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2. The harmonic volume

We recall the harmonic volume for a compact Riemann surface X of genus $g \geq 3$ [4]. We identify the first integral homology group $H_1(X;\mathbb{Z})$ of X with the first integral cohomology group by Poincaré duality, and denote it by H. Moreover we identify H with the space of all the real harmonic 1-forms on X with integral periods. Let K be the kernel of the intersection pairing $(\ ,\ )\colon H \otimes_{\mathbb{Z}} H \to \mathbb{Z}$. For the rest of this paper, we write $\otimes = \otimes_{\mathbb{Z}}$, unless otherwise stated. The Hodge star operator $*$ on the space of all the 1-forms $A^1(X)$ is locally given by $*(f_1(z)dz + f_2(z)d\bar{z}) = -\sqrt{-1}f_1(z)dz + \sqrt{-1}f_2(z)d\bar{z}$ in a local coordinate z and depends only on the complex structure and not on the choice of Hermitian metric. For any $\sum_{i=1}^n a_i \otimes b_i \in K$, there exists a unique $\eta \in A^1(X)$ such that $d\eta = \sum_{i=1}^n a_i \wedge b_i$ and $\int_X \eta \wedge *\alpha = 0$ for any closed 1-form $\alpha \in A^1(X)$. Here a_i and b_i are regarded as real harmonic 1-forms on X. Choose a point $x_0 \in X$.

Definition 2.1. (The pointed harmonic volume [9])
For $\sum_{i=1}^n a_i \otimes b_i \in K$ and $c \in H$, the pointed harmonic volume defined to be

$$I_{x_0}\left(\left(\sum_{i=1}^n a_i \otimes b_i\right) \otimes c\right) = \sum_{i=1}^n \int_\gamma a_i b_i - \int_\gamma \eta \mod \mathbb{Z}.$$

Here $\eta \in A^1(X)$ is associated to $\sum_{i=1}^n a_i \otimes b_i$ in the way stated above and γ is a loop in X with the base point x_0 whose homology class is equal to c. The integral $\int_\gamma a_i b_i$ is Chen's iterated integral [2], that is, $\int_\gamma a_i b_i = \int_{0 \leq t_1 \leq t_2 \leq 1} f_i(t_1) g_i(t_2) dt_1 dt_2$ for $\gamma^* a_i = f_i(t)dt$ and $\gamma^* b_i = g_i(t)dt$. Here t is the coordinate in the interval $[0,1]$.

The harmonic volume is given as a restriction of the pointed harmonic volume I_{x_0}. We denote by $(H^{\otimes 3})'$ the kernel of a natural homomorphism $p\colon H^{\otimes 3} \to H^{\oplus 3}$ defined by $p(a \otimes b \otimes c) = ((a,b)c, (b,c)a, (c,a)b)$. The *harmonic volume* I for X is a linear form on $(H^{\otimes 3})'$ with values in $\mathbb{R}/\mathbb{Z}$ defined by the restriction of I_{x_0} to $(H^{\otimes 3})'$, i.e., $I = I_{x_0}|_{(H^{\otimes 3})'}$. Harris [4] proved that the harmonic volume I is independent of the choice of the base point x_0. We have $I(\sum_i h_{\sigma(1),i} \otimes h_{\sigma(2),i} \otimes h_{\sigma(3),i}) = \mathrm{sgn}(\sigma) I(\sum_i h_{1,i} \otimes h_{2,i} \otimes h_{3,i}) \mod \mathbb{Z}$, where $\sum_i h_{1,i} \otimes h_{2,i} \otimes h_{3,i} \in (H^{\otimes 3})'$ and σ is an element of the third symmetric group S_3. See Harris [4] and Pulte [9] for details.

In general, it is difficult to compute the correction term η in Definition 2.1. If X is a hyperelliptic curve, we have an explicit formula for the 1-form η given by Harris [4]. This allows us to calculate the harmonic volumes for all the hyperelliptic curves (Tadokoro [11]). In this paper, we deal with the case η vanishes.

3. The algebraic cycle $X - X^-$ and an intermediate Jacobian

We review a relation between the algebraic cycle $X - X^-$ and the harmonic volume I.

Let $j_2 \colon H^{\otimes 3} \to \wedge^3 H$ be a natural homomorphism $j_2(a \otimes b \otimes c) = a \wedge b \wedge c$, where $\wedge^3 H$ denotes the third exterior power of H. We have the homomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (H^{\otimes 3})' & \longrightarrow & H^{\otimes 3} & \xrightarrow{p} & H^{\oplus 3} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_3} & & \\ 0 & \longrightarrow & (\wedge^3 H)' & \longrightarrow & \wedge^3 H & \xrightarrow{\bar{p}} & H & \longrightarrow & 0, \end{array}$$

where $j_3(a,b,c) = a+b+c$, $\bar{p}(a \wedge b \wedge c) = (a,b)c + (b,c)a + (c,a)b$ and j_1 is the restriction homomorphism of j_2 to $(H^{\otimes 3})'$. Let $\mathscr{A}_0^k(J)$ be the space of algebraic k-cycles homologous to zero on the Jacobian variety $J = J(X)$, modulo rational equivalence. The Abel-Jacobi map of Griffiths $\Phi_{\mathbb{R}} \colon \mathscr{A}_0^k(J) \to \mathrm{Hom}_{\mathbb{Z}}(H^{2k+1}(J;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$ is defined by

$$\partial W \mapsto \left\{ \omega \mapsto \int_W \omega \right\},$$

where ω is a harmonic $(2k+1)$-form on J with integral periods (Section 4 in [9]). Here, the module $\mathrm{Hom}_{\mathbb{Z}}(H^{2k+1}(J;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$ can be identified with an intermediate Jacobian of $H_{2k+1}(J;\mathbb{Z})$ [9]. From now on, we consider the case $k=1$. Let ν denote the Abel-Jacobi image $\Phi_{\mathbb{R}}(X - X^-)$. Harris (Proposition 2.1 in [6], [4]) proved that $(\wedge^3 H)'$ can be identified with the primitive subgroup of $H^3(J;\mathbb{Z})$ in the sence of Lefchetz, denoted by $H^3_{\mathrm{prim}}(J;\mathbb{Z})$. Using this identification and the natural projection $\mathrm{Hom}_{\mathbb{Z}}(H^3(J;\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H^3_{\mathrm{prim}}(J;\mathbb{Z}), \mathbb{R}/\mathbb{Z})$, we consider ν as an element of $\mathrm{Hom}_{\mathbb{Z}}((\wedge^3 H)', \mathbb{R}/\mathbb{Z})$ (Section 4 and 6 in [9]).

Theorem 3.1. (Harris [4], [6]). *The Abel-Jacobi image ν satisfies the commutative diagram*

$$\begin{array}{ccc} (H^{\otimes 3})' & \xrightarrow{2I} & \mathbb{R}/\mathbb{Z} \\ \downarrow{\scriptstyle j_1} & \nearrow{\scriptstyle \nu} & \\ (\wedge^3 H)'. & & \end{array}$$

We say the algebraic cycle $X - X^-$ is *algebraically equivalent to zero in* J if there exists a topological 3-chain W such that $\partial W = X - X^-$ and W lies on S, where S is an algebraic (or complex analytic) subset of J of complex dimension 2 (Harris [6]). The chain W is unique up to 3-cycles. We denote by $H^{1,0}$ the space of all the holomorphc 1-forms on X. From [5], 2.6 in [6] and 533-534 in [13], we have

Proposition 3.2. *Let $\omega \in \left(H^{1,0}\right)^{\otimes_{\mathbb{C}} 3}$ satisfying that $\omega + \overline{\omega}$ and $(\omega - \overline{\omega})/\sqrt{-1} \in (H^{\otimes 3})'$. If $X - X^-$ is algebraically equivalent to zero in J, then twice the values at both $\omega + \overline{\omega}$ and $(\omega - \overline{\omega})/\sqrt{-1}$ of the harmonic volume are zero modulo $\mathbb{Z}$.*

Proof. Since $X - X^-$ is algebraically equivalent to zero in J, there exist a 3-chain W and an algebraic subset S satisfying the above conditions. Let $H_{\mathbb{C}}$ denote $H \otimes \mathbb{C}$. Theorem 3.1 gives

$$2I(\omega + \overline{\omega}) = \int_W j_1(\omega + \overline{\omega}) \text{ and } 2I((\omega - \overline{\omega})/\sqrt{-1})) = \int_W j_1(\omega - \overline{\omega})/\sqrt{-1}.$$

It is clear that $j_1(\omega)$ and $j_1(\overline{\omega})$ are $(3,0)$ and $(0,3)$-form in $H^3(J;\mathbb{C}) = \wedge^3 H_{\mathbb{C}}$ respectively. Since $\dim_{\mathbb{C}} S = 2$, the restriction of them to S are clearly zero. □

If twice the value at $\omega + \overline{\omega}$ or $(\omega - \overline{\omega})/\sqrt{-1}$ of the harmonic volume is nonzero modulo $\mathbb{Z}$, then $X - X^-$ is not algebraically equivalent to zero in J. See Hain [3], Pirola [10] and their references for the algebraic cycle $X - X^-$ in J.

4. Some values of the harmonic volume for the Klein quartic

We compute some values of the harmonic volume for the Klein quartic to prove the main theorem (Theorem 4.14).

4.1. **A 1-dimensional homology basis of the Klein quartic.** We denote by C the Klein quartic which is, by definition, the plane curve $C\colon = \{(X:Y:Z) \in \mathbb{C}P^2; X^3Y+Y^3Z+Z^3X = 0\}$. It is a compact Riemann surface of genus 3. It is known that the holomorphic automorphism group of C, $\mathrm{Aut}(C)$, is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_7)$. See [7] for the details of the Klein quartic. Let x and y denote $X^3Y^{-2}Z^{-1}+1$ and $-XY^{-1}$ respectively. The equation $X^3Y+Y^3Z+Z^3X=0$ induces $y^7 = x(1-x)^2$. The holomorphic map $\pi : C \to \mathbb{C}P^1$ is defined by $\pi(x,y) = x$, which is a 7-sheeted covering $C \to \mathbb{C}P^1$, branched over 3 branch points $\{0,1,\infty\}$. Let ζ_7 denote $\exp(2\pi\sqrt{-1}/7)$. For $t \in [0,1]$, we define a loop $e_0 : [0,1] \to C$ by $e_0(t) = (t, y_0(t))$, where $y_0(t)$ is a real analytic function $\sqrt[7]{t(1-t)^2}$. Let $\sigma\colon C \to C$ be a holomorphic automorphism $\sigma(x,y) = (x, \zeta_7 y)$. For $k = 0,1,\ldots,6$, we define loops in C by $c_k = \sigma_*^k(e_0)\cdot e_0^{-1}$. We denote $\ell_k = \sigma_*^{k-1}(e_0)\cdot\sigma_*^k(e_0)^{-1}, k = 0,1,\ldots,7$. The loop ℓ_0 can be identified with ℓ_7. By abuse of notation, the homology classes of c_k and ℓ_k are denoted by c_k and $\ell_k \in H_1(C;\mathbb{Z})$ respectively. Let $(\ ,\ ) : H_1(C;\mathbb{Z}) \otimes H_1(C;\mathbb{Z}) \to \mathbb{Z}$ be the intersection pairing, i.e., a non-degenerate bilinear form on $H_1(C;\mathbb{Z})$. Tretkoff and Tretkoff [12] proved

$$(c_1, c_k) = \begin{cases} 0 & \text{if} \quad k = 1,2,4,6, \\ 1 & \text{if} \quad k = 3,5, \end{cases}$$

using the Hurwitz system of the branched covering π. By the definition of ℓ_k, we have

$$(\ell_1, \ell_k) = (c_1, c_k) - (c_1, c_{k-1}) = \begin{cases} 0 & \text{if} \quad k = 1,2, \\ 1 & \text{if} \quad k = 3,5, \\ -1 & \text{if} \quad k = 4,6. \end{cases}$$

Moreover, we obtain that $\sigma_*(\ell_k) = \ell_{k+1}$ and $(\ell_i, \ell_j) = (\sigma_*(\ell_i), \sigma_*(\ell_j)) = (\ell_{i+1}, \ell_{j+1})$. The intersection matrix K' of $\ell_k, k = 1,2,\ldots,6$ is given by

$$\begin{pmatrix} 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ -1 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \end{pmatrix},$$

i.e., its (i,j)-th entry is (ℓ_i, ℓ_j). It is easy to prove $\det K' = 1$ and $\{\ell_k\}_{k=1,2,\ldots,6} \subset H_1(C;\mathbb{Z})$ is a basis of $H_1(C;\mathbb{Z})$.

4.2. **Poincaré dual of the Klein quartic.** Let ω_1', ω_2' and ω_3' be holomorphic 1-forms on C, $(1-x)dx/y^6, (1-x)dx/y^5$ and dx/y^3 respectively. It is known that $\{\omega_i'\}_{i=1,2,3}$ is a basis of the space of all the holomorphic 1-forms on C. The beta function $B(u,v)$ is defined by $\int_0^1 t^{u-1}(1-t)^{v-1}dt$ for $u,v > 0$. We denote $(h_1,h_2,h_3,h_4) = (1/7, 2/7, 4/7, 1/7)$ and $\xi_i = \zeta_7^{7h_i}$. From the equations $\sigma^*\omega_i' = \xi_i\omega_i$ and $\int_{e_0} \omega_i' = B(h_i, h_{i+1})$, we have

Lemma 4.1.

$$\int_{\ell_k} \omega_i' = (\xi_i^{k-1} - \xi_i^k) B(h_i, h_{i+1}).$$

Remark 4.2. These integrals depend only on the cohomology class of ω_j' and the homology class of ℓ_k.

We set $B'_i = B(h_i, h_{i+1})$ and $\omega_i = \omega'_i/B'_i$, $i = 1, 2, 3$. We write $L_k := \sum_{i=1}^{7} \zeta_7^{ik}\ell_k \in H_1(C;\mathbb{C})$ and denote the Poincaré dual by P.D.: $H^1(C;\mathbb{C}) \to H_1(C;\mathbb{C})$.

Proposition 4.3. *We denote $\lambda_i = -1/(\xi_i^3(\xi_i^2+1)) \in \mathbb{C}$. Then, we have*

$$\mathrm{P.D.}(\omega_i) = \lambda_i L_{7h_i}.$$

Proof. Since $\sigma_*(\ell_k) = \ell_{k+1}$, we obtain $\sigma_* L_k = \zeta_7^{-k} L_k$. The eigenvalues and eigenvectors of the action of σ on the $\mathbb{C}$-vector space $H_1(C;\mathbb{C})$ are ζ_7^{-k} and L_k for $k = 1, 2, \ldots, 6$. We have

$$\sigma_*(\mathrm{P.D.}(\omega_i)) = \mathrm{P.D.}((\sigma^{-1})^*\omega_i) = \xi_i^{-1}\mathrm{P.D.}(\omega_i) = \zeta_7^{-7h_i}\mathrm{P.D.}(\omega_i).$$

There exists a constant $\lambda_i \in \mathbb{C}$ such that $\mathrm{P.D.}(\omega_i) = \lambda_i L_{7h_i}$. The result follows from Lemma 4.1 and the equation

$$\int_{\ell_1} \omega_i = (\mathrm{P.D.}(\omega_i), \ell_1) = (\lambda_i L_{7h_i}, \ell_1) = \lambda_i (L_{7h_i}, \ell_1) = -\lambda_i(1-\xi_i)(\xi_i^3(\xi_i^2+1)).$$

□

Remark 4.4. We have $\mathrm{P.D.}(\overline{\omega}_i) = \overline{\lambda}_i \overline{L}_{7h_i}$. It immediately follows $\lambda_1\lambda_2\lambda_3 = -1$.

4.3. **Some values of the harmonic volume for the Klein quartic.** For $t \in [0,1]$, let f_i be a real 1-form on $[0,1]$ defined by $e_0^*\omega'_i = t^{h_i-1}(1-t)^{h_{i+1}-1}dt$, $i = 1, 2, 3$. Let $x_{i,j}$ denote an iterated integral $\int_{e_0} \omega_i\omega_j = \int_\gamma f_i f_j \Big/ (B'_i B'_j)$. Here, γ is the path $[0,1] \ni t \mapsto t \in [0,1]$. We compute the iterated integrals of ω_1, ω_2 and ω_3 along the loop ℓ_k.

Lemma 4.5. *We consider ℓ_k as loops with the base point $(x,y) = (0,0) \in C$. We have*

$$\int_{\ell_k} \omega_i\omega_j = (\xi_i\xi_j)^{k-1}(1-\xi_i\xi_j)x_{i,j} + (\xi_i\xi_j)^{k-1}(\xi_i\xi_j - \xi_j).$$

Remark 4.6. Since ω_i is closed and $\omega_i \wedge \omega_j = 0$, these iterated integrals are invariant under homotopy with fixed endpoints.

Proof. Using the shuffle product formula (Chen [2], 1.6) and the equations

$$0 = \int_{e_0\cdot e_0^{-1}} \omega_i\omega_j = \int_{e_0} \omega_i\omega_j + \int_{e_0^{-1}} \omega_i\omega_j + \int_{e_0}\omega_i \int_{e_0^{-1}} \omega_j \quad \text{and} \quad \int_{e_0}\omega_i = \int_{e_0} \omega'_i \Big/ B'_i = 1,$$

we have

$$\begin{aligned}
\int_{\ell_k} \omega_i\omega_j &= \int_{\sigma_*^{k-1}(e_0)\cdot\sigma_*^k(e_0)^{-1}} \omega_i\omega_j \\
&= \int_{\sigma_*^{k-1}(e_0)} \omega_i\omega_j + \int_{\sigma_*^k(e_0)^{-1}} \omega_i\omega_j + \int_{\sigma_*^{k-1}(e_0)} \omega_i \int_{\sigma_*^k(e_0)^{-1}} \omega_j \\
&= (\xi_i\xi_j)^{k-1} \int_{e_0} \omega_i\omega_j + (\xi_i\xi_j)^k \int_{e_0^{-1}} \omega_i\omega_j - \xi_i^{k-1}\xi_j^k \int_{e_0}\omega_i \int_{e_0}\omega_j \\
&= (\xi_i\xi_j)^{k-1} \int_{e_0} \omega_i\omega_j + (\xi_i\xi_j)^k \left\{ -\int_{e_0} \omega_i\omega_j + \int_{e_0}\omega_i \int_{e_0}\omega_j \right\} - \xi_i^{k-1}\xi_j^k \\
&= (\xi_i\xi_j)^{k-1}(1-\xi_i\xi_j) \int_{e_0} \omega_i\omega_j + (\xi_i\xi_j)^{k-1}(\xi_i\xi_j - \xi_j).
\end{aligned}$$

□

The subset $\mathcal{H}$ of $(H^{\otimes 3})' \otimes \mathbb{R}$ is defined by $\{\omega + \overline{\omega}, (\omega - \overline{\omega})/\sqrt{-1}; \omega \in H^{1,0} \otimes_{\mathbb{C}} H^{1,0} \otimes_{\mathbb{C}} H^{1,0}\}$. We will find some elements of $\mathcal{H} \cap (H^{\otimes 3})'$. Let D and $\overline{D}$ denote $\sum_{\mu\in S_3} \mathrm{sgn}(\mu)\omega_{\mu(1)} \otimes_{\mathbb{C}} \omega_{\mu(2)} \otimes_{\mathbb{C}} \omega_{\mu(3)}$ and $\sum_{\mu\in S_3} \mathrm{sgn}(\mu)\overline{\omega}_{\mu(1)} \otimes_{\mathbb{C}} \overline{\omega}_{\mu(2)} \otimes_{\mathbb{C}} \overline{\omega}_{\mu(3)} \in (H_{\mathbb{C}})^{\otimes_{\mathbb{C}} 3}$ respectively. Using Proposition 4.3 and

Remark 4.4, D and $\overline{D}$ are identified with $-\sum_{\mu\in S_3}\mathrm{sgn}(\mu)L_{7h_{\mu(1)}}\otimes_{\mathbb{C}}L_{7h_{\mu(2)}}\otimes_{\mathbb{C}}L_{7h_{\mu(3)}}$ and $-\sum_{\mu\in S_3}\mathrm{sgn}(\mu)\overline{L}_{7h_{\mu(1)}}\otimes_{\mathbb{C}}\overline{L}_{7h_{\mu(2)}}\otimes_{\mathbb{C}}\overline{L}_{7h_{\mu(3)}}$ respectively. The coefficients of $\ell_p\otimes_{\mathbb{C}}\ell_q\otimes_{\mathbb{C}}\ell_r$ of D and $\overline{D}$ are

$$\alpha_{p,q,r}=-\begin{vmatrix}\zeta_7^p & \zeta_7^{2p} & \zeta_7^{4p}\\ \zeta_7^q & \zeta_7^{2q} & \zeta_7^{4q}\\ \zeta_7^r & \zeta_7^{2r} & \zeta_7^{4r}\end{vmatrix}\quad\text{and}\quad \overline{\alpha}_{p,q,r}=-\begin{vmatrix}\zeta_7^{6p} & \zeta_7^{5p} & \zeta_7^{3p}\\ \zeta_7^{6q} & \zeta_7^{5q} & \zeta_7^{3q}\\ \zeta_7^{6r} & \zeta_7^{5r} & \zeta_7^{3r}\end{vmatrix}$$

respectively. It is trivial that $D+\overline{D}$ and $(D-\overline{D})/\sqrt{-1}\in\mathcal{H}$. Furthermore, we have

Proposition 4.7. $(D+\overline{D})/7$ *and* $(D-\overline{D})/\sqrt{-7}\in(H^{\otimes 3})'$.

Proof. It suffices to prove that $\alpha_{p,q,r}$ belongs to the principal ideal $(\sqrt{-7})\mathbb{Z}[(1+\sqrt{-7})/2]\subset\mathbb{Z}[(1+\sqrt{-7})/2]$. It is well known that $\mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})\cong\{\sigma_i\}_{i=1,2,\dots,6}\cong\mathbb{Z}/6\mathbb{Z}$, where $\sigma_i(\zeta_7)=\zeta_7^i$. Since $[\mathbb{Q}(\sqrt{-7})\colon\mathbb{Q}]=2$, we obtain $\mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q}(\sqrt{-7}))$, the subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$, is generated by σ_2. It is clear that $\alpha_{p,q,r}$ is invariant under the action of σ_2. So, we have $\alpha_{p,q,r}\in\mathbb{Q}(\sqrt{-7})$. On the other hand, it immediately follows $\alpha_{p,q,r}$ belongs to the principal ideal $(\zeta_7-1)\mathbb{Z}[\zeta_7]\subset\mathbb{Z}[\zeta_7]$. Therefore, we have

$$\alpha_{p,q,r}\in\mathbb{Q}(\sqrt{-7})\cap(\zeta_7-1)\mathbb{Z}[\zeta_7]=(\sqrt{-7})\mathbb{Z}[(1+\sqrt{-7})/2]\subset\mathbb{Z}[(1+\sqrt{-7})/2].$$

We have $\alpha_{p,q,r}+\overline{\alpha}_{p,q,r}\in 7\mathbb{Z}$ and $\alpha_{p,q,r}-\overline{\alpha}_{p,q,r}\in\sqrt{-7}\mathbb{Z}$. We complete the proof. □

Remark 4.8. Using the character of $\mathrm{Aut}(C)=PSL_2(\mathbb{F}_7)$, we have $H^0(\mathrm{Aut}(C);(H_{\mathbb{C}})^{\otimes_{\mathbb{C}}3})=\mathbb{C}^2$. This induces $H^0(\mathrm{Aut}(C);H^{\otimes 3})=\mathbb{Z}^2$. We can also prove that $\{(D+\overline{D})/7,(D-\overline{D})/\sqrt{-7}\}$ is a generator of $H^0(\mathrm{Aut}(C);(H^{\otimes 3})')$.

Theorem 4.9. *The values at* $(D+\overline{D})/7$ *and* $(D-\overline{D})/\sqrt{-7}\in(H^{\otimes 3})'$ *for the harmonic volume of the Klein quartic* C *are given by*

$$0 \textit{ and } \frac{28}{\sqrt{-7}}\left(\frac{\zeta_7^2-\zeta_7^6}{\zeta_7+1}x_{1,2}+\frac{\zeta_7^4-\zeta_7^5}{\zeta_7^2+1}x_{2,3}+\frac{\zeta_7-\zeta_7^3}{\zeta_7^4+1}x_{3,1}\right)\bmod\mathbb{Z}$$

respectively.

Proof. All iterated integral parts of $I((D+\overline{D})/7)$ and $I((D-\overline{D})/\sqrt{-7})$ are linear combinations of $\int_{\ell_k}\omega_i\omega_j$ and $\int_{\ell_k}\overline{\omega}_i\overline{\omega}_j=\overline{\int_{\ell_k}\omega_i\omega_j}$. Furthermore, $\omega_i\wedge\omega_j=\overline{\omega}_i\wedge\overline{\omega}_j=0$. So we need no correction terms η in Definition 2.1. Therefore, it suffices to calculate only the iterated integral parts.

By definition, there exist complex constants $\theta_{i,j,k}$ so that $I((D+\overline{D})/7)$ is of the form

$$\sum_{k=1}^{7}\sum_{(i,j)\in U}\theta_{i,j,k}\int_{\ell_k}(\omega_i\omega_j-\omega_j\omega_i)+\sum_{k=1}^{7}\sum_{(i,j)\in U}\overline{\theta}_{i,j,k}\overline{\int_{\ell_k}(\omega_i\omega_j-\omega_j\omega_i)},$$

where U is a set $\{(1,2),(2,3),(3,1)\}$. Using $\mathrm{P.D.}(\omega_i)=\lambda_iL_{7h_i}=\lambda_i\sum_{k=1}^{7}\xi_i^k\ell_k$, it can be written as $(I_{1,2,3}+\overline{I}_{1,2,3})/7\bmod\mathbb{Z}$. Here, we denote

$$I_{1,2,3}=\lambda_3\sum_{k=1}^{7}\xi_3^k\int_{\ell_k}(\omega_1\omega_2-\omega_2\omega_1)+\lambda_1\sum_{k=1}^{7}\xi_1^k\int_{\ell_k}(\omega_2\omega_3-\omega_3\omega_2)+\lambda_2\sum_{k=1}^{7}\xi_2^k\int_{\ell_k}(\omega_3\omega_1-\omega_1\omega_3).$$

Similarly, we obtain

$$I((D-\overline{D})/\sqrt{-7})=(I_{1,2,3}-\overline{I}_{1,2,3})/\sqrt{-7}\bmod\mathbb{Z}.$$

In order to complete the proof, we need two lemmas.

Lemma 4.10. *We have*

$$\int_{\ell_k}(\omega_i\omega_j-\omega_j\omega_i)=2(\xi_i\xi_j)^{k-1}(1-\xi_i\xi_j)x_{i,j}+(\xi_i\xi_j)^{k-1}(\xi_i-1)(\xi_j+1).$$

Proof. We use Lemma 4.1, Lemma 4.5 and the equation

$$\int_{\ell_k}\omega_j\omega_i=-\int_{\ell_k}\omega_i\omega_j+\int_{\ell_k}\omega_i\int_{\ell_k}\omega_j.$$

□

Lemma 4.11. *We have*

$$I_{1,2,3}=14\Big(\frac{\zeta_7^2-\zeta_7^6}{\zeta_7+1}x_{1,2}+\frac{\zeta_7^4-\zeta_7^5}{\zeta_7^2+1}x_{2,3}+\frac{\zeta_7-\zeta_7^3}{\zeta_7^4+1}x_{3,1}-\frac{3}{2}\sqrt{-7}\Big).$$

Proof. Using Lemma 4.10 and $\xi_1\xi_2\xi_3=1$, we calculate the coefficient of $x_{1,2}$ of $I_{1,2,3}$ as follows:

$$\begin{aligned}\lambda_3\sum_{k=1}^{7}\xi_3^k\cdot 2(\xi_1\xi_2)^{k-1}(1-\xi_1\xi_2)=&\frac{-2}{\xi_3^3(\xi_3^2+1)}\sum_{k=1}^{7}(\xi_1\xi_2\xi_3)^{k-1}(\xi_3-1)\\=&\frac{-2}{\zeta_7^{12}(\zeta_7^8+1)}\sum_{k=1}^{7}(\zeta_7^4-1)=14\frac{\zeta_7^2-\zeta_7^6}{\zeta_7+1}.\end{aligned}$$

Similarly, we compute the coefficients of $x_{2,3},x_{3,1}$ and the constant term of $I_{1,2,3}$. For the computation of the constant term, we need $\zeta_7+\zeta_7^2+\zeta_7^4=(-1+\sqrt{-7})/2$. □

The result follows from Lemma 4.11. We remark that all the coefficients of $x_{1,2},x_{2,3},x_{3,1}$ and the constant term of $I_{1,2,3}$ are pure imaginary. □

For the numerical calculation of $x_{i,j}$, we recall the generalized hypergeometric function ${}_3F_2$. We denote the gamma function $\Gamma(\tau)=\int_0^\infty e^{-t}t^{\tau-1}dt$ for $\tau>0$ and $(\alpha,n)=\Gamma(\alpha+n)/\Gamma(\alpha)$ for non-negative integer n. For $x\in\{z\in\mathbb{C};|z|<1\}$ and $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2>-1$, the generalized hypergeometric function ${}_3F_2$ is defined by

$${}_3F_2\Big(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix};x\Big)=\sum_{n=0}^{\infty}\frac{(\alpha_1,n)(\alpha_2,n)(\alpha_3,n)}{(\beta_1,n)(\beta_2,n)(1,n)}x^n.$$

See [8] for example. By straightforward computation, we have

Proposition 4.12. *Let Δ be a 1-simplex $\{(u,v)\in\mathbb{R}^2;0\le v\le1,0\le u\le v\}$. If $a,b,p,q>0,b<1$, then we have*

$$\int_\Delta u^{a-1}(1-u)^{b-1}v^{p-1}(1-v)^{q-1}dudv=\frac{B(a+p,q)}{a}\lim_{\substack{t\to1-0\\ t\in\mathbb{R}}}{}_3F_2\Big(\begin{matrix}a,1-b,a+p\\ 1+a,a+p+q\end{matrix};t\Big).$$

From Proposition 4.12, we have

Lemma 4.13.

$$x_{i,j}=\frac{B(h_i+h_j,h_{j+1})}{h_iB_i'B_j'}\lim_{\substack{t\to1-0\\ t\in\mathbb{R}}}{}_3F_2\Big(\begin{matrix}h_i,1-h_{i+1},h_i+h_j\\ 1+h_i,h_i+h_j+h_{j+1}\end{matrix};t\Big).$$

Theorem 4.14. *Let C be the Klein quartic. Then, the cycle $C-C^-$ is not algebraically equivalent to zero in $J(C)$.*

Proof. By Theorem 4.9, Lemma 4.13, the numerical calculation (Figure 1 in Appendix), we obtain the value

$$2I((D-\overline{D})/\sqrt{-7}) = 0.72270 \pm 1 \times 10^{-5} \bmod \mathbb{Z}.$$

The result follows from Proposition 3.2. □

5. Appendix

In this section, we introduce the MATHEMATICA program [14] in the proof of Theorem 4.14.

```
z = Cos[(2 Pi) / 7] + i Sin[(2 Pi) / 7]
{h[1], h[2], h[3], h[4]} = {1 / 7, 2 / 7, 4 / 7, 1 / 7}
x[i_, j_] :=
 Beta[h[i] + h[j], h[j + 1]] / (h[i] * Beta[h[i], h[i + 1]] * Beta[h[j], h[j + 1]])
  HypergeometricPFQ[{h[i], 1 - h[i + 1], h[i] + h[j]}, {1 + h[i], h[i] + h[j] + h[j + 1]}, 1]
N[2 * (FullSimplify[28 (z^2 - z^6) / (i Sqrt[7] (z + 1))] x[1, 2] +
    FullSimplify[28 (z^4 - z^5) / (i Sqrt[7] (z^2 + 1))] x[2, 3] +
    FullSimplify[28 (z - z^3) / (i Sqrt[7] (z^4 + 1))] x[3, 1]), 20]
```

Figure 1. Numerical calculation program of Theorem 4.14

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