UTMS 2006–30 October 25, 2006

On existence and nonexistence global solutions of reaction-diffusion equations

by

Noriaki Umeda

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

On Existence and Nonexistence Global Solutions of Reaction-Diffusion Equations

Noriaki Umeda

Graduate School of Mathematical Sciences,
The University of Tokyo,
3-8-1, Komaba, Meguro-ku, Tokyo, 153-8914, Japan

Abstract

We consider the initial value problem for the reaction-diffusion equation $u_t = \Delta u + f(u)$. In this paper we show the existence and nonexistence of the global solutions in time. Especially, we extend the condition of the nonlinear terms to more general. We have the results of the existence and the nonexistence for the equation with the nonlinear term f satisfying $\liminf_{s\to 0} f(s)/s^p > 0$ and $\limsup_{s\to 0} f(s)/s^q < \infty$ with some $p > 0$ and $q > 0$.

Keyword and Phrases: reaction-diffusion, blow-up, global existence, nonlinear term.
AMS subject classifications: 35K15, 35K57.

1 Introduction and main theorems

We consider the nonnegative solutions of the initial value problem for the equation

$$\begin{cases} u_t = \Delta u + f(u), & x \in \mathbf{R}^d, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^d, \end{cases} \tag{1}$$

where u_0 is a nonnegative, bounded and continuous function in $\mathbf{R}^d$, f satisfies

$$f \in C([0,\infty)) \text{ and } f(r) \geq 0, f'(r) \geq 0 \text{ for } r \geq 0, \tag{2}$$

$$f \text{ is locally Lipschitz in } (0,\infty). \tag{3}$$

We let X be a set of f satisfying (2) and (3).

Problem (1) has one and over nonnegative and bounded solutions at least locally in time. Let $T^* = T^*(u)$ be the maximal existence time of u. When u is the unique solution of (1), it can be expressed $T^* = T^*(u_0, f)$ with a given initial value u_0 and nonlinear term f. If $T^* = \infty$, the solution exists globally in time. If $T^* < \infty$, the solution does not exist global in time and there exists the solution that blows up in finite time such that

$$\limsup_{t \to T^*} \|u(\cdot, t)\|_\infty = \infty, \tag{4}$$

where $\|u\|_\infty$ denotes the L^∞-norm of u in space variables. We set

$$Z^p = \left\{ g \in X; \limsup_{r \to 0} \frac{g(r)}{r^p} < \infty \right\} \quad \text{and} \quad Z_p = \left\{ g \in X; \liminf_{r \to 0} \frac{g(r)}{r^p} > 0 \right\}.$$

Note that if there exists $f \in Z^{p_1} \cap Z_{p_2}$, then $p_1 \le p_2$. Moreover, if f satisfies $\lim_{r \to 0} f(r)/r^p = C_p$ with some $p > 0$ and $C_p \in (0, \infty)$, then we have $p_1 \le p \le p_2$. For the nonlinear term f we set

$$\begin{aligned} J &= \left\{ g \in X; \int_a^\infty \frac{ds}{g(s)} < \infty \text{ with some } a \in (0, \infty) \right\}, \\ J_c &= \left\{ g \in J; g(r) \text{ is convex for } r \ge r_0 \text{ with some } r_0 \ge 0 \right\}. \end{aligned}$$

For example e^u, $u(\log(u+1))^b$ with $b > 1$ and $u^p + u^q$ with p or $q > 1$ are in J_c (see [9] and [10]). We denote by BC the space of all bounded continuous functions in $\mathbf{R}^d$. For $a \ge 0$ we put

$$\begin{aligned} I^a &= \{ \xi \in BC; \xi(x) \ge 0 \text{ and } \limsup_{|x| \to \infty} |x|^a \xi(x) < \infty \}, \\ I_a &= \{ \xi \in BC; \xi(x) \ge 0 \text{ and } \liminf_{|x| \to \infty} |x|^a \xi(x) > 0 \}, \\ L_a^\infty &= \{ \xi \in BC; \|\xi\|_{\infty, a} = \sup_{x \in \mathbf{R}^d} < x >^a |\xi(x)| < \infty \}, \end{aligned}$$

where $< x >= (|x|^2 + 1)^{1/2}$. It is clear that $I^a \subset L_a^\infty$ for $a \ge 0$. We use the notation $S(t)\xi$ to represent the solution of the heat equation with an initial value $\xi(x)$;

$$S(t)\xi(x) = (4\pi t)^{-d/2} \int_{\mathbf{R}^d} e^{-|x-y|^2/4t} \xi(y) dy. \tag{5}$$

We briefly recall the history of the study on the blow-up and the global existence of the solution to the equation (1). First of this field, the blow-up and the global existence of solutions of the equation (1) in the case $f(u) = u^p$ with $p > 1$,

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbf{R}^d, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbf{R}^d \end{cases} \tag{6}$$

was studied by Fujita[4]. He proved that when $1 < p < 1 + 2/d$, the solution of (6) blows up in finite time for any continuous function $u_0 \not\equiv 0$. On the other hand he also proved that when $p > 1 + 2/d$ the solution of (6) exists globally in time if the initial value u_0 is small and has an exponential decay. The number $p_F = 1 + 2/d$ is called a critical blow-up exponent or Fujita exponent (We call p_F *first cutoff.*) for (6).

Fujita's results were also extended by some researcher. Hayakawa[6], Kobayashi-Sirano-Tanaka[8] and Weissler[14] proved that when $p = p_F$, the solution of (6) blows up in finite time for any continuous $u_0 \geq 0$, $\not\equiv 0$. Lee-Ni[11] considered the condition on the initial value whether the solution blows up or not when $p > p_F$. They proved that if $u_0 \in I_a$ with $a < 2/(p-1)$, then every nontrivial solution of (6) blows up in a finite time. They also had the result that if $u_0 \in I^a$ with $a > 2/(p-1)$ and $\|u_0\|_{\infty,a}$ is small enough, then every solution of (6) is global. The words "nontrivial solution" denotes that $u \not\equiv 0$ in this paper. Here we call the exponent $a = 2/(p-1)$ *second cutoff* for initial value.

For the case $p \leq 1$ Aguirre and Escobedo[1] studied and they got the result that if $p \leq 1$, then every solution of (6) is global in time.

Additionally, Garaktionov-Kurdyumov-Mikhailov-Samarskii[5, CapterIV, §7, **3**, 1 and 2] studied for the case $f(u) = (1+u)[\log(1+u)]^\beta$ with $\beta > 1$, and they have the result that if $\beta < 1 + 2/d$ then the solution with $u_0 \not\equiv 0$ blows up in finite time, and if $\beta > 1 + 2/d$, $\|u_0\|_\infty$ is small enough and u_0 decays fast, then the solution is global in time.

In this paper, we have following results.

Theorem 1. *Assume that $f \in X \backslash J$. Let $u_0 \geq 0$ and $\in BC$. Then every solution of* (1) *is global in time.*

Remark. *It is not necessary that solution of* (1) *is unbounded as much as the case $f(u) = u^p$ with $0 < p \leq 1$ (see [1]). For example, if $f \in Z^{p_1}$ with $p_1 > 1 + 2/d$ and $u_0 \leq C_0 e^{-\nu_0 |x|^2}$ with sufficiently small C_0, then the solution is bounded (see [4] or Theorem 3).*

Theorem 2. *Assume that $f \in J_c$ and $u_0 \geq 0$, $\in BC$ and $\not\equiv 0$. If f and u_0 satisfy the one of the following four conditions;*

(i) $f(0) > 0$.

(ii) $f \in Z_p$ *with* $0 < p \le p_F = 1 + 2/d$.

(iii) $u_0(x) \ge Ce^{-\nu_0|x|^2}$ *for some* $\nu_0 > 0$ *and* $C = C(\nu_0) > 0$ *large enough.*

(iv) $u_0 \in I_a$ *with* $a < 2/(p-1)$.

Then every solution of (1) *blows up in finite time.*

Remark. *In Theorem 2 if (i) is hold, then it is possible to remove the condition* $u_0 \not\equiv 0$.

Theorem 3. *Assume that* $f \in Z^p$ *with* $p > p_F$. *Let* $u_0 \ge 0$. *Suppose that*

$$u_0 \in I^a \quad with \quad a > 2/(p-1) \tag{7}$$

and $\|u_0\|_\infty$ *is small enough. Then, every solution of* (1) *is global. Moreover, if* $\|u_0\|_{\infty,b}$ *is small enough for* $b \in (2/(p-1), a)$, *then we have the estimate that*

$$u(x,t) \le mS(t) < x >^{-b} \tag{8}$$

in $\boldsymbol{R}^d \times (0, \infty)$.

Remark. *By comparison Theorem 3 implies that if* $u_0 \le Ce^{-\nu_0|x|^2}$ *with some* $\nu_0 > 0$ *and* C *sufficiently small, the solution of* (1) *is global (See Theorem 3.1).*

If f satisfies $\lim_{r\to 0} f(r)/r^p = C_p$ and $\lim_{r\to\infty} f(r)/r^q = C_q$ with some $p > 0$, $q > 0$, $C_p > 0$ and $C_q > 0$, then by Theorems 1, 2 and 3, the solution of (1) satisfies following table.

	$p < 1$	$p = 1$	$1 < p < p_F$	$p = p_F$	$p > p_F$
$q < 1$	G	G	G	G	GB
$q = 1$	G	G	G	G	GB
$1 < q < p_F$	N	N	N	N	NB
$q = p_F$	N	N	N	N	NB
$q > p_F$	N	N	N	N	NB

In this table the signs “G”, “N”, “GB” and “NB” denote as following;

G: Every solution is global in time.

N: Any nontrivial solution is not global in time. (Of cource there does not exist bounded nontrivial solution.)

GB: Any solution is global in time and there exist also the bounded solutions.

NB: There exist both bounded solutions and non-global solutions in time. (There may exist the unbounded and global solution in time.)

The rest of the paper is organized as follows. In section 2 we show the local existence of the solution for the equation (1) in time together with the comparison principle. For the proof of the blow-up result, we prepare some tools in section 3. In section 4 we note some preliminary results for the proof of the global existence of the solution of (1). Finally, we give the proof of Theorems 1, 2 and 3 in section 5.

2 Local existence in time

First we show the local existence of the solutions of (1).

Theorem 2.1. *Assume that $f \in X$ and $u_0 \geq 0, \in BC$. Then there exists $T > 0$ such that* (1) *admits a nonnegative and bounded classical solution u in $[0, T) \times \boldsymbol{R}^d$. Moreover, if $f(r)$ is locally Lipschitz function for $r \in [0, \infty)$, then the solution is unique.*

Remark. *If $f(r)$ is locally Lipschitz function for $r \in [0, \infty)$, then the condition $f \in X$ can be changed to only $f \in C([0, \infty))$.*

First, we proof the case $f(r)$ is locally Lipschitz function for $r \in [0, \infty)$.

Proof of Theorem 2.1 (*the case f is locally Lipschitz in* $[0, \infty)$). Although we follow the same argument as in [1, Lemma (1.3)], we give the outline of the proof for reader's convenience. For arbitrary $T > 0$, let

$$E_T = \{u : [0, T] \to L^\infty; \|u\|_{E_T} = \sup_{t \in [0,T]} \|u(\cdot, t)\|_\infty < \infty\}. \tag{9}$$

We consider in E_T the related integral equation

$$u(x, t) = S(t)u_0(x) + \int_0^t S(t - s)f(u(x, s))ds. \tag{10}$$

where $S(t)$ is defined in (5). Note that in the closed subset $P_T = \{u \in E_T; u \geq 0\}$ of E_T, (1) is reduced to (10). First, we show in the case $\lim_{u \to 0} |(f(u) - f(0))/u| < \infty$. Define $\Psi(u)(x, t) = (S(t)u_0(x) + \Phi(u)(x, t))$, where $\Phi(u)(x, t) = \int_0^t S(t - s)f(u(x, s))ds$. Then we can easily obtain that $\|S(\cdot)u_0\|_{E_T} \leq \|u_0\|_\infty$, $\|\Phi(u)\|_{E_T} \leq T\|f(u)\|_{E_T}$.

For some v_1, $v_2 \in B_R = \{u_n \in E_T; \|u_n\|_{E_T} \le R\}$, we have

$$\|\Psi(v_1) - \Psi(v_2)\|_{E_T}(x,t) \le \int_0^t S(t-s)\Big|f(v_1(x,s)) - f(v_2(x,s))\Big|ds.$$

We consider this expression in $B_R \cap P_T$ for R sufficient large. Thus we have $\|\Psi(v_1) - \Psi(v_2)\|_{E_T}(x,t) \le CTF \sup_{s\in[0,t]} \|v_1(\cdot,s) - v_2(\cdot,s)\|_\infty$, where $F = F(R) = \sup_{u_1,u_2\in[0,R]} |(f(u_1) - f(u_2))/(u_1 - u_2)|$. Since f is locally Lipschitz in $[0,\infty)$, F is bounded. Take T is small enough. Then we obtain $\|\Psi(v_1) - \Psi(v_2)\|_{E_T} \le CTF\|v_1 - v_2\|_{E_T} \le \rho\|v_1 - v_2\|_{E_T}$ for some $\rho < 1$. Then Ψ is a strict contraction of $B_R \cap P_T$ into itself, whence there exists a unique fixed point $u \in B_R \cap P_T$ which solves (10). Thus we obtain a unique nonnegative and bounded solution $u(t)$ to (1) in $\mathbf{R}^d \times [0,T)$ for some T. □

Next, we show the more general case with following lemma.

Lemma 2.2. *Assume that $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are solutions of*

$$\begin{cases} \overline{u}_t = \Delta\overline{u} + f(\overline{u}), & x \in \boldsymbol{R}^d, t > 0, \\ \overline{u}(x,0) = \overline{u}_0(x), & x \in \boldsymbol{R}^d, \end{cases} \tag{11}$$

and

$$\begin{cases} \underline{u}_t = \Delta\underline{u} + g(\underline{u}), & x \in \boldsymbol{R}^d, t > 0, \\ \underline{u}(x,0) = \underline{u}_0(x), & x \in \boldsymbol{R}^d, \end{cases} \tag{12}$$

where $\overline{u}_0$, $\underline{u}_0 \ge 0 \in BC$, f, $g \in X$ and f or g is locally Lipschitz in $[a,\infty)$ with some $a \in \mathbf{R}$. If $\overline{u}_0(x) \ge \underline{u}_0(x) \ge a$ for $x \in \boldsymbol{R}^d$ and $f(s) \ge g(s)$ for $s \ge a$, then $\overline{u}(x,t) \ge \underline{u}(x,t)$ for $x \in \boldsymbol{R}^d \times [0, T^*(\overline{u}_0, f))$.

Proof. If f is locally Lipschiz in $[a,\infty)$, then from (11) and (12) we have

$$(\overline{u} - \underline{u})_t = \Delta(\overline{u} - \underline{u}) + f(\overline{u}) - g(\underline{u}) \ge \Delta(\overline{u} - \underline{u}) + f(\overline{u}) - f(\underline{u}).$$

Put $w = \overline{u} - \underline{u}$. Then w satisfies

$$\begin{cases} w_t(x,t) = \Delta w(x,t) + a(x,t)w(x,t), & x \in \mathbf{R}^d, t > 0, \\ w(x,0) = (\overline{u}_0 - \underline{u}_0)(x), & x \in \mathbf{R}^d, \end{cases} \tag{13}$$

where $a(x,t) = \int_0^1 f'(\theta\overline{u} + (1-\theta)\underline{u})(x,t)d\theta$. By the maximum principle (see [13, CHAPTER 3, SECTION 6, THEOREM 10]), we have $w(x,t) \ge 0$. Thus we have $\overline{u} \ge \underline{u}$. When g is locally Lipschiz in $[a,\infty)$, we can show this by same argument. □

Proof of Theorem 2.1 (general case). We put $g_n(r) = c_n r$ $(0 \le r \le 1/2n)$, $= f(r)$ $(r > 1/2n)$, where $c_n = 2nf(1/2n)$. Here $\{g_n\}$ is a sequence of locally

Lipschitz continuous functions in $[0,\infty)$ for any fixed $n>0$. Consider now the approximating problems for (1);

$$\begin{cases} (u_n)_t - \Delta u_n = g_n(u_n), & t>0, x\in \mathbf{R}^d, \\ u_n(x,0) = u_0(x) + 1/n, & x\in \mathbf{R}^d. \end{cases} \tag{14}$$

Define $\Psi_n(u_n)(x,t) = (S(t)u_n(x,0) + \Phi_n(u_n)(x,t))$, where $\Phi_n(u_n)(x,t) = \int_0^t S(t-s)g_n(u_n)(x,s)ds$.

Let $B_R = \{u_n \in E_T; \|u_n\|_{E_T} \le R\}$. If R is large enough and $T>0$ is small enough, Ψ_n is a strict contraction from $B_R \cap P_T$ into itself by using same argument as the case $\lim_{u\to 0} |(f(u)-f(0))/u| < \infty$. Whence there exists a unique fixed point $u_n \in B_R \cap P_T$ which solves

$$u_n(x,t) = S(t)u_n(x,0) + \int_0^t S(t-s)g_n(u_n(x,s))ds, \tag{15}$$

Thus we obtain a unique nonnegative and bounded solution $u_n(t)$ of (14) in $\mathbf{R}^d \times [0,T)$ for some T. Furthermore, we can show $u_n(x,t) \le u_m(x,t)$ for $n \ge m$ by Lemma 2.2, where we use the argument of [1, Lemma (1.3)]. Therefore, the sequences $\{u_n(t)\}$ are nonincreasing with respect to n and bounded below. So, we can define $u(x,t) = \lim_{n\to\infty} u_n(x,t)$. Then we can conclude that $u(t)$ satisfies (10) (see [1]).

To complete the proof of Theorem 2.1, let $u(x,t)$ be the nonnegative and bounded solution of (10) that has been obtained in $[0,T)\times\mathbf{R}^d$ for some $T>0$. By (10), $u(x,t)$ is continuous in $[0,T)\times\mathbf{R}^d$. Moreover, by considering the difference quotients $(1/h)\{u(x+e_jh,t) - u(x,t)\}$ with $h\to 0$, one easily sees that $\partial u(x,t)/\partial x_j$ is locally bounded in $\mathbf{R}^d \times [\tau, T)$ for $j=1,2,\ldots,d$ and any τ such that $0<\tau<T$, where e_j is j-th unit vector of $\mathbf{R}^d$. Then $f(u)$ are locally Hölder continuous functions in space uniformly with respect to time. It then follows from the representation formula (10) that u is a classical solution of (1) in $\mathbf{R}^d \times (0,T)$ (see [3, Chapter 1, Theorem 10]). $\square$

Remark. *By Theorem 2.1, solutions are unique when* $\lim_{u\to 0}(f(u)-f(0))/u < \infty$. *If this assumption is dropped, this result is false in general (see [1]).*

3 Preliminaries for blow-up

In this section we prepare some tools for proving main theorems. First, we consider the case f is more general form. Let $f\in J_c$ and $f(s)>0$ for $s>0$. Then, we can take a convex and strict increasing function $\tilde{f}$ satisfying

$$\tilde{f}(s) > 0 \text{ for } s>0, \quad \tilde{f} \in Z^{\tilde{p}} \cap J \text{ with some } \tilde{p} > 1, \quad f(u) \ge \tilde{f}(u). \tag{16}$$

We consider the equation

$$\begin{cases} \hat{u}_t = \Delta \hat{u} + \tilde{f}(\hat{u}), & x \in \mathbf{R}^d, t > 0, \\ \hat{u}(x,0) = u_0(x), & x \in \mathbf{R}^d. \end{cases} \tag{17}$$

By Lemma 2.2 we see $u \geq \hat{u}$, where u is solution of (1). Put

$$G_\epsilon(t) = \int_{\mathbf{R}^d} \rho_\epsilon(x) \hat{u}(x,t) dx \tag{18}$$

for $\epsilon > 0$, where $\rho_\epsilon(x) = (\epsilon/\pi)^{-d/2} e^{-\epsilon |x|^2}$. Then, we have

$$G'_\epsilon(t) = \int_{\mathbf{R}^d} \rho_\epsilon(x) \left(\Delta \hat{u}(x,t) + \tilde{f}(\hat{u}(x,t)) \right) dx.$$

Since $\Delta \rho_\epsilon(x) \geq -2d\epsilon \rho_\epsilon(x)$ and $\tilde{f}(u)$ is convex, we have

$$G'_\epsilon(t) \geq -2d\epsilon G_\epsilon(t) + \tilde{f}\left(G_\epsilon(t) \right). \tag{19}$$

by Green's inequarity and Jensen's inequarity. We consider the equation

$$\begin{cases} g'_\epsilon(t) = -2d\epsilon g_\epsilon(t) + \tilde{f}(g_\epsilon(t)), \\ g_\epsilon(0) = G_\epsilon(0). \end{cases} \tag{20}$$

We see that $g_\epsilon(t) \leq G_\epsilon(t)$ by comparison. If $g_\epsilon(t)$ blows up in finite time, $G_\epsilon(t)$ does, too.

If $G_\epsilon(0)$ satisfies

$$\tilde{f}(G_\epsilon(0)) > 2d\epsilon G_\epsilon(0), \tag{21}$$

we see that $\lim_{t \to T} g_\epsilon(t) = \infty$ for some $T \in (0, \infty]$. (Because $-2d\epsilon s + \tilde{f}(s)$ is strict increasing function for s satisfying $c_0 \tilde{f}(s) > 2d\epsilon s$.) Next, we estimate T. From (20), we have $T = \int_{G_\epsilon(0)}^{\infty} d\xi / (\tilde{f}(\xi) - 2d\epsilon\xi) < \infty$. Thus $g_\epsilon(t)$ blows up in finite time with $G_\epsilon(0)$ satisfying (21), and $G_\epsilon(t)$ blow up in finite time, too.

This fact shows the following Lemma.

Lemma 3.1. *Let $G_\epsilon(t)$ satisfy differential inequality* (19)*. If* (21) *hold for some $\epsilon > 0$, then $G_\epsilon(t)$ blows up in finite time.*

In fact, for u_0 satisfying (21) for some $\epsilon > 0$ with (18), the solution $\hat{u}(x,t)$ of (17) blows up in finite time.

Since $\tilde{f} \in Z_p$, we can put

$$\hat{a} = \tilde{a}(f, p, c_1) = \sup \left\{ a : f(r) \geq c_1 r^p \text{ for } r \leq a \right\} \tag{22}$$

for some c_1. If we confine to $\epsilon < c_1 \hat{a}^{p-1}/2d$, we can change (21) to

$$G_\epsilon(0) > (2d\epsilon/c_1)^{\frac{1}{p-1}}. \tag{23}$$

with some $c_1 > 0$. Thus, we have following result.

Proposition 3.2. *Let $G_\epsilon(t)$ satisfies defferential inequality* (19)*. If* (23) *is satisfied for some $\epsilon \in (0, c_1\hat{a}^{p-1}/2d)$, then $G_\epsilon(t)$ blows up in finite time, where $\hat{a}$ is defined in* (22)

With this proposition, we have following lemmas.

Lemma 3.3. *Let $\tilde{f}$ be a strict increasing function, $\tilde{f} \in J \cap Z_{\tilde{p}}$ with $1 < \tilde{p} < p_F$, $u_0 \geq 0$, $\not\equiv 0$, $\in BC$. Then, solution of* (17) *blows up in finite time.*

Proof. Since $u_0 \in BC$ and $\not\equiv 0$, we can assume $u_0 \in L^1(\mathbf{R})$ and $\int_{\mathbf{R}^d} u_0(x)dx > 0$. By the Lebesque dominated convergence theorem, we have that there exist $\epsilon_0 \in (0, \hat{a}^{p-1}/2d)$ such that

$$G_\epsilon(0) = \left(\frac{\epsilon}{\pi}\right)^{d/2} \int_{\mathbf{R}^d} u_0(x)e^{-\epsilon|x|^2}dx \geq \frac{1}{2}\left(\frac{\epsilon}{\pi}\right)^{d/2} \int_{\mathbf{R}^d} u_0(x)dx$$

for any $\epsilon \in (0, \epsilon_0]$. Since $\tilde{p} < a + 2/d$ and $2/(\tilde{p}-1) > d$ by assumption, that the codition (23) of Proposition 3.3 is satisfied if ϵ is sufficiently small. Thus $G_\epsilon(t)$ blows up in finite time, and $u(x,t)$ does, too. □

Lemma 3.4. *Let $\tilde{f}$ be a strict increasing function and $\hat{u}(x,t)$, and $\tilde{f} \in Z^{\tilde{p}} \cap J$ with $\tilde{p} > 1$. Suppose the following two conditions.*

(I) $u_0 \in I_a$ with $a < 2/(\tilde{p}-1)$.

(II) $u_0 \geq Ce^{-\nu_0|x|^2}$ for some $\nu > 0$ and some C large enough.

Then, solution of (17) *blows up in finite time.*

Proof. First we show the case (I). From assumption, we have

$$G_\epsilon(0) = \left(\frac{\epsilon}{\pi}\right)^{d/2} \int_{\mathbf{R}^d} u_0(x)e^{-\epsilon|x|^2}dx = \pi^{-d/2} \int_{\mathbf{R}^d} u_0(\epsilon^{-1/2}x)e^{-|x|^2}dx.$$

Then, it follows that

$$\epsilon^{-1/(\tilde{p}-1)}G_\epsilon(0) \geq C\epsilon^{-\frac{1}{\tilde{p}-1}+\frac{a}{2}}\pi^{-d/2} \int_{\mathbf{R}^d} |x|^{-a}e^{-|x|^2}dx > \left(\frac{2d}{c_1}\right)^{1/(p-1)}$$

for $\epsilon \in (0, c_1\hat{a}^{p-1}/2d)$, with $\hat{a}$ defined in (22) . Thus, we see that $G_\epsilon(t)$ blows up in finite time by Proposition 5.2, and $u(x,t)$ blows up, too.

Next, we consider the case (II). We have

$$G_\epsilon(0) \geq C\left(\frac{\epsilon}{\pi}\right)^{d/2} \int_{\mathbf{R}^d} e^{-(\epsilon+\nu_0)|x|^2}dx = C\left(\frac{\epsilon}{\epsilon+\nu_0}\right)^{d/2}.$$

So, if we choose $\epsilon = c_1\hat{a}^{\tilde{p}-1}/4d$ and

$$C > \frac{\hat{a}}{2^{1/(\tilde{p}-1)}}\left(1+\frac{4d\nu_0}{c_1\hat{a}^{\tilde{p}-1}}\right)^{d/2},$$

the condition of Proposition 3.3 is also satisfied in this case. $\square$

Next, we consider the case $\lim_{r\to 0} f(r)/r^{p_F} > C$ with some constant C. For showing this case, we should following propositions.

Proposition 3.5. *Let $u_0 \not\equiv 0$, $\in BC$ and $\hat{u}$ be the solution of* (17) *with initial data u_0. Then for any $\tau > 0$, there exist constants $\nu > 0$ and $C = C(\nu_0, u_0) > 0$ such that $\hat{u}(x,\tau) \geq Ce^{-\nu|x|^2}$.*

Proof. (cf. [2, Lemma 2.4]) Assume for instance that $u_{1,0} \not\equiv 0$. Since $u(x,t) \geq S(t)u_0(x)$, it follows that

$$\hat{u}(x,t) \geq \exp(-\frac{|x|^2}{2t})(4\pi t)^{-d/2}\int_{\mathbf{R}^d}\exp(-|y|^2/2t)u_0(y)dy.$$

Define $\hat{u}(x,t) = u(x,t+\tau_1)$ for some $\tau_1 > 0$. Then, we obtain

$$\bar{u}(x,0) = \hat{u}(x,\tau_1) \geq C\exp(-\nu|x|^2) \tag{24}$$

with

$$\nu = \frac{1}{2\tau_1}, \qquad C = (4\pi\tau_1)^{-d/2}\int_{\mathbf{R}^d}\exp\left(-\frac{|y|^2}{2\tau_1}\right)u_0(y)dy. \tag{25}$$

$\square$

Proposition 3.6. *Let $u_0 \geq 0$, $\not\equiv 0$, $\in BC$ and $\lim_{r\to\infty} f(r)/r^{p_F} > C$. Assume that $p = p_F$. Then the solution of* (17) *satisfying that*

$$\hat{u}(x,t) \geq Ct^{-d/2}e^{-|x|^2/t}\log(t/2b)$$

for $t \in (b, T^*)$ with any $b \in (0, T^*)$ and $C > 0$ sufficientry small.

Proof. By Proposition 3.5, we may assume $u_0(x) \geq Ce^{-\mu|x|^2}$ for some $C > 0$ and $\mu > 0$ without loss of generality. From (10), we have

$$\hat{u}(x,t) \geq S(t)u_0(x) \geq C(4\mu t+1)^{-d/2}e^{-|x|^2/(4t+1/\mu}.$$

By using (10) again, we have

$$\hat{u}(x,t) \geq \int_0^t S(t-s)f(u(x,s))ds \geq C_1 \int_0^t S(t-s)u^p(x,s)ds$$
$$\geq C_2 \int_0^t (4\mu s+1)^{-dp/2} S(t-s)e^{-p|x|^2/(4s+1/\mu)}ds$$

with $p = p_F$. Since

$$S(t)e^{-p|x|^2/(4s+1/\nu)} \geq C_3 \left\{ \frac{2pt}{4s+1/\nu} + 1 \right\}^{-d/2} e^{-|x|^2/2t}.$$

We obtain

$$\hat{u}(x,t) \leq C_4 \int_{t/4}^{t/2} (4s\mu+1)^{-dp/2} e^{|x|^2/2(t-s)}ds \leq C_5 t(t+1)^{-dp/2} e^{-|x|^2/t}.$$

By using (10) one more time. Then we have

$$\hat{u}(x,t) \geq C_6 \int_0^t s^p(s+1)^{-dp^2/2} \left\{ \frac{2p(t-s)}{s} + 1 \right\}^{-d/2} e^{-|x|^2/2(t-s)}ds$$
$$\geq C_7 (t+1)^{-d/2} e^{-|x|^2/t} \int_b^{t/2} s^{\{-d(p^2-1)+2p\}/2}ds$$

for small $b > 0$. Since $d(p^2-1) - 2p = 2$, then we have

$$\hat{u}(x,t) \geq C_8 (t+1)^{-d/2} e^{-|x|^2/t} \log(t/2a).$$

$\square$

Lemma 3.7. *Let $\tilde{f}$ be a strict increasing function satisfying* $\lim_{r\to 0} f(r)/r^{p_F} >$ 0 *with some constant $C > 0$ and $u_0 \geq 0$, $\not\equiv 0$, $\in BC$. The the solution of* (17) *blows up in finite time.*

Proof. From Proposition 3.6, we obtain

$$S(t)u(0,t) \geq C_1 t^{-\frac{d}{2}} \log\left(\frac{t}{2b}\right) \int_{\mathbf{R}^d} e^{-5|x|^2/4}dx \geq C_2 t^{-\frac{d}{2}} \log\left(\frac{t}{2b}\right) \tag{26}$$

in $b < t < T^*$ with $b \in (0, T^*)$.

We should show $T^* < \infty$. Assume that $T^* = \infty$. Then by Proposition 3.2, it hold that

$$F_\epsilon(t) = \left(\frac{\epsilon}{\pi}\right) \int_{\mathbf{R}^d} u(x,t)e^{-\epsilon|x|^2}dx \leq A\epsilon^{1/(p-1)}$$

for any $t \geq 0$ and $\epsilon \in (0, \epsilon_0)$ with some $\epsilon_0 > 0$. Thus, choosing $\epsilon = (4\pi)^{-1}$, we obtain

$$F_{1/4t} = S(t)u(0,t) \leq A(4t)^{-1/(p-1)} = A(4t)^{-d/2} \tag{27}$$

for $t > 1/4\epsilon_0$. If $T^* = \infty$, the contradiction is caused in (26) and (27). Thus we have $T^* < \infty$. □

4 Preliminaries for global existence

In this section, we consider the case $f \in Z^p$ with $p > p_F$. In the case, the solution of (1) can be bounded with some inital value u_0. We put $g(u)$ satisfying $g(u) \geq f(u)$ and some condition. Next lemma, we use this $g(u)$. For $\gamma > 0$ we set

$$\eta_\gamma(x,t) = S(t) < x >^{-\gamma} \tag{28}$$

Lemma 4.1. *Define η_a such as* (28)*. Then*

$$\eta_a(x,t)^p \leq C(1+t)^{(a-p\min\{a,d\})/2}\eta_a(x,t) \quad i\ \boldsymbol{R}^d \times (0,\infty).$$

Proof. See [7, Lemma 4.2] or [12, Lemma 5.2]. □

We define the Banach space $E_{\eta,a}$ of $u(x,t)$ such that $\|u\|_{E_{\eta,a}} \equiv |\|u/\eta_a\||_\infty < \infty$, where $|\|w\||_\infty = \sup_{(x,t)\in\mathbf{R}^d\times(0,\infty)} |w(x,t)|$. We consider the integral equation (10) in $E_{\eta,a}$.

Lemma 4.2. *Let $\check{u}(x,t)$ satisfy*

$$\begin{cases} \check{u}_t = \Delta\check{u} + g(\check{u}), & x \in \boldsymbol{R}^d, t > 0, \\ \check{u}(x,0) = u_0(x), & x \in \boldsymbol{R}^d, \end{cases} \tag{29}$$

where $g(s) \in C([0,\infty))$, $= Cs^p$ for $s \leq 1$ with some $p > p_F$ and $C > 0$. If $u_0 \in I^a$ with $a > 2/(p-1)$ and $\|u_0\|_\infty$ is small enough. Then, every solution $\check{u}(x,t)$ is global. Moreover, if $\|u_0\|_{\infty,b}$ is small enough, we have a decay estimate that $\check{u}(x,t) \leq mS(t) < x >^{-a}$ in $\boldsymbol{R}^d \times (0,\infty)$ with $b \in (2/(p-1), a)$.

Proof. (See [7], [11] or [12].) Let $u = \check{u}$ in this proof. Assume that $u_0 \in I^b$ with $b \in (2/(p-1), a)$ and $\|u_0\|_{\infty,b}$ is small enough. Let $\|u_0\|_{\infty,b} \leq m/2$. Define $M_{\eta,m,b} = \{u \in E_{\eta,b}; \|u\|_{E_{\eta,b}} \leq m\}$ and $P_{\eta,b} = \{u \in E_{\eta,b}; u \geq 0\}$. Let $u \in M_{\eta,m,b}$ with $m < 1$. Put $\Phi(u(x,t)) = \int_0^t S(t-s)g(u)ds$ and $\Psi(u(x,t)) = S(t)u_0(x) + \Phi(u(x,t))$.

First, from the definition of $E_{\eta,b}$ we have

$$\|S(\cdot)u_0\|_{E_{\eta,b}} \leq C'\|u_0\|_{\infty,b}. \tag{30}$$

with some C'.

Next, since $u \in M_{\eta,m,b}$, then $\Phi(u(x,t)) \leq C\int_0^t S(t-s)\eta_b^p(s)ds|\|u/\eta_b|\|_\infty^p$. Since $u \in B_{\eta,m,b}$ with $m < 1$, we have $u < 1$. Then, by Lemma 4.1, we have $\Phi(u(x,t)) \leq C\eta_b(x,t)\int_0^t(1+s)^{(1-p)b/2}ds|\|u/\eta_b|\|_\infty^p$. Since $(p-1)b > 2$ in the assumption, we have $\Phi(u(x,t)) \leq \tilde{C}\eta_b(x,t)|\|u/\eta_b|\|_\infty^p$ for some $\tilde{C}$. Then, we have

$$\|\Phi\|_{E_{\eta,b}} \leq \tilde{C}\|u\|_{E_{\eta,b}}^p. \tag{31}$$

From (30) and (31), we can choose m sufficiently small such that

$$\|\Psi\|_{E_{\eta,b}} \leq \frac{m}{2} + C'm^p < m$$

in $B_{\eta,m,b} \cap P_{\eta,b}$.

We should show that Ψ is a strict contraction of $B_{\eta,m,b} \cap P_{\eta,b}$ for m small enough. For $v_1, v_2 \in M_{\eta,m,b} \cap P_{\eta,b}$ we have

$$|\Psi(v_1) - \Psi(v_2)| \leq \left|C\int_0^t S(t-s)\left(\left(\frac{v_1}{\eta_b}\right)^p - \left(\frac{v_2}{\eta_b}\right)^p\right)\eta_b^p(x,s)ds\right|$$

Since v_1, $v_2 \in M_{\eta,m,b} \cap P_{\eta,b}$ and from Lemma 4.1, we have

$$\begin{aligned}
|\Psi(v_1) - \Psi(v_2)| &\leq Cpm^{p-1}\left|\left|\left|\frac{v_1 - v_2}{\eta_b}\right|\right|\right|_\infty \left|\int_0^t S(t-s)\eta_b^p(x,s)ds\right| \\
&\leq Cpm^{p-1}\left|\left|\left|\frac{v_1 - v_2}{\eta_b}\right|\right|\right|_\infty \left|\int_0^t (1+s)^{(1-p)b/2}\eta_b(x,t)ds\right|
\end{aligned}$$

by definition of η_b. Since $p\min\{b,d\} - b > 2$

$$\|\Psi(v_1) - \Psi(v_2)\|_{E_{\eta,b}} \leq \tilde{C}m^{p-1}\|v_1 - v_2\|_{E_{\eta,b}}$$

with some $\tilde{C}$. If m is sufficiently small, then Ψ is a strict contracion in $P_{\eta,b} \cup M_{\eta,m,b}$. Thus we can take $u(x,t) \leq mS(t) < x >^b$ and $u(x,t)$ is global in time.

Finally, take $a > b$. Then by Lemma 2.2, the condition of u_0 may be replaced with the condition $u_0 \in I^a$ with $a \geq b \geq 2(p-1)$ and $\|u_0\|_\infty$ is sufficiently small. In fact, $b > 2/(p-1)$ is arbitary, we need only $a > 2/(p-1)$ for global existence of the solution in time. $\square$

5 Proof of Thoerems

In this section, we proof Theorems 1, 2 and 3.

First we consider the case

$$\int_m^\infty \frac{ds}{f(s)} = \infty \tag{32}$$

with $m = m(f)$ satisfying $f(s) > 0$ for $s > m$.

Proof of Theorem 1. Let $\bar{v}$ be the solution of (1) with the initial data $\|u_0\|_\infty + m + 1 = M$. From Lemma 2.2, we have $\bar{v}(t) \geq u(x,t)$ for $x \in \mathbf{R}^d \times [0, T_{\bar{v}})$, where $T_{\bar{v}} = T^*(M, f)$. It can be assumed that $\lim_{t\to T_{\bar{v}}} \bar{v}(t) = \infty$ for some $T_{\bar{v}} \in (0, \infty]$. (If this assumption is dropped, $v(t)$ does not blow up in finite time.) But from (32), it seems easily that $T_{\bar{v}} = \int_M^\infty ds/f(s) = \infty$. Thus, $\bar{v}$ is global in time, so u is, too. □

Secondly, in the case $f \in J_c$, we consider the solutions blowing up in finite time.

Proof of Theorem 2. We can put strict increasing function $\tilde{f}$ satisfying $\tilde{f} \in J_c \cap Z_p$ with $1 < p \leq p_F$ such that $f(u) \geq \tilde{f}(u)$ for $u > 0$. We consider the solution $\underline{u}$ of (1) with the nonlinear term exchanged for $\tilde{f}$. From Lemmas 3.3, 3.4 and 3.7, $\underline{u}$ blows up in finite time. Thus from Lemma 2.2, the solution u of (1) blows up in finite time. □

Finally, we estimate the condition of u_0 and $f \in J$ for the solution existing global in time.

Proof of Theorem 3. Let $\hat{f}(s) = \max\{\overline{C}s^{p_1}, f(s)\}$ for $u > 0$, where $\overline{C}$ sufficiently large satisfying $\hat{f}(u) \geq f(u)$ for u near 0. We consider the solution $\overline{u}$ of (1) with the nonlinear term replaced with $\hat{f}$. From Lemma 4.2, $\overline{u}$ is global in time. Thus, from Lemma 2.2, the solution u of (1) is global in time. □

Acknowledgement. Much of the work of the author was done while he visited the Hokkaido University during 2003-2005 and the University of Tokyo during 2005-2006 as a postdoctoral fellow. Its hospitality is gratefully acknowledged of formation of COE "Mathematics of Nonlinear Structures via Singularities" and COE "New Mathematical Development Center to Support Scientific Technology", supported by JSPS.

References

[1] J. Aguirre and M. Escobedo, *A Cauchy problem for $u_t - \Delta u = u^p$ with $0 < p < 1$. Asymptotic behavior of solutions.*, Ann. Fac. Sci. Toulouse Math. **8** (1986-87), 175-203.

[2] M.Escobedo and M. A. Herrero, *Boundedness and blow up for a semilinear reaction-diffusion system,* J. Differential Equations **89** (1991), 176-202.

[3] A. Friedman, *Partial differential equations of parabolic type*, Robert E. Krieger publishing company, Malabar, Florida, 1983.

[4] H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t - \Delta u = u^{1+\alpha}$*,* J. Fac. Sci. Univ. of Tokyo, Sect. I **13** (1966), 65-79.

[5] V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailov and A. A. Samarskii, *Blow-up in quasilinear parabolic equations*, Walter de gruyter, Berlin, New York 1995.

[6] K. Hayakawa, *On nonexistence of global solution of some semilinear parabolic equations*, Proc. Japan. Acad. **49** (1973), 503-505.

[7] Q. Huang and K. Mochizuki, *Existence and behavior of solution for a weakly coupled system of reaction-diffusion equation,* Methods and Application of Analysis **5** (2) 1998, 109-129.

[8] K. Kobayashi, T Sirao and H. Tanaka, *On glowing up problem for semilinear hert equations*, J. Math. Soc. Japan **29** (1977), 407-424.

[9] A. A. Lacey, *The form of blow-up for nonlinear parabolic equations,* Proc. Roy. Soc. Edinburgh Sect. A **98** (1984), no. 1-2, 183–202.

[10] Lacey, A. A. *Global blow-up of a nonlinear heat equation,* Proc. Roy. Soc. Edinburgh Sect. A **104** (1986), no. 1-2, 161–167.

[11] T. Y. Lee and W.-M. Ni, *Global existence, large time behavior and lifespan on solution of a semilinear Cauchy problem,* Tran. Amer. Math. Soc. **333** (1992), 365-378.

[12] K. Mochizuki, *Blow-up, life span and large time behavior of solution of weakly coupled system of reaction-diffusion equation,* Advance in Nonlinear Partial Differential Equation and Stochastics, 175-197, World Scientific Publishing, River Edge, NJ, 1998.

[13] M. H. Protter and H. F. Weinberger, *Maximum principle in differential equations*, Prentice-Hall, INC, Englewood Cliffs, Nwe Jersey.

[14] F. B. Weissler, *Existence and nonexistence of global solutions for semilinear heat equation*, Israel J. Math. **38** (1981), 29-40.

UTMS

2006–19 Adriano Marmora: *Facteurs epsilon p-adiques.*

2006–20 Yukihiro Seki, Ryuichi Suzuki and Noriaki Umeda: *Blow-up directions for quasilinear parabolic equations.*

2006–21 Yoshihiro Sawano and hitoshi Tanaka : *A quarkonial decomposition of Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces.*

2006–22 Victor Isakov, Jenn-Nan Wang and Masahiro Yamamoto: *An inverse problem for a dynamical Lamé system with residual stress.*

2006–23 Oleg Yu. Imanuvilov, Victor Isakov and Masahiro Yamamoto: *New realization of the pseudoconvexity and its application to an inverse problem.*

2006–24 Kazuki Hiroe and Takayuki Oda: *Hecke-Siegel's pull back formula for the Epstein zeta function with a harmonic polynomial.*

2006–25 Takefumi Igarashi and Noriaki Umeda: *Existence and nonexistence of global solutions in time for a reaction-diffusion system with inhomogeneous terms.*

2006–26 Shigeo Kusuoka: *A remark on law invariant convex risk measures.*

2006–27 Ganghua Yuan and Masahiro Yamamoto: *Lipschitz stability in the determination of principal parts of a parabolic equation by boundary observations.*

2006–28 Akishi Kato: *Zonotopes and four-dimensional superconformal field theories.*

2006–29 Yasufumi Osajima: *The asymptotic expansion formula of implied volatility for dynamic SABR model and FX hybrid model.*

2006–30 Noriaki Umeda: *On existence and nonexistence global solutions of reaction-diffusion equations.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012