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Abstract

In this paper under some growth condition we investigate the connection between RBMO and the Morrey spaces. We do not assume the doubling condition which has been a key property of harmonic analysis. We also obtain another type of equivalent norms.

KEYWORDS : Morrey space, Campanato space, equivalent norms

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1 Introduction

In this paper we discuss equivalent norms for the (vector-valued) Morrey spaces with non-doubling measures. We consider the connection between the Morrey spaces and the Campanato spaces with underlying measure μ non-doubling. The Morrey spaces appeared in [4] originally in connection with the partial differential equations and the Campanato spaces in [1] and [2]. We refer to [5] for the result of Morrey spaces coming with the doubling measures. Before we state our main theorem, let us make a brief view of the terminology of measures on $\mathbf{R}^d$. We say that a (positive) Radon measure μ on $\mathbf{R}^d$ satisfies the growth condition if

$$\mu(Q(x,l)) \le C_0\, l^n \text{ for all } x \in \operatorname{supp}(\mu) \text{ and } l > 0, \qquad (1)$$

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where C_0 and $n \in (0, d]$ are some fixed numbers, and μ is said to satisfy the doubling condition if, for some constant $C > 0$,

$$\mu(Q(x, 2l)) \le C\,\mu(Q(x, l)) \text{ for all } x \in \mathbf{R}^d \text{ and } l > 0.$$

A measure μ which satisfies the growth condition is called growth measure while a measure μ with the doubling condition is said to be the doubling measure.

By a "cube" $Q \subset \mathbf{R}^d$ we mean a closed cube having sides parallel to the axes. Its center will be denoted by z_Q and its side length by $\ell(Q)$. By $Q(x, l)$ we will also denote the cube centered at x of sidelength l. For $\rho > 0$, $\rho\,Q$ means a cube concentric to Q with its sidelength $\rho\,\ell(Q)$. Let $\mathcal{Q}(\mu)$ denote the set of all cubes $Q \subset \mathbf{R}^d$ with positive μ-measures. If μ is finite, we include $\mathbf{R}^d$ in $\mathcal{Q}(\mu)$ as well. In [6], the authors defined the Morrey spaces $\mathcal{M}^p_q(k, \mu)$ for non-doubling measures normed by

$$\|f : \mathcal{M}^p_q(k, \mu)\| := \sup_{Q \in \mathcal{Q}(\mu)} \mu(k\,Q)^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q |f|^q \, d\mu \right)^{\frac{1}{q}}, \; 1 \le q \le p < \infty, \; k > 1. \tag{2}$$

The fundamental property of this norm is

$$\|f : \mathcal{M}^p_q(k_1, \mu)\| \le \|f : \mathcal{M}^p_q(k_2, \mu)\| \le C_d \left(\frac{k_1 - 1}{k_2 - 1} \right)^d \|f : \mathcal{M}^p_q(k_1, \mu)\| \tag{3}$$

for $1 < k_1 < k_2 < \infty$. With this relation in mind, we will denote $\mathcal{M}^p_q(\mu) = \mathcal{M}^p_q(2, \mu)$. The aim of this paper is to find some norms equivalent to this Morrey norm.

2 Equivalent norm of doubling type

In this section we investigate an equivalent norm related to the doubling cubes. Although we now envisage the non-homogeneous setting, we are still able to place ourselves in the setting of the doubling cubes. In [9], Tolsa defined the notion of doubling cubes. Let $k, \beta > 1$. We say that $Q \in \mathcal{Q}(\mu)$ is a (k, β)-doubling cube, if $\mu(kQ) \le \beta\,\mu(Q)$. It is well-known that, if $\beta > k^d$, then for μ-almost all $x \in \mathbf{R}^d$ and for all $Q \in \mathcal{Q}(\mu)$ centered at x, we can find a (k, β)-doubling cube R from $k^{-1}Q, k^{-2}Q, \ldots$. In what follows we denote by $\mathcal{Q}(\mu; k, \beta)$ the set of all (k, β)-doubling cubes in $\mathcal{Q}(\mu)$. We fix $k, \beta > 1$ so that they satisfy $\beta > k^d$. Let $1 \le q \le p < \infty$. For $f \in L^1_{loc}(\mu)$ define

$$\|f : \mathcal{M}^p_q(\mu)\|_d := \sup_{Q \in \mathcal{Q}(\mu; k, \beta)} \mu(Q)^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}.$$

Now we present the main theorem in this section.

THEOREM **1** *Let μ be a Radon measure which does not necessarily satisfy the growth condition nor the doubling condition and let $1 \le q < p < \infty$. If $\beta > k^{\frac{dpq}{p-q}}$, then*

$$C^{-1}\,\|f : \mathcal{M}^p_q(\mu)\|_d \le \|f : \mathcal{M}^p_q(\mu)\| \le C\,\|f : \mathcal{M}^p_q(\mu)\|_d, \; f \in \mathcal{M}^p_q(\mu)$$

for some constant $C > 0$.

Before we come to the proof of Theorem 1, two clarifying remarks may be in order.

Remark **2** If $p = q$, this theorem fails in general. However, if we assume the growth condition or the doubling condition, the theorem is still available for $p = q$. In fact, under the growth condition or the doubling condition for any cube $Q \in \mathcal{Q}(\mu)$ we can find a large integer $j \gg 1$ such that $2^j Q \in \mathcal{Q}(\mu; k, \beta)$.

Remark **3** This theorem readily extends to the vector-valued version. Let $1 \le q \le p < \infty$ and $r \in (1, \infty)$. We define the vector-valued Morrey spaces $\mathcal{M}^p_q(l^r, \mu)$ by the set of sequences of μ-measurable functions $\{f_j\}_{j \in \mathbf{N}}$ for which

$$\|f_j \,:\, \mathcal{M}^p_q(l^r, \mu)\| := \sup_{Q \in \mathcal{Q}(\mu)} \mu(2Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \|f_j \,:\, l^r\|^q \, d\mu \right)^{\frac{1}{q}} < \infty. \tag{4}$$

The theorem can be extended to the vector valued version. Let

$$\|f_j \,:\, \mathcal{M}^p_q(l^r, \mu)\|_d := \sup_{Q \in \mathcal{Q}(\mu; k, \beta)} \mu(Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \|f_j(y) \,:\, l^r\|^q \, d\mu(y) \right)^{\frac{1}{q}}.$$

Then $C^{-1} \|f_j \,:\, \mathcal{M}^p_q(l^r, \mu)\|_d \le \|f_j \,:\, \mathcal{M}^p_q(l^r, \mu)\| \le C \|f_j \,:\, \mathcal{M}^p_q(l^r, \mu)\|_d$. The same proof as the scalar-valued spaces works for the vector-valued spaces, so in the actual proof we concentrate on the scalar-valued cases.

Proof. Given $k > 1$, we shall prove

$$C^{-1} \|f \,:\, \mathcal{M}^p_q(\mu)\|_d \le \|f \,:\, \mathcal{M}^p_q(k, \mu)\|, \quad \|f \,:\, \mathcal{M}^p_q(\mu)\| \le C \|f \,:\, \mathcal{M}^p_q(\mu)\|_d$$

for large $\beta > 0$. The left inequality is obvious, so let us prove the right inequality. We have only to show that, for every cube $Q \in \mathcal{Q}(\mu)$,

$$\mu(2Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}} \le C \|f \,:\, \mathcal{M}^p_q(\mu)\|_d.$$

Let $x \in Q \cap \operatorname{supp}(\mu)$ and $Q(x)$ be the largest doubling cube centered at x and having sidelength $k^{-j}\ell(Q)$ for some $j \in \mathbf{N}$. Existence of $Q(x)$ can be ensured for μ-almost all $x \in \mathbf{R}^d$. Set

$$\mathcal{Q}_0(j) := \{Q(x) \,:\, \ell(Q(x)) = k^{-j}\ell(Q)\}, \, j \in \mathbf{N}.$$

By Besicovitch's covering lemma we can take $\mathcal{Q}(j) \subset \mathcal{Q}_0(j)$ so that $\displaystyle\sum_{R \in \mathcal{Q}(j)} \chi_R \le 4^d \chi_{2Q}$ and that $x \in \displaystyle\bigcup_{R \in \mathcal{Q}(j)} R$ for μ-almost all $x \in Q$ with $\ell(Q(x)) = k^{-j}\ell(Q)$. Volume argument gives us that $\sharp(\mathcal{Q}(j)) \le 8^d \, k^{jd}$. Since

$$\left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}} \le \sum_{j=1}^{\infty} \sum_{R \in \mathcal{Q}(j)} \left(\int_R |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}$$

and $\mu(R) \le \beta^{-j}\mu(2Q)$ for all $R \in \mathcal{Q}(j)$, we have

$$\begin{aligned}
&\mu(2Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}} \\
&\le \sum_{j=1}^{\infty} \beta^{j\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{R \in \mathcal{Q}(j)} \mu(R)^{\frac{1}{p}-\frac{1}{q}} \left(\int_R |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}} \\
&\le \sum_{j=1}^{\infty} 8^d \, k^{jd} \, \beta^{j\left(\frac{1}{p}-\frac{1}{q}\right)} \|f \,:\, \mathcal{M}^p_q(\mu)\|_d \\
&= \sum_{j=1}^{\infty} 8^d \, \exp\left\{ j \left(d \log k + \left(\frac{1}{p} - \frac{1}{q} \right) \log \beta \right) \right\} \|f \,:\, \mathcal{M}^p_q(\mu)\|_d \le C \, \|f \,:\, \mathcal{M}^p_q(\mu)\|_d,
\end{aligned}$$

where the constant C is finite, provided $\beta > k^{\frac{dpq}{p-q}}$. ∎

3 Equivalent norms of Campanato type

Throughout the rest of this paper we assume that μ satisfy the growth condition (1). We do not assume that μ is doubling. Before we formulate our theorems, let us recall the definition of the RBMO spaces due to Tolsa [9]. Given two cubes $Q \subset R$ with $Q \in \mathcal{Q}(\mu)$, we denote

$$\delta(Q, R) := \int_{\ell(Q)}^{\ell(Q_R)} \frac{\mu(Q(z_Q, l))}{l^n} \frac{dl}{l}, \quad K_{Q,R} = 1 + \delta(Q, R),$$

where Q_R denotes the smallest cube concentric to Q containing R. Here and below we abbreviate the $(2, 2^{d+1})$-doubling cube to the doubling cube and $\mathcal{Q}(\mu; 2, 2^{d+1})$ by writing $\mathcal{Q}(\mu, 2)$. Given $Q \in \mathcal{Q}(\mu)$, we set Q^* as the smallest doubling cube R of the form $R = 2^j Q$ with $j = 0, 1, \ldots$.[2]

Tolsa defined a new BMO for the growth measures, which is suitable for the Calderón-Zygmund theory. We say that $f \in L^1_{loc}(\mu)$ is an element of RBMO if it satisfies

$$\|f\|_* := \sup_{Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(x) - m_{Q^*}(f)| \, d\mu(x) + \sup_{\substack{Q \subset R \\ Q, R \in \mathcal{Q}(\mu, 2)}} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}} < \infty,$$

where $m_Q(f) := \dfrac{1}{\mu(Q)} \displaystyle\int_Q f(y) \, d\mu(y)$. Further details may be found in [9, Section 2]. The following lemma is due to Tolsa.

Lemma 4 [9, Corollary 3.5] *Let $f \in$RBMO.*

[2] By the growth condition (1) there are a lot of big doubling cubes. Precisely speaking, given a cube $Q \in \mathcal{Q}(\mu)$, we can find $j \in \mathbf{N}$ with $2^j Q \in \mathcal{Q}(\mu, 2)$ (see [9]).

1. *There exist positive constants C and C' independent of f so that, for every $\lambda > 0$ and every cube $Q \in \mathcal{Q}(\mu)$,*

$$\mu\{x \in Q \,:\, |f(x) - m_{Q^*}(f)| > \lambda\} \le C\,\mu\left(\frac{3}{2}Q\right)\,\exp\left(-\frac{C'\lambda}{\|f\|_*}\right).$$

2. *Let $1 \le q < \infty$. Then there exists a constant C independent of f, so that, for every cube $Q \in \mathcal{Q}(\mu)$,*

$$\left(\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q |f(x) - m_{Q^*}(f)|^q \, d\mu(x)\right)^{\frac{1}{q}} \le C\,\|f\|_*.$$

Elementary property of $\delta(\cdot,\cdot)$ Below we list elementary properties of $\delta(\cdot,\cdot)$ used in this paper.

LEMMA **5** *Let and $Q \in \mathcal{Q}(\mu)$. Then the following properties hold :*

(1) *For $\rho > 1$, we have $\delta(Q, \rho Q) \le C_0 \log \rho$.*

(2) $\delta(Q, Q^*) \le C_0\, 2^{n+1} \log 2$.

(3) *Let $k_0 \in \mathbf{N}$ and $\alpha > 0$. Assume, for some $\theta > 0$, $\alpha \le \mu(Q) \le \mu(2^{k_0}Q) \le \theta\,\alpha$. Then $\delta(Q, 2^{k_0}Q) \le 2^n \log 2 \cdot \theta\, C_0\, c_n$, where $c_n := \sum_{k=0}^{\infty} 2^{-nk}$.*

(4) *Given the cubes $P \subset Q \subset R$ with $P \in \mathcal{Q}(\mu)$, then*

$$|\delta(P,R) - (\delta(P,Q) + \delta(Q,R))| \le C,$$

where C is a constant depending only on C_0, n, d.

(5) *Let $Q, R \in \mathcal{Q}(\mu)$. Suppose that, for some constant $c_1 > 1$, $Q \subset R$ and $\ell(R) \le c_1\,\ell(Q)$. Then there exists a doubling cube $S \in \mathcal{Q}(\mu, 2)$ such that $Q^*, R^* \subset S$ and $\delta(Q^*, S), \delta(R^*, S) \le C$, where C is a constant depending only on c_1, C_0, n, d.*

Proof. In [8], we have proved (1)–(4). For reader's convenience the full proof is given here. (1) is obvious. To prove (2) we set $Q^* = 2^{k_0}Q_0$. We may assume that $k_0 \ge 1$. The dyadic argument yields that

$$\delta(Q, 2^{k_0}Q) = \int_{\ell(Q)}^{\ell(2^{k_0}Q)} \frac{\mu(Q(z_Q, l))}{l^n}\,\frac{dl}{l} \le 2^n \log 2 \sum_{k=1}^{k_0} \frac{\mu(2^kQ)}{\ell(2^kQ)^n}.$$

Note that $2^{d+1}\,\mu(2^{k-1}Q) \le \mu(2^kQ)$ for $k = 1, 2, \ldots, k_0$, since $2^{k-1}Q$ is not doubling, which yields, together with the fact that $d \ge n$,

$$\delta(Q, 2^{k_0}Q) \le 2^n \log 2\,\frac{\mu(2^{k_0}Q)}{\ell(2^{k_0}Q)^n}\sum_{k=1}^{k_0}(2^{n-d-1})^{k_0-k} \le C_0\, 2^{n+1} \log 2.$$

We prove (3). It follows by the dyadic argument and the assumption that

$$\delta(Q,2^{k_0}Q) \le 2^n \, \log 2 \sum_{k=1}^{k_0} \frac{\mu(2^k Q)}{\ell(2^k Q)^n} \le 2^n \, \log 2 \cdot \frac{\theta \alpha}{\ell(Q)^n} \sum_{k=0}^{k_0} 2^{-nk} \le 2^n \, \log 2 \cdot \theta \, C_0 \, c_n.$$

Now we prove (4). It suffices to prove that

$$A := |\delta(P_Q,R) - \delta(Q,R)| \le C. \tag{5}$$

We decompose A as

$$\begin{aligned} A \;&=\; \left| \int_{\ell(P_Q)}^{\ell(P_R)} \frac{\mu(Q(z_P,l))}{l^n} \frac{dl}{l} - \int_{\ell(Q)}^{\ell(Q_R)} \frac{\mu(Q(z_Q,l))}{l^n} \frac{dl}{l} \right| \\ &\le\; \int_{\ell(Q)}^{\ell(P_Q)} \frac{\mu(Q(z_Q,l))}{l^n} \frac{dl}{l} + \left| \int_{\ell(P_Q)}^{\min\{\ell(P_R),\ell(Q_R)\}} (\mu(Q(z_P,l)) - \mu(Q(z_Q,l))) \frac{dl}{l^{n+1}} \right| \\ &\quad + \int_{\min\{\ell(P_R),\ell(Q_R)\}}^{\max\{\ell(P_R),\ell(Q_R)\}} \left(\frac{\mu(Q(z_P,l))}{l^n} + \frac{\mu(Q(z_Q,l))}{l^n} \right) \frac{dl}{l} \;\; =: \; A_1 + A_2 + A_3. \end{aligned}$$

By (1) the integrals A_1 and A_3 are easily estimated above by some constant C. So we estimate A_2. Bound A_2 from above by

$$A_2 \le \int_{\ell(P_Q)}^{\infty} \mu(Q(z_P,l) \Delta Q(z_Q,l)) \frac{dl}{l^{n+1}} = \int_{\ell(P_Q)}^{\infty} \int_{\mathbf{R}^d} \chi_{Q(z_P,l)\Delta Q(z_Q,l)}(y) \, d\mu(y) \, \frac{dl}{l^{n+1}}.$$

A simple geometric observation tells us that $\chi_{Q(z_P,l)\Delta Q(z_Q,l)}(y) = 0$ if

$$l \notin [\min\{|y-z_P|_\infty, |y-z_Q|_\infty\}, \max\{|y-z_P|_\infty, |y-z_Q|_\infty\}],$$

where $|y|_\infty := \max\{|y_1|, \dots, |y_d|\}$. This observation and Fubini's theorem yield

$$\begin{aligned} A_2 \;&\le\; C \int_{\mathbf{R}^d \setminus P_Q} \left| \frac{1}{|y-z_P|_\infty{}^n} - \frac{1}{|y-z_Q|_\infty{}^n} \right| d\mu(y) \\ &\le\; C \int_{|y-z_P|_\infty \ge \ell(P_Q)/2} \frac{|z_P - z_Q|_\infty}{|y-z_P|_\infty{}^{n+1}} \, d\mu(y) \le C \, \frac{|z_P - z_Q|_\infty}{\ell(P_Q)} \le C. \end{aligned}$$

This proves (5).

Finally we establish (4). Let $Q^* = 2^j Q$. Then we claim $\delta(R, 2^j R) \le C$. Indeed, by virtue of the fact that $Q \subset R$ we see that if $l \ge \ell(R)$ then $Q(z_R, l) \subset Q(z_q, 2l)$. As a consequence we obtain

$$\begin{aligned} \delta(R, 2^j R) &= \int_{\ell(R)}^{2^j \ell(R)} \frac{\mu(Q(z_R,l))}{l^n} \frac{dl}{l} \\ &\le \int_{\ell(R)}^{2^j \ell(R)} \frac{\mu(Q(z_Q,2l))}{l^n} \frac{dl}{l} \le \int_{\ell(Q)}^{c_1 \, 2^{j+1} \ell(Q)} \frac{\mu(Q(z_Q,l))}{l^n} \frac{dl}{l} \le C. \end{aligned}$$

If we put $S := (2^{j+1}R)^*$, then $\delta(R^*, S) \le C$. (1) and (4) finally give us

$$\delta(Q^*, S) \le \delta(Q^*, 2^{j+1}R) + \delta(2^{j+1}R, S) + C \le C. \quad \blacksquare$$

Scalar-valued Campanato space Having cleared up the definition of RBMO, we will find a relationship between RBMO and the Morrey spaces. With the definition of RBMO in mind, we shall define the Campanato spaces.

Let $f \in L^1_{loc}(\mu)$. We set the norm of the Campanato spaces $\mathcal{C}^p_q(k,\mu)$ by

$$\begin{aligned} \|f : \mathcal{C}^p_q(k,\mu)\| &:= \sup_{Q \in \mathcal{Q}(\mu)} \mu(kQ)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x) - m_{Q^*}(f)|^q \, d\mu(x) \right)^{\frac{1}{q}} \\ &+ \sup_{\substack{Q \subset R \\ Q,R \in \mathcal{Q}(\mu,2)}} \mu(Q)^{\frac{1}{p}} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}}, \ 1 \le q \le p \le \infty, \ k > 1. \end{aligned}$$

Let $k_1, k_2 > 1$. Then $\mathcal{C}^p_q(k_1,\mu)$ and $\mathcal{C}^p_q(k_2,\mu)$ coincide as a set and their norms are mutually equivalent. Speaking more precisely, we have the norm equivalence

$$\|f : \mathcal{C}^p_q(k_1,\mu)\| \sim \|f : \mathcal{C}^p_q(k_2,\mu)\|. \tag{6}$$

To prove (6) we may assume that $k_2 = 2k_1 - 1$ because of the monotonicity of $\|\cdot : \mathcal{C}^p_q(k,\mu)\|$ with respect to k. Then all we have to prove is

$$\mu(k_1Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x) - m_{Q^*}(f)|^q \, d\mu(x) \right)^{\frac{1}{q}} \le C \, \|f : \mathcal{C}^p_q(k_2,\mu)\|$$

for fixed cube $Q \in \mathcal{Q}(\mu)$. Divide equally Q into 2^d cubes and collect those in $\mathcal{Q}(\mu)$. Let us name them $Q_1, Q_2, \ldots, Q_N$, $N \le 2^d$. The triangle inequality reduces the matter to showing

$$\mu(k_1Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_l} |f(x) - m_{Q^*}(f)|^q \, d\mu(x) \right)^{\frac{1}{q}} \le C \, \|f : \mathcal{C}^p_q(k_2,\mu)\|, \ 1 \le l \le N.$$

Note that $k_2Q_l \subset k_1Q$. We apply Lemma 5 (5) to obtain an auxiliary doubling cube R which contains $(Q_l)^*, Q^*$ and satisfies $K_{(Q_l)^*,R}, K_{Q^*,R} \le C$. Thus, we obtain

$$\begin{aligned} &\mu(k_1Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_l} |f(x) - m_{Q^*}(f)|^q \, d\mu(x) \right)^{\frac{1}{q}} \\ \le\ & \mu(k_1Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_l} |f(x) - m_{(Q_l)^*}(f)|^q \, d\mu(x) \right)^{\frac{1}{q}} \\ +\ & \mu(Q_l)^{\frac{1}{p}} |m_{(Q_l)^*}(f) - m_R(f)| \ + \ \mu(Q_l)^{\frac{1}{p}} |m_R(f) - m_{Q^*}(f)| \\ \le\ & C \, \|f : \mathcal{C}^p_q(k_2,\mu)\|. \end{aligned}$$

As a result (6) is proved.

Since $\mathcal{C}^p_q(k_1,\mu)$ and $\mathcal{C}^p_q(k_2,\mu)$ are isomorphic to each other as Banach spaces, no confusion can occur if we denote $\mathcal{C}^p_q(\mu) = \mathcal{C}^p_q(2,\mu)$.

Note that $\mathcal{C}^\infty_q(\mu) = RBMO$, if $1 \le q < \infty$. This is an immediate consequence of Lemma 4. Thus we can say that RBMO is a limit function space of $\mathcal{C}^p_q(\mu)$ as $p \to \infty$ with $q \in [1,\infty)$ fixed.

Next, we observe $\mathcal{Q}(\mu,2)$ can be seen as a net whose order is induced by natural inclusion. With the aid of the following proposition, we shall cope with the ambiguity of constant functions in the semi-norm of the Campanato spaces.

Proposition **6** *Let* $1 \le q \le p < \infty$. *Then the limit* $M(f) := \lim_{Q \in \mathcal{Q}(\mu,2)} m_Q(f)$ *exists for every* $f \in \mathcal{C}^p_q(\mu)$. *That is, given* $\varepsilon > 0$, *we can find a doubling cube* $Q \in \mathcal{Q}(\mu,2)$ *such that*

$$|m_R(f) - m_Q(f)| \le \varepsilon$$

for all $R \in \mathcal{Q}(\mu,2)$ *engulfing* Q. *In particular there exists an increasing sequence of concentric doubling cubes* $I_0 \subset I_1 \subset \ldots \subset I_k \subset \ldots$ *such that*

$$\{m_{I_k}(f)\}_{k\in\mathbf{N}_0} \text{ is Cauchy and } \bigcup_k I_k = \mathbf{R}^d. \tag{7}$$

We remark that the condition like (7) appears in [3]. We are mainly interested in the function $f \in \mathcal{C}^p_q(\mu)$ such that $M(f) = 0$.

Before we come to the proof of Proposition 6, we note that

$$\|\,|f| \,:\, \mathcal{C}^p_1(\mu)\| \le C\,\|f \,:\, \mathcal{C}^p_1(\mu)\|. \tag{8}$$

Indeed, we have

$$\begin{aligned}
&\mu\left(\frac{3}{2}Q\right)^{\frac{1}{p}-1} \int_Q ||f(x)| - m_{Q^*}(|f|)|\,d\mu(x) \\
&= \mu\left(\frac{3}{2}Q\right)^{\frac{1}{p}-1} \frac{1}{\mu(Q^*)} \int_Q \left|\int_{Q^*} |f(x)| - |f(y)|\,d\mu(y)\right| d\mu(x) \\
&\le \mu\left(\frac{3}{2}Q\right)^{\frac{1}{p}-1} \frac{1}{\mu(Q^*)} \int_Q int_{Q^*}\, ||f(x)| - |f(y)||\,d\mu(y)\,d\mu(x) \\
&\le \mu\left(\frac{3}{2}Q\right)^{\frac{1}{p}-1} \frac{1}{\mu(Q^*)} \int_Q \int_{Q^*} |f(x) - f(y)|\,d\mu(y)\,d\mu(x) \\
&\le \mu\left(\frac{3}{2}Q\right)^{\frac{1}{p}-1} \int_Q |f(x) - m_{Q^*}(f)|\,d\mu(x) + \mu(Q^*)^{\frac{1}{p}-1} \int_{Q^*} |m_{Q^*}(f) - f(y)|\,d\mu(y) \\
&\le C\,\|f \,:\, \mathcal{C}^p_1(\mu)\|.
\end{aligned}$$

In the same way we can prove

$$\sup_{\substack{Q \subset R \\ Q,R \in \mathcal{Q}(\mu,2)}} \mu(Q)^{\frac{1}{p}} \frac{|m_Q(|f|) - m_R(|f|)|}{K_{Q,R}} \le C\,\|f \,:\, \mathcal{C}^p_1(\mu)\|.$$

As a consequence (8) is justified.

We now turn to the proof of Proposition 6. By the monotonicity of $\mathcal{C}^p_q(\mu)$ with respect to q, we may assume $q = 1$.

Case 1 μ is infinite. Take a sequence of concentric doubling cubes $\{Q_j\}_{j\in\mathbf{N}}$ such that for all $j \in \mathbf{N}$

$$\mu(Q_1) \geq 1,\ \mu(Q_{j+1}) \geq 2\mu(Q_j), \quad \delta(Q_j, Q_{j+1}) \leq C$$

for some $C > 0$ depending only on C_0. Then by the definition of $\mathcal{C}_1^p(\mu)$ it holds that

$$|m_{Q_j}(f) - m_{Q_{j+1}}(f)| \leq C\, 2^{-\frac{j}{p}} \|f \,:\, \mathcal{C}_1^p(\mu)\|,\ j \in \mathbf{N}.$$

Thus we establish at least that the existence of $M(f) := \lim_{j\to\infty} m_{Q_j}(f)$ is proved. Let $Q \in \mathcal{Q}(\mu)(\mu, 2)$ which contains Q_j and does not contain Q_{j+1}. Set $Q' = ({Q_j}^Q)^*$. Then by using Lemma 5 it is easy to see that $\delta(Q, Q') \leq C$ for some absolute constant $C > 0$. Then we have

$$|m_{Q'}(f) - m_Q(f)|,\ |m_{Q'}(f) - m_{Q_j}(f)| \leq C\, 2^{-\frac{j}{p}} \|f \,:\, \mathcal{C}_1^p(\mu)\|,$$

which implies

$$|m_Q(f) - M(f)| \leq C\, 2^{-\frac{j}{p}} \|f \,:\, \mathcal{C}_1^p(\mu)\|.$$

Thus we finally establish $M(f) = \lim_{Q\in\mathcal{Q}(\mu,2)} m_Q(f)$.

Case 2 μ is finite. In this case, we have only to prove

CLAIM **7** *If μ is finite and $\|f \,:\, \mathcal{C}_1^p(\mu)\| < \infty$, then $f \in L^1(\mu)$.*

In proving Claim 7, (8) allows us to assume f is positive.

We take an increasing sequence of concentric doubling cubes $\{Q_j\}_{j\in\mathbf{N}}$ such that $\delta(Q_1, Q_k) \leq C$ for all $k \in \mathbf{N}$. Then we have

$$m_{Q_k}(f) \leq m_{Q_1}(f) + \mu(Q_1)^{-\frac{1}{p}}(1 + C)\|f \,:\, \mathcal{C}_q^p(\mu)\|.$$

Passage to the limit then gives

$$\int_{\mathbf{R}^d} f\, d\mu \leq \mu(\mathbf{R}^d)\left(m_{Q_1}(f) + \mu(Q_1)^{-\frac{1}{p}}(1 + C)\|f \,:\, \mathcal{C}_q^p(\mu)\|\right).$$

This establishes $f \in L^1(\mu)$. ∎

The main theorem in this section is the following.

THEOREM **8** *Let $1 \leq q \leq p < \infty$. Assume that $f \in \mathcal{C}_q^p(\mu)$ satisfies*

$$M(f) = \lim_{Q\in\mathcal{Q}(\mu,2)} m_Q(f) = 0.$$

Then

$$C^{-1}\, \|f \,:\, \mathcal{C}_q^p(\mu)\| \leq \|f \,:\, \mathcal{M}_q^p(\mu)\| \leq C\, \|f \,:\, \mathcal{C}_q^p(\mu)\|$$

for some constant $C > 0$.

The left inequality is obvious. To prove the right inequality we need a lemma.

LEMMA **9** *Under the assumption of Theorem 8, given $R \in \mathcal{Q}(\mu, 2)$, there exists a sequence of increasing doubling cubes $\{R_k\}_{k=1}^K$ such that*

1. *R_k is concentric and $R_1 = R$.*
2. *If μ is finite, then so is K and $R_K = \mathbf{R}^d$.*
3. *For large $K_0 \in \mathbf{N}$, there exists R_{k_0} so that $R_{k_0} \subset I_{K_0} \subset R_{k_0+1}$.*
4. *$\mu(R_k) \geq 2^{k-1}\mu(R)$, $k < K$.*
5. *$\delta(R_k, R_{k+1}) \leq C$, $k < K$.*

Proof. Take $R_1 \in \mathcal{Q}(\mu, 2)$ so that it is contained in I_0. Suppose we have defined R_k. If $\mu(\mathbf{R}^d) \leq 2^k \mu(R)$, then we set $R_{k+1} = \mathbf{R}^d$ and we stop. Suppose otherwise. We define R_{k+1} as the smallest doubling cube of the form $2^l R_k$ with $l \geq 3$ whose μ-measure exceeds $2^k \mu(R)$. By virtue of Lemma 5 (3) it is easy to verify that $\{R_k\}_{k=1}^K$ obtained in this way satisfies the property of the lemma. ∎

Let us return to the proof of Theorem 8. Let $R \in \mathcal{Q}(\mu)$. We shall estimate

$$\mu(2R)^{\frac{1}{p}-\frac{1}{q}} \left(\int_R |f(x)|^q \, d\mu(x) \right)^{\frac{1}{q}}.$$

The triangle inequality enables us to majorize the above integral by

$$\mu\left(\frac{3}{2}R\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_R |f(x) - m_{R^*}(f)|^q \, d\mu \right)^{\frac{1}{q}} + \mu(R)^{\frac{1}{p}} \, |m_{R^*}(f)|.$$

Consequently we can reduce the matters to the estimate of $\mu(R^*)^{\frac{1}{p}} \, |m_{R^*}(f)|$.

Now we invoke Lemma 9 for K_0 taken so that $\mu(R)^{\frac{1}{p}} \, |m_{I_{K_0}}(f)| \leq \|f \, : \, \mathcal{C}^p_q(\mu)\|$. Using the sequence $\{R_k\}_{k=1}^K$, we obtain

$$\begin{aligned} &\mu(R^*)^{\frac{1}{p}} \, |m_{R_k}(f) - m_{R_{k+1}}(f)| \\ &\leq \; C \, 2^{-\frac{k}{p}} \, \mu(R_k)^{\frac{1}{p}} \, \frac{|m_{R_k}(f) - m_{R_{k+1}}(f)|}{1 + \delta(R_k, R_{k+1})} \; \leq \; C \, 2^{-\frac{k}{p}} \, \|f_j \, : \, \mathcal{C}^p_q(l^r, \mu)\|. \end{aligned} \tag{9}$$

We also have $\mu(R^*)^{\frac{1}{p}} \, |m_{R_{k_0}}(f) - m_{I_{K_0}}(f)| \leq C \, 2^{-\frac{k_0}{p}} \, \|f \, : \, \mathcal{C}^p_q(\mu)\|$, since by the properties *3* and *4* of Lemma 9 we see that $\delta(R_{k_0}, R_{k_0+1}), \delta(I_{K_0}, R_{k_0+1})$ are majorized by some constants dependent only on C_0.

The triangle inequality gives us

$$\begin{aligned} &\mu(R^*)^{\frac{1}{p}} \, |m_{R^*}(f)| \\ &\leq \; \mu(R)^{\frac{1}{p}} \sum_{k=1}^{k_0-1} |m_{R_k}(f) - m_{R_{k+1}}(f)| + \mu(R^*)^{\frac{1}{p}} \left(|m_{R_{k_0}}(f) - m_{I_{K_0}}(f)| + |m_{I_{K_0}}(f)| \right) \\ &\leq \; C \left(\sum_{k=1}^{\infty} 2^{-\frac{k}{p}} \right) \|f \, : \, \mathcal{C}^p_q(\mu)\| + \mu(R^*)^{\frac{1}{p}} \, |m_{I_{K_1}}(f)| \; \leq \; C \, \|f \, : \, \mathcal{C}^p_q(\mu)\|. \end{aligned} \tag{10}$$

The proof of Theorem 8 is therefore complete. ∎

Vector-valued extension Finally we consider the vector-valued extensions of Theorem 8. Let $\|a_j : l^r\|$ denote the l^r-norm of $a = \{a_j\}_{j \in \mathbf{N}}$. If possible confusion can occur, then we write $\|\{a_j\}_{j \in \mathbf{N}} : l^r\|$. For $f \in L^1_{loc}(\mu)$, we define the sharp maximal operator due to Tolsa by

$$M^\sharp f(x) := \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(y) - m_{Q^*}(f)| \, d\mu(y) + \sup_{\substack{x \in Q \subset R \\ Q,R \in \mathcal{Q}(\mu,2)}} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}}.$$

Lemma 4 can be extended to the following vector-valued version.

Lemma **10** [8, Corollary 13] *Let $f_j \in$RBMO for $j = 1, 2, \ldots$. For any cube $Q_0 \in \mathcal{Q}(\mu)$ and $q, r \in (1,\infty)$, there exists a constant C independent of f_j such that*

$$\left(\frac{1}{\mu\left(\frac{3}{2}Q_0\right)} \int_{Q_0} \|f_j(x) - m_{(Q_0)^*}(f_j) : l^r\|^q \, d\mu(x) \right)^{\frac{1}{q}} \le C \sup_{x \in \mathbf{R}^d} \left\| M^\sharp f_j(x) : l^r \right\|. \quad (11)$$

We now define the vector-valued Campanato spaces. Let $1 \le q \le p \le \infty$ and $r \in (1,\infty)$. We say that $\{f_j\}_{j \in \mathbf{N}}$ belongs to the vector-valued Campanato spaces $\mathcal{C}^p_q(l^r, \mu)$ if each f_j is μ-measurable and

$$\begin{aligned} \|f_j : \mathcal{C}^p_q(l^r,\mu)\| \quad &:= \quad \sup_{Q \in \mathcal{Q}(\mu)} \mu(2Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q \|f_j(x) - m_{Q^*}(f_j) : l^r\|^q \, d\mu(x) \right)^{\frac{1}{q}} \\ &+ \quad \sup_{\substack{Q \subset R \\ Q,R \in \mathcal{Q}(\mu,2)}} \mu(Q)^{\frac{1}{p}} \frac{\|m_Q(f_j) - m_R(f_j) : l^r\|}{K_{Q,R}} < \infty. \end{aligned} \quad (12)$$

If we consider the vector-valued spaces, the norm equivalence of the Campanato type still holds.

Theorem **11** *Let $1 \le q \le p < \infty$ and let $\{f_j\}_{j \in \mathbf{N}}$ be a sequence in $\mathcal{C}^p_q(\mu)$. Assume that there exists an increasing sequence of concentric doubling cubes $I_0 \subset I_1 \subset \ldots \subset I_k \subset \ldots$ such that*

$$\lim_{k \to \infty} m_{I_k}(f_j) = 0 \text{ for all } j \text{ and } \bigcup_k I_k = \mathbf{R}^d. \quad (13)$$

Then there exists a constant $C > 0$ independent of $\{f_j\}_{j \in \mathbf{N}}$ such that

$$C^{-1} \|f_j : \mathcal{C}^p_q(l^r,\mu)\| \le \|f_j : \mathcal{M}^p_q(l^r,\mu)\| \le C \|f_j : \mathcal{C}^p_q(l^r,\mu)\|.$$

Using Lemma 10, we can say more about $\mathcal{C}^\infty_q(l^r,\mu)$, which gives us a partial clue to the definition of the vector-valued RBMO spaces. Speaking precisely, we obtain the following proposition.

Proposition 12 *Let $\{f_j\}_{j\in\mathbf{N}}$ be a sequence of $L^1_{loc}(\mu)$ functions. Then*

$$\sup_{\substack{Q\subset R\\ Q,R\in\mathcal{Q}(\mu,2)}} \frac{\|m_Q(f_j)-m_R(f_j)\,:\,l^r\|}{K_{Q,R}} \le c\sup_{x\in\mathbf{R}^d}\|M^\sharp f_j(x)\,:\,l^r\|. \tag{14}$$

In particular, we have

$$\|f_j\,:\,\mathcal{C}^\infty_q(l^r,\mu)\| \le c\sup_{x\in\mathbf{R}^d}\|M^\sharp f_j(x)\,:\,l^r\|. \tag{15}$$

Proof. Fix $Q\subset R$ such that $Q\in\mathcal{Q}(\mu)$. Then $\dfrac{|m_Q(f_j)-m_R(f_j)|}{K_{Q,R}} \le c\,M^\sharp f_j(x)$ for all $x\in Q$. By taking the l^r-norm of both sides we obtain

$$\frac{\|m_Q(f_j)-m_R(f_j)\,:\,l^r\|}{K_{Q,R}} \le c\sup_{x\in Q}\|M^\sharp f_j(x)\,:\,l^r\| \le c\sup_{x\in\mathbf{R}^d}\|M^\sharp f_j(x)\,:\,l^r\|.$$

Now since Q and R are taken arbitrarily, (14) is proved. (15) can be obtained with the help of (11) and (14). ∎

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