

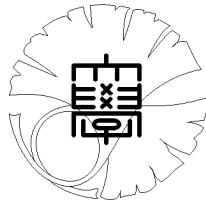
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l^q -valued extension of the
**fractional maximal operators for
non-doubling measures via potential operators**

by

Yoshihiro SAWANO



UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

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Abstract

In this paper we consider fractional maximal operators with Radon measure on \mathbf{R}^d . We do not pose any assumption on the measure μ except that the measure is non-zero. For the proof we construct Riesz-like potential operators for μ .

Keywords covering lemma, fractional integral operator, fractional maximal operator, vector-valued.

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1 Introduction

In this paper we will extend the boundedness of the modified maximal operators on the non-homogeneous space. Let μ be a Radon measure on \mathbf{R}^d . Throughout this paper by “ball” we mean a ball with positive radius, and if μ is finite, we regard \mathbf{R}^d as a special ball with radius ∞ . Denote $B(x, r)$ as an open ball in \mathbf{R}^d with center x and radius $r > 0$ and for a ball $B = B(x, r)$, we set $r(B) := r$ and $\kappa B := B(x, \kappa r)$. Set the centered maximal operator by

$$\tilde{M}f(x) := \sup_{B=B(x,r) \in \mathcal{B}(\mu)} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y).$$

Here and in what follows we denote $\mathcal{B}(\mu)$ as the totality of the balls with positive μ -measure. \tilde{M} is weak-(1, 1) bounded with the aid of Besicovitch’s covering lemma. The proof can be found in [10]. If we set the centered fractional maximal operator by

$$\tilde{M}_\alpha f(x) := \sup_{B=B(x,r) \in \mathcal{B}(\mu)} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)| d\mu(y), \quad 0 \leq \alpha < 1, \quad (1)$$

then the similar proof shows that \tilde{M}_α is weak-(1, $(1 - \alpha)^{-1}$) bounded.

*This work is supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists. Current address Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku Tokyo 153-8914, JAPAN, E-mail: yoshihiro@ms.u-tokyo.ac.jp

What happens for the uncentered maximal operator? We set the uncentered Hardy-Littlewood maximal operator M as

$$Mf(x) := \sup_{x \in B \in \mathcal{B}(\mu)} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y).$$

If the measure is doubling, that is, $\mu(2B) \leq C\mu(B)$ for any ball B with center in $\text{supp}(\mu)$, then M is weak-(1, 1) bounded. But if the measure is not doubling, then it can happen that M is not weak-(1, 1) bounded (see [2]). It is the same that M_α is not weak-(1, $(1 - \alpha)^{-1}$) bounded in general if we replace the uncentered fractional modified maximal operator, where

$$M_\alpha f(x) := \sup_{x \in B \in \mathcal{B}(\mu)} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)| d\mu(y), 0 \leq \alpha < 1.$$

Again if the measure is doubling, we have weak-(1, $(1 - \alpha)^{-1}$) boundedness of M_α .

To overcome this difficulty, in [3], [7], [12], [13] and [17] the modified maximal operator was considered. If we define modified maximal operators by

$$M_{0,\kappa} f(x) := \sup_{x \in B \in \mathcal{B}(\mu)} \frac{1}{\mu(\kappa B)} \int_B |f(y)| d\mu(y), \kappa > 1$$

or more generally

$$M_{\alpha,\kappa} f(x) := \sup_{x \in B \in \mathcal{B}(\mu)} \frac{1}{\mu(\kappa B)^{1-\alpha}} \int_B |f(y)| d\mu(y), \kappa > 1, 0 \leq \alpha < 1, \quad (2)$$

then $M_{\alpha,\kappa}$ is weak-(1, $(1 - \alpha)^{-1}$) bounded. These operators enjoy the boundedness properties with the aid of Besicovitch's covering lemma. For our later purpose we use somehow stronger version, whose proof is similar to [13, Theorem 1.5]. For convenience we prove Lemma 1.1 in Appendix.

Lemma 1.1. *Let $\kappa > 1$ be a fixed number. Suppose that $\mathcal{B} = \{B_\lambda\}_{\lambda \in \Lambda}$ is a family of the balls with bounded radii: We assume $\sup_{\lambda \in \Lambda} r(B_\lambda) < \infty$. Then we can take $\{B_\lambda\}_{\lambda \in \Lambda_1}, \dots, \{B_\lambda\}_{\lambda \in \Lambda_N}$, subfamilies such that the following condition holds. Here, $N = N_\kappa$ depends only on $\kappa > 1$.*

1. *If $\lambda, \lambda' \in \Lambda_j, j = 1, \dots, N_\kappa$ are different, then*

$$\kappa B_\lambda \cap \kappa B_{\lambda'} = \emptyset. \quad (3)$$

2. *For all $\lambda \in \Lambda$ there exists $i(\lambda) \in \bigcup_{j=1}^N \Lambda_j$ such that*

$$B_\lambda \subset \kappa B_{i(\lambda)}. \quad (4)$$

The aim of this paper is to obtain the Fefferman-Stein type extension of $M_{\alpha,\kappa}$ and \tilde{M}_α with $0 < \alpha < 1$ and $\kappa > 1$ for a general non-zero Radon measure μ via Riesz-Potential-like operators for μ . In [1] D. Adams defined the fractional integral operator for Lebesgue measure by

$$(-\Delta)^{-\alpha/2} f(x) := \frac{\Gamma(\frac{\alpha}{2})}{\pi^{\alpha-\frac{d}{2}} \Gamma(\frac{d-\alpha}{2})} \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy, 0 < \alpha < d.$$

$(-\Delta)^{-\alpha/2}$ is known to be $L^p(\mathbf{R}^d)$ - $L^q(\mathbf{R}^d)$ bounded, if $p, q > 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. It is easily seen that $\tilde{M}_\alpha f(x), M_{\alpha, \kappa} f(x) \leq C(-\Delta)^{-d\alpha/2}|f|(x)$ for some big constant $C > 0$. We define a fractional integral operator of this type for general measure μ .

Suppose that μ satisfies the growth condition $\mu(B(x, r)) \leq c_0 r^n$. Then in [5] García-Cuerva and Gatto defined a fractional integral operator J_α by

$$J_\alpha f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y). \quad (5)$$

Carderón-Zygmund theory for J_α with growth measure has been developed. García-Cuerva and Gatto showed the $L^p(\mu)$ - $L^q(\mu)$ boundedness of J_α if $p, q > 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. The $L^p(\mu)$ - $L^q(\mu)$ boundedness of I_α in more general form was firstly proved by V. Kokilashvili in \mathbf{R}^d ($1 < p < q < \infty$) in [8]. In general non-homogeneous spaces in general setting ($1 < p < q < \infty$) it is proved in [9]. See also the monograph by D. Edmunds, V.Kokilashvili and A. Meskhi [4]. Chen and Sawyer showed the boundedness of the commutator $[a, J_\alpha]$, where a is an RBMO function defined by X. Tolsa [17]. For the multilinear version we refer [6]. The example of Calderón-Zygmund theory without growth condition can be found in [7] and [11].

However, the operator J_α is not sufficient for our purpose: Even if μ satisfies the growth condition, J_α can not dominate \tilde{M}_α nor $M_{\alpha, \kappa}$. Thus we shall define linear operators which dominate \tilde{M}_α and $M_{\alpha, \kappa}$ respectively. Our linear operators will be of the form

$$\tilde{I}_\alpha f(x) = \int_{\mathbf{R}^d} \tilde{k}_\alpha(x, y) f(y) d\mu(y) \text{ and } I_{\alpha, \kappa} f(x) = \int_{\mathbf{R}^d} k_{\alpha, \kappa}(x, y) f(y) d\mu(y).$$

\tilde{I}_α and $I_{\alpha, \kappa}$ are different from those appearing in the fore-mentioned papers. Here $k_{\alpha, \kappa}$ and \tilde{k}_α are positive μ -measurable functions. They will satisfy pointwise estimates

$$\tilde{M}_\alpha f(x) \leq C \tilde{I}_\alpha |f|(x) \text{ and } M_{\alpha, \kappa} f(x) \leq C I_{\alpha, \kappa} |f|(x).$$

In what follows we will distinguish the centered-type operators from the uncentered-type operators by $\tilde{\cdot}$ and subindices : The operators with κ such as $M_{\alpha, \kappa}, I_{\alpha, \kappa}, \dots$ are related with the uncentered maximal operators. Meanwhile the operators related to the centered maximal operators are denoted with $\tilde{\cdot}$. For example \tilde{M} is a centered non-modified maximal operator.

We will work on the modified Morrey space defined in [14]. For details we refer [14]. For distinction of weak-type spaces defined in [15] we name them strong-type Morrey spaces.

For $L^q_{loc}(\mu)$ function f the (strong-type) $\mathcal{M}_q^p(\mu)$ norm is given by

$$\|f : \mathcal{M}_q^p(k, \mu)\| := \sup_{B \in \mathcal{B}(\mu)} \mu(kB)^{\frac{1}{p} - \frac{1}{q}} \left(\int_B |f(y)|^q d\mu(y) \right)^{\frac{1}{q}}.$$

Set $\mathcal{M}_q^p(k, \mu)$ as a set of μ -measurable functions with the norm above finite. It is known that if $k_1, k_2 > 1$, then $\mathcal{M}_q^p(k_1, \mu) \sim \mathcal{M}_q^p(k_2, \mu)$. We also have the weak-type Morrey space whose semi-norm is given by

$$\|f : \mathcal{M}_q^p(k, \mu)\|_w := \sup_{\lambda > 0} \sup_{B \in \mathcal{B}(\mu)} \mu(kB)^{\frac{1}{p} - \frac{1}{q}} (\lambda^q \mu\{x \in B : |f(x)| > \lambda\})^{\frac{1}{q}}.$$

The definition can be found in [15]. Set $\mathcal{M}_q^p(k, \mu)_w$ as a set of μ -measurable functions with the semi-norm above finite. In what follows for definiteness we write $\mathcal{M}_q^p(\mu) := \mathcal{M}_q^p(2, \mu)$ and

$\mathcal{M}_q^p(\mu)_w := \mathcal{M}_q^p(2, \mu)_w$. Like the strong-type Morrey spaces the number 2 does not affect the definition of the spaces: It suffices to be strictly greater than 1. We note that $\mathcal{M}_p^p(\mu)_w$ is weak- $L^p(\mu)$ space.

We shall prove the following theorems.

Theorem 1.2. *Let $0 < \alpha < 1$ and \tilde{M}_α be the centered maximal operator given by (1).*

1. *Suppose that $1 < p < \alpha^{-1}$, $1 \leq r \leq \infty$ and that $\frac{1}{q} = \frac{1}{p} - \alpha$. Let $\{f_j\}_{j=1}^\infty$ be a system of μ -measurable functions. Then*

$$\left\| \left(\sum_{j=1}^{\infty} \tilde{M}_\alpha f_j^r \right)^{\frac{1}{r}} : L^q(\mu) \right\| \leq C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} : L^p(\mu) \right\|.$$

Here, C is independent of $\{f_j\}_{j=1}^\infty$.

2. *Suppose that $1 \leq r \leq \infty$. Let $\{f_j\}_{j=1}^\infty$ be a system of μ -measurable functions. Then*

$$\mu \left\{ x \in \mathbf{R}^d : \left(\sum_{j=1}^{\infty} \tilde{M}_\alpha f_j(x)^r \right)^{\frac{1}{r}} > \lambda \right\} \leq \left(\frac{C}{\lambda} \int_{\mathbf{R}^d} \left(\sum_{j=1}^{\infty} |f_j(x)|^r \right)^{\frac{1}{r}} d\mu(x) \right)^{\frac{1}{1-\alpha}}.$$

Here, C is independent of $\{f_j\}_{j=1}^\infty$.

Theorem 1.3. *Let $0 < \alpha < 1$ and $M_{\alpha, \kappa}$ be the uncentered maximal operator given by (2). Let $\{f_j\}_{j=1}^\infty$ be a system of μ -measurable functions.*

1. *Suppose that $1 < q \leq p < \alpha^{-1}$, $1 \leq r \leq \infty$, $\frac{1}{s} = \frac{1}{p} - \alpha$ and that $\frac{t}{s} = \frac{q}{p}$. Then*

$$\left\| \left(\sum_{j=1}^{\infty} M_{\alpha, \kappa} f_j^r \right)^{\frac{1}{r}} : \mathcal{M}_t^s(\mu) \right\| \leq C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} : \mathcal{M}_q^p(\mu) \right\|.$$

Here, C is independent of $\{f_j\}_{j=1}^\infty$.

2. *Suppose that $1 \leq p < \alpha^{-1}$, $1 \leq r \leq \infty$, and that $\frac{1}{s} = \frac{1}{p} - \alpha$. Then*

$$\left\| \left(\sum_{j=1}^{\infty} M_{\alpha, \kappa} f_j^r \right)^{\frac{1}{r}} : \mathcal{M}_{s/p}^s(\mu) \right\|_w \leq C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} : \mathcal{M}_1^p(\mu) \right\|.$$

Here, C is independent of $\{f_j\}_{j=1}^\infty$.

The centered maximal operator \tilde{M} is not bounded from $\mathcal{M}_q^p(\mu)$ to $\mathcal{M}_q^p(\mu)_w$. Let us note that Theorem 1.3 2 is obtained in [14] for $r > 1$. We have obtained the theorem using the vector-valued maximal inequality of Fefferman-Stein type for $M_{0, \kappa}$ with $\kappa > 1$ for Morrey spaces. In [13] and [14] we have proved the vector-valued maximal inequality of Fefferman-Stein type for $M_{0, \kappa}$ with $\kappa > 1$. However, the one for \tilde{M} is still missing so that we cannot obtain Theorem

1.2. The method by the vector-valued maximal function inequality of Fefferman and Stein has another disadvantage. We cannot recover the case $r = 1$. If $r = 1$, then the vector-valued maximal function inequality of Fefferman and Stein fails. For details of this fact we refer [16]. Our present method does not suffer from this shortcoming.

Suppose that, for example, the inequality $\|\tilde{I}_\alpha f : L^q(\mu)\| \leq C \|f : L^p(\mu)\|$ is obtained for $1 < p < \alpha^{-1}$, $\frac{1}{q} = \frac{1}{p} - \alpha$. Then we obtain Theorem 1.2 1 as follows:

By l^r - $l^{r'}$ duality and the pointwise estimate $\tilde{M}_\alpha f_j(x) \leq C \tilde{I}_\alpha |f_j|(x)$, we have

$$\left(\sum_{j=1}^{\infty} \tilde{M}_\alpha f_j(x)^r \right)^{\frac{1}{r}} \leq C \left(\sum_{j=1}^{\infty} \tilde{I}_\alpha |f_j|(x)^r \right)^{\frac{1}{r}} = C \sup_{\substack{a=a(x) : \|a : l^{r'}\| \leq 1 \\ a(x) = \{a_j(x)\}_{j=1}^{\infty} \in l^{r'}}} \sum_{j=1}^{\infty} a_j(x) \tilde{I}_\alpha |f_j|(x).$$

In what follows let us write sup instead of $\sup_{\substack{a=a(x) : \|a : l^{r'}\| \leq 1 \\ a(x) = \{a_j(x)\}_{j=1}^{\infty} \in l^{r'}}$. Since \tilde{I}_α is a linear operator, we can proceed further.

$$\begin{aligned} \left(\sum_{j=1}^{\infty} \tilde{M}_\alpha f_j(x)^r \right)^{\frac{1}{r}} &= C \sup \sum_{j=1}^{\infty} a_j(x) \int_{\mathbf{R}^d} \tilde{k}_\alpha(x, y) |f_j(y)| d\mu(y) \\ &= C \sup \int_{\mathbf{R}^d} \tilde{k}_\alpha(x, y) \left(\sum_{j=1}^{\infty} a_j(x) |f_j(y)| \right) d\mu(y) \\ &\leq C \int_{\mathbf{R}^d} \tilde{k}_\alpha(x, y) \left(\sum_{j=1}^{\infty} |f_j(y)|^r \right)^{\frac{1}{r}} d\mu(y) = C \left[\tilde{I}_\alpha \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right] (x). \end{aligned}$$

As a consequence we obtain

$$\left\| \left(\sum_{j=1}^{\infty} \tilde{M}_\alpha f_j^r \right)^{\frac{1}{r}} : L^q(\mu) \right\| \leq C \left\| I_\alpha \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} : L^q(\mu) \right\| \leq C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} : L^p(\mu) \right\|.$$

The rest of Theorems 1.2 and 1.3 can be obtained similarly.

Thus the rest of this paper is devoted to constructing \tilde{I}_α and $I_{\alpha, \kappa}$ and to studying their properties. To establish the boundedness of these operators we use Lemmas 1.4 and 1.5. The proof can be obtained in the standard way by Besicovitch's covering lemma and interpolation. For details we refer [10].

Lemma 1.4. *Let \tilde{M} be a centered maximal operator. Then \tilde{M} is $L^p(\mu)$ -bounded for $p > 1$ and weak-(1, 1) bounded.*

Unfortunately the centered maximal operator \tilde{M} is not bounded on $\mathcal{M}_q^p(\mu)$ in general. In Section 4 we give an example for which \tilde{M} is not weak- $\mathcal{M}_1^2(\mu)$ bounded.

As for the uncentered maximal operator we use the following result.

Lemma 1.5. *Suppose that $\kappa > 1$. $M_{0, \kappa}$ is the κ -times uncentered modified maximal operator.*

1. [15, Theorem 3.3] Let $1 \leq p < \infty$. Then $\|M_{0,\kappa}f : \mathcal{M}_1^p(\mu)\|_w \leq C \|f : \mathcal{M}_1^p(\mu)\|$, where C is independent of f .
2. [14, Theorem 2.3] Let $1 < q \leq p < \infty$. Then $\|M_{0,\kappa}f : \mathcal{M}_q^p(\mu)\| \leq C \|f : \mathcal{M}_q^p(\mu)\|$, where C is independent of f .

2 Centered fractional maximal operators

In this section we consider centered fractional maximal operators. For this purpose we will define auxiliary numbers $r_k(x)$.

Definition 2.1. Given a Radon measure μ and a point $x \in \mathbf{R}^d$, we define

$$r_k(x) := \sup\{r \geq 0 : \mu(B(x, r)) < 2^k\}$$

for $k \in \mathbf{Z}$ with $k > \log_2 \mu(\{x\})$.

We note that $r_k(x) > 0$ for all $k > k(x)$ and that $r_k(x) \uparrow \infty$ as $k \rightarrow \infty$ by definition.

Now we will define a potential operator for centered maximal operators.

Definition 2.2. Let $0 < \alpha < 1$. Then for μ -measurable positive function f we set

$$\tilde{I}_\alpha f(x) := \sum_{k=[\log_2 \mu(\{x\})]+1}^{\infty} \frac{1}{2^{k(1-\alpha)}} \int_{B(x, r_k(x))} f(y) d\mu(y).$$

Here $[\cdot]$ denotes the Gauss sign. If we define the integral kernel by

$$\tilde{k}_\alpha(x, y) := \sum_{k=[\log_2 \mu(\{x\})]+1}^{\infty} \frac{\chi_{B(x, r_k(x))}(y)}{2^{k(1-\alpha)}},$$

then we can write

$$\tilde{I}_\alpha f(x) = \int_{\mathbf{R}^d} \tilde{k}_\alpha(x, y) f(y) d\mu(y).$$

Before going into details let us see how our kernel looks like for Lebesgue measure.

Example 2.3. If the measure μ is Lebesgue measure, then \tilde{I}_α is pointwise-comparable to $(-\Delta)^{-\frac{d\alpha}{2}}$ in the following sense. For all $x \in \mathbf{R}^d$ the estimate

$$C_0 (-\Delta)^{-d\alpha/2} f(x) \leq \tilde{I}_\alpha f(x) \leq C_1 (-\Delta)^{-d\alpha/2} f(x)$$

holds for all positive Lebesgue measurable function f .

Returning to general Radon measures, we note that the measurability of the function $\tilde{I}_\alpha f(x)$ follows from lower-semicontinuity of $x \in \mathbf{R}^d \mapsto r_k(x) \in \mathbf{R}$.

The following estimate is the key of the boundedness of \tilde{I}_α .

Proposition 2.4. Let $1 \leq p < \alpha^{-1}$, $q > 1$, $0 < \alpha < 1$ and $\frac{1}{q} = \frac{1}{p} - \alpha$. Then

$$\tilde{I}_\alpha f(x) \leq C \tilde{M}f(x)^{\frac{p}{q}} \cdot \|f : L^p(\mu)\|^{1-\frac{p}{q}}$$

for all positive μ -measurable function f .

Proof of Proposition 2.4. Fix $x \in \mathbf{R}^d$. Set $K := \{k \in \mathbf{Z} : r_k(x) > r_{k-1}(x)\}$. We shall estimate

$$I_k := \sum_{\substack{l \in \mathbf{Z} \\ r_l(x) = r_k(x)}} \frac{1}{2^{l(1-\alpha)}} \int_{B(x, r_l(x))} |f(y)| d\mu(y)$$

for $k \in K$. First we note that $\mu(B(x, r_k(x))) \leq 2^{k_j}$ by definition. Consequently Hölder's inequality yields

$$\begin{aligned} I_k &= \sum_{\substack{l \in \mathbf{Z} \\ r_l(x) = r_k(x)}} \frac{1}{2^{l(1-\alpha)}} \int_{B(x, r_l(x))} |f(y)| d\mu(y) \\ &\leq \frac{C}{2^{k(1-\alpha)}} \int_{B(x, r_k(x))} |f(y)| d\mu(y) \leq C 2^{-\frac{k}{s}} \|f : L^p(\mu)\|. \end{aligned} \quad (6)$$

In the same way, using maximal operator \tilde{M} , we have

$$I_k = \sum_{\substack{l \in \mathbf{Z} \\ r_l(x) = r_k(x)}} \frac{1}{2^{l(1-\alpha)}} \int_{B(x, r_l(x))} |f(y)| d\mu(y) \leq C 2^{\alpha k} \tilde{M}f(x). \quad (7)$$

Combine (6) and (7) to obtain

$$I_k \leq C \min\{2^{-\frac{k}{s}} \|f : L^p(\mu)\|, 2^{\alpha k} \tilde{M}f(x)\}.$$

Since $\tilde{I}_\alpha f(x) = \sum_{k \in K} I_k$, we have the desired result. \square

Since \tilde{M} is $L^p(\mu)$ -bounded for $p > 1$, we have

Corollary 2.5. *Let $\frac{1}{q} = \frac{1}{p} - \alpha$ and $1 < p < \alpha^{-1}$. Then*

$$\|\tilde{I}_\alpha f : L^q(\mu)\| \leq C \|f : L^p(\mu)\|$$

for all positive μ -measurable function f . In particular I_α can be extended linearly to a bounded operator from $L^p(\mu)$ to $L^q(\mu)$.

From this corollary, as is noted in Introduction, we can obtain Theorem 1.2.

Finally before investigating the uncentered maximal operator, if the measure μ satisfies the growth condition $\mu(B(x, r)) \leq c_0 r^n$ with $0 < n \leq d$, let us see that \tilde{I}_α dominates $J_{n\alpha}$ defined by (5).

Proposition 2.6. *Assume that the measure μ satisfies the growth condition $\mu(B(x, r)) \leq c_0 r^n$ with $0 < n \leq d$ and that $0 < \alpha < 1$. Then for μ -a.e. $x \in \mathbf{R}^d$*

$$J_{n\alpha}f(x) \leq C \tilde{I}_\alpha f(x),$$

where f is a positive μ -measurable function and C is independent of f .

Proof of Proposition 2.6. It suffices to compare the corresponding kernels. That is, we have only to show that

$$\frac{1}{|x - y|^{n(1-\alpha)}} \leq C k_\alpha(x, y), \quad \mu\text{-a.e. } (x, y) \in \mathbf{R}^d \times \mathbf{R}^d \quad (8)$$

for some $C > 0$.

We may assume that $x \neq y$, since μ does not charge any point in \mathbf{R}^d under the growth condition. Let k be an integer such that $r_{k-1}(x) < |x - y| \leq r_k(x)$. Then

$$|x - y|^{n(1-\alpha)} \geq c_0^{\alpha-1} \mu(B(x, |x - y|))^{1-\alpha} \geq C 2^{k(1-\alpha)},$$

which proves (8). \square

As a corollary of Proposition 2.6, we note that this proposition allows us to transplant Corollary 2.5 and Theorem 1.2 to $J_\beta, 0 < \beta < n$.

3 Uncentered fractional maximal operators

3.1 A family of balls

In this subsection we will consider a family of balls that will be needed to define another fractional maximal operator. The following lemma shows that we can drop the assumption of bounded radii in Lemma 1.1 for some special family of balls.

Lemma 3.1. *Let $b > a > 0$ be fixed positive numbers. Suppose that μ is a Radon measure and*

$$\mathcal{B}(\mu)_{a,b} := \{B \in \mathcal{B}(\mu) : a \leq \mu(\kappa^2 B) \leq b\} \neq \emptyset.$$

Then there exists $N(= N_\kappa)$ subfamilies $\mathcal{B}(\mu)_{a,b,1}, \dots, \mathcal{B}(\mu)_{a,b,N}$ such that

$$\{\kappa B : B \in \mathcal{B}(\mu)_{a,b,j}\} \text{ is disjoint for all } j = 1, \dots, N. \quad (9)$$

and for all $B \in \mathcal{B}(\mu)_{a,b}$ we can find $B' \in \bigcup_{j=1}^N \mathcal{B}(\mu)_{a,b,j}$ such that $B \subset \kappa B'$. Here N_κ does not depend on a nor b .

Proof of Lemma 3.1. We may assume that $\mu(\mathbf{R}^d) > b$. If $\mu(\mathbf{R}^d) \leq b$, we have only to take $\mathcal{B}(\mu)_{a,b,j} = \{\mathbf{R}^d\}, j = 1, \dots, N$. Furthermore we may assume that $\bigcup_{B \in \mathcal{B}(\mu)_{a,b}} \kappa B$ is connected and non-empty. Otherwise we have only to apply this lemma to each connected component and collect the subfamilies from each component.

Let us call a family of balls $\{B_j\}_{j=0}^I \subset \mathcal{B}(\mu)_{a,b}$ an I -chain if $\kappa B_j \cap \kappa B_{j-1} \neq \emptyset$ for all $j = 1, \dots, I$. Fix $B^* \in \mathcal{B}(\mu)_{a,b}$. Since we assume that $\bigcup_{B \in \mathcal{B}(\mu)_{a,b}} \kappa B$ is connected, for all B there exists an I -chain $\{B_j\}_{j=0}^I \subset \mathcal{B}(\mu)_{a,b}$ such that $B_0 = B^*$ and $B_I = B$.

Define $i(B)$ as the smallest integer of such I . Let $\mathcal{B}_I := \{B \in \mathcal{B}(\mu)_{a,b} : i(B) = I\}$. Then by induction and the assumption $\mu(\mathbf{R}^d) > b$, it is easy to see that \mathcal{B}_I is a family of bounded radii.

Now we apply Lemma 1.1 to \mathcal{B}_I . We obtain $\mathcal{B}_I^{(1)}, \dots, \mathcal{B}_I^{(N)}$ with (3) and (4) for \mathcal{B}_I . The definition of $i(B)$ prohibits $\kappa B \cap \kappa B' \neq \emptyset$ with $B \in \mathcal{B}_I^{(j)}$ and $B' \in \mathcal{B}_{I+2}^{(j')}$ for any $j, j' = 1, \dots, N$

and $I \in \mathbf{N}$. Thus our desired family can be obtained by putting

$$\mathcal{B}(\mu)_{a,b,j} := \bigcup_{I \equiv 1 \pmod{2}} \mathcal{B}_I^{(j)}, \mathcal{B}(\mu)_{a,b,j+N} := \bigcup_{I \equiv 0 \pmod{2}} \mathcal{B}_I^{(j+N)}$$

for $j \leq N$. \square

Definition 3.2. Let $k \in \mathbf{Z}$. We apply Lemma 3.1 to the family with $a = 2^{k-1}, b = 2^k$ and $\kappa > 1$ to obtain a (at most) countable balls $\{B_j^{(k)}\}_{j \in J_k}$ such that

1. $\sum_{j \in J_k} \chi_{\kappa B_j^{(k)}}(x) \leq N_\kappa$.
2. For any ball B with $2^{k-1} \leq \mu(\kappa^2 B) \leq 2^k$ we can find $B_j^{(k)}, j \in J_k$ such that $B \subset \kappa B_j^{(k)}$.

Throughout Section 3 we fix $\{B_j^{(k)}\}_{j \in J_k}$ for each $k \in \mathbf{Z}$.

3.2 Fractional integral operators for uncentered maximal operators

Using $\{B_j^{(k)}\}_{j \in J_k, k \in \mathbf{Z}}$ we will construct our potential operators.

Definition 3.3. Let $0 < \alpha < 1$. Let f be a positive μ -measurable function. Then we define

$$I_{\alpha, \kappa} f(x) := \sum_{j, k \in \mathbf{Z}} \left(\frac{1}{2^{k(1-\alpha)}} \int_{\kappa B_j^{(k)}} f(y) d\mu(y) \right) \chi_{\kappa B_j^{(k)}}(x).$$

We can write $I_{\alpha, \kappa} f(x) = \int_{\mathbf{R}^d} k_{\alpha, \kappa}(x, y) f(y) d\mu(y)$ in terms of integral kernel. Here, $k_{\alpha, \kappa}$ is defined as

$$k_{\alpha, \kappa}(x, y) := \sum_{j, k \in \mathbf{Z}} \frac{\chi_{\kappa B_j^{(k)}}(x) \chi_{\kappa B_j^{(k)}}(y)}{2^{k(1-\alpha)}}.$$

$I_{\alpha, \kappa}$ is also pointwise comparable to the operator $(-\Delta)^{-\alpha d/2}$, if μ is Lebesgue measure on \mathbf{R}^d . The similar two-sided estimate like Example 2.3 holds.

Next we shall prove the boundedness of this fractional integral operator. Unlike the centered maximal operator, we use the uncentered maximal operator $M_{0, \kappa}$.

Proposition 3.4. Let $1 \leq p < \infty, s > 1, 0 < \alpha < 1$ and $\frac{1}{s} = \frac{1}{p} - \alpha$. Then

$$I_{\alpha, \kappa} f(x) \leq C M_{0, \kappa} f(x)^{\frac{p}{s}} \|f : \mathcal{M}_1^p(\mu)\|^{1-\frac{p}{s}}$$

for all positive μ -measurable function f .

Proof of Proposition 3.4. We will estimate $\frac{1}{2^{k(1-\alpha)}} \int_{\kappa B_j^{(k)}} f(y) d\mu(y)$ in two ways again. One is obtained by the definition of Morrey norms. Note that $2^{k-1} \leq \mu(\kappa^2 B_j^{(k)}) \leq 2^k$. Hence,

$$\begin{aligned} & \frac{1}{2^{k(1-\alpha)}} \int_{\kappa B_j^{(k)}} f(y) d\mu(y) \\ & \leq \frac{\mu(\kappa^2 B_j^{(k)})^{1-\frac{1}{p}}}{2^{k(1-\alpha)}} \cdot \mu(\kappa^2 B_j^{(k)})^{\frac{1}{p}-1} \left(\int_{\kappa B_j^{(k)}} f(y) d\mu(y) \right) \leq 2^{-\frac{k}{s}+1} \|f : \mathcal{M}_1^p(\mu)\|. \end{aligned} \quad (10)$$

The estimate using the uncentered modified maximal operator is the same as that via the centered modified maximal operator.

$$\frac{1}{2^{k(1-\alpha)}} \int_{\kappa B_j^{(k)}} f(y) d\mu(y) \leq \mu(\kappa^2 B_j^{(k)})^\alpha M_{0,\kappa} f(x) \leq 2^{k\alpha} M_{0,\kappa} f(x). \quad (11)$$

We summarize (10) and (11):

$$\frac{1}{\mu(\kappa^2 B_j^{(k)})^{1-\alpha}} \int_{\kappa B_j^{(k)}} f(y) d\mu(y) \leq 2 \min\{2^{-\frac{k}{s}} \|f : \mathcal{M}_1^p(\mu)\|, 2^{k\alpha} M_{0,\kappa} f(x)\}.$$

Let us recall that by our construction, $\sum_{j \in J_k} \chi_{\kappa B_j^{(k)}}(x) \leq N_\kappa$ for all $x \in \mathbf{R}^d$, where N_κ is independent of j , k and x .

Consequently we have

$$I_{\alpha,\kappa} f(x) \leq C N_\kappa \sum_{k \in \mathbf{Z}} \min\{2^{-\frac{k}{s}} \|f : \mathcal{M}_1^p(\mu)\|, 2^{k\alpha} M_{0,\kappa} f(x)\} \leq C M_{0,\kappa} f(x)^{\frac{p}{s}} \|f : \mathcal{M}_1^p(\mu)\|^{1-\frac{p}{s}}.$$

This is the desired conclusion. \square

As a corollary of Proposition 3.4 we obtain the boundedness of I_α .

Theorem 3.5. *Let $\kappa > 1$, $0 < \alpha < 1$, $1 \leq q \leq p < \alpha^{-1}$, $s > 1$, $\frac{q}{p} = \frac{t}{s}$ and $\frac{1}{s} = \frac{1}{p} - \alpha$. Then*

$$\|I_{\alpha,\kappa} f : \mathcal{M}_t^s(\mu)\| \leq C \|f : \mathcal{M}_q^p(\mu)\|.$$

In particular $I_{\alpha,\kappa}$ can be extended to a bounded linear operator from $\mathcal{M}_q^p(\mu)$ to $\mathcal{M}_t^s(\mu)$.

Theorem 3.6. *Let $\kappa > 1$, $0 < \alpha < 1$, $1 \leq p < \alpha^{-1}$, $s > 1$ and $\frac{1}{s} = \frac{1}{p} - \alpha$. Then*

$$\|I_{\alpha,\kappa} f : \mathcal{M}_{s/p}^s(\mu)\|_w \leq C \|f : \mathcal{M}_1^p(\mu)\|.$$

Finally for the proof of Theorem 1.3 it suffices to prove the following estimate: We can dominate $M_{\alpha,\kappa}$ by $I_{\alpha,\kappa} |f|$ pointwise.

Proposition 3.7. *Suppose that $\kappa > 1$, $0 < \alpha < 1$. Then*

$$M_{\alpha,\kappa^2} f(x) \leq 2I_{\alpha,\kappa} |f|(x)$$

for all $x \in \mathbf{R}^d$ and for all positive μ -measurable function f .

Proof of Proposition 3.7. Fix $x \in \mathbf{R}^d$. Let $B \in \mathcal{B}(\mu)$ contain x . It is enough to show

$$\frac{1}{\mu(\kappa^2 B)^{1-\alpha}} \int_B |f(y)| d\mu(y) \leq 2I_{\alpha,\kappa} |f|(x).$$

Suppose that $2^{k-1} \leq \mu(\kappa^2 B) \leq 2^k$. Then by definition of $\mathcal{B}_{2^{k-1}, 2^k}$ we can take $B_j^{(k)}$ such that

$$B \subset \kappa B_j^{(k)}, \quad 2^{k-1} \leq \mu(\kappa^2 B_j^{(k)}) \leq 2^k.$$

These conditions imply

$$\frac{1}{\mu(\kappa^2 B)^{1-\alpha}} \int_B |f(y)| d\mu(y) \leq \frac{2}{\mu(\kappa^2 B_j^{(k)})^{1-\alpha}} \int_{\kappa B_j^{(k)}} |f(y)| d\mu(y) \leq 2I_{\alpha,\kappa} |f|(x).$$

As a result Proposition 3.7 is proved. \square

Finally we note that $J_{n\alpha}$ is dominated by $I_{\alpha,\kappa}$. The proof is similar to Proposition 2.4. We omit the detail.

Proposition 3.8. *Let $0 < \alpha < 1$ and $\kappa > 1$. There exists $C > 0$ such that*

$$J_{n\alpha}f(x) \leq C I_{\alpha,\kappa}f(x)$$

for μ -almost all $x \in \mathbf{R}^d$ and for all positive μ -measurable function f .

4 Appendices

4.1 Proof of Lemma 1.1

The proof is similar to that in [13, Theorem 1.5]. For convenience for the readers we supply the outline.

Proof of Lemma 1.1. The following claim follows easily.

Claim 4.1. *Assume additionally that $\sup_{\lambda \in \Lambda} r(B_\lambda) \leq \kappa^{\frac{1}{2}} \inf_{\lambda \in \Lambda} r(B_\lambda)$. Then Lemma 1.1 holds.*

The proof is completely the same as [13, Theorem 1.5]. Accepting Claim 4.1, we will treat general case. Set

$$\mathcal{X}_1 := \left\{ B \in \mathcal{B} : r(B) \geq \kappa^{-\frac{1}{2}} \sup_{\lambda \in \Lambda} r(B_\lambda) \right\}.$$

Then we can take subfamilies $\{B_\lambda\}_{\lambda \in \Lambda_1^{(1)}}, \dots, \{B_\lambda\}_{\lambda \in \Lambda_N^{(1)}}$ satisfying the conditions (3) and (4) for \mathcal{X}_1 . Suppose we have defined $\{B_\lambda\}_{\lambda \in \Lambda_1^{(j)}}, \dots, \{B_\lambda\}_{\lambda \in \Lambda_N^{(j)}}$. First we set

$$\mathcal{X}_{j+1} := \left\{ B \in \mathcal{B} : \kappa^{-\frac{j+1}{2}} \sup_{\lambda \in \Lambda} r(B_\lambda) \leq r(B) \leq \kappa^{-\frac{j}{2}} \sup_{\lambda \in \Lambda} r(B_\lambda) \right. \\ \left. \text{None of } \kappa B_\lambda, \lambda \in \bigcup_{m=1}^j \bigcup_{k=1}^N \Lambda_k^{(m)} \text{ contains } \kappa B \right\}.$$

Then we can take subfamilies $\{B_\lambda\}_{\lambda \in \Lambda_1^{(j+1)}}, \dots, \{B_\lambda\}_{\lambda \in \Lambda_N^{(j+1)}}$ satisfying (3) and (4) for \mathcal{X}_{j+1} .

Let $B = B_\lambda, \lambda \in \Lambda_j^{(m)}$ and $B' = B_{\lambda'}, \lambda' \in \Lambda_{j'}^{(m')}$ with $j, j' \leq N$ and $m, m' \in \mathbf{N}$. Then in the same way as [13, Theorem 1.5], we see that if m and m' are apart, say $|m - m'| \geq M$, then $\kappa B \cap \kappa B' = \emptyset$. Thus we have only to put

$$\Lambda_{m,j} = \bigcup_{m' \equiv m \pmod{M}} \Lambda_j^{(m')}$$

for $j = 1, \dots, N, m = 1, \dots, M$ and then rearrange them. □

4.2 A note on centered maximal operators and Morrey spaces

As is announced in Introduction, \tilde{M} is not bounded on Morrey space $\mathcal{M}_q^p(\mu)$, unless $p = q$. Let us see an example of this phenomenon. We are going to construct an example of the measure μ on \mathbf{R} for which \tilde{M} is not weak- $\mathcal{M}_1^2(\mu)$ bounded.

Let $j \in \mathbf{N}$. We set four types of open intervals in \mathbf{R} by

$$A_j := (j, j+1), B_j := \left(j, j + \frac{1}{j}\right), C_j := \left(j, j + \frac{2}{3}\right), D_j := \left(j + \frac{2}{3}, j+1\right), j \in \mathbf{N}.$$

First we define a weight function w by

$$w(x) := \chi_{(-\infty, 0]}(x) + \sum_{j=1}^{\infty} \chi_{C_j}(x) + \sum_{j=1}^{\infty} \frac{-(j+1)}{(j+1-x) \log^3(j+1-x)} \chi_{D_j}(x).$$

Here $\log^r x := (\log x)^r, x > 0$. We define a measure on \mathbf{R} by setting $\mu(x) = w(x) \mathcal{H}^1(x)$, where $\mathcal{H}^1(x)$ is an 1-dimensional Hausdorff measure.

We shall prove that \tilde{M} is *not* weak- $\mathcal{M}_1^2(\mu)$ bounded by showing that the function defined by

$$f_j(x) = \frac{1}{(x-j) \log^2(x-j)} \chi_{B_j}(x), j \geq 2$$

is an element in $\mathcal{M}_1^2(\mu)$ such that

$$\|f_j : \mathcal{M}_1^2(\mu)\| = O(\log^{-2} j) \quad (j \rightarrow \infty) \quad (12)$$

$$\|\tilde{M}f_j : \mathcal{M}_1^2(\mu)\|_w \geq c j^{\frac{1}{2}} \log^{-1} j \quad (j \geq 2). \quad (13)$$

As for (12), more precisely we shall prove

Claim 4.2. *As $j \rightarrow \infty$, we have*

$$\|f_j : \mathcal{M}_1^2(10, \mu)\| = O\left(j^{\frac{1}{2}} \log^{-2} j\right). \quad (14)$$

Hence as $j \rightarrow \infty$, it holds that $\|f_j : \mathcal{M}_1^2(2, \mu)\| = O\left(j^{\frac{1}{2}} \log^{-2} j\right)$.

Proof of Claim 4.2. If I and J are intervals such that I contains J , then $10I \subset 10J$. Thus we see

$$\|f_j : \mathcal{M}_1^2(10, \mu)\| = \sup_{I \in \mathcal{B}(\mu), J \subset B_j} \mu(10I)^{-\frac{1}{2}} \int_I |f(y)| d\mu(y).$$

Let I be an interval in \mathbf{R} contained in B_j . We shall show

$$\mu(10I)^{-\frac{1}{2}} \int_I |f(y)| d\mu(y) \leq c \log^{-2} j$$

for such I . Here c is independent of j and I . In the sequel in this proof we assume that $j \geq 100$.

Case 1: $j \in 2I$. In this case we note that

$$\mu(10I)^{-\frac{1}{2}} \int_I |f(y)| d\mu(y) \leq \mu(10I \cap D_{j-1})^{-\frac{1}{2}} \int_{I \cap B_j} |f(y)| d\mu(y).$$

Simple calculation yields

$$\int_{I \cap B_j} |f(y)| d\mu(y) = -\log^{-1} \mathcal{H}^1(I \cap B_j), \mu(10I \cap D_{j-1})^{\frac{1}{2}} = -\frac{1}{2} j^{\frac{1}{2}} \log^{-1} \mathcal{H}^1(10I \cap D_{j-1}).$$

Since $\mathcal{H}^1(I \cap B_j) \leq \mathcal{H}^1(10I \cap D_{j-1}) \leq c \mathcal{H}^1(I \cap B_j)$ in this case, we have

$$\mu(10I)^{-\frac{1}{2}} \int_I |f(y)| d\mu(y) \leq c j^{-\frac{1}{2}}.$$

Thus we have (14).

Case 2: $j \in 2I$. In this case we use

$$\mu(10I)^{-\frac{1}{2}} \int_I |f(y)| d\mu(y) \leq \mu(I)^{-\frac{1}{2}} \int_I |f(y)| d\mu(y)$$

and we estimate the right-hand-side. Let $I = (j + a, j + b)$ with $0 < \frac{1}{2}b \leq a < b \leq \frac{1}{j} \leq \frac{1}{100}$.

Then we have

$$\mu(I)^{-\frac{1}{2}} \int_I |f(y)| d\mu(y) = \frac{\log b - \log a}{\log a \cdot \log b \cdot (b - a)^{\frac{1}{2}}}.$$

If we take this quantity as a function of b , then simple calculation shows that the maximum can be attained at $b = \min(2a, j^{-1})$. Next fixing $b = \min(2a, j^{-1})$, we regard

$$\mu(I)^{-\frac{1}{2}} \int_I |f(y)| d\mu(y) = \frac{\log b - \log a}{\log a \cdot \log b \cdot (b - a)^{\frac{1}{2}}}.$$

as a function of a , where $0 \leq a \leq \frac{1}{j}$, that is, $2I \subset B_j$. Then again elementary arithmetic shows that this function attains its maximum at $a = \frac{1}{2j}$. Thus we have

$$\mu(I)^{-\frac{1}{2}} \int_I |f(y)| d\mu(y) \leq \frac{(\log 2)(2j)^{\frac{1}{2}}}{(\log j)(\log j + \log 2)} \leq j^{\frac{1}{2}} \log^{-2} j.$$

Consequently (14) holds for I with $2I \subset B_j$, whether $2I$ contains j or not. \square

To disprove the weak- $\mathcal{M}_1^2(\mu)$ boundedness of \tilde{M} , it suffices to estimate $\|\tilde{M}f_j : \mathcal{M}_1^2(2, \mu)\|_w$ from below.

Claim 4.3. *There exists $c > 0$ such that $\|\tilde{M}f_j : \mathcal{M}_1^2(2, \mu)\|_w \geq c j^{\frac{1}{2}} \log^{-1} j$ for all $j \geq 2$.*

Proof of Claim 4.3. Let $E_j = \left(j + \frac{2}{3j}, j + \frac{5}{6j}\right)$. Then $\tilde{M}f_j(x) \geq c_0 j \log^{-1} j$ on E_j , where c_0 is a constant independent of j . Let $\lambda_j = c_0 j \log^{-1} j$. Then

$$\begin{aligned} & \|\tilde{M}f_j : \mathcal{M}_1^2(2, \mu)\|_w \\ & \geq \mu(2E_j)^{-\frac{1}{2}} \lambda_j \mu\{x \in E_j : \tilde{M}f_j(x) > \lambda_j\} \geq (3j)^{\frac{1}{2}} \times c_0 j \log^{-1} j \times (2j)^{-1} = c j^{\frac{1}{2}} \log^{-1} j. \end{aligned}$$

The proof of Claim 4.3 is now complete. \square

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References

- [1] D. Adams, A note on Riesz potentials, *Duke Math. J.* **42** (1975), 765–778.
- [2] J. Journé, Calderón-Zygmund Operators, Pseudo differential operators and the Cauchy integral of Calderón. *Lecture Notes in Mathematics*, **994**, Springer-Verlag, Berlin, 1983.
- [3] W. Chen and E. Sawyer, A note on commutators of fractional integrals with $\text{RBMO}(\mu)$ functions. *Illinois J. Math.* **46** (2002), no. 4, 1287–1298.
- [4] D. Edmunds, V. Kokilashvili and A. Meskhi, Bounded and compact integral operators. *Kluwer Academic Publishers*, Dordrecht, Boston London, 2002, Chapter 6.
- [5] J. García-Cuerva and E. Gatto, Boundedness properties of fractional integral operators associated to non-doubling measures, *Studia Math.* **162** (2004), no. 3, 245–261.
- [6] H. Guoen, M. Yan and Y. Dachun, Multilinear commutators for fractional integrals in non-homogeneous spaces, *Publ. Mat.* **48** (2004), no. 2, 335–367.
- [7] Y. Komori, A note on modified maximal functions associated with general measures, *Int. J. Pure Appl. Math.* **12** (2004), no. 2, 169–175.
- [8] V. Kokilashvili, Weighted estimates for classical integral operators. *Nonlinear analysis, function spaces and applications*, **4** (Roudnice nad Labem, 1990), 86–103, *Teubner-Texte Math.*, 119, Teubner, Leipzig, (1990).
- [9] V. Kokilashvili and A. Meskhi, Fractional integrals on measure spaces, *Fract. Calc. Appl. Anal.* **4** (2001), no. 1, 1–24.
- [10] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, *Cambridge Studies in Advanced Mathematics*, (1995) 44.
- [11] J. Mateu, P. Mattila, A. Nicolau and J. Orobitg, BMO for nondoubling measures, *Duke Math. J.* **102**(2000), no. 3, 533–565.
- [12] F. Nazarov, S. Treil and A. Volberg, Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces, *Internat. Math. Res. Notices* (1998), 463–487.
- [13] Y. Sawano, Sharp estimates of the modified Hardy-Littlewood maximal operator on the nonhomogeneous space via covering lemmas, *Hokkaido Math. J.* **34** (2005) p435-p458.
- [14] Y. Sawano and H. Tanaka, Morrey spaces for non-doubling measures, to appear in *Acta Math. Sinica*.
- [15] Y. Sawano and H. Tanaka, Morrey spaces for non-doubling measures, *Suurikaisekikoukyuuroku*
- [16] E. M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*. With the assistance of Timothy S. Murphy. *Princeton Mathematical Series*, 43. Princeton University Press, Princeton, NJ, 1993.
- [17] X. Tolsa, BMO, H^1 , and Calderón-Zygmund operators for non doubling measures, *Math. Ann.* **319** (2001), 89–149.

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