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Abstract

In this paper, we prove a Carleman estimate for a Kirchhoff plate equation and apply the Carleman estimate to inverse problems of determining spatially varying two Lamé coefficients and the mass density by a finite number of boundary observations.

Our main results are Lipschitz stability estimates for the inverse problems under suitable conditions of initial values and boundary val-

ues, which are satisfied, in particular, by paraboloid initial displacements.

1 Introduction and the main results.

We consider a classical model by Kirchhoff for flexural waves in a thin plate, whose governing equation is given as follows:

$$
\begin{aligned}
(Ly)(t,x) = (L_{\lambda,\mu,\rho}y)(t,x) \equiv \rho(x)\partial_t^2 y(t,x) + (\lambda(x)+\mu(x))\Delta^2 y(t,x) \\
+2\nabla(\lambda+\mu)(x)\cdot\nabla(\Delta y(t,x)) \\
+\Delta(\lambda+\mu)(x)\Delta y(t,x) + 2(\partial_1\partial_2\mu)(x)\partial_1\partial_2 y(t,x) \\
-(\partial_1^2\mu)(x)\partial_2^2 y(t,x) - (\partial_2^2\mu)(x)\partial_1^2 y(t,x) = H(t,x), \quad 0<t<T,\ x\in\Omega.
\end{aligned}
\tag{1.1}
$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $x = (x_1, x_2) \in \mathbb{R}^2$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$, $\nabla = (\partial_1, \partial_2)$, $\Delta = \partial_1^2 + \partial_2^2$. Moreover let $\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}$ with $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{N}\cup\{0\})^2$, $|\alpha| = \alpha_1 + \alpha_2$.

Physically $\rho = \rho(x)$ is the mass density per volume of the plate, and $\lambda = \lambda(x)$, $\mu = \mu(x)$ are Lamé coefficients and for more physical details of the flexural waves, see for example, Lagnese [32], Graff [12], Landau and Lifshitz [33], Lions and Lagnese [38].

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To (1.1) we attach initial and boundary conditions:

$$y(0,x) = a(x), \quad \partial_t y(0,x) = 0, \qquad x \in \Omega, \tag{1.2}$$

and

$$(B_1(\lambda,\mu)y)(t,x) = g(t,x), \quad (B_2(\lambda,\mu)y)(t,x) = h(t,x),$$
$$0 < t < T,\ x \in \partial\Omega, \tag{1.3}$$

where we set

$$B_1 y = B_1(\lambda,\mu)y = (\lambda+\mu)\Delta y + \mu(2\nu_1\nu_2\partial_1\partial_2 y - \nu_2^2\partial_1^2 y - \nu_1^2\partial_2^2 y) \quad \text{on } \partial\Omega, \tag{1.4}$$

and

$$B_2 y = B_2(\lambda,\mu)y = \partial_\nu[(\lambda+\mu)\Delta y] - \partial_1(\mu\partial_2^2 y)\nu_1 - \partial_2(\mu\partial_1^2 y)\nu_2$$
$$+\partial_1(\mu\partial_1\partial_2 y)\nu_2 + \partial_2(\mu\partial_1\partial_2 y)\nu_1$$
$$+\partial_\tau[\mu(\nu_1^2 - \nu_2^2)\partial_1\partial_2 y \ + \mu\nu_1\nu_2(\partial_2^2 y - \partial_1^2 y)] \quad \text{on } \partial\Omega. \tag{1.5}$$

Here and henceforth $\nu(x) = (\nu_1(x), \nu_2(x))$ denote the unit outward normal vector to $\partial\Omega$ at x, while $\tau(x) = (-\nu_2(x), \nu_1(x))$ is a unit tangential vector on $\partial\Omega$ at x, and we set $\partial_\nu y = \nabla y \cdot \nu$, $\partial_\tau y = \nabla y \cdot \tau$. In boundary condition (1.3) g and $-h$ describe the moment of action force and the action force per unit length on the boundary respectively (e.g., Landau and Lifshitz [33]). This kind of plate model can be derived by the principle of virtual work (e.g. Lagnese [32], Landau and Lifshitz [33], Lions and Lagnese [38]) or other kind of method (e.g. Graff [12]).

In this paper, we discuss

Inverse Problem. *Determine all or some of* $\lambda(x)$, $\mu(x)$ *and* $\rho(x)$ *in (1.1) by boundary measurements on* $(0,T)\times\partial\Omega$ *of solutions* y *to (1.1) - (1.3).*

Remark 1. The main results on the inverse problems in this paper are also valid for other kinds of boundary conditions. For example, in the case where boundary conditon (1.3) is replaced by $y = g$ and $\partial_\nu y = h$, we can discuss inverse problems similarly.

The plate equation (1.1) is a basic equation for thin plates such as wings of an airplane, and determination of λ, μ, ρ is practically important for example for the sake of evaluation of strength or the optimal design of the wing. Our main concern is the mathematical analysis, that is, the uniqueness and the stability for the inverse problem. We formulate our inverse problem. Let $y = y(\lambda, \mu, \rho; g, h, a, H)(t, x)$ denote the solution to (1.1) - (1.3) provided that we will specify the class of solutions later. Take k_0, $\mathcal{J} \in \mathcal{N}$ and choose a set of $\{g_j, h_j, a_j, H_j\}_{1\le j\le \mathcal{J}}$ of boundary values, initial values and force terms which are considered as inputs to an unknown system. Then determine $\lambda(x)$, $\mu(x)$, $\rho(x)$, $x \in \Omega$ by $\partial_\nu^k y(\lambda, \mu, \rho; g_j, h_j, a_j, H_j)(t, x)$, $0 < t < T$, $x \in \partial\Omega$, $1 \le j \le \mathcal{J}$, $0 \le k \le k_0$, which are observation data and regarded as outputs. In particular, we will search for stability estimates for the inverse problem: Estimate $\|\lambda_1 - \lambda_2\|_{H^2(\Omega)}$, $\|\mu_1 - \mu_2\|_{H^2(\Omega)}$ and/or $\|\rho_1 - \rho_2\|_{H^1(\Omega)}$ by suitable norms of $\partial_\nu^k y(\lambda_1, \mu_1, \rho_1; g_j, h_j, a_j, H_j) - \partial_\nu^k y(\lambda_2, \mu_2, \rho_2; g_j, h_j, a_j, H_j)$, $1 \le j \le \mathcal{J}$, $1 \le k \le k_0$.

The stability in inverse problems is not only important as mathematical subject but also for example for estimation of convergence rate of Tikhonov's regularization (e.g., Cheng and Yamamoto [11]).

The number $\mathcal{J}$ corresponds to the number of experiments where we suitably choose initial values, boundary values and force terms to execute the vibration processes and make observations. In the case where we want to determine several coefficients among λ, μ, ρ, we can expect that $\mathcal{J} = 1$, a single measurement, may not guarantee the uniqueness as well as the stability and we will look for the minimum $\mathcal{J}$. Moreover, in order that initial and boundary data and force terms are effective for our identification process, we have to require conditions on g_j, h_j, a_j, H_j, $1 \le j \le \mathcal{J}$. For example, we must not choose $a_1 = 0$, $g_1 = h_1 = 0$ and $H_1 = 0$, which can not stimulate the system at all. It is another interest to seek for such generous conditions for g_j, h_j, a_j, H_j.

We will consider two kinds of inverse problems:

Inverse Problem I. *We assume that $\rho = \rho(x)$ is known. Then determine $\lambda(x)$ and $\mu(x)$, $x \in \Omega$.*

Inverse Problem II. *Determine $\lambda(x)$, $\mu(x)$ and $\rho(x)$, $x \in \Omega$.*

Inverse Problem I is practically motivated by the fact that the mass density $\rho(x)$ can be determined by a different way (e.g., some static method) from the method for elastic properties $\lambda(x)$ and $\mu(x)$, and so it is worth an independent research.

For the statements of our main results, we need a set of admissible initial values, boundary values and force terms, and an admissible set of unknown coefficients. Henceforth $\epsilon_0 > 0$, $M > 0$, $x_0 = (x_1^0, x_2^0) \in \mathbb{R}^2 \setminus \overline{\Omega}$ be arbitrarily given and let $\lambda_0, \mu_0 \in C^{14}(\overline{\Omega})$, $\rho_0 \in C^{12}(\overline{\Omega})$ be given such that

$$\begin{cases} \lambda_0 + \mu_0, \mu_0, \rho_0 > \epsilon_0 & \text{on } \overline{\Omega} \\ \nabla \log\left(\frac{\rho_0(x)}{\lambda_0(x)+\mu_0(x)}\right) \cdot (x - x_0) > -1, & x \in \overline{\Omega}. \end{cases} \tag{1.6}$$

We note that if λ_0, μ_0, ρ_0 are positive constants or close to positive constants for example, then (1.6) is satisfied. We define an admissible set of initial values, boundary values and force terms.

$$\mathcal{V} = \{(g, h, a, H); \partial_t H(0, \cdot) = 0 \text{ in } \Omega,$$
$$\|y(\lambda_0, \mu_0, \rho_0; g, h, a, H)\|_{W^{4,\infty}(0,T;C(\overline{\Omega}))\cap W^{2,\infty}(0,T;C^4(\overline{\Omega}))} \le M\}. \tag{1.7}$$

Remark 2. We can write the conditions for $(g, h, a, H) \in \mathcal{V}$ explicitly in terms of sufficient smoothness and compatibility conditions of sufficient orders. This can be proved by results presented for example in Lions and Magenes [39].

For the statement of the compatibility conditions, we set $A = A_{\lambda_0,\mu_0} \equiv L_{\lambda_0,\mu_0,\rho_0} - \rho_0 \partial_t^2$, which is the stationary part of L defined by (1.1). We first define

$$a^{(0)} = a, \qquad b^{(0)} = 0,$$
$$a^{(j)} = \frac{1}{\rho_0}(-Aa^{(j-1)} + \partial_t^{2(j-1)} H(0, x)), \quad j = 1, 2, 3, \cdots$$
$$b^{(j)} = \frac{1}{\rho_0}(-Ab^{(j-1)} + \partial_t^{2j-1} H(0, x)), \quad j = 1, 2, 3, \cdots$$

provided that $a^{(j)}$, $b^{(j)}$ are well-defined in $L^2(\Omega)$. We note that $b^{(1)} = 0$ by

$\partial_t H(0,\cdot) = 0$. Let us assume smoothness conditions:

$$a^{(j)},\, b^{(j)} \in H^4(\Omega), \quad 0 \le j \le 2, \quad a^{(3)} \in H^4(\Omega), \quad b^{(3)} \in H^3(\Omega),$$

$$\begin{cases} g \in H^7(0,T; H^{\frac{3}{2}}(\partial\Omega)) \cap H^{\frac{31}{4}}(0,T; L^2(\partial\Omega)), \\ h \in H^7(0,T; H^{\frac{1}{2}}(\partial\Omega)) \cap H^{\frac{29}{4}}(0,T; L^2(\partial\Omega)), \\ H \in H^7(0,T; L^2(\Omega)),\ \partial_t^3 H \in C([0,T]; H^4(\Omega)) \end{cases}$$

and compatibility conditions of order 6:

$$\begin{cases} B_1(\lambda_0,\mu_0)a^{(j)}(x) = \partial_t^{2j} g(0,x), \\ B_2(\lambda_0,\mu_0)a^{(j)}(x) = \partial_t^{2j} h(0,x), \\ B_1(\lambda_0,\mu_0)b^{(j)}(x) = \partial_t^{2j+1} g(0,x), \\ B_2(\lambda_0,\mu_0)b^{(j)}(x) = \partial_t^{2j+1} h(0,x), \quad 0 \le j \le 3, \quad x \in \partial\Omega. \end{cases} \tag{1.8}$$

Then we can apply Theorem 3.1 of Chapter 5 in Lions and Magenes [39] to $\partial_t^j y$, $0 \le j \le 6$, and we see that $y \in H^6(0,T; H^4(\Omega)) \cap H^3(0,T; H^8(\Omega))$. By the Sobolev embedding, this yields that $y \in W^{4,\infty}(0,T; C(\overline{\Omega})) \cap W^{2,\infty}(0,T; C^5(\overline{\Omega}))$. One can relax the smoothness conditions and the compatibility conditions by refined regularity properties in [39]. However for concentrating on inverse problems, we will not pursue more.

Next we introduce an admissible set of unknown λ and μ for Inverse Problem I:

$$\begin{aligned} \mathcal{U}_I = \{(\lambda,\mu);\ &\|\lambda\|_{C^{14}(\overline{\Omega})},\ \|\mu\|_{C^{14}(\overline{\Omega})} \le M, \\ \lambda+\mu, \mu > \epsilon_0 \quad &\text{on } \overline{\Omega}. \text{ There exists a neighbourhood } U = U(\lambda,\mu) \\ &\text{of } \partial\Omega \text{ such that } \lambda = \lambda_0 \text{ and } \mu = \mu_0 \text{ in } U, \\ &\nabla \log\left(\tfrac{\rho(x)}{\lambda(x)+\mu(x)}\right) \cdot (x - x_0) > -1, \quad x \in \overline{\Omega}\}. \end{aligned} \tag{1.9}$$

Remark 3. The last condition in (1.9):

$$\nabla \log \left(\frac{\rho(x)}{\lambda(x) + \mu(x)} \right) \cdot (x - x_0) > -1, \quad x \in \overline{\Omega} \tag{1.10}$$

restricts an admissible set of unknown coefficients, but the uniqueness as well as the Lipschitz stability are extremely difficult to be established without such a condition. Condition (1.10) is a sufficient condition for our key Carleman estimate, and for hyperbolic operators of the second order, we have to assume similar conditions and as for related discussions, see Amirov and Yamamoto [3], Cheng, Isakov, Yamamoto and Zhou [10], Imanuvilov and Yamamoto [20], Triggiani and Yao [41].

We set

$$H^{4,2}(Q) = L^2(0,T; H^4(\Omega)) \cap H^2(0,T; L^2(\Omega)),$$

$$H^{\ell,2}((0,T) \times \partial\Omega) = L^2(0,T; H^{\ell}(\partial\Omega)) \cap H^2(0,T; L^2(\partial\Omega)),\ \ell \ge 0$$

and

$$\|u\|_{H^{4,2}(Q)} = \|u\|_{L^2(0,T;H^4(\Omega))} + \|u\|_{H^2(0,T;L^2(\Omega))},$$

$$\|u\|_{H^{\ell,2}((0,T)\times\partial\Omega)} = \|u\|_{L^2(0,T;H^{\ell}(\partial\Omega))} + \|u\|_{H^2(0,T;L^2(\partial\Omega))}.$$

We are ready to state the first result on the Lipschitz stability for Inverse Problem I.

Theorem 1. *We assume that* $(g_j, h_j, a_j, 0) \in \mathcal{V}$, $1 \le j \le 6$ *satisfy*

$$det\,(\Delta^2 a_j, \partial_1(\Delta a_j), \partial_2(\Delta a_j), \partial_1^2 a_j, \partial_2^2 a_j, \partial_1\partial_2 a_j)_{1\le j\le 6} \ne 0 \quad on\ \overline{\Omega}. \tag{1.11}$$

Then there exists a constant $C_1 = C_1(\Omega, T, g_j, h_j, a_j, \mathcal{U}_I, \lambda_0, \mu_0, \rho_0) > 0$ *such that*

$$\begin{aligned} & \|\lambda_1 - \lambda_2\|_{H^2(\Omega)} + \|\mu_1 - \mu_2\|_{H^2(\Omega)} \\ \le \;& C_1 \sum_{j=1}^{6} \sum_{k=0}^{3} \|\partial_\nu^k \partial_t^2 y(\lambda_1, \mu_1, \rho_0; g_j, h_j, a_j, 0) \\ - \;& \partial_\nu^k \partial_t^2 y(\lambda_2, \mu_2, \rho_0; g_j, h_j, a_j, 0)\|_{H^{\frac{7}{2}-k,2}((0,T)\times\partial\Omega)} \end{aligned} \tag{1.12}$$

for $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathcal{U}_I$.

Since $\lambda = \lambda_0$ and $\mu = \mu_0$ near $\partial\Omega$ for $(\lambda, \mu) \in \mathcal{U}_I$, as is seen in Remark 2, we can verify that for a_j, $1 \le j \le 6$ satisfying (1.11), we can choose g_j, h_j such that $(g_j, h_j, a_j, 0) \in \mathcal{V}$ and that there exists a constant $M_1 > 0$ such that

$$\|y(\lambda, \mu, \rho_0; g_j, h_j, a_j, 0)\|_{W^{4,\infty}(0,T;C(\overline{\Omega}))\cap W^{2,\infty}(0,T;C^4(\overline{\Omega}))} \le M_1, \quad 1 \le j \le 6, \tag{1.13}$$

for any $(\lambda, \mu) \in \mathcal{U}_I$. In particular, the right hand side of (1.12) is finite.

Theorem 1 asserts that if we choose six input data satisfying (1.11), then we have the Lipschitz stability (1.12) in determining two coefficients λ and μ. Assumption (1.11) is not physical and should be realized artificially in our identification process. In Theorems 2 - 4 below, we have to pose assumptions of similar characters for initial values.

The requirement of the six choices of input implies an overdetermining formulation for the inverse problem. If we choose quadratic functions a_1, a_2, then we prove

Theorem 2. *We set*

$$a_j(x_1,x_2) = \frac{p_{1j}}{2}x_1^2 + \frac{p_{2j}}{2}x_2^2 + p_{3j}x_1x_2, \tag{1.14}$$

where $p_{1j}, p_{2j}, p_{3j} \in \mathbb{R}$, $j = 1,2$ *and*

$$r_1 = \frac{p_{11}}{p_{11}+p_{21}} - \frac{p_{12}}{p_{12}+p_{22}}, \quad r_2 = \frac{p_{21}}{p_{11}+p_{21}} - \frac{p_{22}}{p_{12}+p_{22}},$$
$$r_3 = \frac{p_{31}}{p_{11}+p_{21}} - \frac{p_{32}}{p_{12}+p_{22}},$$

if $p_{11}+p_{21} \neq 0$ *and* $p_{12}+p_{22} \neq 0$. *We assume that there exist* $(g_j, h_j, a_j, 0) \in \mathcal{V}$, $j = 1,2$, *such that*

$$|r_1| + |r_2| \neq 0, \quad p_{11}+p_{21} \neq 0, \quad p_{12}+p_{22} \neq 0, \quad r_1r_2 \neq r_3^2 \tag{1.15}$$

or

$$p_{11} \neq 0, \quad p_{11}+p_{21} = 0, \quad p_{12}+p_{22} \neq 0, \quad p_{11}p_{21} \neq p_{31}^2. \tag{1.16}$$

Then there exists a constant $C_2 = C_2(\Omega, T, g_j, h_j, a_j, \mathcal{U}_I, \lambda_0, \mu_0, \rho_0) > 0$ *such that*

$$\begin{aligned} &\|\lambda_1 - \lambda_2\|_{H^2(\Omega)} + \|\mu_1 - \mu_2\|_{H^2(\Omega)} \\ \leq\ & C_2 \textstyle\sum_{j=1}^2 \sum_{k=0}^3 \|\partial_\nu^k \partial_t^2 y(\lambda_1, \mu_1, \rho_0; g_j, h_j, a_j, 0) \\ -\ & \partial_\nu^k \partial_t^2 y(\lambda_2, \mu_2, \rho_0; g_j, h_j, a_j, 0)\|_{H^{\frac{7}{2}-k,2}((0,T)\times\partial\Omega)} \end{aligned} \tag{1.17}$$

for $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathcal{U}_I$.

The two is the minimum number of choices of inputs because we have two unknown coefficients depending on two independent variables and one boundary data depend on two independent variables $x \in \partial\Omega$ and t. Conditions (1.15) and (1.16) allow the following simple functions which may be realized easily in the realistic experiments at a laboratory:

$$a_1(x) = x_1^2, \qquad a_2(x) = x_2^2. \tag{1.18}$$

If λ_0 and μ_0 are constants near $\partial\Omega$ for example, then compatibility conditions (1.8) are satisfied for $a_1(x)$ if we choose g and h such that

$$g(0,x) = 2(\lambda_0 + \mu_0) - 2\nu_2^2\mu_0,$$

$$\partial_t^j g(0,x) = 0, \quad 1 \le j \le 7,$$

$$\partial_t^k h(0,x) = 0, \quad 0 \le k \le 7, \quad x \in \partial\Omega.$$

Next we will show the stability in Inverse Problem II of determining three coefficients λ, μ, ρ. We introduce a admissible set of λ, μ, ρ.

$$\mathcal{U}_{II} = \{(\lambda, \mu, \rho);\ \|\lambda\|_{C^{14}(\overline{\Omega})},\ \|\mu\|_{C^{14}(\overline{\Omega})},\ \|\rho\|_{C^{12}(\overline{\Omega})} \le M,$$

$$\lambda + \mu, \mu, \rho > \epsilon_0 \quad \text{on } \overline{\Omega}. \text{ There exists a neighbourhood } U = U(\lambda, \mu, \rho)$$

$$\text{of } \partial\Omega \text{ such that } \lambda = \lambda_0, \mu = \mu_0 \text{ and } \rho = \rho_0 \text{ in } U,$$

$$\nabla \log\left(\tfrac{\rho(x)}{\lambda(x)+\mu(x)}\right) \cdot (x - x_0) > -1, \quad x \in \overline{\Omega}\}. \tag{1.19}$$

Now we are ready to state the main results for Inverse Problem II.

Theorem 3. *We assume that* $(g_j, h_j, a_j, H_j) \in \mathcal{V}$, $1 \le j \le 7$ *satisfy*

$$det(\Delta^2 a_j, \partial_1(\Delta a_j), \partial_2(\Delta a_j), \partial_1^2 a_j,$$

$$\partial_2^2 a_j, \partial_1\partial_2 a_j, H_j(0,\cdot))_{1\le j\le 7} \ne 0 \qquad on\ \overline{\Omega}. \tag{1.20}$$

Then there exists a constant $C_3 = C_3(\Omega, T, g_j, h_j, a_j, H_j, \mathcal{U}_{II}, \lambda_0, \mu_0, \rho_0) > 0$ *such that*

$$\begin{aligned}
&\|\lambda_1 - \lambda_2\|_{H^2(\Omega)} + \|\mu_1 - \mu_2\|_{H^2(\Omega)} + \|\rho_1 - \rho_2\|_{H^1(\Omega)} \\
&\qquad \le C_3 \sum_{j=1}^{7} \sum_{k=0}^{3} \|\partial_\nu^k \partial_t^2 y(\lambda_1, \mu_1, \rho_1; g_j, h_j, a_j, H_j) \\
&\qquad -\partial_\nu^k \partial_t^2 y(\lambda_2, \mu_2, \rho_2; g_j, h_j, a_j, H_j)\|_{H^{\frac{7}{2}-k,2}((0,T)\times\partial\Omega)}
\end{aligned} \tag{1.21}$$

for $(\lambda_1, \mu_1, \rho_1), (\lambda_2, \mu_2, \rho_2) \in \mathcal{U}_{II}$.

For the simultaneous determination of λ, μ, ρ, we have to choose also external forces H_j suitably. Like Theorem 1, the 7 choices of inputs make our inverse problem overdetermining. The following theorem gives the Lipschitz stability with special choices of a_1, a_2, a_3 whose number is three, the minimum.

Theorem 4. *We set*

$$a_j(x) = \frac{p_{1j}}{2} x_1^2 + \frac{p_{2j}}{2} x_2^2, \quad p_{1j}, p_{2j} \in \mathbb{R},\ j = 1, 2, 3. \tag{1.22}$$

We assume that there exist $(g_j, h_j, a_j, H_j) \in \mathcal{V}$, $j = 1, 2, 3$ *such that*

$$det(H_j(0,x), p_{1j}, p_{2j})_{1 \le j \le 3} \ne 0, \qquad x \in \overline{\Omega}. \tag{1.23}$$

Then there exists a constant $C_4 = C_4(\Omega, T, g_j, h_j, a_j, H_j, \mathcal{U}_{II}, \lambda_0, \mu_0, \rho_0) > 0$ *such that*

$$\|\lambda_1 - \lambda_2\|_{H^2(\Omega)} + \|\mu_1 - \mu_2\|_{H^2(\Omega)} + \|\rho_1 - \rho_2\|_{H^1(\Omega)}$$

$$\leq C_4 \sum_{j=1}^{3}\sum_{k=0}^{3} \|\partial_\nu^k \partial_t^2 y(\lambda_1, \mu_1, \rho_1; g_j, h_j, a_j, H_j)$$
$$-\partial_\nu^k \partial_t^2 y(\lambda_2, \mu_2, \rho_2; g_j, h_j, a_j, H_j)\|_{H^{\frac{7}{2}-k,2}((0,T)\times\partial\Omega)} \tag{1.24}$$

for $(\lambda_1, \mu_1, \rho_1)$, $(\lambda_2, \mu_2, \rho_2) \in \mathcal{U}_{II}$.

Condition (1.23) can be realized by simple choices. For example, let $H_1(0, x) = 1$, $H_2(0, x) = H_3(0, x) = 0$, $x \in \overline{\Omega}$. Setting

$$a_1(x) = x_1^2, \quad a_2(x) = x_1^2, \quad a_3(x) = x_2^2,$$

we see that the matrix in (1.23) is $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and (1.23) is satisfied.

Since the smoothness and the compatibility conditions of sufficient orders guarantee the sufficient smooth solutions $y(\lambda_\ell, \mu_\ell, \rho_\ell; g_j, h_j, a_j, H_j)$ by Remark 2, for given a_j satisfying (1.15) - (1.16), (1.20) and (1.23) respectively in Theorems 2 - 4, we can choose g_j, h_j, H_j such that $(g_j, h_j, a_j, H_j) \in \mathcal{V}$.

Our formulation is with a finite number of observations and this kind of inverse problems was firstly solved by Bukhgeim and Klibanov [9], whose methodology is based on Carleman estmates. Since then, rich references have been available: Amirov [2], Baudouin and Puel [4], Bellassoued [5], Bellassoued and Yamamoto [6], Bukhgeim [7], Bukhgeim, Cheng, Isakov and Yamamoto [8], Imanuvilov and Yamamoto [16], [17], [18], Isakov [22], Isakov and Yamamoto [23], Khaĭdarov [25] , Klibanov [26], Klibanov and Timonov [28], Klibanov and Yamamoto [29], Kubo [31], Li Shumin [35], Yamamoto

[42]. Inverse problems for isotropic Lamé systems, see Ikehata, Nakamura and Yamamoto [14], Isakov [21], Imanuvilov, Isakov and Yamamoto [15], Imanuvilov and Yamamoto [19], and for similar inverse problems for the Lamé system with residual stresses, see Lin and Wang [36]. On the other hand, as for the corresponding inverse problem for plate equations, to the authors' knowledge, there are no papers. Theorems 1 - 4 are proved by a modification of arguments in Bukhgeim and Klibanov [9], Imanuvilov, Isakov and Yamamoto [15], Imanuvilov and Yamamoto [18], Klibanov and Yamamoto [29]. In particular, an argument in [29] can improve the stability for the inverse problems to establish the Lipschitz stability. That argument is based on an energy estimate which is closely related with an observability inequality (Kazemi and Klibanov [24], Klibanov and Malinsky [27], Klibanov and Timonov [28], Komornik [30], Lions [37]).

This paper is composed of five sections. In Sections 2 and 3, we will prove a key Carleman estimate and an observability inequality respectively. In Sections 4 and 5, we prove Theorems 1-2 and Theorems 3-4 respectively.

2 Carleman Inequalities

We set

$$Q = (0, T) \times \Omega$$

and

$$Lu \equiv \rho \partial_t^2 u + (\lambda + \mu)\triangle^2 u + 2\nabla(\lambda + \mu) \cdot \nabla\left(\triangle u\right) + \triangle(\lambda + \mu)\triangle u$$

$$+2\,(\partial_1\partial_2\mu)\,(\partial_1\partial_2 u) - (\partial_1^2\mu)(\partial_2^2 u) - (\partial_2^2\mu)(\partial_1^2 u)$$

$$= \rho\partial_t^2 u + \Delta((\lambda+\mu)\Delta u)$$

$$+2(\partial_1\partial_2\mu)(\partial_1\partial_2 u) - (\partial_1^2\mu)(\partial_2^2 u) - (\partial_2^2\mu)(\partial_1^2 u), (t,x) \in Q.$$

In Sections 2 and 3, we assume a weaker smoothness assumption that $\rho = \rho(x)$, $\lambda = \lambda(x)$ and $\mu = \mu(x)$ are in $C^2(\overline{\Omega})$ and positive on $\overline{\Omega}$. Moreover

$$L_1 v := \mathrm{i}p(x)\partial_t v + \triangle v, \qquad L_2 v := -\mathrm{i}p(x)\partial_t v + \triangle v, \tag{2.1}$$

$$\varphi(t,x) = e^{\gamma(|x-x_0|^2 - \beta|t-t_0|^2)}, \qquad (t,x) \in Q, \tag{2.2}$$

where $p \in C^1(\overline{\Omega})$, $p(x) > 0$, $x \in \overline{\Omega}$, γ and β are positive constants, $x_0 = (x_0^1, x_0^2) \in \mathbb{R}^2 \backslash \overline{\Omega}$, $t_0 \in (0,T)$ and $\mathrm{i} = \sqrt{-1}$. By $\overline{c}$ we denote the complex conjugate of $c \in \mathbb{C}$, while $\overline{\Omega}$ means the closure of a domain Ω.

First we will show Carleman estimates for the Schrödinger operators L_1 and L_2.

Lemma 2.1. *Let $p \in C^1(\overline{\Omega})$ satisfy $p(x) > 0$, $x \in \overline{\Omega}$ and*

$$\nabla \log p(x) \cdot (x - x_0) > -2, \quad x \in \overline{\Omega}. \tag{2.3}$$

Then there exists a number $\gamma_0 > 0$ such that for arbitrary $\gamma \geq \gamma_0$, we can choose $s_0 \geq 0$ satisfying: there exists a constant $C_5 > 0$ such that

$$\int_Q \left\{ s|\nabla v|^2 + s^3|v|^2 \right\} e^{2s\varphi} dxdt \leq C_5 \int_Q |L_\ell v|^2 e^{2s\varphi} dxdt, \quad \ell = 1, 2, \tag{2.4}$$

for all $v \in C_0^\infty(\Omega)$ and all $s > s_0$.

Here and henceforth $C_j > 0$ denote generic constants which may be dependent on Ω, T, other quantities but independent of s.

Proof. It sufficient to prove (2.4) for $\ell = 1$ because we can prove (2.4) for $\ell = 2$ in the same way. Let $\zeta = (\zeta_0, \zeta_1, \zeta_2)$ and $\zeta' = (\zeta_1, \zeta_2)$, and by $L(x, \zeta)$ we denote the symbol of L_1: $L(x, \zeta) = -p(x)\zeta_0 - \sum_{j=1}^{2} \zeta_j^2$. In terms of Theorem 3.2.1 (p.49) in Isakov [22], it is sufficient to prove the following. If

$$L(x, \zeta) = 0, \quad x \in \overline{\Omega}, \quad \zeta_0 = \xi_0, \quad \zeta_j = \xi_j + 2\mathrm{i}s\gamma(x_j - x_0^j)\varphi, \quad j = 1, 2,$$
$$s \in \mathbb{R}\backslash\{0\}, \quad \xi_0, \quad \xi_1, \quad \xi_2 \in \mathbb{R} \tag{2.5}$$

or

$$L(x, \xi) = 0, \quad \sum_{j=1}^{2} \frac{\partial L}{\partial \xi_j}(x, \xi)\partial_j\varphi = 0, \quad \xi \in \mathbb{R}^3\backslash\{0\} \tag{2.6}$$

implies that

$$J = J(x, \zeta) \equiv \sum_{j,k=1}^{2} (\partial_j\partial_k\varphi)\frac{\partial L}{\partial \zeta_j}(x, \zeta)\overline{\frac{\partial L}{\partial \zeta_k}(x, \zeta)} + \frac{1}{s}Im\sum_{k=1}^{2} \partial_k L(x, \zeta)\overline{\frac{\partial L}{\partial \zeta_k}(x, \zeta)} > 0. \tag{2.7}$$

We see that (2.5) or (2.6) implies that

$$|\xi'|^2 - 4s^2\gamma^2|x - x_0|^2\varphi^2 = -p(x)\xi_0, \quad s \in \mathbb{R}, \quad \xi' \cdot (x - x_0) = 0. \tag{2.8}$$

Therefore using the second equation in (2.8), we have

$$J = 8\gamma\varphi|\xi'|^2 + 64s^2\gamma^4\varphi^3|x - x_0|^4 + 32s^2\gamma^3\varphi^3|x - x_0|^2$$
$$-4(\nabla p(x) \cdot (x - x_0))\gamma\varphi\xi_0. \tag{2.9}$$

By (2.8), we have

$$-p(x)\xi_0 = |\xi'|^2 - 4s^2\gamma^2|x - x_0|^2\varphi^2$$

and we substitute it into the last term, so that

$$-4(\nabla p(x)\cdot(x-x_0))\gamma\varphi\xi_0 = 4(\nabla \log p(x)\cdot(x-x_0))\gamma\varphi(-p(x)\xi_0).$$

Hence

$$J \ge s^2\gamma^4\varphi^3\left\{64|x-x_0|^4 - \frac{16}{\gamma}|x-x_0|^2(\nabla\log p(x)\cdot(x-x_0))\right\} \\ +8\gamma\varphi|\xi'|^2\left\{1+\frac{1}{2}(\nabla\log p(x)\cdot(x-x_0))\right\}. \quad (2.10)$$

Hence by (2.3), there exist positive constants C_6 and γ_0 such that for $\gamma \ge \gamma_0$, we have

$$J \ge C_6\gamma\varphi|\xi'|^2 + C_6\gamma^4 s^2\varphi^3,$$

for all $x \in \overline{\Omega}$ and $\xi' \in \mathbb{R}^2$ satisfying (2.8). If $s \ne 0$, then $J > 0$. If $s = 0$, then (2.8) implies $|\xi'|^2 = -p(x)\xi_0$, which yields $\xi' \ne 0$ by the second case (2.6). Therefore also in the case of $s = 0$, we have $J > 0$. Thus the proof of Lemma 2.1 is complete.

Now we are ready to show a Carleman estimate for the Kirchhoff plate equation.

Theorem 2.1. *We assume that ρ, μ and λ are in $C^2(\overline{\Omega})$ and positive on $\overline{\Omega}$ and that*

$$\nabla \log\left(\sqrt{\frac{\rho}{\lambda+\mu}}\right)\cdot(x-x_0) > -2, \quad x\in\overline{\Omega}. \quad (2.11)$$

Then there exists a number $\gamma_0 > 0$ such that for arbitrary $\gamma \ge \gamma_0$, we can choose $s_0 \ge 0$ satisfying: there exists a constant $C_7 > 0$ such that

$$\int_Q \left\{ s|\nabla\partial_t v|^2 + s|\nabla\triangle v|^2 + s^3|\partial_t v|^2 + s^2\sum_{j,k=1}^{2}|\partial_j\partial_k v|^2 + s^4|\nabla v|^2 + s^6|v|^2\right\} e^{2s\varphi}dxdt$$

$$\leq C_7 \int_Q |Lv|^2 e^{2s\varphi} dxdt \tag{2.12}$$

for every real-valued $v \in H^{4,2}(Q)$ *with compact support in* Q *and all* $s \geq s_0$.

Proof. By the mollifier and Friedrich's lemma (e.g., Lemma 17.1.5 in Hörmander [13 , vol. III]), it suffices to prove (2.12) for $v \in C_0^\infty(Q)$. First we will prove a Carleman estimate for the operator $\widetilde{P}$:

$$\widetilde{P}v := P_1P_2v = \frac{\rho}{\lambda+\mu}\partial_t^2 v + \triangle^2 v + \mathrm{i}\triangle\left(\sqrt{\frac{\rho}{\lambda+\mu}}\right)\partial_t v + 2\mathrm{i}\nabla\left(\sqrt{\frac{\rho}{\lambda+\mu}}\right)\cdot\nabla\partial_t v,$$

where

$$P_1 v = -\mathrm{i}\sqrt{\frac{\rho}{\lambda+\mu}}\partial_t v + \triangle v, \qquad P_2 v = \mathrm{i}\sqrt{\frac{\rho}{\lambda+\mu}}\partial_t v + \triangle v.$$

By virtue of Lemma 2.1 we have

$$\int_Q \left\{ s|\nabla v|^2 + s^3|v|^2 \right\} e^{2s\varphi} dxdt \leq C_8 \int_Q |P_2 v|^2 e^{2s\varphi} dxdt, \quad s \geq s_0, \tag{2.13}$$

and

$$\int_Q \left\{ s|\nabla(P_2 v)|^2 + s^3|P_2 v|^2 \right\} e^{2s\varphi} dxdt \leq C_8 \int_Q |\widetilde{P}v|^2 e^{2s\varphi} dxdt. \tag{2.14}$$

Therefore, noting that v is real-valued, we can see from (2.14) that

$$\int_Q \left\{ s\left|\nabla\left(\sqrt{\frac{\rho}{\lambda+\mu}}\partial_t v\right)\right|^2 + s|\nabla\triangle v|^2 + s^3\left|\sqrt{\frac{\rho}{\lambda+\mu}}\partial_t v\right|^2 + s^3|\triangle v|^2 \right\} e^{2s\varphi} dxdt$$
$$\leq C_9 \int_Q |\widetilde{P}v|^2 e^{2s\varphi} dxdt.$$

Combining (2.13) and (2.14), we have

$$\int_Q \left\{ s^4|\nabla v|^2 + s^6|v|^2 \right\} e^{2s\varphi} dxdt \leq C_{10} \int_Q |\widetilde{P}v|^2 e^{2s\varphi} dxdt.$$

Hence we take sufficiently large s_0, so that

$$\int_Q \{s|\nabla\partial_t v|^2 + s|\nabla\triangle v|^2 + s^3|\partial_t v|^2 + s^3|\triangle v|^2 + s^4|\nabla v|^2 + s^6|v|^2\} e^{2s\varphi} dxdt$$
$$\le C_{10}\int_Q |\widetilde{P}v|^2 e^{2s\varphi} dxdt, \quad s \ge s_0. \tag{2.15}$$

Moreover we have

$$\triangle(ve^{s\varphi}) = (\triangle v)e^{s\varphi} + 2s(\nabla v\cdot\nabla\varphi)e^{s\varphi} + (s\triangle\varphi + s^2|\nabla\varphi|^2)ve^{s\varphi}$$

and

$$ve^{s\varphi}|_{\partial\Omega} = 0.$$

Therefore we apply a usual a priori estimate for the Dirichlet problem for the Laplace operator and integrate over $(0,T)$, so that

$$\sum_{j,k=1}^{2}\int_Q |\partial_j\partial_k(ve^{s\varphi})|^2 dxdt \le C_{11}\int_Q (|\triangle v|^2 + s^2|\nabla v|^2 + s^4 v^2)e^{2s\varphi} dxdt,$$

so that

$$s^2\sum_{j,k=1}^{2}\int_Q |\partial_j\partial_k(ve^{s\varphi})|^2 dxdt \le C_{11}\int_Q (s^2|\triangle v|^2 + s^4|\nabla v|^2 + s^6 v^2)e^{2s\varphi} dxdt.$$

Hence combining (2.14) and (2.15), we can obtain

$$\int_Q \left\{ s|\nabla\partial_t v|^2 + s|\nabla\triangle v|^2 + s^3|\partial_t v|^2 + s^2\sum_{j,k=1}^{2}|\partial_j\partial_k v|^2 + s^4|\nabla v|^2 + s^6|v|^2 \right\} e^{2s\varphi} dxdt$$
$$\le C_{11}\int_Q |\widetilde{P}v|^2 e^{2s\varphi} dxdt. \tag{2.16}$$

Since

$$(\lambda+\mu)\widetilde{P}v = Lv + \mathrm{i}(\lambda+\mu)\left(\triangle\sqrt{\tfrac{\rho}{\lambda+\mu}}\right)\partial_t v + 2\mathrm{i}(\lambda+\mu)\left(\nabla\sqrt{\tfrac{\rho}{\lambda+\mu}}\right)\cdot\nabla(\partial_t v)$$
$$-2\nabla(\lambda+\mu)\cdot\nabla(\triangle v) - \triangle(\lambda+\mu)\triangle v - 2(\partial_1\partial_2\mu)(\partial_1\partial_2 v)$$
$$+(\partial_1^2\mu)(\partial_2^2 v) + (\partial_2^2\mu)(\partial_1^2 v),$$

in (2.16) we can absorb the lower order terms by taking large s_0, so that we obtain (2.12). Thus the proof of Theorem 2.1 is complete.

Remark. As for Carleman estimates, see Hömander [13], Isakov [22]CKlibanov and Timonov [28], Tataru [40], Triggiani and Yao [41].

We conclude this section with a Carleman estimate for a third order partial differential operator.

Lemma 2.2. *Let $(r_1, r_2, r_3) \in \mathbb{R}^3$ satisfy*

$$|r_1| + |r_2| \neq 0, \qquad r_3^2 \neq r_1 r_2. \tag{2.17}$$

Then there exists a number $\gamma_0 > 0$ such that for arbitrary $\gamma \geq \gamma_0$, we can choose $s_0 \geq 0$ satisfying: there exists a constant $C_{12} > 0$ such that

$$s \int_\Omega \sum_{|\alpha| \leq 2} |\partial_x^\alpha u|^2 e^{2s\varphi(0,x)} dx \leq C_{12} \int_\Omega |\nabla(r_1 \partial_1^2 u + r_2 \partial_2^2 u + 2r_3 \partial_1 \partial_2 u)|^2 e^{2s\varphi(0,x)} dx \tag{2.18}$$

for all $v \in H_0^3(\Omega)$ and all $s > s_0$. Here in (2.2) we set $t_0 = 0$, that is, we put

$$\psi(t,x) = |x - x_0|^2 - \beta t^2, \quad \varphi(t,x) = e^{\gamma\psi(t,x)}.$$

Proof. We assume $r_2 \neq 0$ and $r_3^2 \neq r_1 r_2$. Setting

$$(Pu)(x) = r_1 \partial_1^3 u(x) + r_2 \partial_1 \partial_2^2 u(x) + 2r_3 \partial_1^2 \partial_2 u(x), \quad x \in \Omega,$$

we will prove

$$s \int_\Omega \sum_{|\alpha| \leq 2} |\partial_x^\alpha u|^2 e^{2s\varphi(0,x)} dx \leq C_{12} \int_\Omega |Pu|^2 e^{2s\varphi(0,x)} dx.$$

we will apply Theorem 3.2.1 (p.49) in Isakov [22]. We set $m = (m_1, m_2) = (3.3)$ and $|\alpha : m| = \frac{\alpha_1}{3} + \frac{\alpha_2}{3}$ for $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{N} \cup \{0\})^2$. Then we can write P in the form

$$Pu = \sum_{|\alpha:m|=1} a_\alpha \partial_x^\alpha u$$

with suitable $a_\alpha \in \mathbb{R}$. Hence the operator P is treated by [22]. We further set $\zeta' = (\zeta_1, \zeta_2) \in \mathbb{C}^2$ and $P(\zeta') = -\mathrm{i}(r_1\zeta_1^3 + r_2\zeta_1\zeta_2^2 + 2r_3\zeta_1^2\zeta_2)$. By $\partial_j P = 0$ for $j = 1, 2$, for the proof, it suffices to verify that

$$\sum_{j,k=1}^{2} (\partial_j\partial_k\varphi)(0,x)\frac{\partial P}{\partial \zeta_j}(\zeta')\overline{\frac{\partial P}{\partial \zeta_k}(\zeta')} > 0 \tag{2.19}$$

if

$$x \in \overline{\Omega} \text{ and } P(\zeta') = 0, \zeta' = \xi' + 2\mathrm{i}s\gamma(x - x_0)\varphi, \quad s \neq 0, \quad \xi' \in \mathbb{R}^2,$$

or

$$P(\xi') = 0, \qquad \xi' \in \mathbb{R}^2 \setminus \{0\}.$$

Since $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$, it suffices to prove (2.19) for

$$\begin{aligned} &x \in \overline{\Omega} \text{ and } \zeta' \neq 0 \text{ with } P(\zeta') = 0, \\ &\zeta' = \xi' + 2\mathrm{i}s\gamma(x - x_0)\varphi, \quad \xi' \in \mathbb{R}^2. \end{aligned} \tag{2.20}$$

Since $\varphi(0, x) = e^{\gamma|x-x_0|^2}$, we have

$$\begin{aligned} &\textstyle\sum_{j,k=1}^{2}(\partial_j\partial_k\varphi)(0,x)\frac{\partial P}{\partial \zeta_j}(\zeta')\overline{\frac{\partial P}{\partial \zeta_k}(\zeta')} \\ = \gamma\varphi &\textstyle\sum_{j=1}^{2} \partial_j^2\psi(0,x)\left|\frac{\partial P}{\partial \zeta_j}(\zeta')\right|^2 + \gamma^2\varphi\left|(\partial_1\psi)(0,x)\frac{\partial P}{\partial \zeta_1}(\zeta') + (\partial_2\psi)(0,x)\frac{\partial P}{\partial \zeta_2}(\zeta')\right|^2. \end{aligned}$$

Hence, for the verification of (2.19), it is sufficient to prove

$$\frac{1}{2}\sum_{j=1}^{2}\partial_j^2\psi(0,x)\left|\frac{\partial P}{\partial \zeta_j}(\zeta')\right|^2 = |3r_1\zeta_1^2 + r_2\zeta_2^2 + 4r_3\zeta_1\zeta_2|^2 + 4|\zeta_1|^2|r_3\zeta_1 + r_2\zeta_2|^2 > 0 \tag{2.21}$$

if $x \in \overline{\Omega}$ and $\zeta' \neq 0$ satisfies (2.20). First let $\zeta_1 \neq 0$. Then $P(\zeta') = 0$ implies

$$r_1\zeta_1^2 + r_2\zeta_2^2 + 2r_3\zeta_1\zeta_2 = 0. \tag{2.22}$$

We will prove that

$$3r_1\zeta_1^2 + r_2\zeta_2^2 + 4r_3\zeta_1\zeta_2 = 0, \quad r_3\zeta_1 + r_2\zeta_2 = 0 \tag{2.23}$$

and (2.22) are not compatible. Subtracting (2.22) from the first equation in (2.23), we have $r_1\zeta_1^2 + r_3\zeta_1\zeta_2 = 0$, which implies that $r_1\zeta_1 + r_3\zeta_2 = 0$ by $\zeta_1 \neq 0$. In view of (2.17), this and the second equation in (2.23) yield that $\zeta_1 = \zeta_2 = 0$, which contradicts that $\zeta' \neq 0$. Second let $\zeta_1 = 0$. The second equation in (2.23) and $r_2 \neq 0$ yield $\zeta_2 = 0$. This is impossible by $\zeta' = (\zeta_1, \zeta_2) \neq 0$. That is, under (2.20) inequality (2.21) holds true. Finally in (2.17) we assume that $r_1 \neq 0$. Then exchanging ∂_2 by ∂_1 and considering $\hat{P}u = \partial_2(r_1\partial_1^2 u + r_2\partial_2^2 u + 2r_3\partial_1\partial_2 u)$, we can complete the proof of Lemma 2.2.

3 Observability inequalities

In this section, we will derive an observability inequality which may have an independent interest. Let Γ_0 and Γ_1 be relatively open subsets of $\partial\Omega$, Γ_0 be possibly empty, and satisfy

$$\overline{\Gamma_0 \cup \Gamma_1} = \partial\Omega \quad \text{and} \quad \Gamma_0 \cap \Gamma_1 = \emptyset.$$

We consider an initial value/boundary value problem for the Kirchhoff plate equation.

$$Ly \equiv \rho\partial_t^2 y + \Delta((\lambda+\mu)\Delta y) + 2\,(\partial_1\partial_2\mu)\,(\partial_1\partial_2 y) - (\partial_1^2\mu)(\partial_2^2 y) \tag{3.1}$$
$$-(\partial_2^2\mu)(\partial_1^2 y) = H(t,x), \quad (t,x) \in Q,$$

$$y = \tfrac{\partial y}{\partial \nu} = 0, \quad (t,x) \in (0,T) \times \Gamma_0, \tag{3.2}$$

$$B_1 y = B_2 y = 0, \quad (t,x) \in (0,T) \times \Gamma_1, \tag{3.3}$$

$$y(0,x) = y_0(x), \quad \partial_t y(0,x) = y_1(x) \quad x \in \Omega, \tag{3.4}$$

where the boundary operators $B_1 = B_1(\lambda,\mu)$ and $B_2 = B_2(\lambda,\mu)$ are defined by (1.4) and (1.5).

For any $y \in H^{4,2}(Q)$ and $v \in H^{4,2}(Q)$, integrating by parts over Ω, and noting

$$\partial_1 = \nu_1\partial_\nu - \nu_2\partial_\tau, \qquad \partial_2 = \nu_2\partial_\nu + \nu_1\partial_\tau \quad \text{on} \quad (0,T) \times \partial\Omega,$$

we can obtain a formula as follows:

$$\int_\Omega v(t,x)H(t,x)dx = \int_\Omega v(t,x)\partial_t^2 y(t,x)dx + \int_\Omega \{\lambda \triangle y(t,x) \triangle v(t,x)$$
$$+\mu\partial_1^2 y(t,x)\partial_1^2 v(t,x) + \mu\partial_2^2 y(t,x)\partial_2^2 v(t,x) + 2\mu\partial_1\partial_2 y(t,x)\partial_1\partial_2 v(t,x)\}\,dx$$
$$+ \int_{\partial\Omega} (v(t,x)B_2 y(t,x) - \partial_\nu v(t,x)B_1 y(t,x))\,dS, \quad 0 < t < T. \tag{3.5}$$

Now we are ready to state an observability inequality.

Theorem 3.1. *Let (2.11) hold and $H \in L^2(Q)$. We assume that $y \in$*

$H^{4,2}(Q)$ satisfy (3.1) - (3.4). Then there exist a positive constant C_{13} such that

$$\int_\Omega \left\{ |\partial_t y(t,x)|^2 + |y(t,x)|^2 + |\nabla y(t,x)|^2 + \sum_{j,k=1}^2 |\partial_j\partial_k y(t,x)|^2 \right\} dx$$
$$\le C_{13} \left(\|H\|^2_{L^2(Q)} + \sum_{k=0}^3 \|\partial_\nu^k y\|^2_{H^{\frac{7}{2}-k,2}((0,T)\times\partial\Omega)} \right), \qquad 0 \le t \le T.$$

This theorem enables us to estimate initial values by means of lateral Cauchy data, which is called an observability inequality. Observability inequalities are essential also for proving the exact controllability (e.g., Komornik [30], Lions [37]), and for plate equations, see Lions and Lagnese [38], Lasiecka and Triggiani [34].

The method of Carleman estimate was used for proofs of observability inequalities firstly by Klibanov and Malinsky [27] and Kazemi and Klibanov [24]. See also Klibanov and Timonov [28].

Proof. According to [24], [27] and [28], our proof is based on the Carleman estimate in Theorem 2.1. First we will show an energy estimate.

Lemma 3.1. *Let $y_0 \in H^2(\Omega)$, $y_1 \in L^2(\Omega)$ and $H \in L^2(Q)$. We assume that the solution y of (3.1) - (3.4) belongs to $H^{4,2}(Q)$. Then there exists a positive constant C_{14} such that*

$$\int_\Omega \left(|y(t,x)|^2 + |\partial_t y(t,x)|^2 + |\nabla y(t,x)|^2 + \sum_{j,k=1}^2 |\partial_j\partial_k y(t,x)|^2 \right) dx$$
$$\le C_{14}\|H\|^2_{L^2(Q)} + C_{14} \int_\Omega \left(y_0^2 + y_1^2 + \sum_{j,k=1}^2 |\partial_j\partial_k y_0|^2 \right) dx$$

holds for all $t \in (0, T)$.

Proof of Lemma 3.1. Setting $v = 2\partial_t y(t)$ in (3.5) and integrating by parts over the cylindrical domain $Q_t = (0, t) \times \Omega$ for an arbitrary $t \in (0, T)$, by using boundary conditions (3.2) and (3.3), we obtain

$$\int_\Omega \left\{ |\partial_t y(t,x)|^2 + \lambda |\Delta y(t,x)|^2 + \mu \left(\left|\partial_1^2 y(t,x)\right|^2 + \left|\partial_2^2 y(t,x)\right|^2 + 2\left|\partial_1 \partial_2 y(t,x)\right|^2 \right) \right\} dx$$
$$= \int_\Omega \left\{ |y_1(x)|^2 + \lambda |\Delta y_0(x)|^2 + \mu \left(\left|\partial_1^2 y_0(x)\right|^2 + \left|\partial_2^2 y_0(x)\right|^2 \right.\right.$$
$$\left.\left. +2\ \left|\partial_1 \partial_2 y_0(x)\right|^2 \right) \right\} + \int_{Q_t} 2H \partial_t y dx d\tau.$$

Hence, using the Cauchy-Schwarz inequality, we obtain

$$\int_\Omega \left\{ |\partial_t y(t,x)|^2 + \lambda |\Delta y(t,x)|^2 + \mu \left(\left|\partial_1^2 y(t,x)\right|^2 + \left|\partial_2^2 y(t,x)\right|^2 + 2\left|\partial_1 \partial_2 y(t,x)\right|^2 \right) \right\} dx$$
$$\le \int_\Omega \left\{ |y_1(x)|^2 + \lambda |\Delta y_0(x)|^2 + \mu \left(\left|\partial_1^2 y_0(x)\right|^2 + \left|\partial_2^2 y_0(x)\right|^2 + 2\left|\partial_1 \partial_2 y_0(x)\right|^2 \right) \right\}$$
$$+ \int_{Q_t} |H|^2 dx d\tau + \int_{Q_t} |\partial_t y(t,x)|^2 dx dt. \tag{3.6}$$

We also have

$$y(t,x) = y_0(x) + \int_0^t \partial_t y(\tau, x) d\tau.$$

Hence

$$\int_\Omega y^2(t,x) dx \le C_{14} \left(\|y_0\|^2_{L^2(\Omega)} + \|\partial_t y\|^2_{L^2(Q_t)} \right). \tag{3.7}$$

From (3.6) and (3.7), we have

$$\int_\Omega (|\partial_t y|^2 + y^2)(t,x) dx \le C_{14} \int_0^t \int_\Omega (|\partial_t y|^2 + y^2)(t,x) dx dt + C_{14} \int_{Q_t} |H|^2 dx d\tau$$
$$+ C_{14} \int_\Omega \left(|y_1|^2 + |y_0|^2 + \sum_{|\alpha|=2} |\partial_x^\alpha y_0|^2 \right) dx.$$

By the Gronwall inequality,

$$\int_\Omega (|\partial_t y|^2+y^2)(t,x)dx \le C_{15}\int_\Omega \left\{ |y_1|^2+|y_0|^2+\sum_{|\alpha|=2}|\partial_x^\alpha y_0|^2 \right\} dx + C_{15}\int_{Q_t}|H|^2 dx d\tau.$$

This inequality together with (3.6) leads to

$$\begin{aligned} &\int_\Omega \left(y^2+|\partial_t y|^2+\textstyle\sum_{|\alpha|=2}|\partial_x^\alpha y|^2 \right)(t,x)dx \\ \le\ & C_{15}\int_\Omega \left(y_1^2+y_0^2+\textstyle\sum_{|\alpha|=2}|\partial_x^\alpha y_0|^2 \right) dx + C_{15}\int_{Q_t}|H|^2 dx d\tau. \end{aligned}$$

From the interpolation inequality in a Sobolev space (e.g., Adams [1]), we know that there exists a positive constant C_{16} such that

$$\int_\Omega |\nabla y(t,x)|^2 dx \le C_{16}\int_\Omega |y(t,x)|^2 dx + C_{16}\int_\Omega \sum_{|\alpha|=2}|\partial_x^\alpha y(t,x)|^2 dx$$

for all $t \in (0,T)$. From the last two inequalities above, the proof of Lemma 3.1 is complete.

Now we proceed to the completion of the proof of Theorem 3.1. Let $d = \inf_{x\in\Omega}\exp(\gamma|x-x_0|^2)$. We choose $\beta > 0$ such that

$$\sup_{x\in\Omega}\gamma|x-x_0|^2 < \log d + \frac{T^2}{4}\gamma\beta.$$

With this β, we set $\varphi(t,x) = \exp\{\gamma(|x-x_0|^2 - \beta|t-\frac{T}{2}|^2)\}$. Then we have

$$\varphi\left(\frac{T}{2},x\right) \ge d, \quad \varphi(0,x) = \varphi(T,x) < d, \quad x \in \overline{\Omega}.$$

Thus, for given $\epsilon > 0$, we can choose a sufficiently small $\delta = \delta(\epsilon) > 0$ such that

$$\varphi(t,x) \ge d-\epsilon, \quad (t,x) \in \left[\frac{T}{2}-\delta, \frac{T}{2}+\delta\right] \times \overline{\Omega}, \tag{3.8}$$

$$\varphi(t,x) \le d - 2\epsilon, \quad (t,x) \in ([0,2\delta] \cup [T-2\delta, T]) \times \overline{\Omega} \tag{3.9}$$

and

$$T \ge 6\delta. \tag{3.10}$$

We introduce a cut-off function χ satisfying $0 \le \chi \le 1$, $\chi \in C^\infty[0,T]$ and

$$\chi(t) = \begin{cases} 0, & t \in [0,\delta] \cup [T-\delta, T], \\ 1, & t \in [2\delta, T-2\delta]. \end{cases} \tag{3.11}$$

By using the Sobolev extension theorem (e.g., [1]), we can find a function y^* such that

$$\begin{cases} \partial_\nu^j y^* = \partial_\nu^j y \quad on \quad [0,T] \times \partial\Omega, \quad 0 \le j \le 3, \\ \|y^*\|_{H^{4,2}(Q)} \le C_{17} \sum_{k=0}^3 \|\partial_\nu^k y\|_{H^{\frac{7}{2}-k,2}((0,T)\times\partial\Omega)}. \end{cases} \tag{3.12}$$

We set

$$z(t,x) = y(t,x) - y^*(t,x) \quad \text{in } Q \tag{3.13}$$

and

$$\Phi = \sup_{(t,x)\in Q} \varphi(t,x), \quad F^2 = \|H\|^2_{L^2(Q)} + \sum_{k=0}^3 \|\partial_\nu^k y\|^2_{H^{\frac{7}{2}-k,2}((0,T)\times\partial\Omega)}.$$

By (3.1) and (3.11), we can see

$$L(z\chi) = \chi H + \rho(\partial_t^2 \chi) z + 2\rho(\partial_t \chi)\partial_t z - L(\chi y^*).$$

By (3.11) and (3.12) we can see χz has compact support in Q and $\chi z \in H^{4,2}(Q)$. Applying the Carleman estimate in Theorem 2.1 to χz and noting that $\partial_t \chi \ne 0$ only in the case where $\varphi(t,x) \le d - 2\epsilon$, we have

$$\int_Q \left\{ s^3 |\partial_t(\chi z)|^2 + s^2 \sum_{j,k=1}^2 |\partial_j \partial_k (\chi z)|^2 + s^4 |\nabla(\chi z)|^2 + s^6 |\chi z|^2 \right\} e^{2s\varphi} dxdt$$
$$\le C_{18} \left\{ F^2 e^{2s\Phi} + (\|z\|^2_{L^2(Q)} + \|\partial_t z\|^2_{L^2(Q)}) e^{2s(d-2\epsilon)} \right\}.$$

On the other hand, $\chi = 1$ in the case where $2\delta < t < T - 2\delta$ and

$$\left\{t;\ \frac{T}{2} - \delta < t < \frac{T}{2} + \delta\right\} \subseteq \{t;\ 2\delta < t < T - 2\delta\}$$

by (3.10). Therefore, using also (3.10), we obtain

$$\begin{aligned}
&\int_Q \left\{s^3|\partial_t(\chi z)|^2 + s^2\sum_{j,k=1}^2 |\partial_j\partial_k(\chi z)|^2 + s^4|\nabla(\chi z)|^2 + s^6|\chi z|^2\right\} e^{2s\varphi}dxdt \\
&\ge \int_{\left(\frac{T}{2}-\delta,\frac{T}{2}+\delta\right)\times\Omega} \left\{s^3|\partial_t z|^2 + s^2\sum_{j,k=1}^2 |\partial_j\partial_k z|^2 + s^4|\nabla z|^2 + s^6|z|^2\right\} e^{2s\varphi}dxdt \\
&\ge e^{2s(d-\epsilon)} \int_{\left(\frac{T}{2}-\delta,\frac{T}{2}+\delta\right)\times\Omega} \left\{s^3|\partial_t z|^2 + s^2\sum_{j,k=1}^2 |\partial_j\partial_k z|^2 \right. \\
&\qquad \left. + s^4|\nabla z|^2 + s^6|z|^2\right\} e^{2s\varphi}dxdt.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{\left(\frac{T}{2}-\delta,\frac{T}{2}+\delta\right)\times\Omega} \left\{|\partial_t z|^2 + \sum_{j,k=1}^2 |\partial_j\partial_k z|^2 + |\nabla z|^2 + |z|^2\right\} dxdt \\
\le\ & C_{19}F^2e^{2s(\Phi-d+\epsilon)} + C_{19}\left(\|z\|^2_{L^2(Q)} + \|\partial_t z\|^2_{L^2(Q)}\right)e^{-2s\epsilon}, \quad s \ge s_0.
\end{aligned}$$

By (3.12) and (3.13), we can obtain

$$\begin{aligned}
&\int_{\left(\frac{T}{2}-\delta,\frac{T}{2}+\delta\right)\times\Omega} \left\{|\partial_t y|^2 + \sum_{j,k=1}^2 |\partial_j\partial_k y|^2 + |\nabla y|^2 + |y|^2\right\} dxdt \\
\le\ & C_{19}F^2e^{2s(\Phi-d+\epsilon)} + C_{19}\left(\|y\|^2_{L^2(Q)} + \|\partial_t y\|^2_{L^2(Q)}\right)e^{-2s\epsilon}, \quad s \ge s_0.
\end{aligned}$$

Thus by the mean value theorem, there exists a constant $\frac{T}{2} - \delta < t_1 < \frac{T}{2} + \delta$ such that

$$\begin{aligned}
&\int_\Omega \left\{|\partial_t y(t_1,x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha y(t_1,x)|^2\right\} dx \\
&\le \frac{C_{19}}{2\delta}\left(\|\partial_t y\|^2_{L^2(Q)} + \|y\|^2_{L^2(Q)}\right)e^{-2s\epsilon} + C_{19}F^2e^{2s(\Phi-d+\epsilon)}. \qquad (3.14)
\end{aligned}$$

Considering (3.1) in the time intervals (t_1, T) and $(0, t_1)$, in terms of the time reversibility of (3.1), we apply Lemma 3.1 to have

$$\int_\Omega \left(|\partial_t y(t,x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha y(t,x)|^2\right) dx \tag{3.15}$$
$$\le C_{20}\|H\|^2_{L^2(Q)} + C_{20}\int_\Omega \left(|\partial_t y(t_1,x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha y(t_1,x)|^2\right) dx, \quad 0 \le t \le T.$$

Hence

$$\|\partial_t y\|^2_{L^2(Q)} + \|y\|^2_{L^2(Q)} \le C_{21}T\int_\Omega(|\partial_t y(t_1,x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha y(t_1,x)|^2)dx$$
$$+C_{21}T\|H\|^2_{L^2(Q)}. \tag{3.16}$$

Substituting (3.16) into (3.14), we have

$$(1 - C_{22}e^{-2s\epsilon})\int_\Omega \left(|\partial_t y(t_1,x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha y(t_1,x)|^2\right) dx \le C_{22}e^{C_{22}s}F^2.$$

Taking $s > 0$ sufficiently large and fixing, we obtain

$$\int_\Omega \left(|\partial_t y(t_1,x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha y(t_1,x)|^2\right) dx \le C_{23}F^2.$$

Inequality (3.15) completes the proof of Theorem 3.1.

4 Lipschitz stability in determining coefficients λ and μ

Now we are in the position to prove our main results.

Proof of Theorem 1.

We set $y_j^{(1)} = y(\lambda_1, \mu_1, \rho_0; g_j, h_j, a_j, 0)$, $y_j^{(2)} = y(\lambda_2, \mu_2, \rho_0; g_j, h_j, a_j, 0)$, $u_j = y_j^{(2)} - y_j^{(1)}$, $1 \le j \le 6$, $f_1 = \lambda_2 - \lambda_1$, $f_2 = \mu_2 - \mu_1$ and

$$\begin{aligned} L_2 y(t,x) \equiv \rho(x)\partial_t^2 y(t,x) + (\lambda_2(x) + \mu_2(x))\Delta^2 y(t,x) + 2\nabla(\lambda_2 + \mu_2)(x) \cdot \nabla(\Delta y(t,x)) \\ +\Delta(\lambda_2 + \mu_2)\Delta y(t,x) + 2(\partial_1\partial_2\mu_2)(x)\partial_1\partial_2 y(t,x) \\ -(\partial_1^2\mu_2)(x)\partial_2^2 y(t,x) - (\partial_2^2\mu_2)(x)\partial_1^2 y(t,x). \end{aligned}$$

Then from (1.1), we can obtain

$$\begin{aligned} L_2 u_j = -\left(f_1 + f_2\right)\triangle^2 y_j^{(1)} - 2\nabla\left(f_1 + f_2\right) \cdot \nabla\triangle y_j^{(1)} - \triangle\left(f_1 + f_2\right)\triangle y_j^{(1)} \\ -2(\partial_1\partial_2 f_2)\left(\partial_1\partial_2 y_j^{(1)}\right) + (\partial_1^2 f_2)\left(\partial_2^2 y_j^{(1)}\right) + (\partial_2^2 f_2)\left(\partial_1^2 y_j^{(1)}\right) \text{ in Q} . \quad (4.1) \end{aligned}$$

We extend $y_j^{(1)}$ and u_j in $(0,T)\times\Omega$ by $y_j^{(1)}(-t,x) = y_j^{(1)}(t,x)$ and $u_j(-t,x) = u_j(t,x)$, $(t,x) \in (-T,0)\times\Omega$. Then, since $\partial_t y_j^{(1)}(0,\cdot) = 0$ and $\partial_t u_j(0,\cdot) = 0$, we have $\partial_t^3 y_j^{(1)}(0,\cdot) = \partial_t^3 u_j(0,\cdot)(0,\cdot) = 0$, by $H \equiv 0$. Therefore $y_j^{(1)}$, $u_j \in W^{4,\infty}(-T,T;L^\infty(\Omega)) \cap W^{2,\infty}(-T,T;W^{4,\infty}(\Omega))$. We set $d = \inf_{x\in\Omega}\exp(\gamma|x - x_0|^2)$ and choose $\beta > 0$ such that

$$\sup_{x\in\Omega} \gamma|x - x_0|^2 < \log d + T^2\gamma\beta.$$

Let $\varphi(t,x) = \exp\{\gamma(|x - x_0|^2 - \beta t^2)\}$. Then we have

$$\varphi(0,x) \ge d, \quad \varphi(-T,x) = \varphi(T,x) < d, \quad x \in \overline{\Omega}.$$

Thus, for given $\epsilon > 0$, we can choose a sufficiently small $\delta = \delta(\epsilon) > 0$ such that

$$\varphi(t,x) \ge d - \epsilon, \quad (t,x) \in [-\delta,\delta]\times\overline{\Omega}$$

and

$$\varphi(t,x) \le d - 2\epsilon, \quad (t,x) \in ([-T,-T+2\delta] \cup [T-2\delta, T]) \times \overline{\Omega}.$$

We introduce a cut-off function χ satisfying $0 \le \chi \le 1$, $\chi \in C^\infty[-T,T]$ and

$$\chi(t) = \begin{cases} 0, & t \in [-T,-T+\delta] \cup [T-\delta, T], \\ 1, & t \in [-T+2\delta, T-2\delta]. \end{cases} \tag{4.2}$$

By using the Sobolev extension theorem we can find a function u_j^* such that

$$\partial_\nu^k u_j^* = \partial_\nu^k \partial_t^2 u_j \quad \text{on} \quad [-T,T] \times \partial\Omega, \quad k = 0,1,2,3 \tag{4.3}$$

and

$$\|u_j^*\|_{H^{4,2}(Q)} \le C_{23} \sum_{k=0}^{3} \|\partial_\nu^k \partial_t^2 u_j\|_{H^{\frac{7}{2}-k,2}((0,T)\times\partial\Omega)}. \tag{4.4}$$

Here we recall that

$$\|u_j^*\|_{H^{4,2}(Q)} = \|u_j^*\|_{L^2(0,T;H^4(\Omega))} + \|u_j^*\|_{H^2(0,T;L^2(\Omega))}.$$

We set

$$z_j = z_j(t,x) = \partial_t^2 u_j(t,x) - u_j^*(t,x), \quad 1 \le j \le 6. \tag{4.5}$$

Henceforth we set

$$V^2 = \sum_{k=0}^{3} \|\partial_\nu^k \partial_t^2 u_j\|^2_{H^{\frac{7}{2}-k,2}((0,T)\times\partial\Omega)}, \qquad U^2 = \sum_{j=1}^{6} \left(\|z_j\|^2_{L^2(Q)} + \|\partial_t z_j\|^2_{L^2(Q)} + \|\nabla z_j\|^2_{L^2(Q)} \right)$$

and

$$\Phi = \sup_{(t,x)\in Q} \varphi(t,x).$$

Then by using (4.1) and (4.2), we can see

$$L_2(z_j\chi) = \chi\Big\{-(f_1+f_2)\triangle^2\partial_t^2 y_j^{(1)} - 2\nabla(f_1+f_2)\cdot\nabla\triangle\partial_t^2 y_j^{(1)} - \triangle(f_1+f_2)\triangle\partial_t^2 y_j^{(1)} -2(\partial_1\partial_2 f_2)\left(\partial_1\partial_2\partial_t^2 y_j^{(1)}\right) + (\partial_1^2 f_2)\left(\partial_2^2\partial_t^2 y_j^{(1)}\right) +(\partial_2^2 f_2)\left(\partial_1^2\partial_t^2 y_j^{(1)}\right)\Big\} + \rho(\partial_t^2\chi)(\partial_t^2 u_j) + 2\rho(\partial_t\chi)(\partial_t^3 u_j) - L_2\left(u_j^*\chi\right). \tag{4.6}$$

Let

$$\widetilde{Q} = (-T,T)\times\Omega.$$

By definition (1.7) of $\mathcal{V}$, we can choose a constant C_{24} such that

$$\left\|y_j^{(1)}\right\|_{W^{4,\infty}(-T,T;L^\infty(\Omega))\cap W^{2,\infty}(-T,T;W^{4,\infty}(\Omega))} \le C_{24}, \quad 1\le j\le 6. \tag{4.7}$$

By (4.2) and (4.3), we can see that χz_j has compact support in $\widetilde{Q}$ and $\chi z_j \in H^{4,2}(\widetilde{Q})$. Applying Theorem 2.1 to (4.6) and noting (4.4) and (4.7), we can obtain

$$\int_{\widetilde{Q}}\left\{s|\nabla\partial_t(\chi z_j)|^2 + s^3|\partial_t(\chi z_j)|^2 + s^2\sum_{k,\ell=1}^{2}|\partial_k\partial_\ell(\chi z_j)|^2 + s^4|\nabla(\chi z_j)|^2 + s^6|\chi z_j|^2\right\}e^{2s\varphi}dxdt \le C_{25}\int_{\widetilde{Q}}\sum_{|\alpha|\le 2}(|\partial_x^\alpha f_1|^2 + |\partial_x^\alpha f_2|^2)e^{2s\varphi}dxdt +C_{25}V^2e^{2s\Phi} + C_{25}(V^2+U^2)e^{2s(d-2\epsilon)}. \tag{4.8}$$

By noting $z_j(t,x) = \partial_t^2 u_j(t,x) - u_j^*(t,x)$ and using (4.4), we can obtain

$$\int_{\widetilde{Q}}\Big\{s|\nabla\partial_t(\chi\partial_t^2 u_j)|^2 + s^3|\partial_t(\chi\partial_t^2 u_j)|^2 + s\sum_{k,\ell=1}^{2}|\partial_k\partial_\ell(\chi\partial_t^2 u_j)|^2 +s^4|\nabla(\chi\partial_t^2 u_j)|^2 + s^6|\chi\partial_t^2 u_j|^2\Big\}e^{2s\varphi}dxdt \le C_{26}\left(\int_{\widetilde{Q}}\sum_{|\alpha|\le 2}(|\partial_x^\alpha f_1|^2 + |\partial_x^\alpha f_2|^2)e^{2s\varphi}dxdt + s^6V^2e^{2s\Phi} + (V^2+U^2)e^{2s(d-2\epsilon)}\right). \tag{4.9}$$

On the other hand, we have

$$s\int_\Omega \left|\partial_t^2 u_j(0,x)\right|^2 e^{2s\varphi(0,x)}dx = s\int_\Omega \chi(0)^2 \left|\partial_t^2 u_j(0,x)\right|^2 e^{2s\varphi(0,x)}dx$$

$$= s\int_{-T}^0 \frac{\partial}{\partial t}\left(\int_\Omega \chi(t)^2 \left|\partial_t^2 u_j(t,x)\right|^2 e^{2s\varphi(t,x)}dx\right)dt$$

$$= s\int_{-T}^0 \int_\Omega \Big(2\chi(t)\partial_t\chi(t)\left|\partial_t^2 u_j(t,x)\right|^2 e^{2s\varphi(t,x)} \tag{4.10}$$

$$+2\chi(t)^2\partial_t^3 u_j(t,x)\partial_t^2 u_j(t,x)e^{2s\varphi(t,x)} + 2s\chi(t)^2\left|\partial_t^2 u_j(t,x)\right|^2(\partial_t\varphi)e^{2s\varphi(t,x)}\Big)dxdt.$$

By the Cauchy-Schwarz inequality and $\chi\partial_t^3 u_j = \partial_t(\chi\partial_t^2 u_j) - \partial_t\chi\,(\partial_t^2 u_j)$, we have

$$s\int_\Omega \left|\partial_t^2 u_j(0,x)\right|^2 e^{2s\varphi(0,x)}dx \le C_{27}\int_{\widetilde{Q}}\left(s\left|\partial_t\left(\chi\partial_t^2 u_j\right)\right|^2 + s^2\left|\chi\partial_t^2 u_j\right|^2\right)e^{2s\varphi(t,x)}dxdt$$

$$+C_{27}\int_{\widetilde{Q}} s\left|\partial_t\chi\,(\partial_t^2 u_j)\right|^2 e^{2s\varphi(t,x)}dxdt. \tag{4.11}$$

In a similar way, we can obtain

$$\int_\Omega \left|\nabla(\partial_t^2 u_j)(0,x)\right|^2 e^{2s\varphi(0,x)}dx \le C_{28}\int_{\widetilde{Q}}\Big(\left|\partial_t\left(\chi\partial_t^2\nabla u_j\right)\right|^2$$

$$+s\left|\nabla\left(\chi\partial_t^2 u_j\right)\right|^2\Big)e^{2s\varphi(t,x)}dxdt$$

$$+C_{28}\int_{\widetilde{Q}}\left|(\partial_t\chi)\nabla(\partial_t^2 u_j)\right|^2 e^{2s\varphi(t,x)}dxdt. \tag{4.12}$$

Noting that $\partial_t\chi \neq 0$ only in the case where $\varphi(t,x) \le d - 2\epsilon$, we can see from (4.4) and (4.5) that $\nabla(\partial_t^2 u_j) = \nabla z_j + \nabla u_j^*$, and that the second terms on the right hand sides of (4.11) and (4.12) can be estimated by $C(s+1)(V^2 + U^2)e^{2s(d-2\epsilon)}$. This together with (4.9), (4.11) and (4.12) leads to

$$s\int_\Omega \left|(\partial_t^2 u_j)(0,x)\right|^2 e^{2s\varphi(0,x)}dx + \int_\Omega \left|(\nabla\partial_t^2 u_j)(0,x)\right|^2 e^{2s\varphi(0,x)}dx$$

$$\le C_{29}\int_\Omega \sum_{|\alpha|\le 2}|\partial_x^\alpha f_1|^2\left(\int_0^T e^{2s\varphi}dt\right)dx + C_{29}\int_\Omega \sum_{|\alpha|\le 2}|\partial_x^\alpha f_2|^2\left(\int_0^T e^{2s\varphi}dt\right)dx$$

$$+C_{29}s(V^2+U^2)e^{2s(d-2\epsilon)} + C_{29}s^6V^2e^{2s\Phi}, \tag{4.13}$$

for all large $s > 1$. Noting $u_j(0, \cdot) = 0$ and $y_j^{(1)}(0, \cdot) = a_j(\cdot)$, by (4.1) we have

$$\rho\partial_t^2 u_j(0,x) = -(f_1+f_2)\triangle^2 a_j - 2\nabla(f_1+f_2)\cdot\nabla\triangle a_j + (-\triangle f_1 - \partial_1^2 f_2)\,(\partial_1^2 a_j)$$
$$+(-\triangle f_1 - \partial_2^2 f_2)\,(\partial_2^2 a_j) - 2(\partial_1\partial_2 f_2)(\partial_1\partial_2 a_j), \quad x \in \Omega, 1 \le j \le 6. \quad (4.14)$$

By (1.11), we can solve the equations. Then we have

$$-\triangle f_1 - \partial_1^2 f_2 = \frac{det(\triangle^2 a_j, \partial_1(\triangle a_j), \partial_2(\triangle a_j), \rho\partial_t^2 u_j(0,x), \partial_2^2 a_j, \partial_1\partial_2 a_j)}{det(\triangle^2 a_j, \partial_1(\triangle a_j), \partial_2(\triangle a_j), \partial_1^2 a_j, \partial_2^2 a_j, \partial_1\partial_2 a_j)}. \quad (4.15)$$

By (1.7), we see that $a_j \in C^5(\Omega)$. Hence we can write (4.15) by

$$-\triangle f_1 - \partial_1^2 f_2 = \sum_{j=1}^{6} c_{1j}(x)\rho\partial_t^2 u_j(0,x), \quad (4.16)$$

where $c_{1j} \in C^1(\overline{\Omega})$, $1 \le j \le 6$. Similarly there exist $c_{2j} \in C^1(\overline{\Omega})$, $1 \le j \le 6$ such that we can have

$$-\triangle f_1 - \partial_2^2 f_2 = \sum_{j=1}^{6} c_{2j}(x)\rho\partial_t^2 u_j(0,x). \quad (4.17)$$

Subtracting (4.17) from (4.16), we have

$$-\partial_1^2 f_2 + \partial_2^2 f_2 = \sum_{j=1}^{6}(c_{1j}(x) - c_{2j}(x))\rho\partial_t^2 u_j(0,x). \quad (4.18)$$

Because (λ_1, μ_1), $(\lambda_2, \mu_2) \in \mathcal{U}_I$, we have $f_1,\ f_2 \in C_0^3(\Omega)$, and we can apply Lemma 2.2 to (4.18). Therefore we obtain

$$s\sum_{k,\ell=1}^{2}\int_\Omega |\partial_k\partial_\ell f_2|^2 e^{2s\varphi(0,x)}dx + s\int_\Omega |\nabla f_2|^2 e^{2s\varphi(0,x)}dx + s\int_\Omega |f_2|^2 e^{2s\varphi(0,x)}dx$$
$$\le C_{30}\sum_{j=1}^{6}\int_\Omega |\partial_t^2 u_j(0,x)|^2 e^{2s\varphi(0,x)}dx$$
$$+C_{30}\sum_{j=1}^{6}\int_\Omega |\nabla\partial_t^2 u_j(0,x)|^2 e^{2s\varphi(0,x)}dx. \quad (4.19)$$

By virtue of the Carleman estimate for the Laplace operator, in a similar way to the proof of Theorem 2.1, from (4.16) we can derive

$$\begin{aligned}\frac{1}{s}\sum_{k,\ell=1}^{2}\int_\Omega |\partial_k\partial_\ell f_1|^2 e^{2s\varphi(0,x)}dx + s\int_\Omega |\nabla f_1|^2 e^{2s\varphi(0,x)}dx + s^3\int_\Omega |f_1|^2 e^{2s\varphi(0,x)}dx \\ \le C_{31}\sum_{j=1}^{6}\int_\Omega |\partial_t^2 u_j(0,x)|^2 e^{2s\varphi(0,x)}dx \\ + C_{31}\sum_{k,\ell=1}^{2}\int_\Omega |\partial_k\partial_\ell f_2|^2 e^{2s\varphi(0,x)}dx. \qquad (4.20)\end{aligned}$$

By (4.19) and (4.20), we can obtain

$$\begin{aligned}\sum_{k,\ell=1}^{2}\int_\Omega |\partial_k\partial_\ell f_1|^2 e^{2s\varphi(0,x)}dx + s^2\int_\Omega |\nabla f_1|^2 e^{2s\varphi(0,x)}dx + s^4\int_\Omega |f_1|^2 e^{2s\varphi(0,x)}dx \\ \le C_{32}s\sum_{j=1}^{6}\int_\Omega |\partial_t^2 u_j(0,x)|^2 e^{2s\varphi(0,x)}dx \\ + C_{32}\sum_{j=1}^{6}\int_\Omega |\nabla\partial_t^2 u_j(0,x)|^2 e^{2s\varphi(0,x)}dx. \qquad (4.21)\end{aligned}$$

In terms of (4.19) and (4.21), we have

$$\begin{aligned}\sum_{k,\ell=1}^{2}\int_\Omega |\partial_k\partial_\ell f_1|^2 e^{2s\varphi(0,x)}dx + s^2\int_\Omega |\nabla f_1|^2 e^{2s\varphi(0,x)}dx + s^4\int_\Omega |f_1|^2 e^{2s\varphi(0,x)}dx \\ + s\sum_{k,\ell=1}^{2}\int_\Omega |\partial_k\partial_\ell f_2|^2 e^{2s\varphi(0,x)}dx + s\int_\Omega |\nabla f_2|^2 e^{2s\varphi(0,x)}dx + s\int_\Omega |f_2|^2 e^{2s\varphi(0,x)}dx \\ \le C_{33}s\sum_{j=1}^{6}\int_\Omega |\partial_t^2 u_j(0,x)|^2 e^{2s\varphi(0,x)}dx \\ + C_{33}\sum_{j=1}^{6}\int_\Omega |\nabla\partial_t^2 u_j(0,x)|^2 e^{2s\varphi(0,x)}dx. \qquad (4.22)\end{aligned}$$

Applying (4.13) in (4.22), we have

$$\begin{aligned}&\int_\Omega \left(\sum_{|\alpha|\le 2}|\partial_x^\alpha f_1|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha f_2|^2\right) e^{2s\varphi(0,x)}dx \\ &\le C_{34}\int_\Omega \left(\sum_{|\alpha|\le 2}|\partial_x^\alpha f_1|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha f_2|^2\right) e^{2s\varphi(0,x)} \left(\int_0^T e^{2s(\varphi(t,x)-\varphi(0,x))}dt\right) dx \\ &\quad + C_{34}s^6V^2e^{2s\Phi} + C_{34}s(V^2+U^2)e^{2s(d-2\epsilon)}.\end{aligned}$$

Recalling the form of φ and applying the Lebesgue theorem, we have

$$\sup_{x\in\Omega}\left|\int_0^T e^{2s(\varphi(t,x)-2s\varphi(0,x))}dt\right| = \sup_{x\in\Omega}\left|\int_0^T \exp\left(2se^{\gamma|x-x_0|}(e^{-\gamma\beta t^2}-1)\right)dt\right|$$
$$\leq \int_0^T \exp\left(2se^{\gamma\sigma}(e^{-\gamma\beta t^2}-1)\right)dt = o(1),$$

as $s \longrightarrow \infty$, where $\sigma = \inf_{x\in\Omega}|x-x_0|$. Hence, we have

$$(1-o(1))\left[\int_\Omega\left(\sum_{|\alpha|\leq 2}|\partial_x^\alpha f_1|^2+\sum_{|\alpha|\leq 2}|\partial_x^\alpha f_2|^2\right)e^{2s\varphi(0,x)}dx\right]$$
$$\leq C_{35}s^6V^2e^{2s\Phi}+C_{35}s(V^2+U^2)e^{2s(d-2\epsilon)}, \quad s\longrightarrow\infty.$$

By using $\varphi(0,x)\geq d$, we can obtain

$$\int_\Omega\left(\sum_{|\alpha|\leq 2}|\partial_x^\alpha f_1|^2+\sum_{|\alpha|\leq 2}|\partial_x^\alpha f_2|^2\right)dx$$
$$\leq C_{36}s^6V^2e^{2s\Phi}+C_{36}s(U^2+V^2)e^{-4s\epsilon}. \tag{4.23}$$

In terms of (4.1), (1.2) and (1.3), we have

$$L_2\partial_t^2 u_j = -(f_1+f_2)\triangle^2\partial_t^2 y_j^{(1)} - 2\nabla(f_1+f_2)\cdot\nabla\triangle\partial_t^2 y_j^{(1)} - \triangle(f_1+f_2)\triangle\partial_t^2 y_j^{(1)}$$
$$-2(\partial_1\partial_2 f_2)\left(\partial_1\partial_2\partial_t^2 y_j^{(1)}\right)+(\partial_1^2 f_2)\left(\partial_2^2\partial_t^2 y_j^{(1)}\right)+(\partial_2^2 f_2)\left(\partial_1^2\partial_t^2 y_j^{(1)}\right),$$
$$B_1(\lambda_2,\mu_2)\partial_t^2 u_j = B_2(\lambda_2,\mu_2)\partial_t^2 u_j = 0 \text{ on } \partial\Omega\times(0,T),$$
$$\partial_t^2 u_j(0,\cdot)=0, \qquad \partial_t(\partial_t^2 u_j)(0,\cdot)=0 \text{ in } \Omega.$$

Since $u_j \in W^{4,\infty}(-T,T;L^\infty(\Omega))\cap W^{2,\infty}(-T,T;W^{4,\infty}(\Omega))$, we see that $\partial_t^2 u_j \in H^{4,2}(\widetilde{Q})$. By virtue of Theorem 3.1 and (4.7), we can obtain

$$U^2 \leq C_{37}\int_\Omega\left(\sum_{|\alpha|\leq 2}|\partial_x^\alpha f_1|^2+\sum_{|\alpha|\leq 2}|\partial_x^\alpha f_2|^2\right)dx + C_{37}V^2. \tag{4.24}$$

Substituting (4.24) into (4.23) and taking s large enough, we obtain

$$\int_{\Omega}\left(\sum_{|\alpha|\le 2}|\partial_x^\alpha f_1|^2+\sum_{|\alpha|\le 2}|\partial_x^\alpha f_2|^2\right)dx \le C_{38}V^2.$$

Thus we complete the proof of Theorem 1.

Proof of Theorem 2.

To prove Theorem 2, we argue similarly to Theorem 1. From the proof of Theorem 1, we can see it is sufficient to prove (4.19) and (4.20) from (4.14) with $a_j(x_1,x_2)=\frac{p_{1j}}{2}x_1^2+\frac{p_{2j}}{2}x_2^2+p_{3j}x_1x_2$ where p_{1j}, p_{2j}, $p_{3j}\in\mathbb{R}$, $j=1,2$.

Consequently we have

$$-(p_{11}+p_{21})\triangle f_1-p_{11}\partial_1^2 f_2-p_{21}\partial_2^2 f_2-2p_{31}\partial_1\partial_2 f_2=\rho\partial_t^2 u_1(0,x) \quad (4.25)$$

and

$$-(p_{12}+p_{22})\triangle f_1-p_{12}\partial_1^2 f_2-p_{22}\partial_2^2 f_2-2p_{32}\partial_1\partial_2 f_2=\rho\partial_t^2 u_2(0,x). \quad (4.26)$$

We will consider two cases separately:

1). If $p_{11}=-p_{21}$ and $p_{12}\neq -p_{22}$, then

$$p_{11}\partial_1^2 f_2+2p_{31}\partial_1\partial_2 f_2-p_{11}\partial_2^2 f_2=-\rho\partial_t^2 u_1(0,x). \quad (4.27)$$

2). If $p_{11}+p_{21}\neq 0$ and $p_{12}+p_{22}\neq 0$, then we can readily obtain

$$r_1\partial_1^2 f_2+2r_3\partial_1\partial_2 f_2+r_2\partial_2^2 f_2=\frac{-\rho\partial_t^2 u_1(0,x)}{p_{11}+p_{21}}+\frac{\rho\partial_t^2 u_2(0,x)}{p_{12}+p_{22}}, \quad (4.28)$$

where $r_k=\frac{p_{k1}}{p_{11}+p_{21}}-\frac{p_{k2}}{p_{12}+p_{22}}$, $k=1,2,3$.

Thus it suffices to prove (4.19) and (4.20) from either (4.27) or (4.28). We consider only case (4.28), because one can discuss for (4.27) similarly.

By (1.15), we can apply Lemma 2.2 to (4.28), and (4.19) follows. On the other hand, by applying the Carleman estimate for the Laplace operator to (4.26), we obtain (4.20). Thus the proof of Theorem 2 is complete.

5 Lipschitz stability in determining coefficients λ, μ and ρ

Proof of Theorem 3.

We set $y_j^{(1)} = y(\lambda_1, \mu_1, \rho_1; g_j, h_j, a_j, H_j)$, $y_j^{(2)} = y(\lambda_2, \mu_2, \rho_2; g_j, h_j, a_j, H_j)$, $u_j = y_j^{(2)} - y_j^{(1)}$, $1 \le j \le 7$, $f_1 = \lambda_2 - \lambda_1$, $f_2 = \mu_2 - \mu_1$, $f_3 = \rho_2 - \rho_1$ and

$$\begin{aligned} L_2 y(t,x) \equiv \rho_2(x)\partial_t^2 y(t,x) + (\lambda_2(x) + \mu_2(x))\Delta^2 y(t,x) + 2\nabla(\lambda_2 + \mu_2)(x) \cdot \nabla(\Delta y(t,x)) \\ +\Delta(\lambda_2 + \mu_2)\Delta y(t,x) + 2(\partial_1\partial_2\mu_2)(x)\partial_1\partial_2 y(t,x) \\ -(\partial_1^2\mu_2)(x)\partial_2^2 y(t,x) - (\partial_2^2\mu_2)(x)\partial_1^2 y(t,x). \end{aligned}$$

Then from (1.1), we can obtain

$$\begin{aligned} L_2 u_j = -\left(f_1 + f_2\right) \triangle^2 y_j^{(1)} - 2\nabla\left(f_1 + f_2\right) \cdot \nabla \triangle y_j^{(1)} - \triangle\left(f_1 + f_2\right) \triangle y_j^{(1)} \\ -2(\partial_1\partial_2 f_2)\left(\partial_1\partial_2 y_j^{(1)}\right) + (\partial_1^2 f_2)\left(\partial_2^2 y_j^{(1)}\right) \\ +(\partial_2^2 f_2)\left(\partial_1^2 y_j^{(1)}\right) - f_3\partial_t^2 y_j^{(1)} \text{ in Q.} \end{aligned} \tag{5.1}$$

We extend $y_j^{(1)}$ and u_j in $(0,T)\times\Omega$ by $y_j^{(1)}(-t,x) = y_j^{(1)}(t,x)$ and $u_j(-t,x) = u_j(t,x)$, $(t,x) \in (-T,0)\times\Omega$. Then, since $\partial_t y_j^{(1)}(0,\cdot) = 0$ and $\partial_t u_j(0,\cdot) = \partial_t H(0,\cdot) = 0$, we have $\partial_t^3 y_j^{(1)}(0,\cdot) = \partial_t^3 u_j(0,\cdot) = 0$. Therefore $y_j^{(1)}$, $u_j \in$

$W^{4,\infty}(-T,T;L^\infty(\Omega))\cap W^{2,\infty}(-T,T;W^{4,\infty}(\Omega))$. In the same way as the proof of Theorem 1, we choose constants $d>0$, $\beta>0$, $\gamma>0$, $\epsilon>0$, $\delta>0$ and the weight function $\varphi(t,x)=\exp\{\gamma(|x-x_0|^2-\beta t^2)\}$. We choose the cut-off function $\chi(t)$ defined by (4.2). For $\partial_t^2 u_j$, we apply the Sobolev extension theorem to find a function u_j^* such that

$$\partial_\nu^k u_j^* = \partial_\nu^k \partial_t^2 u_j \quad \text{on} \quad [-T,T]\times\partial\Omega, \quad 0\le k\le 3 \tag{5.2}$$

and

$$\|u_j^*\|_{H^{4,2}(Q)} \le C_{39}\sum_{k=1}^{3}\|\partial_\nu^k\partial_t^2 u_j\|_{H^{\frac{7}{2}-k,2}((0,T)\times\partial\Omega)}. \tag{5.3}$$

We set

$$z_j = z_j(t,x) = \partial_t^2 u_j(t,x) - u_j^*(t,x), \quad 1\le j\le 7. \tag{5.4}$$

Henceforth we set

$$V^2=\sum_{k=1}^{3}\|\partial_\nu^k\partial_t^2 u_j\|^2_{H^{\frac{7}{2}-k,2}((0,T)\times\partial\Omega)}, \qquad U^2=\sum_{j=1}^{7}\left(\|z_j\|^2_{L^2(Q)}+\|\partial_t z_j\|^2_{L^2(Q)}+\|\nabla z_j\|^2_{L^2(Q)}\right),$$

and

$$\Phi = \sup_{(x,t)\in Q}\varphi(x,t).$$

Then by (5.1) and (5.2), we can see

$$\begin{aligned} L_2(z_j\chi) = \chi\Big\{&-(f_1+f_2)\triangle^2\partial_t^2 y_j^{(1)} - 2\nabla(f_1+f_2)\cdot\nabla\triangle\partial_t^2 y_j^{(1)} - \triangle(f_1+f_2)\triangle\partial_t^2 y_j^{(1)} \\ &-2(\partial_1\partial_2 f_2)\left(\partial_1\partial_2\partial_t^2 y_j^{(1)}\right) + (\partial_1^2 f_2)\left(\partial_2^2\partial_t^2 y_j^{(1)}\right) + (\partial_2^2 f_2)\left(\partial_1^2\partial_t^2 y_j^{(1)}\right) \\ &-f_3\partial_t^4 y_j^{(1)}\Big\} + \rho_2(\partial_t^2\chi)(\partial_t^2 u_j) + 2\rho_2(\partial_t\chi)(\partial_t^3 u_j) - L_2(u_j^*\chi). \end{aligned} \tag{5.5}$$

In terms of (5.3), (5.4) and

$$\|y_j^{(1)}\|_{W^{4,\infty}(-T,T;L^\infty(\Omega))\cap W^{2,\infty}(-T,T;W^{4,\infty}(\Omega))} \le M, \quad 1\le j\le 7,$$

we apply Theorem 2.1 to (5.5), and similarly to (4.8), we obtain

$$\int_{\widetilde{Q}} \left\{ s|\nabla\partial_t(\chi z_j)|^2 + s^3|\partial_t(\chi z_j)|^2 + s\sum_{k,\ell=1}^{2} |\partial_k\partial_\ell(\chi z_j)|^2 + s^4|\nabla(\chi z_j)|^2 + s^6|\chi z_j|^2 \right\} e^{2s\varphi} dxdt$$

$$\le C_{40} \int_{\widetilde{Q}} \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha f_1|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha f_2|^2 + |f_3|^3 \right) e^{2s\varphi} dxdt$$
$$+ C_{40} V^2 e^{2s\Phi} + C_{40}(V^2 + U^2) e^{2s(d-2\epsilon)}.$$

Therefore, similarly to (4.9), we have

$$\int_{\widetilde{Q}} \left\{ s|\nabla\partial_t(\chi\partial_t^2 u_j)|^2 + s^3|\partial_t(\chi\partial_t^2 u_j)|^2 + s\sum_{k,\ell=1}^{2} |\partial_k\partial_\ell(\chi\partial_t^2 u_j)|^2 \right.$$
$$\left. + s^4|\nabla(\chi\partial_t^2 u_j)|^2 + s^6|\chi\partial_t^2 u_j|^2 \right\} e^{2s\varphi} dxdt$$
$$\le C_{41} \int_{\widetilde{Q}} \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha f_1|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha f_2|^2 + |f_3|^3 \right) e^{2s\varphi} dxdt$$
$$+ C_{41} s^6 V^2 e^{2s\Phi} + C_{41}(V^2 + U^2) e^{2s(d-2\epsilon)}.$$

For u_j, $1 \le j \le 7$, we argue in the same way as (4.10)-(4.12), so that

$$s\int_\Omega |(\partial_t^2 u_j)(0,x)|^2 e^{2s\varphi(0,x)} dx + \int_\Omega |(\nabla\partial_t^2 u_j)(0,x)|^2 e^{2s\varphi(0,x)} dx$$
$$\le C_{42} \int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha f_1|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha f_2|^2 + |f_3|^2 \right) \left(\int_0^T e^{2s\varphi} dt \right) dx$$
$$+ C_{42} s(V^2 + U^2) e^{2s(d-2\epsilon)} + C_{42} s^6 V^2 e^{2s\Phi}, \qquad (5.6)$$

for all large $s > 1$. From (5.1) we have

$$\rho_2 \partial_t^2 u_j(0,x) = -(f_1 + f_2)\triangle^2 a_j - 2\nabla(f_1 + f_2)\cdot\nabla\triangle a_j - 2(\partial_1\partial_2 f_2)(\partial_1\partial_2 a_j)$$
$$+ (-\triangle f_1 - \partial_2^2 f_2)(\partial_2^2 a_j) + (-\triangle f_1 - \partial_1^2 f_2)(\partial_1^2 a_j)$$
$$- f_3 \partial_t^2 y_j^{(1)}(0,x), \quad 1 \le j \le 7. \qquad (5.7)$$

We note

$$\partial_t^2 y_j^{(1)}(0,x) = \frac{1}{\rho_1}\{-(\lambda_1+\mu_1)\triangle^2 a_j - 2\nabla(\lambda_1+\mu_1)\cdot\nabla\triangle a_j - \triangle(\lambda_1+\mu_1)\triangle a_j$$
$$-2(\partial_1\partial_2\mu_1)(\partial_1\partial_2 a_j) + (\partial_1^2\mu_1)(\partial_2^2 a_j) + (\partial_2^2\mu_1)(\partial_1^2 a_j) + H_j(0,x)\}$$
$$\equiv b_j(x). \tag{5.8}$$

Then (5.7) becomes

$$\rho_2\partial_t^2 u_j(0,x) = -(f_1+f_2)\triangle^2 a_j - 2\nabla(f_1+f_2)\cdot\nabla\triangle a_j$$
$$-2(\partial_1\partial_2 f_2)(\partial_1\partial_2 a_j) + (-\triangle f_1 - \partial_2^2 f_2)(\partial_2^2 a_j) + (-\triangle f_1 - \partial_1^2 f_2)(\partial_1^2 a_j)$$
$$-f_3 b_j(x), \quad 1 \le j \le 7. \tag{5.9}$$

Thanks to (1.20), we can solve (5.9) with respect to $-\triangle f_1 - \partial_1^2 f_2$, $-\triangle f_1 - \partial_2^2 f_2$ and f_3. Similarly to the proof of Theorem 1, there exist c_{1j},c_{2j}, $c_{3j} \in C^1(\overline{\Omega})$, $1 \le j \le 7$, such that

$$-\triangle f_1 - \partial_1^2 f_2 = \sum_{j=1}^{7} c_{1j}(x)\rho_2\partial_t^2 u_j(0,x), \tag{5.10}$$

$$-\triangle f_1 - \partial_2^2 f_2 = \sum_{j=1}^{7} c_{2j}(x)\rho_2\partial_t^2 u_j(0,x) \tag{5.11}$$

and

$$f_3 = \sum_{j=1}^{7} c_{3j}(x)\rho_2\partial_t^2 u_j(0,x). \tag{5.12}$$

Now, by the same argument in the proof of Theorem 1, we can obtain

$$\int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha f_1|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha f_2|^2\right) e^{2s\varphi(0,x)}dx$$
$$\le C_{43}\sum_{j=1}^{7}\int_\Omega \left(s|\partial_t^2 u_j(0,x)|^2 + |\nabla\partial_t^2 u_j(0,x)|^2\right) e^{2s\varphi(0,x)}dx. \tag{5.13}$$

As for f_3, we directly see that

$$\int_\Omega (|f_3|^2 + |\nabla f_3|^2)\, e^{2s\varphi(0,x)} dx \le C_{44} \sum_{j=1}^{7} \int_\Omega (|\partial_t^2 u_j(0,x)|^2 + |\nabla \partial_t^2 u_j(0,x)|^2)\, e^{2s\varphi(0,x)} dx, \tag{5.14}$$

for $s > 1$. From (5.6), (5.13) and (5.14), we have

$$\int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha f_1|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha f_2|^2 + |f_3|^2 + |\nabla f_3|^2 \right) e^{2s\varphi(0,x)} dx$$

$$\le C_{45} \int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha f_1|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha f_2|^2 + |f_3|^2 + |\nabla f_3|^2 \right) e^{2s\varphi(0,x)} \left(\int_0^T e^{2s(\varphi(t,x)-\varphi(0,x))} dt \right) dx$$
$$+ C_{45} s (V^2 + U^2) e^{2s(d-2\epsilon)} + C_{45} s^6 V^2 e^{2s\Phi}.$$

By the same argument in the proof of Theorem 1, we can obtain (1.21). Thus the proof of Theorem 3 is complete.

Proof of Theorem 4.

According to the proof of Theorem 3, it is sufficient to derive (5.10)-(5.12) from

$$\rho_2 \partial_t^2 u_j(0,x) = -(f_1 + f_2) \triangle^2 a_j - 2\nabla(f_1 + f_2) \cdot \nabla \triangle a_j - 2(\partial_1\partial_2 f_2)(\partial_1\partial_2 a_j)$$
$$+ (-\triangle f_1 - \partial_2^2 f_2)(\partial_2^2 a_j) + (-\triangle f_1 - \partial_1^2 f_2)(\partial_1^2 a_j)$$
$$- f_3 b_j, \quad j = 1,2,3. \tag{5.15}$$

Since $a_j(x) = \frac{p_{1j}}{2} x_1^2 + \frac{p_{2j}}{2} x_2^2$, p_{1j}, $p_{2j} \in \mathbb{R}$ and b_j are defined by (5.8), we obtain

$$\rho_2 \partial_t^2 u_j(0,x) = (-\triangle f_1 - \partial_2^2 f_2) p_{2j} + (-\triangle f_1 - \partial_1^2 f_2) p_{1j} - f_3 b_j,$$
$$j = 1,2,3.$$

In view of (1.23), the equations are solvable with respect to $-\triangle f_1 - \partial_2^2 f_2$, $-\triangle f_1 - \partial_1^2 f_2$ and f_3. Therefore we can similarly have (5.10)-(5.12), and the proof of Theorem 4 is complete.

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