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INVERSE SOURCE PROBLEM FOR THE NAVIER-STOKES EQUATIONS

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ABSTRACT. We consider an inverse problem of determining a spatially varying factor in a source term in a nonstationary Navier-Stokes equations by observation data in a neighbourhood of the boundary. We prove the Lipschitz stability provided that the t-dependent factor satisfies a non-degeneracy condition. For the proof, we show a Carleman estimate for the vorticity equation of the Navier-Stokes equations.

§1. Introduction and the main results.

We consider the Navier-Stokes equations for an incompressible viscous fluid:

$$\partial_t v(x,t) - \nu\Delta v(x,t) + (v\cdot\nabla)v + \nabla p = R(x,t)f(x),$$
$$x\in\Omega,\ 0<t<T, \tag{1.1}$$

$$\operatorname{div} v(x,t) = 0, \qquad x\in\Omega,\ 0<t<T, \tag{1.2}$$

$$v(x,t) = 0, \qquad x\in\partial\Omega,\ 0<t<T. \tag{1.3}$$

Here $\Omega\subset\mathbb{R}^3$ is a bounded domain with C^2-boundary $\partial\Omega$, $v=(v_1,v_2,v_3)^T$, $\cdot^T$ denotes the transpose of matrices, $\nu>0$ is a constant describing the viscosity, and

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for simplicity we assume that the density is one. Let $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1, 2, 3$, $\Delta = \sum_{j=1}^3 \partial_j^2$, $\nabla = (\partial_1, \partial_2, \partial_3)^T$,

$$(v \cdot \nabla)v = \left(\sum_{j=1}^3 v_j \partial_j v_1, \sum_{j=1}^3 v_j \partial_j v_2, \sum_{j=1}^3 v_j \partial_j v_3 \right)^T.$$

Moreover let $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{N} \cup \{0\})^3$, $\partial_x^\gamma = \partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3}$ and $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$.

Physically v denotes the velocity field of the incompressible fluid and the term $R(x,t)f(x)$ models the density of the external force causing the movement of the fluid. In this paper, we consider the two forms:

$$R(x,t) = (r_1(x,t), r_2(x,t) r_2(x,t))^T, \quad f = f(x),\ r_j = r_j(x,t),\ j = 1,2,3\text{: real-valued.} \tag{1.4}$$

In the forward problem we are required to discuss the unique existence of solutions in suitable senses to (1.1) - (1.3) for a given extrernal source term Rf and there are a vast amount of works (e.g., Ladyzhenskaya [31], Temam [35] and the references therein). The forward problem is important, but any practical studies of the forward problem can be launched only after suitable modelling of physical parameters such as the viscosity ν, the force Rf. The inverse source problems are concerned with such modelling. In our inverse problem, we are mainly discussing the determination of a spatially varying function $f(x)$ for given $R(x,t)$.

Inverse Source Problem. *Let $\omega \subset \Omega$ be a given subdomain such that $\partial\omega \supset \partial\Omega$, $0 < \theta < T$ and let v satisfy (1.1) - (1.3). Then determine $f(x)$, $x \in \Omega$ by observation data $v|_{\omega\times(0,T)}$, $v(x,\theta)$, $x \in \Omega$.*

Inverse problems by this type of observations for the Navier-Stokes equations have been not studied sufficiently by taking into consideration its physical significance. See Imanuvilov and Yamamoto [16]. As for different kinds of inverse problems for the Navier-Stokes equations, see Prilepko, Orlovsky and Vasin [34] and the references therein. In [34], the authors discuss inverse problems by final overdetermining observation data $u(x,T)$, $x \in \Omega$.

In this paper, we study only one case where unknown f is real-valued, and in a more comprehensive forthcoming paper, we will discuss a more general subdomain ω and establish stability estimates in determining vector-valued f in the case where $R(x,t)$ is a 3×3 matrix.

For a non-empty subdomain $\omega_1 \subset \Omega$ such that $\overline{\omega_1} \subset \omega$ and $\partial\omega_1 \supset \partial\Omega$, let $\eta \in C^2(\overline{\Omega})$ satisfy

$$\eta > 0 \quad \text{in } \Omega, \quad \eta|_{\partial\Omega} = 0, \quad |\nabla \eta| > 0 \quad \text{on } \overline{\Omega \setminus \omega_1}. \tag{1.5}$$

As for the existence of η, see Fursikov and Imanuvilov [10], Imanuvilov [12].

Example of η. Let $\Omega = \{x \in \mathbb{R}^3; \rho_1 < |x| < \rho_2\}$ with $0 < \rho_1 < \rho_2$ and $\omega_1 = \{x \in \mathbb{R}^3; \rho_2 - \delta < |x| < \rho_2\}$ where $\delta > 0$ is sufficiently small. Then we can directly verify that

$$\eta(x) = (\rho_2^{2m} - |x|^{2m})(|x|^{2m} - \rho_1^{2m})$$

satisfies (1.5) if $m \in \mathbb{N}$ is sufficiently large for $\delta > 0$.

In fact, we have

$$\nabla \eta(x) = 2mx|x|^{2m-2}(\rho_1^{2m} + \rho_2^{2m} - 2|x|^{2m})$$

and $\left(\frac{\rho_1^{2m}+\rho_2^{2m}}{2}\right)^{\frac{1}{2m}} > |x| \ge \rho_1$ implies $|\nabla\eta(x)| > 0$. Since $\lim_{m\to\infty} \left(\frac{\rho_1^{2m}+\rho_2^{2m}}{2}\right)^{\frac{1}{2m}} = \rho_2$ by $\rho_2 > \rho_1$, we see that for small $\delta > 0$, we can choose large $m \in \mathbb{N}$ such that $|\nabla\eta(x)| > 0$ if $x \in \overline{\Omega \setminus \omega_1}$.

We set

$$Q = \Omega \times (0,T)$$

and

$$W_2^{1,2}(Q) = \{w;\ \partial_t w, \partial_x^\gamma w \in L^2(Q), |\gamma| \le 2\}.$$

Throughtout this paper, we assume that the velocity field v and the pressure p are sufficiently regular and bounded:

$$\begin{aligned} &\partial_t^j v, \partial_t^j(\mathrm{rot}\, v) \in W_2^{1,2}(Q), \quad \partial_t^j p \in L^2(0,T;H^1(\Omega)), \quad j=0,1,2, \\ &\sum_{j=0}^{2} \|\partial_t^j v\|_{L^\infty(Q)} + \|\nabla v\|_{L^\infty(Q)} + \|\partial_t \nabla v\|_{L^\infty(Q)} \le M. \end{aligned} \tag{1.6}$$

Remark. We can relax the regularity if we will use a Carleman estimate involving H^{-1}-norms. In this paper, however, for a simpler treatment, we will use a conventional Carleman estimate without Sobolev norms of negative orders.

We are ready to state our first main result.

Theorem 1. *Let ω be a subdomain of Ω such that $\partial\omega \supset \partial\Omega$. Let $0 < \theta < T$ and let $R(x,t) = (r_1(x,t), r_2(x,t), r_3(x,t))^T$ satisfy*

$$R(\cdot,\theta) \in C^2(\overline{\Omega}), \quad \partial_t^j R, \partial_t^j rot R \in L^\infty(Q), \quad j=0,1,2$$

and $f(x)$ be a real-valued function. We assume that

$$f|_\omega = 0, \quad R(x,\theta) \times \nabla\eta(x) \ne 0, \quad x \in \overline{\Omega \setminus \omega}. \tag{1.7}$$

Then there exists a constant $C = C(\Omega, T, \theta, R, M) > 0$ such that

$$\begin{aligned} &\|f\|_{H^1(\Omega)} \le C(\|rot v\|_{H^3(0,T;L^2(\omega))} + \|v\|_{H^2(0,T;H^3(\omega))} \\ &+\|rot v(\cdot,\theta)\|_{H^3(\Omega)} + \|v(\cdot,\theta)\|_{H^2(\Omega)}). \end{aligned} \tag{1.8}$$

For determination of f, we have to assume the non-degeneracy condition on R given by (1.7). In Theorem 1, we notice that for ω, we need the geometric constraint $\partial\omega \supset \partial\Omega$, which seems strange in view of the parabolicity of the equation. In fact, in the corresponding inverse parabolic problem (e.g., Imanuvilov and Yamamoto [15]), we need not any constraints for ω. However, when we do not use data of the pressure $p(x,t)$, our inverse problem is involved with a first-order equation $\mathrm{rot}\, Rf = g$ with given g, so that we have to assume some geometric conditions for ω. Here, for simplicity, we assume that $\partial\omega \supset \partial\Omega$.

Example of (1.7). Let $\Omega = \{x \in \mathbb{R}^3; \rho_1 < |x| < \rho_2\}$ with $0 < \rho_1 < \rho_2$ and $\omega = \{x \in \mathbb{R}^3; \rho_2 - \delta < |x| < \rho_2\}$ with sufficiently small $\delta > 0$. Then in the previous example, if $m \in \mathbb{N}$ is sufficiently large and $x \times R(x,\theta) \neq 0$, $x \in \overline{\Omega \setminus \omega}$, then (1.7) is satisfied.

With (1.7), our observation data yield the Lipschitz stability. We note that $\theta > 0$. If $\theta = 0$, then our inverse problem is exactly an inverse problem to the forward problem, that is, the initial/ boundary value problem. However, as the corresponding inverse problem for a parabolic equation is open in the case of $\theta = 0$ (cf. Isakov [23], [24]), our inverse problem with $\theta = 0$ is an open problem.

Our main methodology is based on Bukhgeim and Klibanov [7] which introduced the application of a Carleman estimate to inverse problems (also see Isakov [22], Klibanov [28], [29]). Our proof is by Imanuvilov and Yamamoto [15] which modified the method in [7].

As for similar inverse problems, we refer to the following works: Amirov and Yamamoto [1], Baudouin and Puel [2], Bellassoued [3], [4], Bellassoued and Yamamoto [5], Bukhgeim [6], Imanuvilov, Isakov and Yamamoto [14], Imanuvilov and Yamamoto [17] - [21], Isakov [22] - [24], Isakov and Yamamoto [25], Khaĭdarov [26], [27], Klibanov and Timonov [30], Li [32], Li and Yamamoto [33], Yamamoto [36]. This list is far from the complete and the readers can consult the references therein.

Our proof uses also a Carleman estimate for the Navier-Stokes equations, for which we refer to Fernández-Cara, Guerrero, Imanuvilov and Puel [8], [9]. See also Fursikov and Imanuvilov [10], Imanuvilov [13].

§2. Key Carleman estimate.

We establish a key Carleman estimate in the case where $\partial\omega \supset \partial\Omega$. We consider

$$\partial_t v(x,t) - \nu\Delta v + (q(x,t)\cdot\nabla)v + \nabla p = F(x,t), \quad x \in \Omega,\, 0 < t < T, \tag{2.1}$$

$$\operatorname{div} v(x,t) = 0, \qquad x \in \Omega,\, 0 < t < T, \tag{2.2}$$

$$v(x,t) = 0, \qquad x \in \partial\Omega,\, 0 < t < T. \tag{2.3}$$

Here $q = (q_1, q_2, q_3)^T \in L^\infty(0,T; W^{1,\infty}(\Omega))$ with $\|q\|_{L^\infty(Q)}, \|\nabla q\|_{L^\infty(Q)} \le M$. Let ω, ω_1 be subdomains such that $\overline{\omega_1} \subset \omega$ and $\omega_1 \ne \emptyset$ and let $\eta \in C^2(\overline{\Omega})$ satisfy (1.5). We set

$$\alpha(x,t) = \frac{e^{\lambda\eta(x)} - e^{2\lambda\|\eta\|_{L^\infty(\Omega)}}}{t(T-t)}, \tag{2.4}$$

$$\varphi(x,t) = \frac{e^{\lambda\eta(x)}}{t(T-t)} \tag{2.5}$$

with large parameter $\lambda > 0$, and

$$Q_\omega = \omega \times (0,T).$$

We can state our key Carleman estimate:

Theorem 2.1. *Let $\partial\omega \supset \partial\Omega$. Then there exists a constant $\Lambda = \Lambda(\Omega,\omega,T) > 0$ such that for $\lambda > \Lambda$, we can choose constants $C = C(\lambda, M) > 0$ and $s_0 = s_0(\Lambda, M) > 0$ such that*

$$\int_Q \left(\frac{s^5}{t^5(T-t)^5}|v|^2 + \frac{s^3}{t^3(T-t)^3}|\nabla v|^2 + \frac{s^4}{t^4(T-t)^4}|rot\, v|^2 \right.$$
$$\left. + \frac{s^2}{t^2(T-t)^2}|\nabla rot\, v|^2 \right) e^{2s\alpha} dxdt$$
$$\le C \int_Q \frac{s}{t(T-t)}|rot\, F|^2 e^{2s\alpha} dxdt + Ce^{Cs}(\|\partial_t(rot\, v)\|^2_{L^2(Q_\omega)} + \|v\|^2_{L^2(0,T;H^3(\omega))})$$

for all $s \ge s_0$ and all $v \in W^{1,2}_2(Q)$ such that $rot\, v \in W^{1,2}_2(Q)$ and v satisfies (2.1) - (2.3) and $\|v\|_{L^\infty(Q)}, \|rot\, v\|_{L^\infty(Q)} \le M$.

The Carleman estimate for the Navier-Stokes equations has been studied for the controllability and see Fernández-Cara, Guerrero, Imanuvilov and Puel [8], [9], Fursikov and Imanuvilov [10], Imanuvilov [13]. In the case where $\partial\omega \supset \partial\Omega$, we can derive a Carleman estimate on the basis of the vorticity equation (i.e., the parabolic equation in $\mathrm{rot}\, v$), so that we need not treat ∇p which is different from the Carleman estimate by [8], [9].

For the proof, we show two Carleman estimates.

Lemma 2.1. *Let*

$$P_0 y = \partial_t y - \nu\Delta y + A(x,t)\cdot\nabla y + A_0 y = g \quad in\ Q, \tag{2.6}$$

where $\|A\|_{L^\infty(Q)}, \|A_0\|_{L^\infty(Q)} \le M$, $g \in L^2(Q)$. Then there exists a constant $\widehat{\lambda} > 0$ such that for $\lambda > \widehat{\lambda}$, we can choose constants $C_1 = C_1(\Omega,\omega,T,\lambda,M) > 0$ and

$s_0 = s_0(\lambda) > 0$ *such that*

$$\int_Q \left(\frac{s^2}{t^2(T-t)^2}|\nabla y|^2 + \frac{s^4}{t^4(T-t)^4}|y|^2 \right) e^{2s\alpha} dxdt$$
$$\le C_1 \int_Q \frac{s}{t(T-t)}|g|^2 e^{2s\alpha} dxdt + C_1 e^{C_1 s}(\|\partial_t y\|^2_{L^2(Q_\omega)} + \|y\|^2_{L^2(0,T;H^2(\omega))}) \tag{2.7}$$

for all $s \ge s_0$ *and all* $y \in W_2^{1,2}(Q)$.

Here and henceforth C, C_j denote generic constants which are dependent on $\Omega, \omega, T, \lambda, M$, but independent of s.

This is a Carleman estimate which is global in Q and is with a singular weight function. As for the proof, see Fursikov and Imanuvilov [10], Imanuvilov [12]. In fact, if $y|_{\partial\Omega\times(0,T)} = 0$, then the conclusion follows directly from [10], [12]. Let $y|_{\partial\Omega\times(0,T)} \ne 0$. Henceforth, without loss of generality, we may assume that $\partial\omega$ is sufficiently smooth. If not, then we can take a subdomain $\omega' \subset \Omega$ such that $\partial\omega' \supset \partial\Omega$ and $\partial\omega'$ is smooth. Therefoe, by the extension theorem, we can find a function $\widetilde{y}$ such that $\widetilde{y} = y$ in Q_ω and

$$\|\partial_t \widetilde{y}\|_{L^2(Q)} + \|\widetilde{y}\|_{L^2(0,T;H^2(\Omega))} \le C(\|\partial_t y\|_{L^2(Q_\omega)} + \|y\|_{L^2(0,T;H^2(\omega))}). \tag{2.8}$$

Set $v = y - \widetilde{y}$. Then, noting that $\partial\omega \supset \partial\Omega$, we see that $P_0 v = g - P_0\widetilde{y} = g - (\partial_t\widetilde{y} - \nu\Delta\widetilde{y} + A\cdot\nabla\widetilde{y} + A_0\widetilde{y})$ and $v|_{\partial\Omega\times(0,T)} = 0$ and $v|_{Q_\omega} = 0$. Therefore the Carleman estimate in [10], [12] yields

$$\int_Q \left(\frac{s^2}{t^2(T-t)^2}|\nabla v|^2 + \frac{s^4}{t^4(T-t)^4}|v|^2 \right) e^{2s\alpha} dxdt$$
$$\le C\int_Q \frac{s}{t(T-t)}|g|^2 e^{2s\alpha}dxdt + C\int_Q \frac{s}{t(T-t)}|\partial_t\widetilde{y} - \nu\Delta\widetilde{y} + A\cdot\nabla\widetilde{y} + A_0\widetilde{y}|^2 e^{2s\alpha}dxdt$$
$$\le C\int_Q \frac{s}{t(T-t)}|g|^2 e^{2s\alpha}dxdt + Ce^{Cs}\int_Q (|\partial_t\widetilde{y}|^2 + |\Delta\widetilde{y}|^2 + |\nabla\widetilde{y}|^2 + |\widetilde{y}|^2)dxdt.$$

This and (2.8) yield the conclusion (2.7).

Next we show a conventional Carleman estimate for Δ.

Lemma 2.2. *There exists a number $\widetilde{\lambda} > 0$ such that for any $\lambda > \widetilde{\lambda}$, we can choose constants $s_0 = s_0(\lambda) > 0$ and $C_2 = C_2(\Omega, \omega, \lambda) > 0$ such that*

$$\begin{aligned}&\int_Q \left(\frac{s^3}{t^3(T-t)^3} |\nabla v(x,t)|^3 + \frac{s^5}{t^5(T-t)^5} |v(x,t)|^2 \right) e^{2s\alpha} dxdt \\ \le & C_2 \int_Q \frac{s^2}{t^2(T-t)^2} |\Delta v(x,t)|^2 e^{2s\alpha} dxdt + C_2 e^{C_2 s} \|v\|^2_{L^2(0,T;H^2(\omega))} \end{aligned} \tag{2.9}$$

for all $v \in L^2(0,T;H^2(\Omega))$ and all $s \ge s_0$.

Proof of Lemma 2.2. By the Sobolev extenstion theorem, we can find $\widetilde{v}(x,t)$ such that $\widetilde{v} = v$ in Q_ω and

$$\|\widetilde{v}\|_{L^2(0,T;H^2(\Omega))} \le C_2 \|v\|_{L^2(0,T;H^2(\omega))}. \tag{2.10}$$

Setting $V = v - \widetilde{v}$, we have $V(\cdot, t) \in H^2_0(\Omega)$ for almost all $t \in (0,T)$ by $\partial\omega \supset \partial\Omega$. Therefore, by $|\nabla \eta| \ne 0$ on $\overline{\Omega \setminus \omega_1}$, we can apply a classical Carleman estimate for Δ (e.g., Hörmander [11]) in terms of $V = 0$ in Q_ω, so that

$$\begin{aligned}&\int_\Omega (\tau |\nabla V(x,t)|^2 + \tau^3 |V(x,t)|^2) \exp(2\tau e^{\lambda\eta(x)}) dx \\ =&\int_{\Omega\setminus\overline{\omega}} (\tau |\nabla V(x,t)|^2 + \tau^3 |V(x,t)|^2) \exp(2\tau e^{\lambda\eta(x)}) dx \\ \le & C_2 \int_{\Omega\setminus\overline{\omega}} |\Delta v(x,t) - \Delta\widetilde{v}(x,t)|^2 \exp(2\tau e^{\lambda\eta(x)}) dx\end{aligned}$$

for $\tau \ge \tau_0$: a constant. Hence

$$\begin{aligned}&\int_\Omega (\tau |\nabla v(x,t)|^2 + \tau^3 |v(x,t)|^2) \exp(2\tau e^{\lambda\eta(x)}) dx \\ \le & C_2 \int_\Omega |\Delta v(x,t)|^2 \exp(2\tau e^{\lambda\eta(x)}) dx \\ + & C_2 \int_\Omega (|\Delta\widetilde{v}(x,t)|^2 + \tau |\nabla\widetilde{v}(x,t)|^2 + \tau^3 |\widetilde{v}(x,t)|^2) \exp(2\tau e^{\lambda\eta(x)}) dx.\end{aligned}$$

We fix $s_1 > 0$ sufficiently large, so that $\frac{s_1}{t(T-t)} \ge \frac{4}{T^2} s_1 > \tau_0$. Then, if $s \ge s_2 \equiv$

$\max\{s_0, s_1\}$, then $\tau = \frac{s}{t(T-t)} > \tau_0$. Therefore

$$\int_\Omega \left(\frac{s^3}{t^3(T-t)^3} |\nabla v(x,t)|^2 + \frac{s^5}{t^5(T-t)^5} |v(x,t)|^2 \right) e^{2s\varphi(x,t)} dx$$
$$\le C_2 \int_\Omega \frac{s^2}{t^2(T-t)^2} |\Delta v(x,t)|^2 e^{2s\varphi} dx$$
$$+ C_2 \int_\Omega \left(\frac{s^2}{t^2(T-t)^2} |\Delta \widetilde{v}(x,t)|^2 + \frac{s^3}{t^3(T-t)^3} |\nabla \widetilde{v}(x,t)|^2 + \frac{s^5}{t^5(T-t)^5} |\widetilde{v}(x,t)|^2 \right) e^{2s\varphi} dx$$

for all $s \ge s_2$. Multiplying the both hand sides with $\exp\left(-2s\frac{e^{2\lambda\|\eta\|_{L^\infty(\Omega)}}}{t(T-t)}\right)$, noting that C_2 is independent of t and integrating in $t \in (0,T)$, we obtain

$$\int_Q \left(\frac{s^3}{t^3(T-t)^3} |\nabla v(x,t)|^2 + \frac{s^5}{t^5(T-t)^5} |v(x,t)|^2 \right) e^{2s\alpha} dxdt$$
$$\le C_2 \int_Q \frac{s^2}{t^2(T-t)^2} |\Delta v(x,t)|^2 e^{2s\alpha} dxdt$$
$$+ C_2 s^5 \int_\Omega \left(\frac{e^{2s\alpha}}{t^2(T-t)^2} |\Delta \widetilde{v}(x,t)|^2 + \frac{e^{2s\alpha}}{t^3(T-t)^3} |\nabla \widetilde{v}(x,t)|^2 + \frac{e^{2s\alpha}}{t^5(T-t)^5} |\widetilde{v}(x,t)|^2 \right) dxdt.$$

Since

$$\max_{(x,t)\in\overline{Q}, k=1,3,5} \left| \frac{1}{t^k(T-t)^k} e^{2s\alpha(x,t)} \right| \equiv C_3 < \infty,$$

the last term at the right hand side is bounded by $C_2C_3 s^5 \|\widetilde{v}\|^2_{L^2(0,T;H^2(\Omega))}$. In terms of (2.10), the proof of the lemma is complete.

We proceed to

Proof of Theorem 2.1. Set $z = \mathrm{rot}\, v$. Then, by noting that $\mathrm{rot}\,\mathrm{rot}\, w = -\Delta w + \nabla \mathrm{div}\, w$, (2.1) - (2.3) imply

$$\partial_t z - \nu \Delta z + \sum_{j=1}^3 \nabla q_j \times \partial_j v + (q\cdot\nabla) z = \mathrm{rot}\, F \quad \text{in } Q, \tag{2.11}$$

and

$$\Delta v = -\mathrm{rot}\, z \qquad \text{in } Q, \tag{2.12}$$

$$v(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{2.13}$$

Applying Lemma 2.1 to (2.11), we have

$$\begin{aligned}&\int_Q\left(\frac{s^2}{t^2(T-t)^2}|\nabla z|^2+\frac{s^4}{t^4(T-t)^4}|z|^2e^{2s\alpha}dxdt\right.\\ \le& C\int_Q\frac{s}{t(T-t)}|\nabla v|^2|\nabla q|^2e^{2s\alpha}dxdt\\ +&C\int_Q\frac{s}{t(T-t)}|\mathrm{rot}\,F|^2e^{2s\alpha}dxdt+Ce^{Cs}(\|\partial_t\mathrm{rot}\,v\|^2_{L^2(Q_\omega)}+\|v\|^2_{L^2(0,T;H^3(\omega))})\end{aligned}\tag{2.14}$$

for all large $s>0$.

On the other hand, applying Lemma 2.2 to (2.12) and (2.13), we have

$$\begin{aligned}&\int_Q\left(\frac{s^3}{t^3(T-t)^3}|\nabla v(x,t)|^2+\frac{s^5}{t^5(T-t)^5}|v(x,t)|^2\right)e^{2s\alpha}dxdt\\ \le& C_2\int_Q\frac{s^2}{t^2(T-t)^2}|\nabla z|^2e^{2s\alpha}dxdt+C_2e^{C_2s}\|v\|^2_{L^2(0,T;H^2(\omega))}\end{aligned}\tag{2.15}$$

for all large $s>0$. Inequalities (2.14) and (2.15) yield

$$\begin{aligned}&\int_Q\left(\frac{s^3}{t^3(T-t)^3}|\nabla v|^2+\frac{s^5}{t^5(T-t)^5}|v|^2+\frac{s^2}{t^2(T-t)^2}|\nabla z|^2+\frac{s^4}{t^4(T-t)^4}|z|^2\right)e^{2s\alpha}dxdt\\ \le& CM^2\int_Q\frac{s}{t(T-t)}|\nabla v|^2e^{2s\alpha}dxdt+C\int_Q\frac{s}{t(T-t)}|\mathrm{rot}\,F|^2e^{2s\alpha}dxdt\\ +&Ce^{Cs}(\|\partial_t\mathrm{rot}\,v\|^2_{L^2(Q_\omega)}+\|v\|^2_{L^2(0,T;H^3(\omega))})\end{aligned}$$

for $s\ge s_0$. Here we used also $\|\nabla q\|_{L^\infty(Q)}\le M$. Taking $s>0$ large, we can absorb the first term at the right hand side into the left hand side, so that we complete the proof of Theorem 2.1.

We conclude this section with a Carleman estimate of a first-order equation.

Lemma 2.3. *Let $\alpha_0\in C^2(\overline{\Omega})$ and*

$$Lf(x)=\sum_{j=1}^3a_j(x)\partial_jf(x),\qquad x\in\Omega,$$

where $a_j\in C^1(\overline{\Omega})$, $1\le j\le 3$, and let us set $\mu(x)=\sum_{j=1}^3a_j(x)\partial_j\alpha_0(x)$, $x\in\Omega$. Then there exists a number $\widetilde{\lambda}>0$ such that for any $\lambda>\widetilde{\lambda}$, we can choose $s_3=$

$s_3(\lambda) > 0$ *satisfying: there exists a constant* $C = C(\Omega, \omega, \lambda) > 0$ *such that*

$$
\begin{aligned}
& s^2 \int_\Omega \left(\mu^2(x) - \frac{C}{s}\right)(|f|^2 + |\nabla f|^2) e^{2s\alpha_0(x)} dx \\
\le & C \int_\Omega (|\nabla(Lf)|^2 + |Lf|^2) e^{2s\alpha_0(x)} dx
\end{aligned}
$$

for all $s \ge s_3(\lambda)$ *and all* $f \in H_0^2(\Omega)$.

Proof of Lemma 2.3. For simplicity, we set $g = e^{s\alpha_0} f$ and $L_0 g = e^{s\alpha_0} L(e^{-s\alpha_0} g)$.

Then

$$\int_\Omega |Lf|^2 e^{2s\alpha_0} dx = \int_\Omega |L_0 g|^2 dx.$$

Direct calculations show that $L_0 g(x) = Lg(x) - s\mu(x) g(x)$. Therefore, by integrations by parts, we obtain

$$
\begin{aligned}
& \|L_0 g\|^2_{L^2(\Omega)} = \|Lg\|^2_{L^2(\Omega)} + s^2 \|\mu g\|^2_{L^2(\Omega)} - 2s \int_\Omega \sum_{j=1}^3 a_j (\partial_j g) \mu g dx \\
\ge & s^2 \int_\Omega \mu^2(x) g^2(x) dx - s \int_\Omega \sum_{j=1}^3 a_j \mu \partial_j (g^2) dx \\
= & s^2 \int_\Omega \mu^2(x) g^2(x) dx + s \int_\Omega \sum_{j=1}^3 \partial_j (a_j \mu) g^2 dx.
\end{aligned}
$$

Therefore

$$s^2 \int_\Omega \left(\mu^2(x) - \frac{C}{s}\right) |f|^2 e^{2s\alpha_0} dx \le C \int_\Omega |Lf|^2 e^{2s\alpha_0} dx. \tag{2.16}$$

Setting $Lf = h$, we have $L(\partial_k f) = \partial_k h - \sum_{j=1}^3 (\partial_k a_j) \partial_j f$. Since $\partial_k f = 0$ on $\partial\Omega$, we repeat the above argument to obtain

$$
\begin{aligned}
& s^2 \int_\Omega \left(\mu^2(x) - \frac{C}{s}\right) |\partial_k f|^2 e^{2s\alpha_0} dx \\
\le & C \int_\Omega |\nabla h|^2 e^{2s\alpha_0} dx + C \int_\Omega |\nabla f|^2 e^{2s\alpha_0} dx, \quad 1 \le k \le 3,
\end{aligned}
$$

that is,

$$s^2 \int_\Omega \left(\mu^2(x) - \frac{C}{s}\right) |\nabla f|^2 e^{2s\alpha_0} dx \le C \int_\Omega (|\nabla h|^2 + |\nabla f|^2) e^{2s\alpha_0} dx.$$

Hence

$$s^2\int_\Omega\left(\mu^2(x)-\frac{C}{s}\right)|\nabla f|^2e^{2s\alpha_0}dx \le s^2\int_\Omega\left(\mu^2(x)-\frac{C}{s}-\frac{C}{s^2}\right)|\nabla f|^2e^{2s\alpha_0}dx$$
$$\le C\int_\Omega|\nabla h|^2e^{2s\alpha_0}dx$$

for all large $s>0$. This and (2.16) completes the proof of the lemma.

§3. Proof of Theorem 1.

Without loss of generality, we may assume that $\theta=\frac{T}{2}$. Because we can choose small $\kappa>0$ such that $0<\theta-\kappa<\theta+\kappa<T$ and we can discuss the whole problem in the time interval $(\theta-\kappa,\theta+\kappa)$. Regarding $\theta-\kappa$ and $\theta+\kappa$ as 0 and T respectively, we can argue.

Let us set $w_1=\partial_t v$ and $w_2=\partial_t^2 v$. Then

$$\partial_t v-\nu\Delta v+(v\cdot\nabla)v+\nabla p=Rf$$

$$\partial_t w_1-\nu\Delta w_1+(v\cdot\nabla)w_1+(w_1\cdot\nabla)v+\nabla(\partial_t p)=(\partial_t R)f$$

$$\partial_t w_2-\nu\Delta w_2+(v\cdot\nabla)w_2+2(w_1\cdot\nabla)w_1+(w_2\cdot\nabla)v$$
$$+\nabla(\partial_t^2 p)=(\partial_t^2 R)f$$

$$\operatorname{div} v=\operatorname{div} w_1=\operatorname{div} w_2=0\quad\text{in } Q$$

and

$$v=w_1=w_2=0\qquad\text{on }\partial\Omega\times(0,T).$$

Here and henceforth we set

$$\mathcal{D}=\sum_{k=1}^3\|\partial_t^k\mathrm{rot}\, v\|_{L^2(Q_\omega)}+\|v\|_{H^2(0,T;H^3(\omega))}. \tag{3.1}$$

Therefore applications of Theorem 2.1 to v, w_1, w_2 yield

$$\begin{aligned}
&\int_Q\Bigg(\frac{s^4}{t^4(T-t)^4}\sum_{j=0}^2|\partial_t^j\mathrm{rot}\,v|^2+\frac{s^2}{t^2(T-t)^2}\sum_{j=0}^2|\nabla\partial_t^j\mathrm{rot}\,v|^2\\
&+\frac{s^5}{t^5(T-t)^5}\sum_{j=0}^2|\partial_t^jv|^2+\frac{s^3}{t^3(T-t)^3}\sum_{j=0}^2|\partial_t^j\nabla v|^2\Bigg)e^{2s\alpha}dxdt\\
\le& C\int_Q\frac{s}{t(T-t)}(|\mathrm{rot}\,((w_1\cdot\nabla)v)|^2+|\mathrm{rot}\,((w_1\cdot\nabla)w_1)|^2+|\mathrm{rot}\,((w_2\cdot\nabla)v)|^2)e^{2s\alpha}dxdt\\
&+C\int_Q\frac{s}{t(T-t)}\left|\mathrm{rot}\,\left(\sum_{j=0}^2\partial_t^jR\right)f\right|^2e^{2s\alpha}dxdt+Ce^{Cs}\mathcal{D}^2.
\end{aligned}$$

Here

$$\mathrm{rot}\,((w_1\cdot\nabla)v)=(\partial_tv\cdot\nabla)\mathrm{rot}\,v+\sum_{j=1}^3\nabla(\partial_tv_j)\times\partial_jv,$$

$$\mathrm{rot}\,((w_1\cdot\nabla)w_1)=(\partial_tv\cdot\nabla)\partial_t\mathrm{rot}\,v+\sum_{j=1}^3\nabla(\partial_tv_j)\times\partial_t\partial_jv,$$

$$\mathrm{rot}\,((w_2\cdot\nabla)v)=(\partial_t^2v\cdot\nabla)\mathrm{rot}\,v+\sum_{j=1}^3\nabla(\partial_t^2v_j)\times\partial_jv$$

and

$$\begin{aligned}
&|\mathrm{rot}\,((w_1\cdot\nabla)v)|^2+|\mathrm{rot}\,((w_1\cdot\nabla)w_1)|^2+|\mathrm{rot}\,((w_2\cdot\nabla)v)|^2\\
\le& CM^2(|\nabla\mathrm{rot}\,v|^2+|\nabla(\partial_t\mathrm{rot}\,v)|^2+|\nabla\partial_tv|^2+|\nabla\partial_t^2v|^2)
\end{aligned}$$

by the bounds in (1.6). Therefore

$$\begin{aligned}
&\int_Q\Bigg(\frac{s^4}{t^4(T-t)^4}\sum_{j=0}^2|\partial_t^j\mathrm{rot}\,v|^2+\frac{s^2}{t^2(T-t)^2}\sum_{j=0}^2|\nabla\partial_t^j\mathrm{rot}\,v|^2\\
&+\frac{s^5}{t^5(T-t)^5}\sum_{j=0}^2|\partial_t^jv|^2+\frac{s^3}{t^3(T-t)^3}\sum_{j=0}^2|\partial_t^j\nabla v|^2\Bigg)e^{2s\alpha}dxdt\\
\le& CM^2\int_Q\frac{s}{t(T-t)}(|\nabla\mathrm{rot}\,v|^2+|\nabla(\partial_t\mathrm{rot}\,v)|^2+|\nabla\partial_tv|^2+|\nabla\partial_t^2v|^2)e^{2s\alpha}dxdt\\
&+C\int_Q\frac{s}{t(T-t)}(|f|^2+|\nabla f|^2)e^{2s\alpha}dxdt+Ce^{Cs}\mathcal{D}^2 \qquad (3.2)
\end{aligned}$$

for all large $s > 0$.

The first term at the right hand side can be absorbed into the left hand side of (3.2) by taking $s > 0$ sufficiently large, so that

$$\begin{aligned}
&\int_Q \frac{s^2}{t^2(T-t)^2} \sum_{j=0}^{2} \sum_{|\gamma|\le 1} |\partial_t^j \partial_x^\gamma \mathrm{rot}\, v|^2 e^{2s\alpha} dxdt \\
\le& C \int_Q \frac{s}{t(T-t)} (|f|^2 + |\nabla f|^2) e^{2s\alpha} dxdt + Ce^{Cs}\mathcal{D}^2
\end{aligned} \tag{3.3}$$

for all large $s > 0$.

Noting that $e^{2s\alpha(x,0)} = 0$ for $x \in \overline{\Omega}$, we have

$$\begin{aligned}
&\int_\Omega \sum_{|\gamma|\le 1} \left| \partial_t \partial_x^\gamma \mathrm{rot}\, v \left(x, \frac{T}{2}\right)\right|^2 e^{2s\alpha(x,T/2)} dx \\
=& \int_\Omega \frac{\partial}{\partial t} \left(\int_0^{T/2} \sum_{|\gamma|\le 1} |\partial_t \partial_x^\gamma \mathrm{rot}\, v|^2 e^{2s\alpha} dt \right) dx \\
=& \int_\Omega \int_0^{T/2} \left\{ 2 \sum_{|\gamma|\le 1} (\partial_t \partial_x^\gamma \mathrm{rot}\, v \cdot \partial_t^2 \partial_x^\gamma \mathrm{rot}\, v) + 2s(\partial_t \alpha) \sum_{|\gamma|\le 1} |\partial_t \partial_x^\gamma \mathrm{rot}\, v|^2 \right\} e^{2s\alpha} dxdt \\
\le& C \int_Q \left\{ \sum_{|\gamma|\le 1} |\partial_t \partial_x^\gamma \mathrm{rot}\, v|^2 + |\partial_t^2 \partial_x^\gamma \mathrm{rot}\, v|^2 \right\} e^{2s\alpha} dxdt \\
&+ C \int_Q \frac{s}{t^2(T-t)^2} \sum_{|\gamma|\le 1} |\partial_t \partial_x^\gamma \mathrm{rot}\, v|^2 e^{2s\alpha} dxdt.
\end{aligned}$$

Here we used

$$|\partial_t \alpha(x,t)| = \left| \frac{2t - T}{t^2(T-t)^2} (e^{\lambda\eta(x)} - e^{2\lambda\|\eta\|_{L^\infty(\Omega)}}) \right| \le \frac{C}{t^2(T-t)^2}, \quad (x,t) \in \overline{Q}.$$

Hence (3.3) implies

$$\begin{aligned}
&\int_\Omega \sum_{|\gamma|\le 1} \left| \partial_t \partial_x^\gamma \mathrm{rot}\, v \left(x, \frac{T}{2}\right)\right|^2 e^{2s\alpha(x,T/2)} dx \\
\le& C \int_Q \frac{s}{t(T-t)} (|f|^2 + |\nabla f|^2) e^{2s\alpha} dxdt + Ce^{Cs}\mathcal{D}^2
\end{aligned} \tag{3.4}$$

for all large $s > 0$.

On the other hand, operating rot to (1.1), similarly to (2.11), we have

$$\mathrm{rot}\,(R(x,T/2)f(x)) = \partial_t \mathrm{rot}\, v(x,T/2) - \nu\Delta \mathrm{rot}\, v(x,T/2)$$
$$+\sum_{j=1}^{3} \nabla v_j(x,T/2) \times \partial_j v(x,T/2) + (v\cdot\nabla)\mathrm{rot}\, v(x,T/2), \quad x\in\Omega.$$

We set $\alpha_0(x) = \alpha(x,T/2)$. Therefore

$$|\nabla \mathrm{rot}\,(R(x,T/2)f)|e^{s\alpha_0} \le |\partial_t\nabla(\mathrm{rot}\, v)(x,T/2)|e^{s\alpha_0} + Ce^{s\alpha_0}\Bigg\{|\nabla(\Delta\mathrm{rot}\, v(x,T/2))|$$
$$+\sum_{j=1}^{3} |\nabla(\nabla v_j(x,T/2)\times\partial_j v(x,T/2))| + |\nabla((v\cdot\nabla)\mathrm{rot}\, v(x,T/2))|\Bigg\}, \quad x\in\Omega.$$

Hence

$$\int_\Omega |\nabla\mathrm{rot}\,(R(x,T/2)f(x))|^2 e^{2s\alpha_0}dx$$
$$\le \int_\Omega |\partial_t\nabla(\mathrm{rot}\, v)(x,T/2)|^2e^{2s\alpha_0}dx + Ce^{Cs}\left(\|\mathrm{rot}\, v(\cdot,T/2)\|^2_{H^3(\Omega)} + \|v(\cdot,T/2)\|^2_{H^2(\Omega)}\right).$$

For $\int_\Omega |\mathrm{rot}\,(R(x,T/2)f(x))|^2e^{2s\alpha_0}dx$, we can similarly argue and apply (3.4) to obtain

$$\int_\Omega (|\nabla\mathrm{rot}\,(R(x,T/2)f)|^2 + |\mathrm{rot}\,(R(x,T/2)f)|^2)e^{2s\alpha_0}dx$$
$$\le C\int_Q \frac{s}{t(T-t)}(|f|^2+|\nabla f|^2)e^{2s\alpha}dxdt + Ce^{Cs}\mathcal{D}^2 + Ce^{Cs}\mathcal{E}^2 \tag{3.5}$$

for all large $s > 0$. Here and henceforth we set

$$\mathcal{E} = \|\mathrm{rot}\, v(\cdot,T/2)\|_{H^3(\Omega)} + \|v(\cdot,T/2)\|_{H^2(\Omega)}.$$

On the other hand, setting $R(x,T/2)\equiv a(x) = (a_1(x),a_2(x),a_3(x))^T$, we have

$$\mathrm{rot}\,(R(x,T/2)f(x)) = \nabla f(x)\times a(x) + f(x)\mathrm{rot}\, a(x)$$
$$=((a_3\partial_2 f - a_2\partial_3 f),(a_1\partial_3 f - a_3\partial_1 f),(a_2\partial_1 f - a_1\partial_2 f))^T + f(x)\mathrm{rot}\, a(x)$$
$$\equiv (L_1 f, L_2 f, L_3 f)^T + f(x)\mathrm{rot}\, a(x).$$

Note that

$$\partial_j \alpha_0 = \frac{4\lambda}{T^2} e^{\lambda\eta} \partial_j \eta, \quad j = 1, 2, 3.$$

Denote

$$\mu_1(x) = \begin{pmatrix} 0 \\ a_3 \\ -a_2 \end{pmatrix} \cdot \begin{pmatrix} \partial_1\alpha_0 \\ \partial_2\alpha_0 \\ \partial_3\alpha_0 \end{pmatrix} = \frac{4\lambda}{T^2} e^{\lambda\eta}(a_3\partial_2\eta - a_2\partial_3\eta),$$

$$\mu_2(x) = \begin{pmatrix} -a_3 \\ 0 \\ a_1 \end{pmatrix} \cdot \begin{pmatrix} \partial_1\alpha_0 \\ \partial_2\alpha_0 \\ \partial_3\alpha_0 \end{pmatrix} = \frac{4\lambda}{T^2} e^{\lambda\eta}(a_1\partial_3\eta - a_3\partial_1\eta),$$

$$\mu_3(x) = \begin{pmatrix} a_2 \\ -a_1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_1\alpha_0 \\ \partial_2\alpha_0 \\ \partial_3\alpha_0 \end{pmatrix} = \frac{4\lambda}{T^2} e^{\lambda\eta}(a_2\partial_1\eta - a_1\partial_2\eta).$$

That is,

$$(\mu_1(x), \mu_2(x), \mu_3(x))^T = \frac{4\lambda}{T^2} e^{\lambda\eta}(\nabla\eta \times a(x)).$$

Applying Lemma 2.3 to the first-order differential operators L_1, L_2, L_3, we have

$$\begin{aligned} & s^2 \int_\Omega \left(\mu_1^2(x) + \mu_2^2(x) + \mu_3^2(x) - \frac{3C}{s} \right) (|f(x)|^2 + |\nabla f(x)|^2) e^{2s\alpha_0} dx \\ \leq & C \int_\Omega (|\nabla(\mathrm{rot}\, R(x, T/2) f(x))|^2 + |\mathrm{rot}\, R(x, T/2) f(x)|^2) e^{2s\alpha_0} dx \\ + & C \int_\Omega (|\nabla(f(x) \mathrm{rot}\, a(x))|^2 + |f(x) \mathrm{rot}\, a(x)|^2) e^{2s\alpha_0} dx. \end{aligned}$$

Therefore (3.5) and $f|_\omega = 0$ yield

$$\begin{aligned} & s^2 \int_{\Omega\setminus\omega} \left(\frac{16\lambda^2}{T^4} e^{2\lambda\eta} |\nabla\eta \times a|^2 - \frac{3C}{s} \right) (|f(x)|^2 + |\nabla f(x)|^2) e^{2s\alpha_0} dx \\ \leq & C \int_Q \frac{s}{t(T-t)} (|f(x)|^2 + |\nabla f(x)|^2) e^{2s\alpha} dx dt \\ + & C \int_{\Omega\setminus\overline{\omega}} (|f(x)|^2 + |\nabla f(x)|^2) e^{2s\alpha_0} dx + Ce^{Cs}(\mathcal{D}^2 + \mathcal{E}^2). \end{aligned}$$

By (1.7), we can take $s > 0$ sufficiently large, so that we can absorb the second term at the right hand side into the left hand side. Hence

$$\begin{aligned} & s^2 \int_{\Omega\setminus\omega} (|f(x)|^2 + |\nabla f(x)|^2) e^{2s\alpha_0} dx \\ \leq & Cs \int_{\Omega\setminus\omega} \int_0^T \frac{1}{t(T-t)} (|f(x)|^2 + |\nabla f(x)|^2) e^{2s\alpha} dx dt + Ce^{Cs}(\mathcal{D}^2 + \mathcal{E}^2). \end{aligned} \tag{3.6}$$

We set $\ell(t) = t(T-t)$. By (2.4), we have $\partial_t \alpha(x, T/2) = 0$ for $x \in \Omega$ and

$$\partial_t^2 \alpha(x,t) = \frac{2\ell'(t)^2 - \ell(t)\ell''(t)}{\ell^3(t)}(e^{\lambda\eta(x)} - e^{2\lambda\|\eta\|_{L^\infty(\Omega)}}),$$
$$\partial_t^3 \alpha(x,t) = \frac{6(\ell(t)\ell''(t) - \ell'(t)^2)}{\ell^4(t)}\ell'(t)(e^{\lambda\eta(x)} - e^{2\lambda\|\eta\|_{L^\infty(\Omega)}})$$

for $(x,t) \in Q$, so that

$$\partial_t^2 \alpha(x,t) \le -\frac{c_0}{\ell^3(t)}, \qquad (x,t) \in Q$$

with some constant $c_0 > 0$, and

$$\partial_t^3 \alpha(x,t) \begin{cases} \ge 0, & 0 \le t \le \frac{T}{2}, \\ \le 0, & \frac{T}{2} \le t \le T, \quad x \in \Omega. \end{cases}$$

Therefore by the mean value theorem, we can choose a constant $\kappa = \kappa(x,t) \in (0, T/2)$ for $(x,t) \in Q$ such that κ is between t and $\frac{T}{2}$, and

$$\alpha(x,t) = \alpha(x, T/2) + \frac{1}{2}\partial_t^2 \alpha(x,t)\left(t - \frac{T}{2}\right)^2 + \frac{1}{6}\partial_t^3 \alpha(x, \kappa(x,t))\left(t - \frac{T}{2}\right)^3$$
$$\le \alpha(x, T/2) - \frac{c_0}{2t^3(T-t)^3}\left(t - \frac{T}{2}\right)^2, \qquad (x,t) \in Q.$$

Consequently, by $c_0 > 0$ and $\frac{-1}{t(T-t)} \le -\frac{4}{T^2}$, we obtain

$$\int_0^T \frac{1}{t(T-t)} e^{2s\alpha} dt \le C e^{2s\alpha_0(x)} \int_0^T \frac{1}{t(T-t)} \exp\left(-\frac{sc_0}{t^3(T-t)^3}\left(t - \frac{T}{2}\right)^2\right) dt$$
$$\le C e^{2s\alpha_0} \int_0^T \frac{1}{t(T-t)} \exp\left(-\frac{c_0}{t^3(T-t)^3}\left(t - \frac{T}{2}\right)^2\right) \exp\left(-\frac{(s-1)c_0}{t^3(T-t)^3}\left(t - \frac{T}{2}\right)^2\right) dt$$
$$\le C e^{2s\alpha_0} \int_0^T \exp\left(-\left(\frac{4}{T^2}\right)^3 c_0(s-1)\left((t - \frac{T}{2}\right)^2\right) dt$$
$$\le C e^{2s\alpha_0} \int_{-\infty}^{\infty} \exp\left(-\left(\frac{4}{T^2}\right)^3 c_0(s-1)\xi^2\right) d\xi = C\left(\frac{T^2}{4}\right)^{\frac{3}{2}} \sqrt{\frac{\pi}{c_0}} \frac{1}{\sqrt{s-1}} e^{2s\alpha_0(x)}.$$

Hence

$$s\int_{\Omega\setminus\omega}\int_0^T \frac{1}{t(T-t)}(|f(x)|^2 + |\nabla f(x)|^2)e^{2s\alpha} dx dt$$
$$= s\int_{\Omega\setminus\omega}(|f(x)|^2 + |\nabla f(x)|^2)\left(\int_0^T \frac{1}{t(T-t)} e^{2s\alpha} dt\right) dx$$
$$\le \frac{Cs}{\sqrt{s-1}} \int_{\Omega\setminus\omega}(|f(x)|^2 + |\nabla f(x)|^2)e^{2s\alpha_0} dx. \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain

$$(s^2 - C\sqrt{s}) \int_{\Omega\setminus\omega} (|f(x)|^2 + |\nabla f(x)|^2) e^{2s\alpha_0} dx \le Ce^{Cs}(\mathcal{D}^2 + \mathcal{E}^2).$$

Taking $s > 0$ sufficiently large, we see the conclusion of Theorem 1.

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