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Abstract

For the solution to $\partial_t^2 u(x,t) - \triangle u(x,t) + q(x)u(x,t) = \delta(x_1)\delta'(t)$ and $u|_{t<0} = 0$, we consider an inverse problem of determining $q(x)$, $x \in \Omega$ from data $f = u|_{S_T}$ and $g = \frac{\partial u}{\partial \nu}|_{S_T}$. Here $\Omega \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 > 0\}$, $n \geq 2$, is a bounded domain, $S_T = \{(x,t);\ x \in \partial\Omega, x_1 < t < T + x_1\}$ and $T > 0$. For suitable $T > 0$, we prove an $L^2(\Omega)$-size estimation of q:

$$\|q\|_{L^2(\Omega)} \leq C \left\{ \|f\|_{H^1(S_T)} + \|g\|_{L^2(S_T)} \right\},$$

provided that q satisfies a priori uniform boundedness conditions. We use an inequality of Carleman type in our proof.

1 Introduction and main results

We consider an inverse problem of determining a coefficient in a hyperbolic equation by an impulsive source located outside the domain where a coefficient is unknown. Let $u(x,t)$, $x = (x_1, \ldots, x_n)$, $n \geq 2$ solve the Cauchy problem

$$\partial_t^2 u(x,t) - \triangle u(x,t) + q(x)u(x,t) = \delta(x_1)\delta'(t), \quad u|_{t<0} = 0, \tag{1.1}$$

where δ and δ' are the Dirac delta function and the t-derivative:

$$\langle \delta(x_1), \psi \rangle = \psi\left(0, x_2, \ldots, x_n, t\right),$$

and

$$\langle \delta'(t), \psi \rangle = -\partial_t \psi(x, 0), \ \forall \psi \in C_0^\infty\left(\mathbb{R}^{n+1}\right).$$

Let $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 > 0\}$ and let $\Omega \subset \mathbb{R}_+^n$ be a bounded domain with C^1-piecewise smooth boundary $\partial\Omega$. Furthermore let $T > 0$ be suitably given. Set

$$G_T = \{(x,t);\ x \in \Omega,\ x_1 < t < T + x_1\}, \tag{1.2}$$

$$\Sigma_0 = \{(x,t);\ x \in \Omega,\ t = x_1 + 0\},$$

$$\Sigma_T = \{(x,t);\ x \in \Omega,\ t = T + x_1\},$$

$$S_T = \{(x,t);\ x \in \partial\Omega,\ x_1 < t < T + x_1\}.$$

We consider:

Inverse problem. Let Cauchy data of the solution u to (1.1) be given on S_T:

$$u(x,t) = f(x,t), \quad \frac{\partial u}{\partial \nu}(x,t) = g(x,t), \quad (x,t) \in S_T, \tag{1.3}$$

where $\nu = \nu(x)$ is the unit outward normal vector to $\partial\Omega$ at $x \in \partial\Omega$. Then determine $q(x)$, $x \in \Omega$ from given data (1.3).

In order to state the main result, we introduce the notations. Let $r =$(diam $\Omega)/2$. Assume that

$$\begin{aligned} &\Omega \subseteq B\left(x^0, r\right) = \left\{x \in \mathbb{R}^n;\ \left|x - x^0\right| < r\right\} \\ &\text{where } x^0 = \left(x_1^0, 0, \ldots, 0\right) \in \mathbb{R}_+^n \text{ and } x_1^0 > r > 0. \end{aligned} \tag{1.4}$$

Set

$$K = K\left(x^0, T, r\right) = \left\{(x,t);\ |x_1| < t < \left(T + x_1^0 + 2r\right) - \left|x - x^0\right|\right\}.$$

Noting that $x_1 > 0$ and $T + x_1 \leq T + x_1^0 + r \leq \left(T + x_1^0 + 2r\right) - \left|x - x^0\right|$ for $x \in \Omega$, we see that $G_T \subseteq K$. Denote by

$$P = P\left(x^0, T, r\right) = \left\{x \in \mathbb{R}^n;\ |x_1| < \left(T + x_1^0 + 2r\right) - \left|x - x^0\right|\right\}$$

the projection of K on the space $\mathbb{R}^n$. Throughout this paper, $H^1(S_T)$, $H^{n+2}(P)$, etc. denote usual Sobolev spaces (e.g. Adams [1]), and $[\alpha]$ denotes the greatest integer not exceeding α. We set

$$\mathcal{U}_M = \left\{ q \in H^{n+2}(P) \,\middle|\, \|q\|_{H^{n+2}(P)} \leq M \right\} \tag{1.5}$$

for any fixed $M > 0$. Furthermore, we take a constant β such that

$$0 < \beta < 1 \text{ and } 0 < \beta \left(r\beta + x_1^0 + 2r\right)^2 < (x_1^0 - r)^2. \tag{1.6}$$

Now we state the main result.

Theorem 1.1. *Assume that Ω satisfies (1.4). Let*

$$T > 2r + \frac{4(x_1^0 + 2r)}{\beta} \tag{1.7}$$

where β satisfies (1.6). Furthermore, let u be the solution to (1.1) with $q \in \mathcal{U}_M$ and let Cauchy data of u be given by (1.3). Then there exists a constant $C = C\left(\Omega, T, x^0, r, M\right) > 0$ such that

$$\|q\|_{L^2(\Omega)} \leq C \left\{ \|f\|_{H^1(S_T)} + \|g\|_{L^2(S_T)} \right\}. \tag{1.8}$$

In our inverse problem we assume that the initial values are identically zero and we are requested to determine a coefficient by a single measurement on the boundary.

If we can be allowed to repeat infinitely many measurements, then the Dirichlet to Neumann map can guarantee the uniqueness and the stability also with the zero initial condition (e.g., Sun [20]).

If we can assume the positivity condition $u(\cdot, 0) > 0$ on $\overline{\Omega}$, then the method on the basis of a Carleman estimate which was discussed first in Bukhgeim and Klibanov [2], implies the uniqueness. As for the stability, see Imanuvilov and Yamamoto [5, 6], Khaĭdarov [10], Yamamoto [21], and we refer also to Isakov [7, 8, 9], Klibanov [11], Klibanov and Timonov [12].

The above results by a Carleman estimate or the Dirichlet to Neumann map, hold without smallness assumptions of unknown coefficients or the spatial domain Ω under consideration.

On the other hand, the infinitely many repeat of the measurements are not realistic and the positivity of the initial displacement may be difficult to be realized in practise even though a single measurement can guarantee the uniqueness and the stability in the inverse problem.

In (1.1) we take impulsive inputs $\delta(x_1)\delta'(t)$ and the initial values can be zero. The impulsive input is acceptable from the practical viewpoint.

In the case where the spatial dimension is greater than 1, it is a hard open problem that in the inverse problem for (1.1), one can establish the uniqueness without any smallness conditions on the coefficients or Ω. In Romanov and Yamamoto [17], if $\|p\|_{H^{n+2}(P)}$ and $\|q\|_{H^{n+2}(P)}$ are sufficiently small, then with suitable T, we can prove the Lipschitz stability for $\|p-q\|_{L^2(\Omega)}$ by means of the boundary data. As related results, see Glushkov [3], Glushkov and Romanov [4], Romanov [13, 14, 15, 16], Romanov and Yamamoto [17, 18, 19].

To the above long standing open problem, Theorem 1.1 is a partial answer : we can estimate the difference between a not necessarily small q and $p(x) \equiv 0$ by means of the boundary data. We note that in Romanov and Yamamoto [17], one has the estimate between two sufficiently small coefficients p and q. We can interpret Theorem 1.1 as $L^2(\Omega)$-size estimation of the coefficient by means of boundary output.

Our proof is inspired by the argument in §4.1 in [16] and [17], but we will use an inequality of Carleman type.

2 Proof of Theorem 1.1

First we show a lemma, which is different from Lemma 4.1.4 in [16]. For $T > 0$, $x_1^0 > 0$ and $\beta \in (0,1)$, we define a function $\varphi = \varphi(x,t)$ by

$$\varphi(x,t) = \frac{1}{4}|x|^2 - \frac{1}{8}\beta\left(t - x_1^0 - \frac{T}{2}\right)^2. \tag{2.1}$$

Furthermore, we set

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_j = \frac{\partial}{\partial x_j}, \quad 1 \le j \le n, \quad \nabla_x = (\partial_1, \ldots, \partial_n),$$

$$\nabla_{x,t} = (\partial_1, \ldots, \partial_n, \partial_t), \quad \nabla_{x'} = (\partial_2, \ldots, \partial_n), \quad \Box y = \partial_t^2 y - \triangle y.$$

Lemma 2.1. *Let $v \in H^2(G_T)$. Assume (1.4), (1.6) and (1.7). Then there exists a constant $\vartheta > 0$ such that for $T \in \left(2r + 4\left(x_1^0 + 2r\right)/\beta, 2r + 4\left(x_1^0 + 2r\right)/\beta + \vartheta\right)$ there exist $s_0 > 0$ and $C_1 = C_1(s_0, T, x^0, r, \beta) > 0$ such that*

$$\begin{aligned}
&\int_{G_T}\left(s\left|\nabla_{x,t}v\right|^2 + s^3v^2\right)\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
&+\int_{\Sigma_0\cup\Sigma_T}\left[s\left(\partial_t v + \partial_1 v\right)^2 + s\left|\nabla_{x'}v\right|^2 + s^3v^2\right]\mathrm{e}^{2s\varphi}\mathrm{d}x \\
&\le C_1\left\{\int_{G_T}(\Box v)^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t + \int_{S_T}\left[s\left|\nabla_{x,t}v\right|^2 + s^3v^2\right]\mathrm{e}^{2s\varphi}\mathrm{d}s\mathrm{d}t\right\}
\end{aligned} \tag{2.2}$$

for all $s \ge s_0$.

We shall prove Lemma 2.1 in §3.

In [16, 17], the following proposition is proved.

Proposition 2.2 ([16] or [17])**.** *Let $q \in \mathcal{U}_M$. Then the solution to (1.1) can be represented in the form*

$$u(x,t) = \frac{1}{2}\delta\left(t - |x_1|\right) + \widehat{u}(x,t)\theta_0\left(t - |x_1|\right) \tag{2.3}$$

where $\widehat{u} \in H^m(K)$, $m = \left[\frac{n+1}{2}\right]+1$, $\theta_0(t)$ is the Heaviside step function: $\theta_0(t) = 1$ if $t \ge 0$ and $\theta_0(t) = 0$ if $t < 0$. Moreover

$$\widehat{u}(x, |x_1| + 0) = -\frac{1}{4}(\mathit{sign}\ x_1)\int_0^{x_1} q(\xi, x')\mathrm{d}\xi, \quad x \in P \tag{2.4}$$

with $x' = (x_2, \dots, x_n)$*, and there exists a constant* $C_2 = C_2(T, x^0, r, M) > 0$ *such that*

$$|\widehat{u}(x,t)| \le C_2 M, \quad (x,t) \in K. \tag{2.5}$$

The constant C_2 *is a non-decreasing function of Parameters* T*,* r*,* M*.*

Remark 2.1 ([16] or [17]). The representation (2.3) means that the regular part of the solution $u(x,t)$ coincides with $\widehat{u}(x,t)$ for $(x,t) \in K$. Moreover $u \in H^1(S_T)$ and $\frac{\partial u}{\partial \nu} \in L^2(S_T)$ by the trace theorem (e. g., [1]), because $\partial\Omega$ is piecewise C^1 smooth and $u \in H^m(K)$ with $m \ge 2$.

Now we prove Theorem 1.1.

Proof of Theorem 1.1. For any $T > 0$ satisfying (1.7), we set

$$\widetilde{T} = \min\left\{T, 2r + \frac{4\left(x_1^0 + 2r\right)}{\beta} + \frac{\vartheta}{2}\right\}, \tag{2.6}$$

where ϑ is given by Lemma 2.1. Therefore, estimate (2.2) holds in $G_{\widetilde{T}}$.

Let u be the solution to problem (1.1) with $q \in \mathcal{U}_M$ and f, g be the data in (1.3) for u. By (1.1), we have

$$\Box u(x,t) + q(x)u(x,t) = 0, \quad (x,t) \in G_{\widetilde{T}}. \tag{2.7}$$

By $q \in \mathcal{U}_M$ and the embedding theorem, we see that $q \in C^m(P)$ with $m = \left[\frac{n+1}{2}\right] + 1$ and there exists a constant $C_{*l} = C_{*l}(T, x^0, r) > 0$ such that

$$\|q\|_{C^l(P)} \le C_{*l}\|q\|_{H^{n+2}(P)} \le C_{*l}M, \quad l = 0, 1, \dots, m. \tag{2.8}$$

By Proposition 2.2 and Remark 2.1, we have $u \in H^2(G_{\widetilde{T}})$. It follows from (2.7) and (2.8) that

$$\left(\Box u(x,t)\right)^2 \le C_{*0}^2 M^2 u^2(x,t), \quad (x,t) \in G_{\widetilde{T}}. \tag{2.9}$$

Then, by Lemma 2.1, there exists $s_0 > 0$ such that

$$\begin{aligned}
&\int_{G_{\widetilde{T}}} \left(s\left|\nabla_{x,t}u\right|^2 + s^3u^2\right) \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
&+\int_{\Sigma_0\cup\Sigma_{\widetilde{T}}} \left(s\left(\partial_t u + \partial_1 u\right)^2 + s\left|\nabla_{x'}u\right|^2 + s^3u^2\right) \mathrm{e}^{2s\varphi}\mathrm{d}x \\
&\le C_1\left(\int_{G_{\widetilde{T}}} (\Box u)^2\,\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t + \int_{S_{\widetilde{T}}} \left(s\left|\nabla_{x,t}u\right|^2 + s^3u^2\right) \mathrm{e}^{2s\varphi}\mathrm{d}s\mathrm{d}t\right) \\
&\le C_1\left(C_{*0}^2M^2\int_{G_{\widetilde{T}}} u^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t + \int_{S_{\widetilde{T}}} \left(s\left|\nabla_{x,t}u\right|^2 + s^3u^2\right) \mathrm{e}^{2s\varphi}\mathrm{d}s\mathrm{d}t\right)
\end{aligned} \tag{2.10}$$

for all $s > s_0$, where φ is defined by (2.1). In the last inequality in (2.10), we have used (2.9).

By relation (2.4) in Proposition 2.2, we have

$$\partial_t u + \partial_1 u = -\frac{1}{4}q(x), \quad (x,t) \in \Sigma_0. \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$\begin{aligned}
&\int_{G_{\widetilde{T}}} \left[s\left|\nabla_{x,t}u\right|^2 + \left(s^3 - C_1C_{*0}^2M^2\right)u^2\right] \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\
&+\frac{s}{16}\int_\Omega q^2\mathrm{e}^{2s\varphi(x,x_1)}\mathrm{d}x \\
\le\;& C_1\int_{S_{\widetilde{T}}} \left(s\left|\nabla_{x,t}u\right|^2 + s^3u^2\right) \mathrm{e}^{2s\varphi}\mathrm{d}s\mathrm{d}t
\end{aligned} \tag{2.12}$$

for all $s > s_0$. We can take $s_0 > 0$ sufficiently large such that

$$\begin{aligned}
&\int_{G_{\widetilde{T}}} \left[s\left|\nabla_{x,t}u\right|^2 + \frac{1}{2}s^3u^2\right] \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t + \frac{s}{16}\int_\Omega q^2\mathrm{e}^{2s\varphi(x,x_1)}\mathrm{d}x \\
&\le C_1\int_{S_{\widetilde{T}}} \left(s\left|\nabla_{x,t}u\right|^2 + s^3u^2\right) \mathrm{e}^{2s\varphi}\mathrm{d}s\mathrm{d}t
\end{aligned} \tag{2.13}$$

for all $s > s_0$. We take $s > s_0$ and fix it. By (1.4) and (2.1), we have

$$\frac{1}{4}(x_1^0 - r)^2 - \frac{\beta}{8}\left(\frac{\widetilde{T}}{2} + r\right)^2 \le \varphi \le \frac{1}{4}(x_1^0 + r)^2, \quad (x,t) \in G_{\widetilde{T}}. \tag{2.14}$$

Therefore, by (1.3), (2.6), (2.13) and (2.14), we can obtain (1.8). We have completed the proof of Theorem 1.1. $\Box$

3 Proof of Lemma 2.1

First of all, we note that the following inequalities hold:

$$0 < x_1^0 - r \le x_1 \le x_1^0 + r, \quad 0 < |x|^2 \le \left(r + x_1^0\right)^2, \qquad x \in \Omega, \tag{3.1}$$

and
$$-r - \frac{T}{2} \le t - x_1^0 - \frac{T}{2} \le \frac{T}{2} + r, \qquad (x,t) \in G_T. \tag{3.2}$$

In fact, the first inequality in (3.1) follows from (1.4). The second inequality in (3.1) can be proved as follows:

$$|x|^2 = |x - x^0|^2 + x_1^2 - (x_1 - x_1^0)^2 \le r^2 + 2x_1x_1^0 - (x_1^0)^2 \le r^2 + 2x_1^0(x_1^0 + r) - (x_1^0)^2.$$

(3.2) can be proved by (1.2) and (3.1).

By (1.6), there exists a constant ϑ such that

$$0 < \frac{1}{16}\beta^3 \left[4r + \frac{4\left(x_1^0 + 2r\right)}{\beta} + \vartheta\right]^2 < (x_1^0 - r)^2. \tag{3.3}$$

By (1.7), we can assume that $T \in \left(2r + 4\left(x_1^0 + 2r\right)/\beta, 2r + 4\left(x_1^0 + 2r\right)/\beta + \vartheta\right)$. Then we have

$$4r + \frac{4\left(x_1^0 + 2r\right)}{\beta} < T + 2r < 4r + \frac{4\left(x_1^0 + 2r\right)}{\beta} + \vartheta. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$(x_1^0 - r)^2 > \frac{1}{16}\beta^3 \left(T + 2r\right)^2. \tag{3.5}$$

Therefore we can take a constant $\rho > 0$ such that

$$0 < 2\beta < \rho < \min\left\{2, \frac{64\left(x_1^0 - r\right)^2}{\beta^2\left(T + 2r\right)^2} - 2\beta\right\}. \tag{3.6}$$

Furthermore, by (1.7), we can get

$$\frac{\beta^2}{16}\left(\frac{T}{2} - r\right)^2 > \frac{1}{4}\left(2r + x_1^0\right)^2 \qquad \text{and} \qquad \frac{\beta T}{4} - \frac{\beta r}{2} - x_1^0 > 2r. \tag{3.7}$$

Let $s > 0$, $w = \mathrm{e}^{s\varphi}v$ and $Lw = \mathrm{e}^{s\varphi}\Box\left(\mathrm{e}^{-s\varphi}w\right)$. Then we can obtain that

$$\begin{aligned} Lw = &\left\{\Box w + s^2\left[(\partial_t\varphi)^2 - |\nabla_x\varphi|^2\right]w + \tfrac{1}{4}s\rho w\right\} \\ &+ s\left\{\left[-\Box\varphi - \tfrac{1}{4}\rho\right]w - 2\left(\partial_t\varphi\right)\left(\partial_t w\right) + 2\left(\nabla_x\varphi \cdot \nabla_x w\right)\right\} \\ = &\left(\Box w + s^2 dw + \tfrac{1}{4}s\rho w\right) + s\left[cw + b\left(\partial_t w\right) + a \cdot \nabla_x w\right] \end{aligned}$$

where $a = 2\nabla_x \varphi = x$, $b = -2\left(\partial_t \varphi\right) = \beta\left(t - x_1^0 - T/2\right)/2$, $c = -\Box\varphi - \rho/4 = \beta/4 + n/2 - \rho/4$ and $d = \left(\partial_t \varphi\right)^2 - \left|\nabla_x \varphi\right|^2 = \beta^2\left(t - x_1^0 - T/2\right)^2/16 - |x|^2/4$. We note that c is a constant. Furthermore, by $\rho < 2$, we have

$$c > \frac{\beta}{4} + \frac{n}{2} - \frac{1}{2} \geq \frac{\beta}{4}. \tag{3.8}$$

Using the inequality: $(\alpha + \gamma)^2 \geq 2\alpha\gamma$, we have

$$(Lw)^2 \geq 2s\left(\Box w + s^2 dw + \frac{1}{4}s\rho w\right)\left[cw + b\left(\partial_t w\right) + a \cdot \nabla_x w\right]. \tag{3.9}$$

Noting that

$$a = x, \quad b = \frac{1}{2}\beta\left(t - x_1^0 - \frac{T}{2}\right), \quad \text{and} \quad c = \frac{\beta}{4} + \frac{n}{2} - \frac{\rho}{4}, \tag{3.10}$$

we can verify that

$$2\left(\Box w\right)\left[cw + b\left(\partial_t w\right) + a \cdot \nabla_x w\right] = \partial_t P + \nabla_x \cdot Q + R$$

where

$$P = b\left[\left(\partial_t w\right)^2 + \left|\nabla_x w\right|^2\right] + 2\left(\partial_t w\right)\left(a \cdot \nabla_x w + cw\right), \tag{3.11}$$

$$Q = \left[\left|\nabla_x w\right|^2 - \left(\partial_t w\right)^2\right] a - 2\left[a \cdot \nabla_x w + b\left(\partial_t w\right) + cw\right]\left(\nabla_x w\right), \tag{3.12}$$

$$R = \frac{1}{2}(\rho - 2\beta)\left(\partial_t w\right)^2 + \frac{1}{2}(4 - \rho)\left|\nabla_x w\right|^2. \tag{3.13}$$

Therefore,

$$\begin{aligned} &2\int_{G_T}\left(\Box w\right)\left[cw + b\left(\partial_t w\right) + a \cdot \nabla_x w\right] \mathrm{d}x\mathrm{d}t \\ &= \int_{\Sigma_T}\left(P - Q_1\right)\mathrm{d}x + \int_{\Sigma_0}\left(Q_1 - P\right)\mathrm{d}x + \int_{S_T} Q \cdot \nu \mathrm{d}\sigma\mathrm{d}t + \int_{G_T} R\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.14}$$

By (3.11) and (3.12), we can obtain that

$$\begin{aligned} P - Q_1 = &\left(b + a_1\right)\left(\partial_t w + \partial_1 w\right)^2 + \left(b - a_1\right)\left|\nabla_{x'} w\right|^2 \\ &+2\left(\partial_t w + \partial_1 w\right)\left(a' \cdot \nabla_x w + cw\right), \end{aligned} \tag{3.15}$$

where $a' = (0, a_2, \dots, a_n)$. Then by $x_1^0 > 0$ and the inequality: $|A \cdot B| \le |A|\,|B|$, we have

$$\begin{aligned} P - Q_1 \ge{}& (b + a_1)(\partial_t w + \partial_1 w)^2 + (b - a_1)|\nabla_{x'} w|^2 \\ & -2x_1^0(\partial_t w + \partial_1 w)^2 - \frac{1}{2x_1^0}(a' \cdot \nabla_x w)^2 + 2c(\partial_t w + \partial_1 w)w \\ \ge{}& \left(b + a_1 - 2x_1^0\right)(\partial_t w + \partial_1 w)^2 + \left(b - a_1 - \frac{1}{2x_1^0}|a'|^2\right)|\nabla_{x'} w|^2 \\ & +2c(\partial_t w + \partial_1 w)w. \end{aligned}$$

By (3.1) and (3.7), we have

$$\begin{aligned} & b + a_1 - 2x_1^0 = \frac{1}{2}\beta\left(x_1 - x_1^0 + \frac{T}{2}\right) + x_1 - 2x_1^0 \\ & \ge \frac{1}{2}\beta\left(\frac{T}{2} - r\right) + x_1^0 - r - 2x_1^0 = \frac{1}{4}\beta T - \frac{1}{2}\beta r - x_1^0 - r \\ & > r, \qquad (x,t) \in \Sigma_T. \end{aligned}$$

By (1.4), (3.1) and (3.7), we have

$$\begin{aligned} & b - a_1 - \frac{1}{2x_1^0}|a'|^2 = \frac{1}{2}\beta\left(x_1 - x_1^0 + \frac{T}{2}\right) - x_1 - \frac{1}{2x_1^0}\sum_{j=2}^{n} {x_j}^2 \\ & \ge \frac{1}{2}\beta\left(\frac{T}{2} - r\right) - x_1^0 - r - \frac{r^2}{2r} = \frac{1}{4}\beta T - \frac{1}{2}\beta r - x_1^0 - \frac{3}{2}r \\ & > \frac{r}{2}, \qquad (x,t) \in \Sigma_T. \end{aligned}$$

Therefore,

$$P - Q_1 \ge r(\partial_t w + \partial_1 w)^2 + \frac{1}{2}r|\nabla_{x'} w|^2 + \partial_1\left[cw^2\big|_{\Sigma_T}\right], \qquad (x,t) \in \Sigma_T. \quad (3.16)$$

Similarly, by (3.15), we have

$$\begin{aligned} Q_1 - P ={}& (-b - a_1)(\partial_t w + \partial_1 w)^2 + (a_1 - b)|\nabla_{x'} w|^2 \\ & -2(\partial_t w + \partial_1 w)(a' \cdot \nabla_x w + cw) \\ \ge{}& \left(-b - a_1 - \frac{r}{2}\right)(\partial_t w + \partial_1 w)^2 + \left(a_1 - b - \frac{2}{r}|a'|^2\right)|\nabla_{x'} w|^2 \\ & -2c(\partial_t w + \partial_1 w)w. \end{aligned}$$

By (3.1) and (3.7), we have

$$\begin{aligned} & -b - a_1 - \frac{r}{2} = \frac{1}{2}\beta\left(x_1^0 - x_1 + \frac{T}{2}\right) - x_1 - \frac{r}{2} \\ & \ge \frac{1}{2}\beta\left(\frac{T}{2} - r\right) - x_1^0 - r - \frac{r}{2} = \frac{1}{4}\beta T - \frac{1}{2}\beta r - x_1^0 - \frac{3}{2}r \\ & > \frac{r}{2}, \qquad (x,t) \in \Sigma_0. \end{aligned}$$

By (1.4), (3.1) and (3.7), we have

$$\begin{aligned}
&a_1 - b - \frac{2}{r}\left|a'\right|^2 = x_1 - \frac{1}{2}\beta\left(x_1 - x_1^0 - \frac{T}{2}\right) - \frac{2}{r}\sum_{j=2}^{n}{x_j}^2 \\
&\geq x_1^0 - r - \frac{1}{2}\beta\left(x_1^0 + r - x_1^0 - \frac{T}{2}\right) - \frac{2}{r}r^2 = \frac{1}{4}\beta T - \frac{1}{2}\beta r + x_1^0 - 3r \\
&\geq \left(2r + x_1^0\right) + x_1^0 - 3r = 2x_1^0 - r > r, \qquad (x,t) \in \Sigma_0.
\end{aligned}$$

Therefore,

$$Q_1 - P \geq \frac{1}{2}r\left(\partial_t w + \partial_1 w\right)^2 + r\left|\nabla_{x'} w\right|^2 - \partial_1\left[\left.cw^2\right|_{\Sigma_0}\right], \qquad (x,t) \in \Sigma_0. \quad (3.17)$$

By (3.12), we have

$$Q \cdot \nu = \left[\left|\nabla_x w\right|^2 - \left(\partial_t w\right)^2\right](a \cdot \nu) - 2\left[a \cdot \nabla_x w + b\left(\partial_t w\right) + cw\right]\left[\left(\nabla_x w\right) \cdot \nu\right].$$

Then by (3.1), (3.2) and (3.10), we have

$$|Q \cdot \nu| \leq C_3\left(\left|\nabla_{x,t} w\right|^2 + w^2\right), \quad (x,t) \in S_T. \qquad (3.18)$$

Here and henceforth, $C_k (k = 3, 4, \dots)$ denote generic positive constants which may depend on x_1^0, r, T, n, β, ρ, s_0, and s_1, but are independent of s. It follows from (3.14), (3.16), (3.17) and (3.18) that

$$\begin{aligned}
&2\int_{G_T}(\Box w)\left[cw + b\left(\partial_t w\right) + a \cdot \nabla_x w\right]\mathrm{d}x\mathrm{d}t \\
&\geq \frac{1}{2}r\int_{\Sigma_0 \bigcup \Sigma_T}\left[\left(\partial_t w + \partial_1 w\right)^2 + \left|\nabla_{x'} w\right|^2\right]\mathrm{d}x \\
&\quad -c\int_{\partial\Sigma_0 \bigcup \partial\Sigma_T} w^2 \mathrm{d}\sigma - C_3\int_{S_T}\left(\left|\nabla_{x,t} w\right|^2 + w^2\right)\mathrm{d}\sigma\mathrm{d}t \\
&\quad +\frac{1}{2}\int_{G_T}\left[(\rho - 2\beta)\left(\partial_t w\right)^2 + (4 - \rho)\left|\nabla_x w\right|^2\right]\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $\partial\Sigma_0$ and $\partial\Sigma_T$ denote the boundaries of Σ_0 and Σ_T, respectively, and $\mathrm{d}\sigma$ is an area element of $\partial\Omega$. Furthermore, as (4.1.40) in [16], we can show that

$$\int_{\partial\Sigma_0 \bigcup \partial\Sigma_T} w^2 \mathrm{d}\sigma \leq T\int_{S_T}\left(w_t^2 + \frac{3}{T^2}w^2\right)\mathrm{d}\sigma\mathrm{d}t.$$

Therefore,

$$
\begin{aligned}
&2\int_{G_T}(\Box w)\left[cw+b\left(\partial_t w\right)+a\cdot\nabla_x w\right]\mathrm{d}x\mathrm{d}t\\
&\geq \frac{1}{2}r\int_{\Sigma_0\bigcup\Sigma_T}\left[\left(\partial_t w+\partial_1 w\right)^2+\left|\nabla_{x'}w\right|^2\right]\mathrm{d}x\\
&\quad -C_4\int_{S_T}\left(\left|\nabla_{x,t}w\right|^2+w^2\right)\mathrm{d}\sigma\mathrm{d}t\\
&\quad +\frac{1}{2}\int_{G_T}\left[\left(\rho-2\beta\right)\left(\partial_t w\right)^2+\left(4-\rho\right)\left|\nabla_x w\right|^2\right]\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.19}
$$

Moreover, we can verify that

$$
\begin{aligned}
&2dw\left[cw+b\left(\partial_t w\right)+a\cdot\nabla_x w\right]\\
&=\nabla_x\cdot\left(dw^2 a\right)+\partial_t\left(dbw^2\right)-w^2\left[\nabla_x\cdot(da)\right]-w^2\partial_t\left(bd\right)+2dcw^2.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\quad 2\int_{G_T}dw\left[cw+b\left(\partial_t w\right)+a\cdot\nabla_x w\right]\mathrm{d}x\mathrm{d}t\\
&=\int_{\Sigma_T}dw^2\left(b-a_1\right)\mathrm{d}x+\int_{\Sigma_0}dw^2\left(a_1-b\right)\mathrm{d}x\\
&\quad+\int_{S_T}dw^2\left(a\cdot\nu\right)\mathrm{d}\sigma\mathrm{d}t+\int_{G_T}w^2\left[2dc-\nabla_x\cdot(da)-\partial_t\left(bd\right)\right]\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.20}
$$

By (1.4), (3.1) and (3.7), we have

$$
\begin{aligned}
&d=\frac{1}{16}\beta^2\left(x_1-x_1^0+\frac{T}{2}\right)^2-\frac{1}{4}|x|^2\geq\frac{1}{16}\beta^2\left(\frac{T}{2}-r\right)^2-\frac{1}{4}\left(r+x_1^0\right)^2\\
&>\frac{1}{4}\left(2r+x_1^0\right)^2-\frac{1}{4}\left(r+x_1^0\right)^2=\frac{1}{4}\left(3r^2+2x_1^0r\right)\geq\frac{5}{4}r^2,\quad (x,t)\in\Sigma_T.
\end{aligned}
$$

By (3.1) and (3.7), we have

$$
b-a_1=\frac{1}{2}\beta\left(x_1-x_1^0+\frac{T}{2}\right)-x_1\geq\frac{1}{2}\beta\left(\frac{T}{2}-r\right)-\left(x_1^0+r\right)>r,\quad (x,t)\in\Sigma_T.
\tag{3.21}
$$

Therefore,

$$
dw^2\left(b-a_1\right)\geq\frac{5}{4}r^3w^2,\quad (x,t)\in\Sigma_T.
\tag{3.22}
$$

Similarly, we have

$$
\begin{aligned}
&d=\frac{1}{16}\beta^2\left(x_1-x_1^0-\frac{T}{2}\right)^2-\frac{1}{4}|x|^2\\
&\geq\frac{1}{16}\beta^2\left(\frac{T}{2}-r\right)^2-\frac{1}{4}\left(r+x_1^0\right)^2\geq\frac{5}{4}r^2,\quad (x,t)\in\Sigma_0,
\end{aligned}
$$

and

$$
\begin{aligned}
a_1-b &= x_1-\frac{1}{2}\beta\left(x_1-x_1^0-\frac{T}{2}\right)\geq\left(x_1^0-r\right)-\frac{1}{2}\beta\left(r-\frac{T}{2}\right)\\
&> \left(x_1^0-r\right)+\left(2r+x_1^0\right)=2x_1^0+r\geq 3r,\quad (x,t)\in\Sigma_0.
\end{aligned}
\tag{3.23}
$$

Therefore,

$$
dw^2\left(a_1-b\right)\geq\frac{15}{4}r^3w^2,\quad (x,t)\in\Sigma_0. \tag{3.24}
$$

Furthermore we can verify that

$$
\begin{aligned}
&2dc-\nabla_x\cdot(da)-\partial_t(bd)\\
&=\frac{1}{2}\left[|x|^2-\frac{1}{16}\rho\beta^2\left(t-x_1^0-\frac{T}{2}\right)^2+\frac{1}{4}\rho|x|^2-\frac{1}{8}\beta^3\left(t-x_1^0-\frac{T}{2}\right)^2\right].
\end{aligned}
$$

Then by (3.1), (3.2) and (3.6), we have

$$
\begin{aligned}
&2dc-\nabla_x\cdot(da)-\partial_t(bd)\\
&\geq\frac{1}{2}\left\{\left(x_1^0-r\right)^2-\frac{1}{16}\beta^2\left(\frac{T}{2}+r\right)^2\left[\frac{64\left(x_1^0-r\right)^2}{\beta^2(T+2r)^2}-2\beta\right]\right.\\
&\left.+\frac{1}{4}\rho\left(x_1^0-r\right)^2-\frac{1}{8}\beta^3\left(\frac{T}{2}+r\right)^2\right\}\\
&=\frac{1}{8}\rho\left(x_1^0-r\right)^2,\quad (x,t)\in G_T.
\end{aligned}
\tag{3.25}
$$

It follows from (3.20), (3.22), (3.24) and (3.25) that

$$
\begin{aligned}
&2\int_{G_T}dw\left[cw+b\left(\partial_tw\right)+a\cdot\nabla_xw\right]\mathrm{d}x\mathrm{d}t\\
&\geq\frac{5}{4}r^3\int_{\Sigma_0\bigcup\Sigma_T}w^2\mathrm{d}x+\frac{1}{8}\rho\left(x_1^0-r\right)^2\int_{G_T}w^2\mathrm{d}x\mathrm{d}t-C_5\int_{S_T}w^2\mathrm{d}\sigma\mathrm{d}t.
\end{aligned}
\tag{3.26}
$$

Furthermore, by (3.10), (3.20), (3.21) and (3.23), we have

$$
\begin{aligned}
&2\int_{G_T}w\left[cw+b\left(\partial_tw\right)+a\cdot\nabla_xw\right]\mathrm{d}x\mathrm{d}t\\
&=\int_{\Sigma_T}\left(b-a_1\right)w^2\mathrm{d}x+\int_{\Sigma_0}\left(a_1-b\right)w^2\mathrm{d}x\\
&\quad+\int_{S_T}w^2\left(a\cdot\nu\right)\mathrm{d}\sigma\mathrm{d}t+\int_{G_T}\left(2c-\nabla_x\cdot a-\partial_tb\right)w^2\mathrm{d}x\mathrm{d}t\\
&\geq r\int_{\Sigma_0\bigcup\Sigma_T}w^2\mathrm{d}x-\frac{1}{2}\rho\int_{G_T}w^2\mathrm{d}x\mathrm{d}t-C_6\int_{S_T}w^2\mathrm{d}\sigma\mathrm{d}t.
\end{aligned}
\tag{3.27}
$$

Hence, by (3.9), (3.19), (3.26) and (3.27), there exists $s_1 > 0$ such that, for all $s \geq s_1$,

$$\begin{aligned}
&\int_{G_T} (\Box v)^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t = \int_{G_T} (Lw)^2 \,\mathrm{d}x\mathrm{d}t \\
&\geq \min\left(\frac{r}{2}, \frac{5}{4}r^3\right) \int_{\Sigma_0 \bigcup \Sigma_T} \left[s\left(\partial_t w + \partial_1 w\right)^2 + s\left|\nabla_{x'} w\right|^2 + s^3 w^2 \right] \mathrm{d}x \\
&\quad + \frac{1}{2} \int_{G_T} \left[(\rho - 2\beta) s \left(\partial_t w\right)^2 + (4-\rho) s \left|\nabla_x w\right|^2 \right. \\
&\qquad \left. + \frac{1}{8}\rho \left(x_1^0 - r\right)^2 s^3 w^2 \right] \mathrm{d}x\mathrm{d}t - C_7 \int_{S_T} \left[s\left|\nabla_{x,t} w\right|^2 + s^3 w^2 \right] \mathrm{d}\sigma\mathrm{d}t.
\end{aligned} \tag{3.28}$$

Furthermore, by (1.4) and (3.6), we see that

$$\min\left(\frac{r}{2}, \frac{5}{4}r^3\right) > 0, \quad \rho - 2\beta > 0, \quad 4 - \rho > 0, \quad \text{and} \quad \rho\left(x_1^0 - r\right)^2 > 0. \tag{3.29}$$

On the other hand, by $w = \mathrm{e}^{s\varphi} v$, we have

$$\nabla_{x,t} w = s\mathrm{e}^{s\varphi} v \left(\nabla_{x,t}\varphi\right) + \mathrm{e}^{s\varphi}\left(\nabla_{x,t} v\right) = s\left(\nabla_{x,t}\varphi\right) w + \mathrm{e}^{s\varphi}\left(\nabla_{x,t} v\right).$$

Therefore, we have

$$\left(\partial_t v + \partial_1 v\right)^2 \mathrm{e}^{2s\varphi} \leq 2s^2 \left(\partial_t \varphi + \partial_1 \varphi\right)^2 w^2 + 2\left(\partial_t w + \partial_1 w\right)^2,$$

$$\left(\partial_t v\right)^2 \mathrm{e}^{2s\varphi} \leq 2s^2 \left(\partial_t \varphi\right)^2 w^2 + 2\left(\partial_t w\right)^2,$$

$$\left|\nabla_x v\right|^2 \mathrm{e}^{2s\varphi} \leq 2s^2 \left|\nabla_x \varphi\right|^2 w^2 + 2\left|\nabla_x w\right|^2,$$

$$\left|\nabla_{x'} v\right|^2 \mathrm{e}^{2s\varphi} \leq 2s^2 \left|\nabla_{x'} \varphi\right|^2 w^2 + 2\left|\nabla_{x'} w\right|^2,$$

and

$$\left|\nabla_{x,t} w\right|^2 \leq 2\left|\nabla_{x,t} v\right|^2 \mathrm{e}^{2s\varphi} + 2s^2 \left|\nabla_{x,t}\varphi\right|^2 \mathrm{e}^{2s\varphi} v^2.$$

Using (3.28), (3.29), and the above inequalities, we have

$$
\begin{aligned}
&\int_{G_T} \left[s\left|\nabla_{x,t} v\right|^2 + s^3 v^2 \right] \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Sigma_0 \bigcup \Sigma_T} \left[s\left(\partial_t v + \partial_1 v\right)^2 + s\left|\nabla_{x'} v\right|^2 + s^3 v^2 \right] \mathrm{e}^{2s\varphi} \mathrm{d}x \\
&\le C_8 \left\{ \int_{G_T} \left[s\left|\nabla_{x,t} w\right|^2 + s^3 w^2 \right] \mathrm{d}x\mathrm{d}t \right. \\
&\quad \left. + \int_{\Sigma_0 \bigcup \Sigma_T} \left[s\left(\partial_t w + \partial_1 w\right)^2 + s\left|\nabla_{x'} w\right|^2 + s^3 w^2 \right] \mathrm{d}x \right\} \\
&\le C_9 \left\{ \int_{S_T} \left[s\left|\nabla_{x,t} w\right|^2 + s^3 w^2 \right] \mathrm{d}\sigma\mathrm{d}t + \int_{G_T} \left(\Box v\right)^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \right\} \\
&\le C_{10} \left\{ \int_{S_T} \left[s\left|\nabla_{x,t} v\right|^2 + s^3 v^2 \right] \mathrm{e}^{2s\varphi} \mathrm{d}\sigma\mathrm{d}t + \int_{G_T} \left(\Box v\right)^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \right\}
\end{aligned}
$$

We have completed the proof of Lemma 2.1. □

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