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Hecke-Siegel's pull back formula for the Epstein zeta function with a harmonic polynomial

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abstruct. In this paper, we discuss the generalization of Hecke's integration formula for the Epstein zeta functions. We treat the Epstein zeta function as the Eisenstein series come from the degenerate principal series. For a real quadratic field, Siegel considered the Hecke's formula as the constant term of a certain Fourier expansion of the Epstein zeta function and obtained the other Fourier coefficients as the Dedekind zeta functions with Grössencharacters of the real quadratic field. We apply this to our Eisenstein series. Then We obtain the Dedekind zeta functions with Grössencharacters for arbitrary number fields.

Introduction.

In the classical paper [3], E. Hecke wrote the zeta function of general number field F as the pull-back integration of the Epstein zeta function of degree $[F : \mathbb{Q}]$. In [9], Siegel introduced more general Epstein zeta functions with harmonic polynomials. The purpose of this paper is to discuss the pull-back integrations of them.

Firstly we give the overview of the Hecke's integration formula (see also [5, Section 1], [12, Vol I, Section 1.4]). For a number field F, we have the natural embedding of the algebraic torus $T_F = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ to $GL(n)$ where $\mathrm{Res}_{F/\mathbb{Q}}$ is the restriction of scalars of Weil.

In the down to earth way, this is defined as follows. Let r_1 and r_2 be the numbers of real and complex places of F. Then these real and complex conjugations give a natural mapping

$$\sigma\colon F \ni \alpha \mapsto (\alpha^{(1)}, \cdots, \alpha^{(r_1)}, \alpha^{(r_1+1)}, \cdots, \alpha^{(r_1+r_2)}) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}.$$

This σ can be extended to the isomorphism $F_{\mathbb{R}} = F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Next,

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we embed $(a_1, \cdots, a_{r_1}, b_1, \cdots, b_{r_2}) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ into $M(n, \mathbb{R})$ as

$$\begin{pmatrix} a_1 &&&&&&&& \\ & \ddots &&&&&&& \\ && a_{r_1} &&&&&& \\ &&& \alpha_1 & -\beta_1 &&&& \\ &&& \beta_1 & \alpha_1 &&&& \\ &&&&& \ddots &&& \\ &&&&&& \alpha_{r_2} & -\beta_{r_2} \\ &&&&&& \beta_{r_2} & \alpha_{r_2} \end{pmatrix}$$

where $b_i = \alpha_i + \sqrt{-1}\beta_i$ for $i = 1, \cdots, r_2$. Then we have the group homomorphism

$$i \colon T_F(\mathbb{R}) = (F \otimes_{\mathbb{Q}} \mathbb{R})^{\times} \hookrightarrow GL(n, \mathbb{R}).$$

On the other hand, let us take an integral ideal $\mathfrak{b}$ of F and fix an $\mathbb{Z}$-basis of $\mathfrak{b}$:

$$\mathfrak{b} = \mathbb{Z}\,\omega_1 + \cdots + \mathbb{Z}\,\omega_n.$$

This is also a basis of F over $\mathbb{Q}$, hence we have another identification $\omega \colon F_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}^n$ by this basis. We can find a transformation matrix $W_{\mathfrak{b}} \in GL(n, \mathbb{R})$ from the coordinate of $F_{\mathbb{R}}$ given by ω to the one given by σ as $\mathbb{R}$-vector space. Then ω gives another embedding

$$j \colon T_F(\mathbb{R}) \ni x \longmapsto W_{\mathfrak{b}} \cdot i(x) \cdot W_{\mathfrak{b}}^{-1} \in GL(n, \mathbb{R}).$$

By using these embeddings, we pull the Epstein zeta function back to the number field F in the following way. The Epstein zeta function is defined for a positive definite symmetric matrix Y by

$$Z(s, Y) = \sum_{\mathbf{m} \in \mathbb{Z}^n \setminus \{0\}} (\mathbf{m} Y {}^t\mathbf{m})^{-s} \quad (s \in \mathbb{C}).$$

We denote the space of such matrices which have determinant 1 by $\mathcal{P}_1$. We also restrict $T_F(\mathbb{R})$ to the norm 1 part $T_F^{(1)}(\mathbb{R})$ such that

$$T_F^{(1)}(\mathbb{R}) = \{x \in T_F(\mathbb{R}) \mid N(x) = 1\},$$

where the norm is defined by $N(x) = \det i(x)$ for $x \in T_F(\mathbb{R})$. Take the point $Y_{\mathfrak{b}} = W_{\mathfrak{b}}'{}^t W_{\mathfrak{b}}'$ in $\mathcal{P}_1$ where $W_{\mathfrak{b}}' = (\det W_{\mathfrak{b}})^{-\frac{1}{n}} W_{\mathfrak{b}}$. We restrict $Z(s, Y)$ to the $T_F^{(1)}(\mathbb{R})$-orbit $\mathcal{Q}_{F,\mathfrak{b}}$ of $Y_{\mathfrak{b}}$ where $x \in T_F^{(1)}(\mathbb{R})$ acts on $Y \in \mathcal{P}_1$ by $j(x) Y {}^t j(x)$. Then the function $Z(s, Y)$ $(Y \in \mathcal{Q}_{F,\mathfrak{b}})$ is periodic with respect to norm 1 part of the integer ring $\mathcal{O}_F$, say $\mathcal{O}_F^{(1)}$, and define a function on the compact double coset:

$$\mathcal{O}_F^{(1)} \backslash T_F^{(1)}(\mathbb{R}) / (T_F^{(1)}(\mathbb{R}) \cap SO(Y_{\mathfrak{b}}))$$

where $SO(Y_{\mathfrak{b}})$ is the stabilizer of $Y_{\mathfrak{b}}$ in $SL(n, \mathbb{R})$ which is isomorphic to $SO(n)$.

Now we can consider the Fourier expansion of the pull-back $Z(s, *)|_{\mathcal{Q}_{F,\mathfrak{b}}}$:

$$Z(s, Y) = \sum_{\psi \in \widehat{\mathcal{O}_F^{(1)} \backslash T_F^{(1)}(\mathbb{R})}} a_{\psi}(s) \psi(Y).$$

Choose a fundamental domain $D(Y_{\mathfrak{b}})$ in $T_F^{(1)}(\mathbb{R})/(T_F^{(1)}(\mathbb{R})\cap SO(Y_{\mathfrak{b}}))$ with respect to $\mathcal{O}_F^{(1)}$. On these settings, Hecke's result corresponds to the constant term of this Fourier series

$$a_0=\int_{D(Y_{\mathfrak{b}})} Z(s,Y)\,dv(Y).$$

Theorem 0.1 (Hecke's integration formula)**.** *We have the following equation,*

$$\int_{D(Y_{\mathfrak{b}})} Z(\frac{ns}{2},Y)\,dv(Y)=\omega_F\frac{2^{-r_2 s}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}}{2^{r_1-1}nR\Gamma(ns/2)}\zeta_F(s,A),$$

where R is the regulator of F, ω_F the number of roots of unity in F and A is the ideal class of $\mathfrak{b}^{-1}$.

Meanwhile Siegel obtained all the Fourier coefficients a_ψ for a real quadratic field F in [9]. In his result, zeta functions with Grössencharacters appear . Combining with the functional equation of the Epstein zeta function, this gives functional equations and analytic continuation for these zeta functions by the properties of Epstein zeta functions.

Our goal of this paper is to give all Fourier coefficients explicitly in more general settings, i.e., for the generalized Epstein zeta function with a harmonic polynomial in the sense of Siegel [9]. This zeta function can be interpreted as a maximal parabolic Eisenstein series of $G=SL(n,\mathbb{R})$ with a non-trivial K-type where $K=SO(n)$ is a maximal compact subgroup of G. This is an automorphic form on $SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})$. For $n=3$, Oda and Ishii discussed the Fourier expansion of this Eisenstein series in [4]. For the general case with a trivial K-type, the Fourier expansion was determined by Terras [12]. We will show the analog of Hecke and Siegel's result for this Eisenstein series. Then our result is the following. Let $\chi_{\mathbf{m},\gamma}$ be a character of the group of all fractional ideals of F, write I,

$$\chi_{\mathbf{m},\gamma}\colon I\to\mathbb{C}^1,\quad \mathbf{m}\in\mathbb{Z}^r, \gamma=(\delta_1,\cdots,\delta_{r_1},l_1,\cdots,l_{r_2})\in\{0,1\}^{r_1}\times\mathbb{Z}^{r_2}$$

which satisfies

$$\chi_{\mathbf{m},\gamma}((\alpha))=\prod_{i=1}^{r}(|\alpha^{(i)}|^{-1}|N(\alpha)|^{\frac{1}{n}})^{-2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r}_i}\prod_{j=1}^{r_1}(\alpha^{(j)})^{\delta_j}\prod_{k=1}^{r_2}(\alpha^{(k)})^{l_k}$$

for a principal ideal (α) where $r=r_1+r_2-1$ is the rank of $\mathcal{O}_F^\times$ and $\mathbf{r}_i$ comes from the regulator of F (the precise definition is in Section 3). We take a harmonic polynomial f_γ such that

$$f_\gamma(x)=\prod_{i=1}^{r_1}x_i^{\delta_i}\prod_{j=1}^{r_2}c_{l_j}(x_{r_1+2j-1},x_{r_1+2j})$$

where

$$c_l(x,y)=\begin{cases}(x-\sqrt{-1}y)^l & \text{if } l\geq 0\\ (x+\sqrt{-1}y)^{|l|} & \text{if } l\leq 0\end{cases}.$$

For this f_γ and $\nu\in\mathbb{C}$, we denote the maximal parabolic Eisenstein series by $E(\nu,g;f_\gamma),\ g\in G$.

Theorem 0.2 (=Theorem3.5)**.** *Set* $2s = \frac{\nu}{n-1} + \frac{n}{2}$*. Then we have the following equation.*

$$\pi^{-s}\Gamma\left(s+\frac{d}{2}\right)\int_{[0,1]^r}\widehat{E}(\nu, W_{\mathfrak{b}}i(u(\mathbf{t})); f_\gamma)\exp\left(-2\pi\sqrt{-1}\mathbf{m}^t\mathbf{t}\right)d\mathbf{t}$$
$$= \frac{1}{2^r n}\omega_F|d_F|^{\frac{s}{n}}R_F^{-1}\chi_{\mathbf{m},\gamma}(\mathfrak{b})\zeta_\infty(\frac{2s}{n},\chi_{\mathbf{m},\gamma})\zeta_F(\frac{2s}{n},A,\chi_{\mathbf{m},\gamma}).$$

Here ω_F*,* d_F *and* R_F *are the number of roots of unity, the discriminant and the regulator of* F *respectively. Also* $\zeta_F(s, A, \chi)$ *is the partial zeta function of the ideal class* A *of* $\mathfrak{b}^{-1}$ *with a character* χ*, i.e.,* $\zeta_F(s, A, \chi) = \sum_{\substack{\mathfrak{a}\neq(0)\in A \\ \mathfrak{a}:integral}}\chi(\mathfrak{a})N_F(\mathfrak{a})^{-s}$ *where* N_F *is the ideal norm of* F*.*

For the other notations used in the above theorem, see Section 3.

Now we review the contents of this paper. In Section 1, we give the definition of the Epstein zeta function with a spherical function and some properties of this function. In Section 2, we define the maximal parabolic Eisenstein series of G. This series comes from a degenerate principal series of G with respect to the maximal parabolic subgroup whose K-types are described as spherical harmonics on S^{n-1}. We show this Eisenstein series is identified with our Epstein zeta function. Also the algebraic sum of the spaces of these Eisenstein series has a natural $(\mathfrak{g}, K)$-module structure induced from the degenerate principal series representation (Proposition 2.4). In Section 3, we show that combining with the embedding $F^\times \hookrightarrow GL(n, \mathbb{R})$, the action of the maximal compact subgroup of $T_F(\mathbb{R})$ on the space of the harmonic polynomials H_d of degree d gives Grössencharacters of F which have only rotation-part. Finally we give the Fourier coefficients as zeta functions with Grössencharacters (Theorem 3.5). As Siegel's result, our result gives another proof of the functional equation for zeta function with a Grössencharacter of a number field, which is different from the well-known proof of E. Hecke, that is normally refereed through the doctoral thesis of J. Tate.

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1 Siegel's Epstein zeta functions with harmonic polynomials.

In this section, we review Siegel's definition of the generalized Epstein zeta function and some its properties.

Let $Q(x_1, x_2, \cdots, x_n) = Q(x)$ be a positive definite quadratic form over $\mathbb{R}^n$ and Q its associated real symmetric matrix.

Definition 1.1. *We call* $h(x)$ *a harmonic polynomial of degree* d *with respect to* Q*, if* $h(x)$ *be a homogeneous polynomial of degree* d *and it satisfies*

$$\Delta_{Q^{-1}}h(x) = 0.$$

Here $\Delta_{Q^{-1}}$ is the Laplacian defined by

$$\left(\frac{\partial}{\partial x_1} \quad \cdots \quad \frac{\partial}{\partial x_n}\right) \cdot Q^{-1} \cdot \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}.$$

For a positive definite quadratic form $Q(x)$ and a harmonic polynomial $h(x)$ of degree d, Siegel introduced the generalized Epstein zeta function with a harmonic polynomial

$$Z(s,Q,h) = \sum_{x \in \mathbb{Z}^n - \{0\}} \frac{h(x)}{Q(x)^{s+\frac{d}{2}}}.$$

This zeta function converge absolutely for $\operatorname{Re} s > \frac{n}{2}$.

Definition 1.2. *Let $Q(x)$ be a positive definite quadratic form over $\mathbb{R}^n$. Let $H_d(Q)$ is the space of harmonic polynomials of degree d with respect to Q. Then we define the space of Epstein zeta functions with harmonic polynomials of degree d as $\mathcal{Z}^{(d)} = \{Z(s,Q,h) \,|\, h \in H_d(Q)\}$.*

Theorem 1.3 (Siegel [9])**.** *The function $Z(s,Q,h)$ has an analytic continuation to the whole s-plane, which is an entire function of s if $d > 0$. If $d = 0$, $Z(s,Q,h)$ is holomorphic in the s-plane except for a simple pole at $s = \frac{n}{2}$. In both cases, $Z(s,Q,h)$ satisfies the functional equation*

$$\begin{aligned} &\pi^{-s}\Gamma(s+\frac{d}{2})Z(s,Q,h) \\ &\qquad = (\sqrt{-1})^{-d} \det Q^{-\frac{1}{2}} \pi^{-(\frac{n}{2}-s)} \Gamma(\frac{n}{2}-s+\frac{g}{2}) Z(\frac{n}{2}-s, Q^{-1}, h^*) \end{aligned}$$

where $h^*(x) = h(Q^{-1}x)$ for $h(x) \in H_d(Q)$.

2 Degenerate principal series representations of $SL(n,\mathbb{R})$ with respect to $P_{n-1,1}$.

The generalized Epstein zeta function defined in the previous section can be seen as an automorphic form on $SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})$ like the well-known ordinary Epstein zeta function. This is the Eisenstein series induced from a degenerate principal series of G. The definition of the Eisenstein series which we will explain here can be extended for more general semisimple or reductive Lie groups (see [2], [6] and for the adelic setting [7]).

Let $G = SL(n,\mathbb{R})$, $K = SO(n)$ the maximal compact subgroup of G and $P_{n-1,1}$ the maximal parabolic subgroup of the form

$$\left\{ \begin{array}{cc} \left(\begin{array}{ccc|c} & & & * \\ & * & & \vdots \\ & & & * \\ \hline 0 & \cdots & 0 & * \end{array}\right) & \begin{array}{l} \uparrow \\ n-1 \\ \downarrow \\ \updownarrow 1 \end{array} \end{array} \in G \right\}.$$

To define the degenerate principal series representation with respect to $P_{n-1,1}$, we firstly specify a Langlands decomposition $P_{n-1,1} = MAN$ of $P_{n-1,1}$ by

$$\begin{aligned}
M &= \left\{ \left(\begin{array}{c|c} & 0 \\ h & \vdots \\ & 0 \\ \hline 0 \quad \cdots \quad 0 & \det h^{-1} \end{array} \right) \in G \;\middle|\; h \in GL(n-1, \mathbb{R}),\ \det h \in \{\pm 1\} \right\} \\
&\cong SL(n-1, \mathbb{R}) \times \{\pm 1\}, \\
A &= \{\mathrm{diag}(r, \cdots, r, r^{-(n-1)}) \in G \mid r \in \mathbb{R}_{>0}\}, \\
N &= \left\{ n(x_1, \cdots, x_{n-1}) = \left(\begin{array}{c|c} & x_1 \\ I_{n-1} & \vdots \\ & x_{n-1} \\ \hline 0 \quad \cdots \quad 0 & 1 \end{array} \right) \in G \right\}.
\end{aligned}$$

Let $\sigma \in \widehat{M}$ be a unitary character of M, and $\nu \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C}) = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$ a linear form on $\mathfrak{a} = \mathrm{Lie}(A)$ which is identified with a complex number by evaluation at the element $H = \mathrm{diag}(1, \cdots, 1, -(n-1)) \in \mathfrak{a}$, i.e., $\nu \mapsto \nu(H) \in \mathbb{C}$. Set $\rho = \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|_{\mathfrak{n}})$. We have $\mathrm{ad}(H)n(x_1, \cdots, x_{n-1}) = n(nx_1 \cdots, nx_{n-1})$, hence $\rho = \frac{1}{2}n(n-1)$. Thus for $a_r = \mathrm{diag}(r, \cdots, r, r^{-(n-1)}) \in A$, we have $e^{(\nu+\rho)(\log a_r)} = a_r^{\nu+\rho} = r^{\nu + \frac{1}{2}n(n-1)}$.

Definition 2.1. *Put*

$$\pi(\sigma, \nu) = \mathrm{Ind}_{P_{n-1,1}}^{G}(\sigma \otimes e^{\nu+\rho} \otimes 1_N).$$

Then the representation space of $\pi(\sigma, \nu)$ is given by the completion of a dense subspace

$$\begin{aligned}
H_{\sigma,\nu} = \{f \colon G \to \mathbb{C},\ \textit{continuous} \mid f(manx) = \sigma(m)a^{\nu+\rho}f(x) \\
\textit{for } (x, m, a, n) \in G \times M \times A \times N\},
\end{aligned}$$

with respect to the norm

$$\|f\|^2 = \int_K |f(k)|^2 \, dk.$$

Here $g \in G$ acts on this space by the right regular representation, i.e., $\pi(\sigma, \nu)(g)f(x) = f(xg)$ for $f \in H_{\sigma,\nu}$. We call this the degenerate principal series representation with respect to $P_{n-1,1}$.

The representation space of $\pi(\sigma, \nu)$ is isomorphic to

$$L^2_\sigma(K) = \{f \in L^2(K) \mid f(mk) = \sigma(m)f(k) \text{ for all } m \in M \cap K,\ k \in K\}$$

as a K-module. For $m = \begin{pmatrix} h & \mathbf{0} \\ \mathbf{0} & \det h^{-1} \end{pmatrix} \in M$, we define the character $\det_M$ of M by

$$\det{}_M(m) = \det h.$$

We note that $\widehat{M} = \{1_M, \det_M\}$.

We show that $L_\sigma(K)$ is written as the space of the spherical harmonics on the unit sphere S^{n-1}. We fix the unit vector $e_n = (0, \cdots, 0, 1)$. Then the map

$$g \in SO(n) \longmapsto e_n \cdot g \in S^{n-1}$$

is a surjective map from the special orthogonal group $SO(n)$ of degree n to the $(n-1)$-dimensional unit sphere S^{n-1}. The stabilizer of e_n in $SO(n)$ is given by

$$\left\{ \begin{pmatrix} h & \\ & 1 \end{pmatrix} \;\middle|\; h \in SO(n-1) \right\}$$

and S^{n-1} is naturally identified with the quotient space

$$SO(n-1)\backslash SO(n)$$

as an $SO(n)$-set. If we notice $M \cap K \cong SO(n-1) \times \{\pm 1\}$, we have the isomorphism $L^2_\sigma(K)$ to

$$\begin{aligned} L^2_{\text{even}}(S^{n-1}) &= \{f \in L^2(S^{n-1}) \mid f(-x) = f(x),\ x \in S^{n-1}\} \text{ if } \sigma = 1_M, \\ L^2_{\text{odd}}(S^{n-1}) &= \{f \in L^2(S^{n-1}) \mid f(-x) = -f(x),\ x \in S^{n-1}\} \text{ if } \sigma = \det{}_M \end{aligned}$$

as a K-module. Here the action of $SO(n)$ on $L^2(S^{n-1})$ is given by the right quasi-regular action,

$$g \in SO(n) \longmapsto \left\{ f(x) \mapsto f(xg)\ (x \in S^{n-1}) \right\}.$$

Let H_d be the space of harmonic polynomials of degree d on $\mathbb{R}^n$, i.e., the homogeneous functions of degree d which are annihilated by the Laplace operator $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ of $\mathbb{R}^n$. By the inclusion map $\iota\colon S^{n-1} \hookrightarrow \mathbb{R}^n$, we have the restriction map $\iota^*\colon C^\infty(\mathbb{R}^n) \to C^\infty(S^{n-1})$. Then harmonic polynomials can be treated as functions on the unit sphere S^{n-1}. We call these functions spherical harmonics and denote the space of spherical harmonics by $\mathcal{H}_d = \iota^* H_d$.

It is well-known that both H_d and $\mathcal{H}_d$ are irreducible $SO(n)$-modules of highest weight $(d, 0, \cdots, 0)$, and $\mathcal{H}_d$ is an $SO(n)$-invariant subspace of $L^2(S^{n-1})$. By the theory of spherical functions (cf.[11]), we have the unique spectral decomposition as Hilbert space direct sum

$$L^2(S^{n-1}) = \widehat{\bigoplus_{d=0}^{\infty}} \mathcal{H}_d$$

of a unitary $SO(n)$-module. Thus we also have the spectral decompositions

$$\begin{aligned} L^2_{\text{even}}(S^{n-1}) &= \widehat{\bigoplus_{d:\text{even}}^{\infty}} \mathcal{H}_d, \\ L^2_{\text{odd}}(S^{n-1}) &= \widehat{\bigoplus_{d:\text{odd}}^{\infty}} \mathcal{H}_d. \end{aligned}$$

Now we can define the Eisenstein series of $\pi(\sigma, \nu)$ with a spherical harmonic $h \in \mathcal{H}_d$,

$$E(\nu, g; h) = \sum_{\gamma \in \Gamma_\infty \backslash SL(n,\mathbb{Z})} \sigma(m(\gamma g)) a(\gamma g)^{\nu+\rho} h(e_n k(\gamma g)), \text{ for } g \in G.$$

Here $\Gamma_\infty = P_{n-1,1} \cap SL(n,\mathbb{Z})$, and each $m(g)$, $a(g)$ and $k(g)$ is M-part, A-part and K-part of the decomposition $g \in G = MANK$. Note that the degree of the spherical harmonic d is even (resp. odd) if σ is 1_M (resp. $\det_M$). For the latter use, we normalize this Eisenstein series as follows

$$\widehat{E}(\nu, g; h) = \zeta\left(\frac{\nu}{n-1} + \frac{n}{2}\right) E(\nu, g; h).$$

Here $\zeta(s)$ is the Riemann zeta function.

Proposition 2.2. *For $g \in G$, the matrix $Q = g^t g$ is a positive definite symmetric matrix. Let $h \in \mathcal{H}_d$ be a spherical harmonic. Then, for the harmonic polynomial $\bar{h}(x) = h(xg)$ of degree d with respect to Q we have the following equation*

$$\widehat{E}(\nu, g; h) = Z\left(\frac{\nu}{2(n-1)} + \frac{n}{4}, Q, \bar{h}\right).$$

Proof. Even case. Let $\sigma = 1_M$. Then we have

$$\begin{aligned} E(\nu, g; h) &= \sum_{\gamma \in \Gamma_\infty \backslash SL(n,\mathbb{Z})} a(\gamma g)^{\nu+\rho} h(e_n k(\gamma g)) \\ &= \sum_{\gamma \in \Gamma_\infty \backslash SL(n,\mathbb{Z})} |e_n \cdot \gamma g|^{-\frac{1}{n-1}(\nu + \frac{n(n-1)}{2})} h(e_n k(\gamma g)) \text{ for } g \in G. \end{aligned}$$

We consider the decomposition of $g = mank \in G = MANK$,

$$\begin{aligned} m &= \begin{pmatrix} h & \\ & \det h^{-1} \end{pmatrix} \in M, \\ n &= \begin{pmatrix} I_{n-1} & {}^t\mathbf{n} \\ & 1 \end{pmatrix} \in N \text{ for } \mathbf{n} \in \mathbb{R}^{n-1}, \\ a &= \mathrm{diag}(r, \cdots, r, r^{-(n-1)}) \in A, \\ k &= \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{pmatrix} \in K. \end{aligned}$$

Then we have

$$g = \begin{pmatrix} r \cdot h & {}^t\mathbf{n}' \\ & r^{-(n-1)} \det h^{-1} \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{pmatrix}$$

for $\mathbf{n}' \in \mathbb{R}^{n-1}$, where $r \cdot h$ means the scalar multiple of $h \in GL(n-1, \mathbb{R})$ by $r \in \mathbb{R}_{>0}$. This implies

$$\frac{e_n \cdot g}{r^{-(n-1)} \det h^{-1}} = (k_{n1}, k_{n2}, \cdots, k_{nn}) = e_n \cdot k.$$

Recalling that $r^{-(n-1)} = |e_n \cdot g|$, we have

$$e_n \cdot k = \frac{e_n \cdot g}{\det h^{-1} |e_n \cdot g|}.$$

Noting that degree of h is even, this equation gives

$$\begin{aligned}
\widehat{E}(\nu,g;h) &= \zeta\left(\frac{\nu}{n-1}+\frac{n}{2}\right)\sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})}|e_n\cdot\gamma g|^{-\left(\frac{\nu}{(n-1)}+\frac{n}{2}+d\right)}\cdot h(e_n\cdot\gamma g)\\
&= \zeta\left(\frac{\nu}{n-1}+\frac{n}{2}\right)\sum_{\substack{\mathbf{x}\in\mathbb{Z}^n-\{\mathbf{0}\}\\ \mathbf{x}:\text{coprime}}}|\mathbf{x}g|^{-\left(\frac{\nu}{(n-1)}+\frac{n}{2}+d\right)}\cdot h(\mathbf{x}g)\\
&= \sum_{\mathbf{x}\in\mathbb{Z}^n-\{\mathbf{0}\}}\frac{\bar{h}(\mathbf{x})}{Q(\mathbf{x})^{\frac{1}{2}\left(\frac{\nu}{(n-1)}+\frac{n}{2}+d\right)}}.
\end{aligned}$$

Here we use the same symbol for the harmonic polynomial of $\mathbb{R}^n$ as the associated spherical harmonic $h\in\mathcal{H}_d$. We can easily check that $\bar{h}$ is the harmonic polynomial with respect to Q defined in Section 1. Thus we have the identity

$$\widehat{E}(\nu,g;h) = Z\left(\frac{\nu}{2(n-1)}+\frac{n}{4},Q,\bar{h}\right).$$

Odd case. In this case the Eisenstein series can be written as follows,

$$E(\nu,g;h) = \sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})}\det{}_M(m(\gamma g))\cdot a(\gamma g)^{\nu+\rho}\cdot h(k(\gamma g)^{-1}e_n).$$

Since the degree of h is odd, we have

$$\begin{aligned}
&\widehat{E}(\nu,g;h)\\
&= \zeta\left(\frac{\nu}{n-1}+\frac{n}{2}\right)\sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})}\det{}_M(m(\gamma g))|e_n\cdot\gamma g|^{-\left(\frac{\nu}{(n-1)}+\frac{n}{2}+d\right)}\cdot h(e_n\cdot\gamma g)\\
&= \zeta\left(\frac{\nu}{n-1}+\frac{n}{2}\right)\sum_{\substack{\mathbf{x}\in\mathbb{Z}^n-\{\mathbf{0}\}\\ \mathbf{x}:\text{coprime}}}|\mathbf{x}g|^{-\left(\frac{\nu}{(n-1)}+\frac{n}{2}+d\right)}\cdot h(\mathbf{x}g)\\
&= \sum_{\mathbf{x}\in\mathbb{Z}^n-\{\mathbf{0}\}}\frac{\bar{h}(\mathbf{x})}{Q(\mathbf{x})^{\frac{1}{2}\left(\frac{\nu}{(n-1)}+\frac{n}{2}+d\right)}}.
\end{aligned}$$

Thus we have the identity. □

As Definition 1.2, we consider the space of the Eisenstein series with spherical harmonics $\mathcal{E}^{(d)} = \{\widehat{E}(\nu,g;h)\mid h\in\mathcal{H}_d\}$ of degree d.

Definition 2.3. *We define the spaces of Eisenstein series with spherical harmonics of even and odd degree respectively by*

$$\begin{aligned}
\mathcal{E}^{(even)} &= \bigoplus_{\substack{d\in\mathbb{Z}_{\geq 0}\\ d:even}}\mathcal{E}^{(d)}\subset C^\infty(SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})),\\
\mathcal{E}^{(odd)} &= \bigoplus_{\substack{d\in\mathbb{Z}_{\geq 0}\\ d:odd}}\mathcal{E}^{(d)}\subset C^\infty(SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})).
\end{aligned}$$

Proposition 2.4. *Assume* $\operatorname{Re}\nu > \frac{n(n-1)}{2}$. *Denote* $\mathfrak{g} = Lie(G)$ *the Lie algebra of* G *.*

1. *The space of Eisenstein series $\mathcal{E}^{(even)}$ is a $(\mathfrak{g}, K)$-module which is canonically isomorphic to the $(\mathfrak{g}, K)$-module of the degenerate principal series $\pi(\nu, 1_M)$.*
2. *The space of Eisenstein series $\mathcal{E}^{(odd)}$ is a $(\mathfrak{g}, K)$-module which is canonically isomorphic to the $(\mathfrak{g}, K)$-module of the degenerate principal series $\pi(\nu, \det_M)$.*

Proof. We denote simply $\mathcal{E}$ for $\mathcal{E}^{(\mathrm{even})}$ or $\mathcal{E}^{(\mathrm{odd})}$ according with σ is 1_M or $\det_M$. Let $H^K_{\sigma,\nu}$ be the set of K-finite vectors of the representation space $H_{\sigma,\nu}$ of $\pi(\sigma,\nu)$. It becomes a $(\mathfrak{g}, K)$-module by the admissibility of $\pi(\sigma,\nu)$. For $f \in H^K_{\sigma,\nu}$, we consider the series

$$\sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})} f(\gamma g) = \sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})} \sigma(m(\gamma g))a(\gamma g)^{\nu+\rho} f(k(\gamma g)).$$

By the Peter-Weyl theorem, we have a K-module isomorphism

$$H^K_{\nu,\sigma} \cong \begin{cases} \displaystyle\bigoplus_{\substack{d\in\mathbb{Z}_{\geq 0}\\ d:\mathrm{even}}} \mathcal{H}_d & \text{if } \sigma = 1_M, \\ \displaystyle\bigoplus_{\substack{d\in\mathbb{Z}_{\geq 0}\\ d:\mathrm{odd}}} \mathcal{H}_d & \text{if } \sigma = \det_M. \end{cases}$$

Then it follows that the series is the element of $\mathcal{E}$. By the assumption, this series converges absolutely and uniformly for $g \in G$. Therefore if we define the map $\phi \colon H^K_{\sigma,\nu} \to \mathcal{E}$ such that

$$\phi(f) = \zeta(\frac{\nu}{n-1} + \frac{n}{2}) \sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})} f(\gamma g),$$

it is well defined K-module isomorphism. We see this is also $\mathfrak{g}$-module isomorphism. For $X \in \mathfrak{g}$, we have

$$\begin{aligned} X &\cdot \sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})} f(\gamma g) \\ &= \frac{d}{dt} \sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})} f(\gamma g \exp(tX))\Big|_{t=0} \\ &= \sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})} \frac{d}{dt} f(\gamma g \exp(tX))\Big|_{t=0} \\ &= \sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})} \pi_{\nu,\sigma}(X) f(\gamma g) \in \mathcal{E}. \end{aligned}$$

Thus ϕ is also $\mathfrak{g}$-module isomorphism. To show that $\mathcal{E}$ is a $(\mathfrak{g}, K)$-module, we only need to see the compatibility of the $\mathfrak{g}$-action and the K-action. By the compatibility condition of $H^K_{\nu,\sigma}$, we have

$$\begin{aligned} k \cdot (X \cdot (\sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})} f(\gamma g))) &= \sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})} \pi(k)\pi(X) f(\gamma g) \\ = \sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})} \pi(\mathrm{Ad}(k)X) f(\gamma g) &= \mathrm{Ad}(k)X \cdot (\sum_{\gamma\in\Gamma_\infty\backslash SL(n,\mathbb{Z})} f(\gamma g)). \end{aligned}$$

Hence we have that the map $\phi\colon H_{\nu,\sigma}^K \to \mathcal{E}$ is a $(\mathfrak{g}, K)$-module isomorphism.

$\square$

3 The pull-back formula

3.1 Embeddings of $F^\times$ into $GL(n, \mathbb{R})$

Let F be an algebraic extension of degree n over $\mathbb{Q}$ and r_1 (resp. r_2) denote the number of real (resp. complex) places. Denote

$$\sigma_i\colon F \longrightarrow \mathbb{C} \quad \text{for} \quad i = 1, \cdots, n$$

real embeddings for $i = 1, \cdots, r_1$ and complex embeddings for $i = r_1+1, \cdots, r_1+2r_2$ so that $\sigma_j = \bar{\sigma}_{j+r_2}$ for $j = r_1 + 1, \cdots, r_1 + r_2$. We use the notation $a^{(i)} = \sigma_i(a)$ $(i = 1, \cdots, n)$ for $a \in F$. Then we have the product mapping

$$\sigma\colon F \longrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

given by

$$\sigma(\alpha) = (\alpha^{(1)}, \cdots, \alpha^{(r_1)}, \alpha^{(r_1+1)}, \cdots, \alpha^{(r_1+r_2)})$$

for $\alpha \in F$. This is naturally extended to the isomorphism of $F_\mathbb{R} = F \otimes_\mathbb{Q} \mathbb{R}$ to $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. We also have the embedding $i\colon \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \hookrightarrow M(n, \mathbb{R})$,

$$i((a_1, \cdots, a_{r_1}, b_1, \cdots, b_{r_2})) = \begin{pmatrix} a_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & a_{r_1} & & & & & & \\ & & & \alpha_1 & -\beta_1 & & & & \\ & & & \beta_1 & \alpha_1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & \alpha_{r_2} & -\beta_{r_2} \\ & & & & & & \beta_{r_2} & \alpha_{r_2} \end{pmatrix},$$

where $(a_1, \cdots, a_{r_1}, b_1, \cdots, b_{r_2}) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ and $b_i = \alpha_i + \sqrt{-1}\beta_i$ for $i = 1, \cdots, r_2$. Combining these maps, we obtain an embedding $\lambda = i \circ \sigma\colon F_\mathbb{R} \hookrightarrow M(n, \mathbb{R})$. Here we note that $N(\alpha) = \det \lambda(\alpha)$ is equals to the norm of $F/\mathbb{Q}$ for $\alpha \in F^\times$.

On the other hand, we can also construct the other embedding of F into $M(n, \mathbb{R})$. Let us take an integral ideal $\mathfrak{b}$ of F, and $\omega_1, \cdots, \omega_n$ be a system of $\mathbb{Z}$-basis of $\mathfrak{b}$. Then we have a mapping $\omega\colon F \to \mathbb{R}^n$ such that $\omega(\alpha) = (a_1, \cdots, a_n)$ for $\alpha = a_1\omega_1 + \cdots a_n\omega_n \in F$. This map gives another identification $F_\mathbb{R} \cong \mathbb{R}^n$. For an element $a \in F_\mathbb{R}$, the multiplication $a \cdot x$ for $x \in F_\mathbb{R} \cong \mathbb{R}^n$ is a linear transform of $\mathbb{R}^n$. This gives another embedding $F_\mathbb{R}$ into $M(n, \mathbb{R})$. For these two embeddings, the transformation matrix as an $\mathbb{R}$-vector space is given by

$$W_{\mathfrak{b}} =$$
$$\begin{pmatrix} \omega_1^{(1)} & \cdots & \omega_1^{(r_1)} & \operatorname{Re}\omega_1^{(r_1+1)} & \operatorname{Im}\omega_1^{(r_1+1)} & \cdots & \operatorname{Re}\omega_1^{(r_1+r_2)} & \operatorname{Im}\omega_1^{(r_1+r_2)} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \omega_n^{(1)} & \cdots & \omega_n^{(r_1)} & \operatorname{Re}\omega_n^{(r_1+1)} & \operatorname{Im}\omega_n^{(r_1+1)} & \cdots & \operatorname{Re}\omega_n^{(r_1+r_2)} & \operatorname{Im}\omega_n^{(r_1+r_2)} \end{pmatrix}$$
$$\in GL(n, \mathbb{R}).$$

Consequently, we obtain the following commutative diagram

$$\begin{array}{ccccc} F_{\mathbb{R}} & \xrightarrow{\sigma} & \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} & \xrightarrow{i} & M(n, \mathbb{R}) \\ \Big\| & & {\scriptstyle W_{\mathfrak{b}}}\Big\uparrow & & {\scriptstyle \mathrm{ad}(W_{\mathfrak{b}})}\Big\uparrow \\ F_{\mathbb{R}} & \xrightarrow{\omega} & \mathbb{R}^n & \xrightarrow{j} & M(n, \mathbb{R}) \end{array} .$$

Take the norm 1 part $F_{\mathbb{R}}^{(1)} = \{\alpha \in F_{\mathbb{R}}^{\times} \mid N(\alpha) = 1\}$ where $N(x) = \det \lambda(x)$. By using these embeddings, we restrict the Eisenstein series to the $F_{\mathbb{R}}^{(1)}$-orbit of $W'_{\mathfrak{b}}$,

$$E(\nu, W'_{\mathfrak{b}}\lambda(\alpha); h) \text{ for } \alpha \in F_{\mathbb{R}}^{(1)}.$$

Here we choose the basis $\omega_1, \cdots, \omega_n$ so that $\det W_{\mathfrak{b}} > 0$ and $W'_{\mathfrak{b}} = (\det W_{\mathfrak{b}})^{-\frac{1}{n}} W_{\mathfrak{b}} \in SL(n, \mathbb{R})$. For the simplicity, we use the notation $E_{F,\mathfrak{b}}(\nu, \alpha; h) = E(\nu, W'_{\mathfrak{b}}\lambda(\alpha); h)$ for $\alpha \in F_{\mathbb{R}}^{(1)}$.

Proposition 3.1. *Take $\nu \in \mathbb{C}$ and $h \in \mathcal{H}_d$. The Eisenstein series $E_{F,\mathfrak{b}}(\nu, \alpha; h)$ is invariant under the action of $\mathcal{O}_F^{(1)} = \{\varepsilon \in \mathcal{O}_F^{\times} \mid N(\varepsilon) = 1\}$, i.e.,*

$$E_{F,\mathfrak{b}}(\nu, \varepsilon\alpha; h) = E_{F,\mathfrak{b}}(\nu, \alpha; h)$$

for $\varepsilon \in \mathcal{O}_F^{(1)}$ and $\alpha \in F_{\mathbb{R}}^{(1)}$.

Proof. Because we have $(\varepsilon)\mathfrak{b} = \mathfrak{b}$ there exist integers c_j^i for $1 \leq i, j \leq n$ which satisfy $\varepsilon\omega_i = c_1^i\omega_1 + \cdots + c_n^i\omega_n$ for $1 \leq i, j \leq n$. This implies

$$\begin{pmatrix} c_1^1 & \cdots & c_n^1 \\ \vdots & & \vdots \\ c_n^1 & \cdots & c_n^n \end{pmatrix} W_{\mathfrak{b}} = W_{\mathfrak{b}}\lambda(\varepsilon),$$

and

$$C = \begin{pmatrix} c_1^1 & \cdots & c_n^1 \\ \vdots & & \vdots \\ c_n^1 & \cdots & c_n^n \end{pmatrix} \in SL(n, \mathbb{Z}).$$

Then we have

$$\begin{aligned} E(\nu, W'_{\mathfrak{b}}\lambda(\varepsilon\alpha)) =& E(\nu, CW'_{\mathfrak{b}}\lambda(\alpha)) \\ =& E(\nu, W'_{\mathfrak{b}}\lambda(\alpha)) \end{aligned}$$

by the modularity of the Eisenstein series. □

This proposition tells us that $E_{F,\mathfrak{b}}(\nu, \alpha; h)$ is a function on $\mathcal{O}_F^{(1)} \backslash F_{\mathbb{R}}^{(1)}$.

3.2 Grössencharacters and harmonic polynomials

Now we consider the action of $F^\times$ on H_d. We will see here that this action gives a character of $F^\times$ which appears in Theorem 3.5 as a Grössencharacter of F. Let $T_F = \lambda(F_{\mathbb{R}}^\times) \subset GL(n, \mathbb{R})$. Then T_F has a maximal compact subgroup $C_F \subset O(n)$ of the form

$$\left\{ \begin{pmatrix} \eta_1 & & & & & \\ & \ddots & & & & \\ & & \eta_{r_1} & & & \\ & & & r(\theta_1) & & \\ & & & & \ddots & \\ & & & & & r(\theta_{r_2}) \end{pmatrix} \middle| \begin{array}{l} \eta_i \in \{\pm 1\},\ i = 1, \cdots, r_1, \\ \\ r(\theta_j) = \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}, \\ j = 1, \cdots, r_2 \end{array} \right\} \subset O(n).$$

The space of harmonic polynomials H_d has a natural action of $T_F \subset GL(n, \mathbb{R})$ via

$$\rho(g)f(x) = f(xg)$$

for $g \in T_F$ and $f(x) \in H_d$. Then we decompose H_d as the irreducible representation of C_F. Because C_F is a compact abelian group, the irreducible representation of C_F is 1-dimensional, i.e., all irreducible representation of C_F are written as characters. It is easily checked that for any character $\chi\colon C_F \to \mathbb{C}^1$, there exits $\gamma = (\delta_1, \cdots, \delta_{r_1}, l_1, \cdots, l_{r_2}) \in \{0, 1\}^{r_1} \times \mathbb{Z}^{r_2}$ and $\chi = \chi_\gamma$,

$$\chi_\gamma(\mathrm{diag}(\eta_1, \cdots, \eta_{r_1}, r(\theta_1), \cdots, r(\theta_{r_2}))) = \prod_{i=1}^{r_1} \eta_i^{\delta_i} \prod_{j=1}^{r_2} e^{\sqrt{-1} l_j \theta_j}.$$

Hence we have a decomposition $H_d = \bigoplus_\gamma V_\gamma$, where $V_\gamma = \{f \in H_d \mid f(xt) = \chi_\gamma(t) f(x),\ t \in C_F\}$.

We want to know the explicit descriptions of the weight vectors $f_\gamma \in V_\gamma$. If $\gamma = (\delta_1, \cdots, \delta_{r_1}, l_1, \cdots, l_{r_2}) \in \{0, 1\}^{r_1} \times \mathbb{Z}^{r_2}$ satisfies $\sum_{i=1}^{r_1} \delta_i + \sum_{j=1}^{r_2} |l_j| = d$, we can choose the nice element f_γ in V_γ such that

$$f_\gamma(x) = \prod_{i=1}^{r_1} x_i^{\delta_i} \prod_{j=1}^{r_2} c_{l_j}(x_{r_1+2j-1}, x_{r_1+2j}),$$

where

$$c_l(x, y) = \begin{cases} (x - \sqrt{-1}y)^l & \text{if } l \geq 0 \\ (x + \sqrt{-1}y)^{|l|} & \text{if } l \leq 0 \end{cases}.$$

If we consider the decomposition

$$\lambda(\alpha) = \begin{pmatrix} |\alpha^{(1)}| & & & & & \\ & \ddots & & & & \\ & & |\alpha^{(r_1)}| & & & \\ & & & |\alpha^{(r_1+1)}|I_2 & & \\ & & & & \ddots & \\ & & & & & |\alpha^{(r_1+r_2)}|I_2 \end{pmatrix}$$
$$\times \begin{pmatrix} \mathrm{sgn}(\alpha^{(1)}) & & & & & \\ & \ddots & & & & \\ & & \mathrm{sgn}(\alpha^{(r_1)}) & & & \\ & & & r(\arg(\alpha^{(r_1+1)})) & & \\ & & & & \ddots & \\ & & & & & r(\arg(\alpha^{(r_1+r_2)})) \end{pmatrix},$$

the action of $\alpha \in F^\times$ on f_γ is written as

$$f_\gamma(x\lambda(\alpha)) = \hat{\chi}_\gamma(\alpha) f_\gamma(xi(|\alpha|))$$

where $|\alpha| = (|\alpha^{(1)}|, \cdots, |\alpha^{(r_1+r_2)}|) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Here the character $\hat{\chi}_\gamma \colon F^\times \to \mathbb{C}^1$ is defined by

$$\hat{\chi}_\gamma(\alpha) = \prod_{i=1}^{r_1} \mathrm{sgn}\,(\alpha^{(i)})^{\delta_i} \prod_{j=1}^{r_2} \exp(\sqrt{-1} l_j \arg \alpha^{(j)})$$

for $\alpha \in F^\times$.

Because $\mathcal{O}_F^\times$ is the product of a finite group and a finitely generated free $\mathbb{Z}$-module we can find $\gamma \in \{0,1\}^{r_1} \times \mathbb{Z}^{r_2}$ which satisfies $\hat{\chi}_\gamma(\varepsilon) = 1$ for $\varepsilon \in \mathcal{O}_F^\times$. In the remaining of the section, we fix such $\gamma \in \{0,1\}^{r_1} \times \mathbb{Z}^{r_2}$ and $f_\gamma \in H_d$ constructed for $d = \sum_{i=1}^{r_1} \eta_i + \sum_{j=1}^{r_2} |l_j|$ as above.

3.3 The Hecke-Siegel pull-back formula of the Epstein zeta function

Let C_F be the maximal compact subgroup of T_F as in the section 3.2 and take the pull-back of C_F by λ, denote $\tilde{C}_F = \lambda^{-1}(C_F) \subset F_\mathbb{R}^\times$. We consider the Fourier analysis of $E_{F,\mathfrak{b}}(\nu, \alpha; f_\gamma)$ on the compact double coset $\mathcal{O}_F^{(1)} \backslash F_\mathbb{R}^{(1)} / (F_\mathbb{R}^{(1)} \cap \tilde{C}_F)$.

We define the map $|\cdot| \colon F_\mathbb{R}^{(1)} \to \mathbb{R}_{>0}^{r_1+r_2}$ such that $|\alpha| = (|\alpha^{(1)}|, \cdots, |\alpha^{(r_1+r_2)}|)$ for $\alpha \in F_\mathbb{R}^{(1)}$. Then we have $F_\mathbb{R}^{(1)} / (F_\mathbb{R}^{(1)} \cap \tilde{C}_F) \cong S = \{(a_1, \cdots, a_{r_1+r_2}) \in \mathbb{R}_{>0}^{r_1+r_2} \mid \prod_{i=1}^{r_1+r_2} a_i = 1\}$. If we define the action of $F_\mathbb{R}^{(1)}$ on S by $\alpha \cdot a = (|\alpha^{(1)}|a_1, \cdots, |\alpha^{(r_1+r_2)}|a_{r_1+r_2})$ for $\alpha \in F_\mathbb{R}^{(1)}$ and $a = (a_1, \cdots, a_{r_1+r_2}) \in S$, This becomes $F_\mathbb{R}^{(1)}$-isomorphism. By the Dirichlet unit theorem, we can find the fundamental domain of $S \cong F_\mathbb{R}^{(1)} / (F_\mathbb{R}^{(1)} \cap \tilde{C}_F)$ by the action of $\mathcal{O}_F^{(1)}$,

$$E = \{(\prod_{i=1}^{r} |\varepsilon_i^{(1)}|^{t_i}, \cdots, \prod_{i=1}^{r} |\varepsilon_i^{(r_1+r_2)}|^{t_i}) \in S \mid 0 \le t_i < 1,\ i = 1, \cdots, r\}.$$

Here $\varepsilon_1, \cdots, \varepsilon_r$ be a system of fundamental units in F with $r = r_1 + r_2 - 1$ the rank of $\mathcal{O}_F^\times$. Hence for any $u \in S$, there exits $\mathbf{t} = (t_1, \cdots, t_r) \in \mathbb{R}^r$ such that $u = u(\mathbf{t}) = (\prod_{i=1}^r |\varepsilon_i^{(1)}|^{t_i}, \cdots, \prod_{i=1}^r |\varepsilon_i^{(r_1+r_2)}|^{t_i})$.

The Eisenstein series $\widehat{E}(\nu, W'_{\mathfrak{b}} i(u); f_\gamma)$ as the function of $u \in S$ becomes the function of $\mathbf{t} \in \mathbb{R}^r$. We denote it by $g(\nu, \mathbf{t}) = \widehat{E}(\nu, W'_{\mathfrak{b}} i(u(\mathbf{t})); f_\gamma)$.

Lemma 3.2. *The function $g(\nu, \mathbf{t})$ is periodic with respect to $\mathbf{t} \in \mathbb{R}^r$, i.e.,*

$$g(\nu, \mathbf{t} + \mathbf{a}) = g(\nu, \mathbf{t}),$$

for $\mathbf{a} = (a_1, \cdots, a_r) \in \mathbb{Z}^r$.

Proof. By the Dirichlet unit theorem, there exists $\varepsilon \in \mathcal{O}_F^\times$ such that

$$|\varepsilon| = (|\varepsilon^{(1)}|, \cdots, |\varepsilon^{(r_1+r_2)}|) = (\prod_{i=1}^r |\varepsilon_i^{(1)}|^{a_i}, \cdots, \prod_{i=1}^r |\varepsilon_i^{(r_1+r_2)}|^{a_i})$$

for $\mathbf{a} \in \mathbb{Z}^r$. Hence we have

$$\begin{aligned} g(\nu, (\mathbf{t} + \mathbf{a})) &= \widehat{E}(\nu, W'_{\mathfrak{b}} i(u(\mathbf{t} + \mathbf{a})); f_\gamma) \\ &= \widehat{E}(\nu, W'_{\mathfrak{b}} i(|\varepsilon| u(\mathbf{t})); f_\gamma) \\ &= \sum_{\mathbf{x} \in \mathbb{Z}^n - \{\mathbf{0}\}} |\mathbf{x} W'_{\mathfrak{b}} i(|\varepsilon| u(\mathbf{t}))|^{-\left(\frac{\nu}{(n-1)} + \frac{n}{2} + d\right)} \cdot f_\gamma(\mathbf{x} W'_{\mathfrak{b}} i(|\varepsilon|) i(u(\mathbf{t}))) \\ &= \sum_{\mathbf{x} \in \mathbb{Z}^n - \{\mathbf{0}\}} |\mathbf{x} W'_{\mathfrak{b}} \lambda(\varepsilon) i(u(\mathbf{t})))|^{-\left(\frac{\nu}{(n-1)} + \frac{n}{2} + d\right)} \cdot f_\gamma(\mathbf{x} W'_{\mathfrak{b}} \lambda(\varepsilon) i(u(\mathbf{t}))). \end{aligned}$$

The last equality comes from the equation $f_\gamma(\mathbf{x}\lambda(\varepsilon)) = \hat{\chi}_\gamma(\varepsilon) f_\gamma(\mathbf{x} i(|\varepsilon|)) = f_\gamma(\mathbf{x} i(|\varepsilon|))$. It follows from the proof of Proposition 3.2 that there exits $h \in SL(n, \mathbb{R})_\pm = \{g \in GL(n, \mathbb{R}) \mid \det g \in \{\pm 1\}\}$ and we have

$$h W_{\mathfrak{b}} = W_{\mathfrak{b}} \lambda(\epsilon).$$

The modularity of the Epstein zeta function shows that

$$\begin{aligned} g(\nu, (\mathbf{t} + \mathbf{a})) &= \widehat{E}(\nu, W'_{\mathfrak{b}} \lambda(\varepsilon) i(u(\mathbf{t})); f_\gamma) \\ &= \widehat{E}(\nu, h W'_{\mathfrak{b}} i(u(\mathbf{t})); f_\gamma) = \widehat{E}(\nu, W'_{\mathfrak{b}} i(u(\mathbf{t})); f_\gamma) = g(\nu, \mathbf{t}). \end{aligned}$$

□

This lemma enables us to consider the Fourier series expansion of $g(\nu, (\mathbf{t}))$,

$$g(\nu, \mathbf{t}) = \sum_{\mathbf{m} \in \mathbb{Z}^r} (g, e_{\mathbf{m}}) e_{\mathbf{m}}(\mathbf{t}),$$

where $e_{\mathbf{m}}(\mathbf{t}) = \exp(2\pi\sqrt{-1}\mathbf{m}^t \mathbf{t})$ for $\mathbf{t} \in \mathbb{R}^r$, $\mathbf{m} \in \mathbb{Z}^r$, and

$$(g, e_{\mathbf{m}}) = \int_{[0,1]^r} g(\nu, \mathbf{y}) \overline{e_{\mathbf{m}}(\mathbf{y})} \, d\mathbf{y}.$$

Here we prepare some lemmas before the main theorem. Let R denote the matrix

$$\begin{pmatrix} \log|\varepsilon_1^{(1)}| & \cdots & \log|\varepsilon_r^{(1)}| \\ \vdots & & \vdots \\ \log|\varepsilon_1^{(r)}| & \cdots & \log|\varepsilon_r^{(r)}| \end{pmatrix},$$

and $\mathbf{r}_i$ for $i = 1, \cdots, r$ the row vectors of ${}^tR^{-1}$, i.e., $R^{-1} = ({}^t\mathbf{r}_1, \cdots, {}^t\mathbf{r}_r)$.

Now we choose a character of the ideal group I of F,

$$\chi_{\mathbf{m},\gamma} \colon I \to \mathbb{C}^1$$

such that

$$\chi_{\mathbf{m},\gamma}((\alpha)) = \prod_{i=1}^{r}(|\alpha^{(i)}|^{-1}|N(\alpha)|^{\frac{1}{n}})^{-2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r_i}}\hat{\chi_\gamma}((\alpha))$$

for a principal ideal (α). Here $\hat{\chi}_\gamma$ is the character defined in Section 3.2. This definition makes sense by the following lemma,

Lemma 3.3. *For $\varepsilon \in \mathcal{O}_F^\times$, the character $\chi_{\mathbf{m},\gamma}$ satisfies*

$$\chi_{\mathbf{m},\gamma}((\varepsilon)) = 1.$$

Proof. For $\varepsilon \in \mathcal{O}_F^\times$, there exits $\mathbf{a} = (a_1, \cdots, a_r) \in \mathbb{Z}^r$ such that $|\varepsilon| = (|\varepsilon^{(1)}|, \cdots, |\varepsilon^{(r_1+r_2)}|) = (\prod_{i=1}^r |\varepsilon_i^{(1)}|^{a_i}, \cdots, \prod_{i=1}^r |\varepsilon_i^{(r_1+r_2)}|^{a_i})$. Then we have the equation

$$\begin{aligned} \log\,(\prod_{i=1}^{r}(|\varepsilon^{(i)}|^{-1})^{-2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r}_i} &= \sum_{i=1}^{r} 2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r}_i \log|\varepsilon^{(i)}| \\ &= 2\pi\sqrt{-1}\mathbf{m}^t\mathbf{a} \in 2\pi\sqrt{-1}\mathbb{Z}. \end{aligned}$$

This implies

$$\begin{aligned} \chi_{\mathbf{m},\gamma}((\varepsilon)) &= (\prod_{i=1}^{r}(|\varepsilon^{(i)}|^{-1}))^{-2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r}_i} \\ &= \exp(2\pi\sqrt{-1}\mathbf{m}^t\mathbf{a}) \\ &= 1. \end{aligned}$$

□

Lemma 3.4. *For $u = (u_1, \cdots, u_{r_1+r_2}) \in S$, there exists $\mathbf{t} = (t_1, \cdots, t_r) \in \mathbb{Z}^r$ such that $(u_1, \cdots, u_{r_1+r_2}) = (\prod_{i=1}^r |\varepsilon_i^{(1)}|^{t_i}, \cdots, \prod_{i=1}^r |\varepsilon_i^{(r_1+r_2)}|^{t_i})$. Then we have*

$$e_{\mathbf{m}}(\mathbf{t}) = \prod_{i=1}^{r} u_i^{2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r}_i},$$

for $\mathbf{m} \in \mathbb{Z}^r$.

Proof. We have

$$(\log u_1, \cdots, \log u_r) = R^t\mathbf{t}.$$

Then

$$\mathbf{m}R^{-1}(\log u_1, \cdots, \log u_r) = \mathbf{m}^t\mathbf{t}.$$

Taking exponentials on both sides, we have the equation as required. □

Theorem 3.5. *We use the same notation as above with $2s = \frac{\nu}{n-1} + \frac{n}{2}$. Then we have the following equation.*

$$\pi^{-s}\Gamma\left(s+\frac{d}{2}\right)(g, e_{\mathbf{m}}) = \frac{1}{2^r n}\omega_F |d_F|^{\frac{s}{n}} R_F^{-1}\chi_{\mathbf{m},\gamma}(\mathfrak{b})\zeta_\infty(\frac{2s}{n}, \chi_{\mathbf{m},\gamma})\zeta_F(\frac{2s}{n}, A, \chi_{\mathbf{m},\gamma}).$$

Here ω_F, d_F and R_F are the number of roots of unity, the discriminant and the regulator of F respectively. Also $\zeta_F(s, A, \chi)$ is the partial zeta function of the ideal class A of $\mathfrak{b}^{-1}$ with a character χ, i.e., $\zeta_F(s, A, \chi) = \sum_{\substack{\mathfrak{a}\neq(0)\in A \\ \mathfrak{a}:integral}} \chi(\mathfrak{a}) N_F(\mathfrak{a})^{-s}$ where N_F is the ideal norm of F. The function $\zeta_\infty(s, \chi_{\mathbf{m},\gamma})$ is the gamma factor given by

$$\begin{aligned}
&\zeta_\infty(s, \chi_{\mathbf{m},\gamma}) \\
&\quad = \pi^{-\frac{ns}{2}} \prod_{i=1}^{r_1} \Gamma\left(\frac{s}{2} + \frac{\delta_i}{2} - \sqrt{-1}\pi(\mathbf{m}^t\mathbf{r}_i - \frac{1}{n}\sum_{j=1}^{r}\mathbf{m}^t\mathbf{r}_j)\right) \\
&\quad \times \prod_{j=1}^{r_2} \Gamma\left(s + \frac{|l_j|}{2} - \sqrt{-1}\pi(\mathbf{m}^t\mathbf{r}_i - \frac{1}{n}\sum_{j=1}^{r}\mathbf{m}^t\mathbf{r}_j)\right).
\end{aligned}$$

Remark 3.6. *Our choice of the ideal character $\chi_{\mathbf{m},\gamma}$ is not unique. But we only consider the partial zeta function here. Hence the R.H.S. of the equation of the theorem is independent of the choice of the character $\chi_{\mathbf{m},\gamma}$.*

Proof. For $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{Z}^n$, we have $\beta \in \mathfrak{b}$ such that $\beta = \sum_{i=1}^n x_i\omega_i$. This implies

$$\mathbf{x}W_{\mathfrak{b}} = (\beta^{(1)}, \cdots, \beta^{(r_1)}, \operatorname{Re}\beta^{(r_1+1)}, \operatorname{Im}\beta^{(r_1+1)}, \cdots, \operatorname{Re}\beta^{(r_1+r_2)}, \operatorname{Im}\beta^{(r_1+r_2)}).$$

Then

$$\begin{aligned}
&(g, e_{\mathbf{m}}) \\
&= \int_{[0,1]^r} g(\nu, \mathbf{t})\overline{e_{\mathbf{m}}(\mathbf{t})}\, d\mathbf{t} = \int_{[0,1]^r} \widehat{E}(\nu, W'_{\mathfrak{b}} i(u(\mathbf{t})); f_\gamma)\overline{e_{\mathbf{m}}(\mathbf{t})}\, d\mathbf{t} \\
&= \int_{[0,1]^r} \sum_{\mathbf{x}\in\mathbb{Z}^n-\{\mathbf{0}\}} |\mathbf{x}W'_{\mathfrak{b}} i(u(\mathbf{t}))|^{-(2s+d)} \cdot f_\gamma(\mathbf{x}W'_{\mathfrak{b}} i(u(\mathbf{t})))\overline{e_{\mathbf{m}}(\mathbf{t})}\, d\mathbf{t} \\
&= (\det W_{\mathfrak{b}})^{\frac{2s}{n}} \int_{[0,1]^r} \sum_{\substack{\beta\in\mathfrak{b} \\ \beta\neq 0}} \sum_{i=1}^{r_1+r_2} (|\beta^{(i)}|u_i)^{-(2s+d)} \\
&\times \prod_{j=1}^{r_1} |\beta^j|^{\delta_j}(\operatorname{sgn}\beta^{(j)})^{\delta_j} u_j^{\delta_i} \prod_{k=1}^{r_2} |\beta^{(r_1+k)}|^{|l_k|} \exp(\sqrt{-1}l_k \arg\beta^{(r_1+k)}) u_k^{|l_k|}\overline{e_{\mathbf{m}}(\mathbf{t})}\, d\mathbf{t},
\end{aligned}$$

where u_i $(i = 1, \cdots, r_1+r_2)$ denote i-th component of $u(\mathbf{t}) \in S$. By Lemma 3.4

$$(g, e_{\mathbf{m}}) = (\det W_{\mathfrak{b}})^{\frac{2s}{n}} \int_{[0,1]^r} \sum_{\substack{\beta\in\mathfrak{b}\\ \beta\neq 0}} \sum_{i=1}^{r_1+r_2} (|\beta^{(i)}|u_i)^{-(2s+d)}$$

$$\times \prod_{j=1}^{r_1} |\beta^j|^{\delta_j} (\operatorname{sgn}\beta^{(j)})^{\delta_j} u_j^{\delta} \prod_{k=1}^{r_2} |\beta^{(r_1+k)}|^{|l_k|} \exp(\sqrt{-1} l_k \arg \beta^{(r_1+k)}) u_k^{|l_k|}$$

$$\times \prod_{i=1}^{r} u_i^{-2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r}_i}\, d\mathbf{t}$$

Now we change the variables as $y_i = |\beta^{(i)}||N_F(\beta)|^{\frac{1}{n}} u_i$ $(i = 1, \cdots, r_1 + r_2)$. Note that we have $\prod_{i=1}^{r_1} y_i \prod_{j=1}^{r_2} y_{r_1+j}^2 = \prod_{i_1}^{r_1} u_i \prod_{j=1}^{r_2} u_j^2$=1. Then

$$(g, e_{\mathbf{m}}) = (\det W_{\mathfrak{b}})^{\frac{2s}{n}} R_F^{-1} \times \int_D \sum_{\substack{\beta\in\mathfrak{b}\\ \beta\neq 0}} |N_F(\beta)|^{-\frac{2s}{n}}$$

$$\times \prod_{i=1}^{r_1} (\operatorname{sgn}\beta^{(i)})^{\delta_i} \prod_{j=1}^{r_2} \exp(\sqrt{-1} l_k \arg\beta^{(r_1+k)}) \prod_{l=1}^{r} (|\beta^{(l)}|^{-1} N(\beta)^{\frac{1}{n}})^{-2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r}_i}$$

$$\times (\sum_{k=1}^{r_1+r_2} y_k^2)^{-s-\frac{d}{2}} \prod_{i=1}^{r_1} y_i^{\delta_i} \prod_{j=1}^{r_2} y_j^{|l_j|} \times \prod_{l=1}^{r} y_l^{-2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r}_l} \prod_{m=1}^{r} \frac{dy_m}{y_m},$$

where the domain D is $\{(y_1, \cdots, y_{r_1+r_2}) \mid (u_1, \cdots, u_{r_1+r_2}) \in E\}$ for the fundamental domain E of S. By the Dirichlet unit theorem, the integral equals to

$$\omega_F (\det W_{\mathfrak{b}})^{\frac{2s}{n}} R_F^{-1} \sum_{\substack{(\beta)\subset\mathfrak{b}\\ (\beta)\neq(0)}} |N_F((\beta))|^{-\frac{2s}{n}} \chi_{\mathbf{m},\gamma}((\beta))$$

$$\times \int_{(\mathbb{R}_{>0})^r} (\sum_{k=1}^{r_1+r_2} y_k^2)^{-s-\frac{d}{2}} \prod_{i=1}^{r_1} y_i^{\delta_i} \prod_{j=1}^{r_2} y_j^{|l_j|} \times \prod_{l=1}^{r} y_l^{-2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r}_l} \prod_{m=1}^{r} \frac{dy_m}{y_m}.$$

To complete the calculation, we consider the following integral

$$\Gamma\left(s + \frac{d}{2}\right) \int_{(\mathbb{R}_{>0})^r} (\sum_{k=1}^{r_1+r_2} y_k^2)^{-s-\frac{d}{2}} \prod_{i=1}^{r_1} y_i^{\delta_i} \prod_{j=1}^{r_2} y_j^{|l_j|} \prod_{l=1}^{r} y_l^{-2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r}_l} \prod_{m=1}^{r} \frac{dy_m}{y_m}.$$

By the Mellin transform formula for the gamma function, the integral is equals to

$$\int_{(\mathbb{R}_{>0})} \exp(-t) t^{s+\frac{d}{2}} \frac{dt}{t}$$

$$\times \int_{(\mathbb{R}_{>0})^r} (\sum_{k=1}^{r_1+r_2} y_k^2)^{-s-\frac{d}{2}} \prod_{i=1}^{r_1} y_i^{\delta_i} \prod_{j=1}^{r_2} y_j^{|l_j|} \prod_{l=1}^{r} y_l^{-2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r}_l} \prod_{m=1}^{r} \frac{dy_m}{y_m}$$

$$= \int_{(\mathbb{R}_{>0})}^{r+1} \exp\left(-t \sum_{k=1}^{r_1+r_2} y_k^2\right) t^{s+\frac{d}{2}} \prod_{i=1}^{r_1} y_i^{\delta_i} \prod_{j=1}^{r_2} y_j^{|l_j|} \prod_{l=1}^{r} y_l^{-2\pi\sqrt{-1}\mathbf{m}^t\mathbf{r}_l} \frac{dt}{t} \prod_{m=1}^{r} \frac{dy_m}{y_m}.$$

If we set $p_i = ty_i^2$ $(i = 1, \cdots, r_1 + r_2)$, then we have $\prod_{i=1}^{r_1} p_i \prod_{j=1}^{r_2} p_{r_1+j}^2 = t^n$ and $\frac{dt}{t} \prod_{i=1}^{r} \frac{dy_i}{y_i} = 2^{-r} \frac{1}{n} \prod_{j=1}^{r_1+r_2} \frac{dp_j}{p_j}$. Then we have

$$\frac{1}{2^r n} \int_{(\mathbb{R}_{>0})^{r+1}} \exp\left(-\sum_{i=1}^{r_1+r_2} p_i \right) \prod_{j=1}^{r_1} p_j^{\frac{s}{n}+\frac{\delta_j}{2}-\sqrt{-1}\pi(\mathbf{m}^t\mathbf{r}_j - \frac{1}{n}\sum_{l=1}^{r}\mathbf{m}^t\mathbf{r}_l)}$$
$$\times \prod_{k=1}^{r_2} p_k^{\frac{2s}{n}+\frac{|l_k|}{2}-\sqrt{-1}\pi(\mathbf{m}^t\mathbf{r}_k - \frac{1}{n}\sum_{l=1}^{r}\mathbf{m}^t\mathbf{r}_l)} \prod_{m=1}^{r_1+r_2} \frac{dp_m}{p_m}$$

$$= \frac{1}{2^r n} \prod_{i=1}^{r_1} \Gamma\left(\frac{s}{n} + \frac{\delta_i}{2} - \sqrt{-1}\pi(\mathbf{m}^t\mathbf{r}_i - \frac{1}{n}\sum_{k=1}^{r}\mathbf{m}^t\mathbf{r}_k) \right)$$
$$\times \prod_{j=1}^{r_2} \Gamma\left(\frac{2s}{n} + \frac{|l_j|}{2} - \sqrt{-1}\pi(\mathbf{m}^t\mathbf{r}_j - \frac{1}{n}\sum_{k=1}^{r}\mathbf{m}^t\mathbf{r}_k) \right).$$

Finally let us recall that $|\det W_{\mathfrak{b}}| = N_F(\mathfrak{b})\sqrt{|d_F|}$. Then we have:

$$(\det W_{\mathfrak{b}})^{\frac{2s}{n}} \omega_F \sum_{\substack{(\beta)\subset\mathfrak{b} \\ (\beta)\neq(0)}} \chi_{\mathbf{m},\gamma}((\beta))|N_F((\beta))|^{-\frac{2s}{n}}$$
$$= \omega_F |d_F|^{\frac{s}{n}} \sum_{\substack{(\beta)\subset\mathfrak{b} \\ (\beta)\neq(0)}} \chi_{\mathbf{m},\gamma}(\mathfrak{b})\chi_{\mathbf{m},\gamma}((\beta)\mathfrak{b}^{-1})N_F((\beta)\mathfrak{b}^{-1})^{-\frac{2s}{n}}$$
$$= \omega_F R_F^{-1} |d_F|^{\frac{s}{n}} \chi_{\mathbf{m},\gamma}(\mathfrak{b}) \sum_{\substack{\mathfrak{a}\in A \\ \mathfrak{a}\ \text{integral},\mathfrak{a}\neq(0)}} \chi_{\mathbf{m},\gamma}(\mathfrak{a})N_F(\mathfrak{a})^{-\frac{2s}{n}},$$

Combining these equations, we obtain the equation as required. □

Corollary 3.7. *We use the same notation as the theorem. Then the function*

$$\hat{\zeta}_F(s, A, \chi_{\mathbf{m},\gamma}) = |d_F|^{\frac{s}{2}} \zeta_\infty(s, \chi_{\mathbf{m},\gamma})\zeta(s, A, \chi_{\mathbf{m},\gamma})$$

satisfies the functional equation

$$\hat{\zeta}_F(s, A, \chi_{\mathbf{m},\gamma}) = (\sqrt{-1})^{-d}\chi_{\mathbf{m},\gamma}(\mathfrak{d})\hat{\zeta}_F(1-s, A^*, \overline{\chi}_{\mathbf{m},\gamma})$$

where $\mathfrak{d}$ is the different of $F/\mathbb{Q}$ and A^* is the ideal class which satisfies $AA^* = [\mathfrak{d}]$ the ideal class of $\mathfrak{d}$.

Proof. By Theorem 1.3, we have the functional equation for the Eisenstein series

$$\pi^{-s}\Gamma(s+\frac{d}{s})\hat{E}(\nu, g; f_\gamma) = (\sqrt{-1})^{-d}\pi^{\frac{n}{2}-s}\Gamma(\frac{n}{2}-s+\frac{d}{2})\hat{E}(-\nu, {}^tg^{-1}; f_\gamma).$$

We apply the proof of the theorem for $g'(-\nu, t) = \hat{E}(-\nu, {}^tW_{\mathfrak{b}}'^{-1}i(u(t))^{-1}; f_\gamma)$. For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$, We have

$$\mathrm{Tr}\,(i(\mathbf{x}W_{\mathfrak{b}}) \cdot {}^ti(\mathbf{y}^tW_{\mathfrak{b}}^{-1})) \in \mathbb{Z}.$$

This also says that

$$\mathrm{Tr}\,(\lambda(\beta)\cdot {}^t i(\mathbf{y}^t W_{\mathfrak{b}}^{-1}))\in\mathbb{Z}$$

for $\beta\in\mathfrak{b}$. Then this leads that for $\mathbf{y}\in\mathbb{Z}^n$ there exists $\alpha\in(\mathfrak{b}\mathfrak{d})^{-1}$ such that $i(\mathbf{y}^t W_{\mathfrak{b}}^{-1})=\overline{\lambda(\alpha)}$ (cf. [8, Chapter VII §5 Lemma 5.7.]), i.e.,

$$\mathbf{x}^t W_{\mathfrak{b}}^{-1}= \\ (\alpha^{(1)},\cdots,\alpha^{(r_1)},\mathrm{Re}\,\alpha^{(r_1+1)},-\mathrm{Im}\,\alpha^{(r_1+1)},\cdots,\mathrm{Re}\,\alpha^{(r_1+r_2)},-\mathrm{Im}\,\alpha^{(r_1+r_2)}).$$

Noting this fact, we can obtain that

$$\pi^{\frac{n}{2}-s}\Gamma(\frac{n}{2}-s+\frac{d}{2})(g',e_{\mathbf{m}})= \\ \frac{1}{2^r n}\omega_F R_F^{-1}\overline{\chi_{\mathbf{m},\gamma}}((\mathfrak{b}\mathfrak{d})^{-1})\hat{\zeta}_F(1-\frac{2s}{n},A^*,\overline{\chi_{\mathbf{m},\gamma}})$$

as well as Theorem 3.5. Combining this formula and the above functional equation of the Eisenstein series, we have the corollary. □

4 Postscript

Here are some supplementary remarks.

Scholium 4.1. *The crucial reason of the validity of our pull-back formula is the following. Let $SL(n,\mathbb{A}_{\mathbb{Q}})$ be the adelization of $SL(n)$ over $\mathbb{Q}$, and let $T_F^{(1)}$ be the norm 1 subgroup of $T_F = Res\,\mathbb{G}_m$ and $T_F(\mathbb{A}_{\mathbb{Q}})$ be the associated adelization. Let π be the automorphic representation of $SL(n,\mathbb{A}_{\mathbb{Q}})$ generated by the Epstein zeta function. Then the meaning of our result is that the representation of π to $T_F^{(1)}(\mathbb{A}_{\mathbb{Q}})\cong\mathbb{A}_F^{(1)}$ is multiplicity-free, i.e., the representation of $\mathbb{A}_F^{(1)}$ are interpreted as Grössencharacter χ, and the pull-back of π, $i^*(\pi)$,*

$$i^*(\pi)=\int_{X(\mathbb{A}_F^{(1)})}^{\oplus} m(\chi)\cdot\chi\,dc(\chi)$$

is a direct integral of Grössencharacters χ with some multiplicities $m(\chi)$. Then for our π, this direct integral is reduced to a discrete sum with the multiplicities $m(\chi)$ of each χ is at most one. This is caused because our representation π is quite small. In fact at the real place, its $\mathbb{R}$-component $\pi_{\mathbb{R}}$ has Gelfand-Kirillov dimension $n-1$ as $SL(n,\mathbb{R})$-module, and we may expect their p-adic number part π_p of π should also be quite small.

Remark 4.2. *In this paper, we consider everything over the integer ring $\mathcal{O}_F$ of F. However similarly as Barner [1], we can replace $\mathcal{O}_F$ by an arbitrary order $\mathcal{O}$ in F, and can consider Grössencharacters for ring class groups. But to handle general ray class groups, one has to start from Epstein zeta functions belonging to congruence subgroups of $SL(n,\mathbb{R})$. This amounts to discuss the ramified primes p dividing the levels of congruence subgroups and it is another job.*

Remark 4.3. *By using Epstein zeta functions with harmonic polynomials, one might replace the complicated argument of R. Sczech [10] utilizing conditional convergence by a similar argument to use only absolute convergence.*

Remark 4.4. *In [4], at least for $n = 3$ it is checked that the validity of local multiplicity one theorem for our degenerate principal series. Compare with a computational result in Terras [12]*

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