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Abstract

The aim of this paper is to develop a theory of decomposition of the functions belonging to the modulation space $M^{p,q}$ with $0 < p, q \leq \infty$. We shall define molecules for the modulation spaces. The main difficulty in developing our theory lies in the synthesis part, that is, we shall tackle the norm estimate when we are given a sum of molecules under some suitable coefficient conditions. As an application we give a simple proof of the boundedness of the pseudo-differential operators.

Keywords modulation space, molecule decomposition, pseudo-differential operator, dual space

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1 Introduction

This paper is devoted to some decomposition results in the modulation spaces. And as applications we shall investigate the boundedness property of the pseudo-differential operators and then we specify the dual space.

As is well-known, the modulation spaces are frequently used in signal analysis. Based on the standard notation of signal analysis, we adopt the following notations.

$$
\begin{aligned}
T_a f(x) &:= f(x-a),\ M_b f(x) := e^{ib\cdot x} f(x),\ a,b \in \mathbb{R}^n,\ f \in \mathcal{S}' \\
\mathcal{F}f(\xi) &:= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) \exp(-ix\cdot\xi)\,dx, \\
\mathcal{F}^{-1}f(x) &:= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi) \exp(ix\cdot\xi)\,d\xi.
\end{aligned}
$$

It will be helpful to use the notation from [9] as well. Let $f \in \mathcal{S}'$ and $\tau \in \mathcal{S}$. Then we write

$$
\tau(D)f := \mathcal{F}^{-1}(\tau \cdot \mathcal{F}f) = (2\pi)^{\frac{n}{2}} \mathcal{F}^{-1}\tau * f. \tag{1}
$$

First let us recall the modulation norm $\|\cdot : M^{p,q}\|$ with $1 \le p,q \le \infty$. To do this let us pick a compactly supported function $\psi \in \mathcal{S}$ so that it satisfies

$$
\mathrm{supp}(\psi) \subset Q(2),\ \sum_{m\in\mathbb{Z}^n} T_m\psi(x) \equiv 1.
$$

Then we define

$$
\|f : M^{p,q}\| := \|\, T_m\psi(D)f : l^q(L^p)\| := \left(\sum_{m\in\mathbb{Z}^n} \|\,[\mathcal{F}^{-1}T_m\psi] * f : L^p\|^q \right)^{\frac{1}{q}} \tag{2}
$$

for $f \in \mathcal{S}'$. Here we have defined

$$
\|f_m : l^q(L^p)\| := \left(\sum_{m\in\mathbb{Z}^n} \|f_m : L^p\|^q \right)^{\frac{1}{q}}
$$

for a family of measurable functions $\{f_m\}_{m\in\mathbb{Z}^n}$. A different choice of ψ will give us an equivalent norm. As for the Fourier multipliers and the multiplication operators we prefer to avoid superfluous bracket. We shall list some typical examples appearing in this paper:

$$
T_a\phi(D)f := [T_a\phi](D)f,\ M_b\phi(D)f := [M_b\phi](D)f,\ M_b\psi * f := [M_b\psi] * f,\ a,b \in \mathbb{R}^n.
$$

If possible confusion can occur, we bind the function on which the operator acts on.

The above definition proved to be still valid whenever $0 < p,q \le \infty$. For details we refer to [2, 5]. Thus, for $0 < p,q \le \infty$, we can define the modulation space $M^{p,q}$ as the set of all Schwartz distributions $f \in \mathcal{S}'$ for which the quasi-norm $\|f : M^{p,q}\|$, given by (2), is finite.

In this present paper we will discuss the decomposition method of the function space $M^{p,q}$. In [2], the atomic decomposition was investigated. However, the atomic decomposition suffers from some disadvantages in analyzing the pseudo-differential operators. Indeed, it is not the case that the image of a compactly supported function by a pseudo-differential operator is compactly supported. To overcome this disadvantage, we introduce the molecules for the modulation spaces.

Definition 1.1 (Molecule)**.** Let $K, N \in \mathbb{N}$ be large enough and fixed. A C^K function $\tau : \mathbb{R}^n \to \mathbb{C}$ is said to be an (m, l)-molecule, if it satisfies

$$|\partial^\alpha (e^{-im\cdot x}\tau(x))| \le \langle x - l\rangle^{-N}, \; x \in \mathbb{R}^n$$

for $|\alpha| \le K$. Also set

$$\mathcal{M} := \{M = \{mol_{ml}\}_{m,l\in\mathbb{Z}^n} \subset C^K \; : \; mol_{ml} \text{ is an } (m,l)\text{-molecule for each } m,l \in \mathbb{Z}^n \text{ modulo multiplicative constants}\}.$$

It will turn out that K and N will do in order to develop our theory provided

$$K, N \ge 10\left[\frac{n}{\min(1,p,q)}\right] + 10.$$

Next, we introduce a sequence space $m^{p,q}$ to describe the condition of the coefficients of the molecule decomposition.

Definition 1.2 (Sequence space $m^{p,q}$)**.** Given $\lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n}$, define

$$\|\lambda \, : \, m^{p,q}\| := \left(\sum_{m\in\mathbb{Z}^n}\left(\sum_{l\in\mathbb{Z}^n}|\lambda_{ml}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}, \; 0 < p, q \le \infty.$$

Here a natural modification is made when p or / and q is not finite. $m^{p,q}$ is a set of doubly indexed sequence $\lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n}$ for which the quasi-norm $\|\lambda \, : \, m^{p,q}\|$ is finite.

Observe that

$$\|\lambda \, : \, m^{p,q}\| = \left\| \left\{\sum_{l\in\mathbb{Z}^n}\lambda_{ml}\chi_{Q_l}\right\}_m \, : \, l^q(L^p)\right\|,$$

where $Q_l := [l_1, l_1+1] \times [l_2, l_2+1] \times \ldots \times [l_n, l_n+1]$ for $l \in \mathbb{Z}^n$.

With these definitions in mind, we shall present our main theorem in this paper.

Theorem 1.3. *Let $0 < p, q \le \infty$ and denote $\{x \in \mathbb{R}^n \, : \, \max(|x_1|, |x_2|, \ldots, |x_n|) \le r\}$ by $Q(r)$.*

1. (decomposition) *Let $\kappa \in \mathcal{S}$ be taken so that $\chi_{Q(3)} \le \kappa \le \chi_{Q(3+1/100)}$. Set $mol_{ml} := T_l M_m[\mathcal{F}^{-1}\kappa]$. Then we have $\{mol_{ml}\}_{m,l\in\mathbb{Z}^n} \in \mathcal{M}$. Furthermore, any $f \in M^{p,q}$ admits the following decomposition.*

$$f = \sum_{m,l\in\mathbb{Z}^n}\lambda_{ml}\cdot mol_{ml}, \; \lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n} \in m^{p,q}.$$

Here the mapping $f \in M^{p,q} \mapsto \lambda \in m^{p,q}$ is linear and bounded.

2. (synthesis) *Let* $M = \{mol_{ml}\}_{m,l\in\mathbb{Z}^n} \in \mathcal{M}$ *and* $\lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n} \in m^{p,q}$. *Then the series*

$$f := \sum_{m,l\in\mathbb{Z}^n} \lambda_{ml} \cdot mol_{ml}$$

converges unconditionally in the topology of $\mathcal{S}'$. *Furthermore* f *belongs to* $M^{p,q}$ *and satisfies the norm estimate*

$$\|f \,:\, M^{p,q}\| \le C\, \|\lambda \,:\, m^{p,q}\|.$$

Finally we describe the organization of this paper. In the next section, which is the heart of this paper, we investigate the molecule decomposition of the modulation spaces. Since a decomposition result is obtained in [2], we do not have to consider the decomposition of $f \in M^{p,q}$ into the sum of atoms. However, the decomposition we shall need is not so difficult to prove and for the sake of convenience for readers we include one in Section 2. In this paper we are mainly concerned with the synthesis result. In Section 3 we investigate the pseudo-differential operators whose symbol belongs to $S^0_{0,0}$. Recall that a symbol class $S^m_{\rho,\delta}$ with $m \in \mathbb{R}$ and $0 \le \rho, \delta \le 1$ is a set of $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ functions a satisfying

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha,\beta} \, \langle \xi \rangle^{m-\delta|\alpha|+\rho|\beta|}.$$

We remark that $M^{p,q}$-boundedness of the pseudo-differential operators generated by $S^m_{\rho,\delta}$-symbols is obtained in [2, 4, 7] with $1 \le p, q \le \infty$. As an application of $M^{p,q}$-boundedness of this result and the decomposition result in Section 2 we shall prove that the pseudo-differential operator generated by $M^{\infty,1}(\mathbb{R}^n \times \mathbb{R}^n)$ is bounded on $M^{p,q}$. We remark that in [3, 4] Gröchenig and Heil proved this result in the case when $1 \le p, q \le \infty$. As for the limit case they used the dual argument. What is new about this result is the fact that we have proved the counterpart for general parameters $0 < p, q \le \infty$ and the point that we do not have to rely on the dual argument. Finally in Section 4 we exhibit another application of the results in Section 2. In [6] the first author investigated the dual space of $M^{p,q}$ with $0 < p, q < \infty$. However, the definitive result when $0 < p \le 1 \le q < \infty$ was missing. In this present paper we shall supplement this missing part.

2 Molecule decomposition in $M^{p,q}$

In this section we deal with the molecule decomposition, in particular, the synthesis property. We assume that $\psi \in \mathcal{S}$ is a positive function satisfying

$$\mathrm{supp}(\psi) \subset Q(2), \quad \sum_{m\in\mathbb{Z}^n} T_m\psi(x) \equiv 1. \tag{3}$$

As preliminaries we collect two important results on the band-limited distributions.

Lemma 2.1. [8, Chapter 1] *Let $0 < \eta < \infty$. Then there exists $c > 0$ such that*

$$\sup_{y \in \mathbb{R}^n} \frac{|f(x-y)|}{1+|y|^{\frac{n}{\eta}}} \leq c\, M^{(\eta)} f(x)$$

for all $f \in \mathcal{S}'$ with $\mathrm{diam}(\mathrm{supp}(\mathcal{F}f)) \leq 10$, *where $M^{(\eta)}$ is a powered maximal operator given by*

$$M^{(\eta)} f(x) := \sup_{\substack{x \in Q \\ Q\,:\,cube}} \left(\frac{1}{|Q|} \int_Q |f(y)|^\eta \, dy \right)^{\frac{1}{\eta}}. \tag{4}$$

We note that under our notation the well-known maximal inequality reads

$$\|M^{(\eta)} f \,:\, L^p\| \leq c\, \|f \,:\, L^p\|, \; 0 < \eta < p \leq \infty. \tag{5}$$

Let $M \in \mathbb{N}$. Denote by $W^{M,2}$ the Sobolev spaces consisting of $f \in L^2$ satisfying

$$\|f \,:\, W^{M,2}\| := \| \langle * \rangle^M \cdot \mathcal{F}f \,:\, L^2\| < \infty.$$

Lemma 2.2. [8, Chapter 1] *Let $0 < p \leq \infty$ and $M \in \mathbb{N}$ with*

$$M > \frac{n}{\min(1,p)} + \frac{3n}{2}.$$

Set

$$H(D)f(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} H(\xi) \mathcal{F}f(\xi) e^{ix\cdot\xi} \, d\xi.$$

Then the integral defining $H(D)f(x)$ converges for a.e. $x \in \mathbb{R}^n$ and coincides with the usual definition given by (1) *if $H \in \mathcal{S}$. Furthermore, there exists a constant $c > 0$ independent of $R > 0$ so that*

$$\|H(D)f \,:\, L^p\| \leq c\, \|H(R \cdot *) \,:\, W^{M,2}\| \cdot \|f \,:\, L^p\|,$$

whenever $H \in W^{M,2}$ and $f \in L^p \cap \mathcal{S}'$ with $\mathrm{diam}(\mathrm{supp}(\mathcal{F}f)) \leq R$.

The following well-known lemma is used to prove the decomposition results. For example, we refer for the proof to the paper [1] due to M. Frazier and B. Jawerth, who took originally a full advantage of this equality.

Lemma 2.3. [1] *Let $f \in \mathcal{S}'$ with* $\mathrm{supp}(\mathcal{F}f) \subset Q(2)$. *Then*

$$f = (2\pi)^{-\frac{n}{2}} \sum_{l \in \mathbb{Z}^n} f(l) \cdot \mathcal{F}^{-1}\kappa(* - l). \tag{6}$$

It is convenient to transform (6) to the form in which we use in this present paper :

$$f = \sum_{m \in \mathbb{Z}^n} T_m \psi(D) f = (2\pi)^{-\frac{n}{2}} \sum_{m \in \mathbb{Z}^n} \left(\sum_{l \in \mathbb{Z}^n} T_m \psi(D) f(l) \cdot T_l M_m [\mathcal{F}^{-1}\kappa] \right). \tag{7}$$

2.1 Decomposition result

Now we shall prove a simple result. That is, we are going to prove a decomposition part in Theorem 1.3.

As for the first assertion of Theorem 1.3 *1*, $\{mol_{ml}\}_{m,l\in\mathbb{Z}^n} \in \mathcal{M}$ is clear, once we fix K sufficiently large in the definition of molecules (Definition 1.1).

Let $f \in M^{p,q}$. Then we expand f according to (7):

$$f = (2\pi)^{-\frac{n}{2}} \sum_{m\in\mathbb{Z}^n} \left(\sum_{l\in\mathbb{Z}^n} T_m\psi(D)f(l) \cdot T_l M_m[\mathcal{F}^{-1}\kappa] \right).$$

Thus, if we set $\lambda_{ml} := T_m\psi(D)f(l)$, $mol_{ml} := T_l M_m[\mathcal{F}^{-1}\kappa]$ then we obtain a decomposition of f

$$f = \sum_{m,l\in\mathbb{Z}^n} \lambda_{ml} \cdot mol_{ml}. \tag{8}$$

Let us check that this decomposition fulfills the desired property in Theorem 1.3. Indeed, as we have remarked in Introduction, we have the following expression of the sequence norm:

$$\left(\sum_{m\in\mathbb{Z}^n} \left(\sum_{l\in\mathbb{Z}^n} |\lambda_{ml}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} = \left\| \left\{ \sum_{l\in\mathbb{Z}^n} \lambda_{ml}\chi_{Q_l} \right\}_m \, : \, l^q(L^p) \right\|.$$

Because we are going to utilize the maximal inequality (5), the expression in right-hand side is agreeable.

Lemma 2.1 gives us a pointwise estimate of each summand

$$\left| \sum_{l\in\mathbb{Z}^n} \lambda_{ml}\chi_{Q_l}(x) \right| \le c\, M^{(\eta)}[T_m\psi(D)f](x)$$

with η slightly less than $\min(1,p)$. Now that η is less than $\min(1,p)$, we can remove the maximal operator to obtain

$$\left(\sum_{m\in\mathbb{Z}^n} \left(\sum_{l\in\mathbb{Z}^n} |\lambda_{ml}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \le c\, \| M^{(\eta)}[T_m\psi(D)f] \, : \, l^q(L^p) \| \le c\, \| T_m\psi(D)f \, : \, l^q(L^p) \|.$$

Thus, putting these observations together, we finally have

$$\left(\sum_{m\in\mathbb{Z}^n} \left(\sum_{l\in\mathbb{Z}^n} |\lambda_{ml}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \le c\, \| f \, : \, M^{p,q} \|. \tag{9}$$

(9) together with (8) concludes the proof of the decomposition part of Theorem 1.3.

2.2 An equivalent norm

Having obtained a decomposition result, we are now going to be oriented to the synthesis part. To do this we need an equivalent norm.

Theorem 2.4. *Let $0 < p, q \leq \infty$ and $\psi \in \mathcal{S}$ a positive function satisfying a non-degenerate condition :*

$$\mathcal{F}\psi \neq 0 \text{ on } Q(2).$$

Then there exists a constant $c > 0$ such that

$$c^{-1} \|f : M^{p,q}\| \leq \left(\sum_{k \in \mathbb{Z}^n} \| M_k \psi * f : L^p \|^q \right)^{\frac{1}{q}} \leq c \|f : M^{p,q}\|$$

for all $f \in M^{p,q}$.

To prove the theorem we need one more calculation.

Lemma 2.5. *Let $\tau, \theta \in \mathcal{S}$. Suppose that θ is compactly supported. Then for all $M \in \mathbb{N}$ there exists $c_{M,\alpha}$ depending only on τ, θ, α and M such that there holds*

$$|\partial^\alpha (T_l\theta \cdot T_m\tau)(x))| \leq c_{M,\alpha} \langle l - m \rangle^{-M} \text{ for all } x, l, m \in \mathbb{R}^n, \tag{10}$$

where $\langle x \rangle := \sqrt{|x|^2 + 1}$.

Proof. By the Leibnitz rule and the well-known inequality $\langle a + b \rangle \leq 2 \langle a \rangle \cdot \langle b \rangle$, we have

$$|\partial^\alpha (T_l\theta \cdot T_m\tau)(x))| \leq c_{M,\alpha} \langle x - l \rangle^{-M} \langle x - m \rangle^{-M} \leq c_{M,\alpha} \langle l - m \rangle^{-M},$$

proving (10). ∎

With Lemmas 2.1, 2.2 and 2.5 in mind, let us complete the proof of Theorem 2.4.

Proof of Theorem 2.4. We shall first prove

$$\left(\sum_{k \in \mathbb{Z}^n} \| M_k \psi * f : L^p \|^q \right)^{\frac{1}{q}} \leq c \|f : M^{p,q}\| \tag{11}$$

and then

$$\|f : M^{p,q}\| \leq c \left(\sum_{k \in \mathbb{Z}^n} \| M_k \psi * f : L^p \|^q \right)^{\frac{1}{q}}. \tag{12}$$

We can assume by replacing ϕ, if necessary, even that

$$\sum_{l \in \mathbb{Z}^n} T_l \phi \equiv 1. \tag{13}$$

For the proof of (11) we decompose $M_k\psi * f$ by using (13)

$$M_k\psi * f = \sum_{l\in\mathbb{Z}^n} M_k\psi * [T_l\phi(D)f]. \tag{14}$$

$M_k\psi * f$ having been decomposed in (14), we are to estimate each summand. To do this, we rewrite the summand as

$$\begin{aligned} M_k\psi * [T_l\phi(D)f](x) &= \mathcal{F}^{-1}(T_k[\mathcal{F}\psi]\cdot\mathcal{F}(T_l\phi(D)f))(x) \\ &= T_k[\mathcal{F}\psi](D)T_l\phi(D)f(x) \\ &= [T_k[\mathcal{F}\psi]\cdot T_l\tilde{\kappa}](D)T_l\phi(D)f(x) \\ &= (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}\mathcal{F}^{-1}[T_k[\mathcal{F}\psi]\cdot T_l\tilde{\kappa}](y)T_l\phi(D)f(x-y)\,dy. \end{aligned}$$

where $\tilde{\kappa}\in\mathcal{S}$ is an auxiliary compactly supported function that equals 1 on $\operatorname{supp}(\phi)$. By virtue of Lemma 2.5 we have

$$|\mathcal{F}^{-1}[T_k[\mathcal{F}\psi]\cdot T_l\tilde{\kappa}](y)| \le c_N\,\langle l-k\rangle^{-N}\cdot\langle y\rangle^{-N}, \tag{15}$$

where N is taken arbitrarily large. Let $\eta := \dfrac{\min(1,p)}{2}$. From Lemma 2.1 we have

$$|T_l\phi(D)f(x-y)| \le c\,M^{(\eta)}[T_l\phi(D)f](x)\cdot\langle y\rangle^{\frac{n}{\eta}}. \tag{16}$$

Recall that N is still at our disposal. Thus, if we take N large enough and combine (15) and (16), we obtain

$$|M_k\psi * T_l\phi(D)f(x)| \le c\,\langle l-k\rangle^{-2N}\cdot M^{(\eta)}[T_l\phi(D)f](x).$$

Therefore, inserting this estimate and using the boundedness of $M^{(\eta)}$, we have

$$\begin{aligned} \|\,M_k\psi * f\ :\ L^p\|^{\min(1,p)} &\le \sum_{l\in\mathbb{Z}^n}\|\,M_k\psi * T_l\phi(D)f\ :\ L^p\|^{\min(1,p)} \\ &\le c\sum_{l\in\mathbb{Z}^n}\langle l-k\rangle^{-2N\min(1,p)}\cdot\|M^{(\eta)}[T_l\phi(D)f]\ :\ L^p\|^{\min(1,p)} \\ &\le c\sum_{l\in\mathbb{Z}^n}\langle l-k\rangle^{-2N\min(1,p)}\cdot\|\,T_l\phi(D)f\ :\ L^p\|^{\min(1,p)}. \end{aligned}$$

Suppose that $q\ge\min(1,p)$. In this case $u := \dfrac{q}{\min(1,p)} \ge 1$ and then we use the Hölder inequality to obtain

$$\begin{aligned} &\|\,M_k\psi * f\ :\ L^p\|^{\min(1,p)} \\ &\quad\le c\left(\sum_{l\in\mathbb{Z}^n}\langle l-k\rangle^{-Nu'\min(1,p)}\right)^{\frac{1}{u'}}\cdot\left(\sum_{l\in\mathbb{Z}^n}\langle l-k\rangle^{-Nq}\cdot\|\,T_l\phi(D)f\ :\ L^p\|^q\right)^{\frac{1}{u}} \\ &\quad\le c\left(\sum_{l\in\mathbb{Z}^n}\langle l-k\rangle^{-Nq}\cdot\|\,T_l\phi(D)f\ :\ L^p\|^q\right)^{\frac{1}{u}}. \end{aligned}$$

Therefore, if we arrange this inequality, we are led to

$$\| M_k\psi * f \ : \ L^p\|^q \le c \sum_{l\in\mathbb{Z}^n} \langle l-k\rangle^{-Nq} \cdot \| T_l\phi(D)f \ : \ L^p\|^q. \tag{17}$$

Suppose instead that $q \le \min(1,p)$. Then the matter is simpler than before. A trivial inequality

$$(a+b)^v \le a^v + b^v, \ 0 < v \le 1, \ a,b > 0 \tag{18}$$

suffices. Indeed,

$$\begin{aligned} &\| M_k\psi * f \ : \ L^p\|^q \\ &\quad \le c \left(\sum_{l\in\mathbb{Z}^n} \langle l-k\rangle^{-2N\min(1,p)} \cdot \| T_l\phi(D)f \ : \ L^p\|^{\min(1,p)} \right)^{\frac{q}{\min(1,p)}} \\ &\quad \le c \sum_{l\in\mathbb{Z}^n} \langle l-k\rangle^{-2Nq} \cdot \| T_l\phi(D)f \ : \ L^p\|^q. \end{aligned}$$

Thus, whether q is larger than $\min(1,p)$, the key estimate (17) is always available.

If we add (17) over $m \in \mathbb{Z}^n$, then we obtain (11).

Now we prove (12). To do this, we pick a smooth cutoff function $\eta_0 : \mathbb{R} \to \mathbb{R}$ with

$$\chi_{(-\infty,1)} \le \eta_0 \le \chi_{(-\infty,2)}.$$

Set

$$\eta(x) := \eta_K(x) := \eta_0(K^{-1}x_1)\eta_0(K^{-1}x_2)\ldots\eta_0(K^{-1}x_n)$$

with K large. We let

$$\eta^\sharp := \eta(2^{-1}*)$$

and

$$M := \left[\frac{n}{\min(1,p)} + \frac{3n}{2} \right] + 1.$$

Then we have, taking into account the size of the supports of functions,

$$\|f \ : \ M^{p,q}\| = \left(\sum_{k\in\mathbb{Z}^n} \| T_k\eta^\sharp(D)T_k\eta(D)T_k\phi(D)f \ : \ L^p\|^q \right)^{\frac{1}{q}}.$$

Since $\mathcal{F}\psi$ never vanishes on $\mathrm{supp}(\phi)$, the function

$$\Phi := \frac{\phi}{\mathcal{F}\psi}$$

is well-defined. Note that

$$T_k\phi(D)f = T_k\Phi(D)[M_k\psi * f].$$

Thus, using this decomposition and the translation invariance of $W^{M,2}$, we obtain

$$\begin{aligned}
\|f \,:\, M^{p,q}\| &= \left(\sum_{k \in \mathbb{Z}^n} \|T_k \eta^\sharp(D) T_k \Phi(D) T_k \eta(D)[M_k \psi * f] \,:\, L^p\|^q \right)^{\frac{1}{q}} \\
&\le c \left(\sum_{k \in \mathbb{Z}^n} \|\eta^\sharp \cdot \Phi \,:\, W^{M,2}\|^q \cdot \| T_k \eta(D)[M_k \psi * f] \,:\, L^p\|^q \right)^{\frac{1}{q}} \qquad (19) \\
&\le c\, K^{M+n} \left(\sum_{k \in \mathbb{Z}^n} \| T_k \eta(D)[M_k \psi * f] \,:\, L^p\|^q \right)^{\frac{1}{q}} \\
&\le c\, K^{M+n} \left(\sum_{k \in \mathbb{Z}^n} \| M_k \psi * f \,:\, L^p\|^q \right)^{\frac{1}{q}} \\
&\quad + c\, K^{M+n} \left(\sum_{k \in \mathbb{Z}^n} \| M_k \psi * [(1 - T_k \eta(D)) f] \,:\, L^p\|^q \right)^{\frac{1}{q}}. \qquad (20)
\end{aligned}$$

Here for (19) we have invoked Lemma 2.2. Our strategy for the proof is to establish that the second term of (20) can be made small enough, if we take K sufficiently large. Recall that we have proved (17), that is, for every $g \in \mathcal{S}'$

$$\| M_k \psi * g \,:\, L^p\|^q \le c \sum_{m \in \mathbb{Z}^n} \langle k - m \rangle^{-Nq} \cdot \|T_m \phi(D) g \,:\, L^p\|^q.$$

If we apply the above inequality with $g = (1 - T_l \eta(D)) f$, then we obtain

$$\| M_k \psi * (1 - T_k \eta(D)) f \,:\, L^p\|^q \le c \sum_{m \in \mathbb{Z}^n} \langle k - m \rangle^{-Nq} \cdot \| T_m \phi(D) (1 - T_k \eta(D)) f \,:\, L^p\|^q.$$

Taking into account the support condition of η again, we are led to

$$\| M_k \psi * (1 - T_k \eta(D)) f \,:\, L^p\|^q \le c \sum_{\substack{k \in \mathbb{Z}^n \\ |k-m| \ge K-2}} \langle k - m \rangle^{-Nq} \cdot \| T_m \phi(D) f \,:\, L^p\|^q$$

As a result if we add this inequality over $m \in \mathbb{Z}^n$, then we have

$$\begin{aligned}
\sum_{k \in \mathbb{Z}^n} \| M_k \psi * (1 - T_k \eta(D)) f \,:\, L^p\|^q &\le c \sum_{\substack{k,m \in \mathbb{Z}^n \\ |k-m| \ge K-2}} \langle k - m \rangle^{-Nq} \cdot \| T_m \phi(D) f \,:\, L^p\|^q \\
&\le c\, K^{-Nq+n} \|f \,:\, M^{p,q}\|^q.
\end{aligned}$$

If we insert this estimate to (20), then we obtain

$$\|f \,:\, M^{p,q}\| \le c\, K^{M+n} \left(\sum_{k \in \mathbb{Z}^n} \| M_k \psi * f \,:\, L^p\|^q \right)^{\frac{1}{q}} + c\, K^{M+n+\frac{n}{q}-N} \|f \,:\, M^{p,q}\|. \qquad (21)$$

By assumption, we have $f \in M^{p,q}$. Consequently, if we fix N so large that $N > M + n + \frac{n}{q}$ and then choose K large enough, then we can bring the second term of the right-hand side in (21) to the left-hand side. The result is

$$\|f \,:\, M^{p,q}\| \le c \left(\sum_{k \in \mathbb{Z}^n} \| M_k \psi * f \,:\, L^p \|^q \right)^{\frac{1}{q}},$$

proving (12). ∎

2.3 Synthesis results

First we verify that the sum converges very nicely.

Lemma 2.6. *Assume $\Lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n}$ is a bounded sequence, namely, $\Lambda \in m^{\infty,\infty}$ and that a family of functions $M = \{mol_{ml}\}_{m,l \in \mathbb{Z}^n}$ belongs to $\mathcal{M}$. Then the series*

$$\sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot mol_{ml}$$

is convergent unconditionally in $\mathcal{S}'$.

Proof. Fix a test function $\phi \in \mathcal{S}$. For the sake of brevity we set

$$\Phi_{ml}(x) := e^{-im \cdot x} mol_{ml}(x), \ m, l \in \mathbb{Z}^n.$$

Then $\{\Phi_{ml}\}_{m,l \in \mathbb{Z}^n} \subset C^K$ fulfills the following differential inequality

$$\sup_{x \in \mathbb{R}^n} \langle x - l \rangle^{-N} |\partial^\alpha \Phi_{ml}(x)| < \infty$$

for all multi-indices α with $|\alpha| \le K$. Therefore we have

$$\begin{aligned}\int_{\mathbb{R}^n} \phi(x) mol_{ml}(x)\, dx &= \int_{\mathbb{R}^n} \phi(x) \Phi_{ml}(x) \exp(im \cdot x)\, dx \\ &= \langle m \rangle^{-2K_0} \int_{\mathbb{R}^n} [(1 - \Delta)^{K_0} (\phi(x) \Phi_{ml}(x))] \exp(im \cdot x)\, dx.\end{aligned}$$

Here $K_0 := \left[\frac{K}{2}\right]$. Therefore it follows that

$$\begin{aligned}\left| \int_{\mathbb{R}^n} \phi(x) mol_{ml}(x)\, dx \right| &\le \langle m \rangle^{-2K_0} \int_{\mathbb{R}^n} \left| [(1 - \Delta)^{K_0} (\phi(x) \Phi_{ml}(x))] \right| dx \\ &\le c \left(\langle m \rangle \cdot \langle l \rangle \right)^{-2K_0}.\end{aligned}$$

From this we can readily deduce the desired convergence. ∎

A calculation in the proof of [5, Theorem 3.6] essentially shows the following.

Lemma 2.7. *Any (m,l)-molecule belongs to $M^{p,q}$, provided K is large enough.*

Keeping these lemmas in mind, we complete the proof of the synthesis part of Theorem 1.3.

Proof of Theorem 1.3 2. Lemmas 2.6 and 2.7 together with Fatou's lemma reduce the matters to showing

$$\left(\sum_{k\in\mathbb{Z}^n}\|\, M_k\psi * f \,:\, L^p\|^q\right)^{\frac{1}{q}} \le c\,\|\Lambda \,:\, m^{p,q}\|, \tag{22}$$

where ψ is a smooth function supported on a small ball $B(r)$ and the elements in Λ are zero with finite exceptions. Let $k,l,m\in\mathbb{Z}^n$ be fixed. We estimate

$$M_k\psi * mol_{ml}(x) = e^{ik\cdot x}\int_{\mathbb{R}^n} e^{i(m-k)\cdot y}\psi(x-y)\cdot(e^{-im\cdot y}mol_{ml}(y))\,dy.$$

First insert $(1-\Delta)^M e^{i(m-k)\cdot y} = \langle m-k\rangle^{2M}e^{i(m-k)\cdot y}$ and carry out the integration by parts. Then we obtain

$$\begin{aligned}&M_k\psi * mol_{ml}(x)\\ &\quad= \frac{e^{ik\cdot x}}{\langle m-k\rangle^{2M}}\int_{\mathbb{R}^n} e^{i(m-k)\cdot y}(1-\Delta_y)^M\left\{\psi(x-y)(e^{-im\cdot y}mol_{ml}(y))\right\}\,dy.\end{aligned}$$

Thus, since $\{mol_{ml}\}_{m,l\in\mathbb{Z}^n}\in\mathcal{M}$ and ψ is a function supported on $B(r)$, we are led to

$$|M_k\psi * mol_{ml}(x)| \le \frac{c}{\langle m-k\rangle^{2M}}\int_{B(x,r)}\langle y-l\rangle^{-2M}\,dy \le c\,(\langle m-k\rangle\cdot\langle x-l\rangle)^{-2M}.$$

Inserting this estimate, we obtain

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^n}\left\{\int_{\mathbb{R}^n}\left(\sum_{m,l\in\mathbb{Z}^n}|\lambda_{ml}\cdot M_k\psi * mol_{ml}(x)|\right)^p dx\right\}^{\frac{q}{p}}\\ &\qquad\le c\sum_{k\in\mathbb{Z}^n}\left\{\int_{\mathbb{R}^n}\left(\sum_{m,l\in\mathbb{Z}^n}|\lambda_{ml}|\cdot(\langle m-k\rangle\cdot\langle x-l\rangle)^{-2M}\right)^p dx\right\}^{\frac{q}{p}}.\end{aligned} \tag{23}$$

With the aid of the powered maximal operator $M^{(\eta)}$ with η slightly less than $\min(1,p)$, we have

$$\sum_{l\in\mathbb{Z}^n}|\lambda_{ml}|\cdot\langle x-l\rangle^{-2M} \le c\,M^{(\eta)}\left[\sum_{l\in\mathbb{Z}^n}\lambda_{ml}\chi_{Q_l}\right](x).$$

If we insert this to (23), then we obtain

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^n}\left\{\int_{\mathbb{R}^n}\left(\sum_{m,l\in\mathbb{Z}^n}|\lambda_{ml}\cdot M_k\psi * mol_{ml}(x)|\right)^p dx\right\}^{\frac{q}{p}}\\ &\qquad\le c\sum_{k\in\mathbb{Z}^n}\left\{\int_{\mathbb{R}^n}\left(\sum_{m\in\mathbb{Z}^n}M^{(\eta)}\left[\sum_{l\in\mathbb{Z}^n}\lambda_{ml}\chi_{Q_l}\right](x)\cdot\langle m-k\rangle^{-2M}\right)^p dx\right\}^{\frac{q}{p}}.\end{aligned}$$

Going through the same argument as before using (18) or the Hölder inequality, we are led to

$$\sum_{k\in\mathbb{Z}^n}\left\{\int_{\mathbb{R}^n}\left(\sum_{m,l\in\mathbb{Z}^n}|\lambda_{ml}\cdot M_k\psi * mol_{ml}(x)|\right)^p dx\right\}^{\frac{q}{p}}$$

$$\le c\sum_{k\in\mathbb{Z}^n}\left\{\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}M^{(\eta)}\left[\sum_{l\in\mathbb{Z}^n}\lambda_{ml}\chi_{Q_l}\right](x)^p\cdot\langle m-k\rangle^{-pM}\,dx\right\}^{\frac{q}{p}}$$

$$\le c\left\{\int_{\mathbb{R}^n}\sum_{k,m\in\mathbb{Z}^n}M^{(\eta)}\left[\sum_{l\in\mathbb{Z}^n}\lambda_{ml}\chi_{Q_l}\right](x)^p\cdot\langle m-k\rangle^{-pM/2}\,dx\right\}^{\frac{q}{p}}$$

$$= c\left\{\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}M^{(\eta)}\left[\sum_{l\in\mathbb{Z}^n}\lambda_{ml}\chi_{Q_l}\right](x)^p\,dx\right\}^{\frac{q}{p}}.$$

Thus, if we use the maximal inequality with $\eta<\min(1,p)$, we obtain

$$\left(\sum_{k\in\mathbb{Z}^n}\|\,M_k\psi * f\,:\,L^p\|^q\right)^{\frac{1}{q}}\le\sum_{k\in\mathbb{Z}^n}\left\{\int_{\mathbb{R}^n}\left(\sum_{m,l\in\mathbb{Z}^n}|\lambda_{ml}\cdot M_k\psi * mol_{ml}(x)|\right)^p dx\right\}^{\frac{q}{p}}$$

$$\le c\left\{\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}\left|\sum_{l\in\mathbb{Z}^n}\lambda_{ml}\chi_{Q_l}(x)\right|^p dx\right\}^{\frac{q}{p}}$$

$$= c\,\|\Lambda\,:\,m^{p,q}\|^q,$$

which is exactly the result (22) we wish to prove. ∎

3 Pseudo-differential operators

In this section, as an application of Theorem 1.3, we prove the boundedness of the pseudo-differential operators.

Given $a\in S^m_{\rho,\delta}, m\in\mathbb{R},\ 0\le\delta,\rho\le 1$, we define

$$a(x,D)f(x):=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}a(x,\xi)\mathcal{F}f(\xi)\exp(ix\cdot\xi)\,d\xi, \tag{24}$$

for $f\in\mathcal{S}$. Following [9], we denote $\mathbb{N}_0:=\{0,1,2,\ldots\}$. As is easily seen by carrying out the integration by parts, $a(x,D)$ is a continuous linear operator on $\mathcal{S}$. If we define $a^\sharp(x,D)$, the adjoint operator of $a(x,D)$ by

$$a^\sharp(x,D)g(x):=(2\pi)^{-n}\iint_{\mathbb{R}^n}a(y,\xi)g(y)e^{i(y\cdot\xi-x\cdot\xi)}\,dy\,d\xi \tag{25}$$

in the sense of oscillatory integral, then we see that $a^\sharp(x,D)$ is also a continuous linear operator on $\mathcal{S}$. Therefore, we can extend it to a continuous linear operator on $\mathcal{S}'$ by defining, for $f \in \mathcal{S}'$

$$\langle a(x,D)f, \phi\rangle := \langle f, a^\sharp(x,D)\phi\rangle, \ \phi \in \mathcal{S}. \tag{26}$$

3.1 Symbol class $S^0_{0,0}$

In this section we shall prove $M^{p,q}$-boundedness by means of molecule decomposition of pseudo-differential operators generated by $S^0_{0,0}$ symbols.

Theorem 3.1. *Let $a \in S^0_{0,0}$, namely, assume that $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the differential inequalities*

$$\sup_{x,\xi \in \mathbb{R}^n} |\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| < \infty$$

for all $\alpha, \beta \in {\mathbb{N}_0}^n$. Then, the operator $a(x,D)$, defined initially on $\mathcal{S}$ by (24), can be extended continuously to a bounded linear operator on $M^{p,q}$.

By Theorem 1.3, Theorem 3.1 essentially reduces to establishing the following.

Lemma 3.2. *Let $\kappa \in \mathcal{S}$ be a compactly supported function. We define $mol_{ml} \in \mathcal{S}$ for $m, l \in \mathbb{Z}^n$ by setting $mol_{ml}(x) := T_l M_m[\mathcal{F}^{-1}\kappa](x)$. Then we have*

$$\{a(x,D)mol_{ml}\}_{m,l \in \mathbb{Z}^n} \in \mathcal{M}.$$

Proof. To prove this, we write $a(x,D)mol_{ml}$ out in full. As is easily verified, we have

$$\mathcal{F}mol_{ml} = M_{-l}T_m\kappa$$

and hence

$$\begin{aligned} a(x,D)mol_{ml}(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} a(x,\xi)e^{-il\cdot\xi}\kappa(\xi - m)e^{i\xi\cdot x}\,d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} a(x,\xi+m)e^{i(\xi+m)\cdot(x-l)}\kappa(\xi)\,d\xi. \end{aligned}$$

Therefore, what we have to estimate is the following function:

$$e^{-im\cdot x}a(x,D)mol_{ml}(x) = (2\pi)^{-\frac{n}{2}}e^{-im\cdot l}\int_{\mathbb{R}^n} a(x,\xi+m)e^{i\xi\cdot(x-l)}\kappa(\xi)\,d\xi. \tag{27}$$

By using $(1-\Delta_\xi)^N e^{i\xi\cdot(x-l)} = \langle x-l\rangle^{2N}e^{i\xi\cdot(x-l)}$, it is not so hard to see

$$|e^{-im\cdot x}a(x,D)mol_{ml}(x)| \le c\,\langle x-l\rangle^{-2N}.$$

Since a similar argument works for any partial derivative of $e^{-im\cdot x}a(x,D)mol_{ml}(x)$ in view of (27), the proof of this lemma is now complete. ∎

Having proved Lemma 3.2, we turn to the proof of Theorem 3.1.

Proof. Given $f \in M^{p,q} \subset M^{\infty,\infty}$, we expand it again according to (7) along with the coefficient condition :

$$f = (2\pi)^{-\frac{n}{2}} \sum_{m \in \mathbb{Z}^n} \left(\sum_{l \in \mathbb{Z}^n} T_m \psi(D) f(l) \cdot T_l M_m [\mathcal{F}^{-1} \kappa] \right)$$

$$\| \{ T_m \psi(D) f(l) \}_{m,l \in \mathbb{Z}^n} \, : \, m^{p,q} \| \le c \, \| f \, : \, M^{p,q} \|. \tag{28}$$

With this formula in mind, we define

$$a(x,D) f := (2\pi)^{-\frac{n}{2}} \sum_{m \in \mathbb{Z}^n} \left(\sum_{l \in \mathbb{Z}^n} T_m \psi(D) f(l) \cdot a(x,D) \left[T_l M_m [\mathcal{F}^{-1} \kappa] \right] \right). \tag{29}$$

Since (28) is valid for $f \in \mathcal{S}$, (29) is an extension of $a(x,D)$ from $\mathcal{S}$ to $M^{p,q}$. By virtue of (26) and the convergence of (28) and (29) in $M^{p,q}$, we see that the extension is unique. Now we are in the position of using the synthesis part of Theorem 1.3. As we have verified in Lemma 3.2, we have $\left\{ a(x,D) \left[T_l M_m [\mathcal{F}^{-1} \kappa] \right] \right\}_{m,l \in \mathbb{Z}^n} \in \mathcal{M}$. Thus, the estimate of the coefficients yield that $f \mapsto a(x,D) f$ is a continuous operator on $M^{p,q}$. ∎

Remark 3.3. It is worth pointing out that we can say more. Let $0 < p, q \le \infty$. Then there is a large integer N, which depends on p and q, so that the pseudo-differential operator $a(x,D)$ is bounded on $M^{p,q}$ whenever a is a C^N function satisfying

$$|||a|||_N := \sup_{\substack{x,\xi \in \mathbb{R}^n \\ \alpha,\beta \in {\mathbb{N}_0}^n, \ |\alpha|,|\beta| \le N}} |\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| < \infty.$$

Reexamine Definition 1.1 and the proof of Theorem 3.1 together with Lemma 3.2. Then we see

$$\|a(x,D)\|_{M^{p,q}} := \sup_{f \in M^{p,q} \setminus \{0\}} \frac{\|a(x,D) f \, : \, M^{p,q}\|}{\|f \, : \, M^{p,q}\|} \le c \, |||a|||_N,$$

provided N is large enough.

3.2 Symbol class $M^{\infty,1}$

In this section we deal with the symbol class $M^{\infty,1}$, which contains $S^0_{0,0}$ strictly. The crux of the proof is the decomposition result we have obtained in Section 2. As is easily shown, $M^{\infty,1}$ can be embedded into L^∞. More precisely it can be shown

$$M^{p,p} \subset L^p \subset M^{p,p'}, 1 \le p \le 2$$
$$M^{p,p'} \subset L^p \subset M^{p,p}, 2 \le p \le \infty$$

Meanwhile $M^{\infty,1}$ is known to contain a non-smooth function. Thus, we can say Theorem 3.1 can be widely extended to the theorem below.

Theorem 3.4. *Let $a \in M^{\infty,1}(\mathbb{R}^n \times \mathbb{R}^n)$. Then, the operator $a(x, D)$, defined initially on $\mathcal{S}$ by (24), can be extended continuously to $M^{p,q}$. Furthermore, we have*

$$\|a(x, D)\|_{M^{p,q}} \leq c \, \|a \,:\, M^{\infty,1}(\mathbb{R}^n \times \mathbb{R}^n)\|.$$

Proof. Let $a \in M^{\infty,1}(\mathbb{R}^n \times \mathbb{R}^n)$. As we have discussed in Theorem 1.3, we take an auxiliary function $\kappa : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\chi_{Q(3)} \leq \kappa \leq \chi_{Q(3+1/100)}.$$

In order to apply Theorem 1.3, we shall adopt an auxiliary function κ^* of tensored type. Speaking precisely, we replace κ with κ^* given by

$$\kappa^*(x, \xi) := \kappa(x)\kappa(\xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}.$$

The fact that κ is of tensored type gives us

$$a(x, \xi) = \sum_{\alpha,\beta,m,l \in \mathbb{Z}^n} \lambda_{\alpha,\beta,m,l} \cdot T_\beta M_\alpha[\mathcal{F}^{-1}\kappa](x) T_l M_m[\mathcal{F}^{-1}\kappa](\xi) \tag{30}$$

with a coefficient condition

$$\sum_{m,\alpha \in \mathbb{Z}^n} \left(\sup_{l,\beta \in \mathbb{Z}^n} |\lambda_{\alpha,\beta,m,l}| \right) < \infty. \tag{31}$$

Keeping (30) and (31) in mind, we define

$$a_{m,\alpha}(x, \xi) := \sum_{\beta,l \in \mathbb{Z}^n} \lambda_{\alpha,\beta,m,l} \cdot T_\beta M_\alpha[\mathcal{F}^{-1}\kappa](x) T_l M_m[\mathcal{F}^{-1}\kappa](\xi).$$

Then we have

$$a_{m,\alpha}(x, \xi) = e^{i\alpha\cdot x} \left(\sum_{\beta,l \in \mathbb{Z}^n} e^{-i(l\cdot m + \alpha\cdot\beta)} \cdot \lambda_{\alpha,\beta,m,l} \cdot T_\beta[\mathcal{F}^{-1}\kappa](x) T_l[\mathcal{F}^{-1}\kappa](\xi) \right) e^{im\cdot\xi}.$$

Thus, if we set

$$\begin{aligned}
a^{(1)}_{m,\alpha}(x, \xi) &:= e^{i\alpha\cdot x} \\
a^{(2)}_{m,\alpha}(x, \xi) &:= \sum_{\beta,l \in \mathbb{Z}^n} e^{-i(l\cdot m + \alpha\cdot\beta)} \cdot \lambda_{\alpha,\beta,m,l} \cdot T_\beta[\mathcal{F}^{-1}\kappa](x) T_l[\mathcal{F}^{-1}\kappa](\xi) \\
a^{(3)}_{m,\alpha}(x, \xi) &:= e^{im\cdot\xi},
\end{aligned}$$

then the pseudo-differential operator is factorized into three pseudo-differential operators :

$$a_{m,\alpha}(x, D) = a^{(1)}_{m,\alpha}(x, D) \circ a^{(2)}_{m,\alpha}(x, D) \circ a^{(3)}_{m,\alpha}(x, D).$$

It is easy to see that $a^{(1)}_{m,\alpha}$ is a multiplication operator which is actually an isomorphism on $M^{p,q}$ and that $a^{(3)}_{m,\alpha}$ is a translation operator which is also an isometry on $M^{p,q}$. Note that the operator norm is uniformly bounded over m and α. Thus, the matters are reduced to investigating the operator norm of $a^{(2)}_{m,\alpha}$.

Now it is high time to apply Remark 3.3. Assuming

$$\sup_{l,\beta \in \mathbb{Z}^n} |\lambda_{\alpha,\beta,m,l}| < \infty,$$

it is not so hard to see

$$|||a^{(2)}_{m,\alpha}|||_N \le c \sup_{l,\beta \in \mathbb{Z}^n} |\lambda_{\alpha,\beta,m,l}|,$$

provided N is large enough. Thus, we have obtained

$$\|a_{m,\alpha}(x,D)\|_{M^{p,q}} \le c \sup_{l,\beta \in \mathbb{Z}^n} |\lambda_{\alpha,\beta,m,l}|. \tag{32}$$

Since $a(x,D) = \sum_{m,\alpha \in \mathbb{Z}^n} a_{m,\alpha}(x,D)$ and we have (31), adding (32) over m and α, we see that $a(x,D)$ is bounded on $M^{p,q}$. ∎

4 Another application

Finally in this paper we will apply our decomposition results to specify the dual spaces of $M^{p,q}$. We remark that in [6] we have obtained some results even for $0 < p, q < \infty$. Our approach here is taking full advantage of Theorem 1.3 to prove the following. Given $p \in (0,\infty]$, we define $p' := \dfrac{p}{p-1}$ if $p > 1$ and $p' := \infty$ if $p \le 1$.

Theorem 4.1. *Let* $0 < p, q < \infty$.

1. *Let* $f \in M^{p',q'}$. *Then the functional*

$$g \in \mathcal{S} \mapsto \langle f, g \rangle \in \mathbb{C}$$

can be extended to a continuous linear functional on $M^{p,q}$.

2. *Conversely any continuous linear functional on* $M^{p,q}$ *can be realized with* $f \in M^{p',q'}$.

Proof. The proof of *1* is straightforward and we omit the detail. We shall prove *2* only in the case when $0 < p \le 1 \le q < \infty$, the rest being proved in [6]. Let ζ be a continuous functional on $M^{p,q}$. Then we can define a continuous operator R from $m^{p,q}$ to $M^{p,q}$ as follows: Let

$$R(\lambda)(x) := \sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot T_l M_m [\mathcal{F}^{-1}\kappa](x),$$

where κ is a function appearing in Theorem 1.3. Set $\psi := \zeta \circ R : m^{p,q} \to \mathbb{C}$. Then ψ is a continuous functional on $m^{p,q}$. As is well-known, any continuous functional on $m^{p,q}$ can be realized with a coupling, that is, $\psi(\lambda)$ can be expressed as

$$\psi(\lambda) = \sum_{m,l \in \mathbb{Z}^n} \rho_{ml} \cdot \lambda_{ml},\ \lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n} \text{ with } \|\rho : m^{\infty,q'}\| \le c\,\|\zeta \circ R\|_*,$$

where $\rho = \{\rho_{ml}\}_{m,l \in \mathbb{Z}^n} \in m^{\infty,q'}$ and $\|\cdot\|_*$ denotes the dual norm. Setting

$$S(g) := \{T_m \phi(D) g(l)\}_{l,m \in \mathbb{Z}^n},\ g \in M^{p,q},$$

we obtain a linear mapping $S : M^{p,q} \to m^{p,q}$ satisfying

$$\|S(g) : m^{p,q}\| \le c\,\|g : M^{p,q}\|,\ \zeta = \zeta \circ R \circ S = \psi \circ S.$$

Thus, we have

$$\zeta(g) = \psi(\{T_m \phi(D) g(l)\}_{l,m \in \mathbb{Z}^n}) = \sum_{m,l \in \mathbb{Z}^n} \rho_{ml} \cdot T_m \phi(D) g(l)$$

for all $g \in M^{p,q}$. Now we set

$$f := (2\pi)^{-\frac{n}{2}} \sum_{m,l \in \mathbb{Z}^n} \rho_{ml} \cdot T_l M_{-m}[\mathcal{F}\phi].$$

Then the synthesis part of Theorem 1.3 gives us

$$f \in M^{\infty,q'},\ \|f : M^{\infty,q'}\| \le c\,\|\rho : m^{\infty,q'}\| \le c\,\|\zeta \circ R\|_* \le c\,\|\zeta\|_*.$$

A simple calculation yields

$$\langle f, g \rangle = \zeta(g) \text{ for all } g \in \mathcal{S}.$$

Therefore, *2* is proved. ∎

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