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Abstract The author presents high-speed (sixth-order) methods to approximate expectations of diffusion processes, one of the most important values in mathematical finance, in the spirit of Kusuoka approximation.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $B = (B^1, \dots, B^d)$ be a d-dimensional Brownian motion. Let $B_t^0 := t,\ t \in [0, \infty)$ and $V_0, \dots, V_d \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$. Here $C_b^\infty(\mathbf{R}^N; \mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined in $\mathbf{R}^N$ whose derivatives of any order are bounded. We regard an element in $C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$ as a vector field on $\mathbf{R}^N$. Now we consider a Stratonovich stochastic differential equation

$$\begin{cases} dX(t,x) = \displaystyle\sum_{i=0}^{d} V_i(X(t,x)) \circ dB_t^i, \\ X(0,x) = x. \end{cases}$$

Let $L_\infty(\mathbf{R}^N)$ denote the set of bounded measurable functions defined in $\mathbf{R}^N$. Let us define a norm $\|\cdot\|_\infty$ on $L_\infty(\mathbf{R}^N)$ by

$$\|f\|_\infty := \sup_{x \in \mathbf{R}^N} |f(x)|.$$

Let us define an operator P_T for $T \geq 0$ on $L_\infty(\mathbf{R}^N)$ by

$$P_T g(x) := E[g(X(T,x))],\ g \in L_\infty(\mathbf{R}^N),\ x \in \mathbf{R}^N.$$

We often need to calculate $P_T g(x)$ in mathematical finance problems [11, 12]. Hence it is important to construct high-speed methods to approximate $P_T g(x)$.

Now we introduce a criterion for the speed of methods to approximate P_T. Let $\{Q_n\}_{n \in \mathbf{N}}$ be a family of bounded linear operators on $L_\infty(\mathbf{R}^N)$ which approximates P_T. If there exists a constant $C > 0$ and $k \in \mathbf{N}$ such that

$$\|(P_T - Q_n)g\|_\infty \leq \frac{C}{n^k} \tag{1.1}$$

for any $n \in \mathbf{N}$, then the method constructing $\{Q_n\}_{n \in \mathbf{N}}$ is called a k-th order methods to approximate P_T. Clearly from (1.1), we can expect faster approximation for higher order methods. Here we consider the orders of some known approximation methods. As in the Euler-Maruyama method, the most common approximation method, the method is first-order when the test function $g \in C_b^\infty(\mathbf{R}^N)$ or $g \in L_\infty(\mathbf{R}^N)$ with the vector fields satisfying the Hörmander condition [2, 5, 6]. Next consider the order of the Ninomiya-Victoir method, which is one of the methods of Kusuoka approximation. This is third-order when the test function $g \in C_b^\infty(\mathbf{R}^N)$ or $g \in L_\infty(\mathbf{R}^N)$ with the vector fields satisfying a condition that is weaker than the Hörmander condition [11]. With

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this condition, there are also second-order methods [10, 12]. In this paper, we present sixth-order methods.

2. Algebraic Calculation

Let R be a noncommutative algebra, $d \in \mathbf{N}$ and $x,\ y,\ x_i \in R$ for $i \in \{0, \dots, d\}$. Then we define $\overset{\curvearrowright}{\prod}_{i=0}^{d} x_i,\ \overset{\curvearrowleft}{\prod}_{i=0}^{d} x_i$ and x^d by

$$\overset{\curvearrowright}{\prod_{i=0}^{d}} x_i := x_0 \cdots x_d,\ \overset{\curvearrowleft}{\prod_{i=0}^{d}} x_i := x_d \cdots x_0,\ x^d := \underbrace{x \cdots x}_{d}.$$

Definition 2.1. Let $A(d) := \{A_0, \dots, A_d\}$ be an alphabet, $\mathbf{R}\langle A(d)\rangle$ be the $\mathbf{R}$-algebra of noncommutative polynomials on $A(d)$ and $\mathbf{R}\langle\langle A(d)\rangle\rangle$ be the $\mathbf{R}$-algebra of noncommutative formal series on $A(d)$. Let $M^l(d)$ be the set of all elements of $\mathbf{R}\langle A(d)\rangle$ homogeneous of order $l \in \mathbf{N}$. Let $j_l(d)$ be the canonical projection from $\mathbf{R}\langle\langle A(d)\rangle\rangle$ to $M^l(d)$. We define $M_{\leq l}(d) := \bigoplus_{k=0}^{l} M^k(d)$. Let $j_{\leq l}(d)$ be the canonical projection from $\mathbf{R}\langle\langle A(d)\rangle\rangle$ to $M_{\leq l}(d)$. For $x \in \mathbf{R}\langle\langle A(d)\rangle\rangle$, let us define $\exp(x)$ by

$$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

For $x \in \prod_{k>0} M^k(d)$, let us define $\log(1+x)$ by

$$\log(1+x) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k.$$

When $j_{\leq l}(d)(x-y) = 0$ for $x, y \in \mathbf{R}\langle\langle A(d)\rangle\rangle$, we write $x \overset{l}{=} y$.

Definition 2.2. For $t \in \mathbf{R},\ \theta \in \mathbf{N}$, we define the following elements of $\mathbf{R}\langle\langle A(d)\rangle\rangle$:

$$\hat{P}_t(d) := \exp\left(t \sum_{i=0}^{d} A_i\right),$$

$$\hat{Q}_t^{(i)} := \exp(tA_i), i \in \{0, \dots, d\},$$

$$\bar{Q}_{(t)}^{[\theta]}(d) := \left(\overset{\curvearrowright}{\prod_{i=0}^{d}} \hat{Q}_{t/\theta}^{(i)}\right)^{\theta},$$

$$\check{Q}_{(t)}^{[\theta]}(d) := \left(\overset{\curvearrowleft}{\prod_{i=0}^{d}} \hat{Q}_{t/\theta}^{(i)}\right)^{\theta},$$

$$\hat{Q}_{(t)}^{[\theta]}(d) := \frac{1}{2}\left(\bar{Q}_{(t)}^{[\theta]}(d) + \check{Q}_{(t)}^{[\theta]}(d)\right).$$

We shall occasionally omit the d from the notation. When $t=1$, we shall furthermore omit the subscript t. For $U,\ V \in \prod_{k>0} M^k$ and $l \in \mathbf{N}$, we define $H_l(U,V) := j_l(\log(\exp(U)\exp(V)))$. Here $\tau_{i,d}$, which we shall occasionally abbreviate as τ_i, denotes $j_i(d)(\log(\bar{Q}^{[1]}(d)))$.

Lemma 2.1. *We have*

$$\log\left(\check{Q}^{[1]}(d)\right) = \sum_{i=1}^{\infty}(-1)^{i+1}\tau_{i,d}.$$

Proof. Proof by induction on d. We prove the case $d=2$ by using Lemma 2.15.3 in [14] and induction on i. Suppose $d>2$. It is sufficient then that we prove $j_l(\log\left(\check{Q}^{[1]}(d)\right)) = (-1)^{l+1}\tau_{l,d}$ for $l \in \mathbf{N}$. Let $p \in \mathbf{N}$, and $\{r_i\}_{i\in\{1,\dots,p\}},\ \{s_i\}_{i\in\{1,\dots,p\}} \subset \mathbf{N}$. Let us define a map $H : \mathbf{R}\langle\langle A\rangle\rangle \times \mathbf{R}\langle\langle A\rangle\rangle \to \mathbf{R}\langle\langle A\rangle\rangle$ by

$$H(U,V) := U^{r_1}V^{s_1}\cdots U^{r_p}V^{s_p}.$$

We have

$$j_l\left(H\left(A_d, \sum_{i=1}^{l}(-1)^{i+1}\tau_{i,d-1}\right)\right) = (-1)^{\sum_{i=1}^p s_i + l - \sum_{i=1}^p r_i} j_l\left(H\left(A_d, \sum_{i=1}^{l}\tau_{i,d-1}\right)\right).$$

By the inductive hypothesis we also have

$$\log\left(\check{Q}^{[1]}(d-1)\right) = \sum_{i=1}^{\infty}(-1)^{i+1}\tau_{i,d-1}.$$

Then

$$\begin{aligned}
j_l\left(\log\left(\check{Q}^{[1]}(d)\right)\right) &= j_l\left(\log\left(\exp(A_d)\exp\left(\log\left(\check{Q}^{[1]}(d-1)\right)\right)\right)\right)\\
&= \sum_{m=1}^{l} j_l\left(H_m\left(A_d, \sum_{i=1}^{l}(-1)^{i+1}\tau_{i,d-1}\right)\right)\\
&= \sum_{m=1}^{l}(-1)^{m+l} j_l\left(H_m\left(A_d, \sum_{i=1}^{l}\tau_{i,d-1}\right)\right)\\
&= (-1)^{l+1}\sum_{m=1}^{l} j_l\left(H_m\left(\sum_{i=1}^{l}\tau_{i,d-1}, A_d\right)\right)\\
&= (-1)^{l+1}\tau_{l,d}.
\end{aligned}$$

■

Proposition 2.2. *For $i,\ d \in \mathbf{N}$, there exists $c_{i,d} \in \prod_{k=2i+1}^{\infty} M^k(d)$ such that for all $\theta \in \mathbf{N}$,*

$$\hat{Q}^{[\theta]}(d) = \hat{P}(d) + \sum_{i=1}^{\infty}\frac{c_{i,d}}{\theta^{2i}}.$$

Proof. We have

$$\log\bar{Q}^{[\theta]}(d) = \log\left(\bar{Q}^{[1]}_{1/\theta}(d)\right)^{\theta} = \sum_{i=1}^{\infty}\frac{1}{\theta^{i-1}}\tau_{i,d}.$$

Moreover by Lemma 2.1,

$$\log \check{Q}^{[\theta]}(d) = \sum_{i=1}^{\infty} \frac{1}{(-\theta)^{i-1}} \tau_{i,d}.$$

Then we have

$$\hat{Q}^{[\theta]}(d) = \frac{1}{2} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{i=1}^{\infty} \frac{1}{\theta^{i-1}} \tau_{i,d} \right)^k + \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{i=1}^{\infty} \frac{1}{(-\theta)^{i-1}} \tau_{i,d} \right)^k \right),$$

giving the assertion, since $\tau_{1,d} = \sum_{i=0}^{d} A_i$. ■

The following corollary is straightforward.

Corollary 2.3. We have

$$\hat{P} \overset{2}{=} \hat{Q}^{[\theta]},\ \theta \in \mathbf{N}, \tag{2.1}$$

$$\hat{Q}^{[2]} - \frac{1}{4}\hat{Q}^{[1]} + \frac{3}{4}\hat{P} \overset{4}{=} 0, \tag{2.2}$$

$$\hat{Q}^{[3]} - \frac{1}{9}\hat{Q}^{[1]} + \frac{8}{9}\hat{P} \overset{4}{=} 0, \tag{2.3}$$

$$\hat{P} \overset{6}{=} \frac{81}{40}\hat{Q}^{[3]} - \frac{16}{15}\hat{Q}^{[2]} + \frac{1}{24}\hat{Q}^{[1]}. \tag{2.4}$$

3. Approximations of operators

Definition 3.1. (1) Let us define a semi-norm $\|\cdot\|_k$ for $k \in \mathbf{N}$ on $C_b^\infty(\mathbf{R}^d)$ by

$$\|g\|_k := \sup_{i \le k} \|\nabla^i g\|_\infty.$$

(2) Let $\mathfrak{B}_k$ denote the space of bounded linear operators on $C_k := (C_b^\infty(\mathbf{R}^d),\ \|\cdot\|_k)$. We can regard $\mathfrak{B}_k$ as a normed space with the operator norm.

The following proposition is well-known [4].

Proposition 3.1. *(1) The family $\{P_t\}_{t\in(0,\infty]}$ is a uniform bounded subset of $\mathfrak{B}_k$.*
(2) We have $\|P_t g\|_\infty \le \|g\|_\infty$ for $g \in C_b^\infty(\mathbf{R}^d)$.
(3) Let $\mathcal{A}$ be the differential operator defined by

$$\mathcal{A} := V_0 + \frac{1}{2}\sum_{j=1}^{d} V_j^2.$$

For $g \in C_b^\infty(\mathbf{R}^d)$,

$$P_t g(x) = \sum_{k=0}^{N} \frac{t^k}{k!}\mathcal{A}^k + \frac{1}{N!}\int_0^t (t-s)^N P_s \mathcal{A}^{N+1} g(x) ds.$$

For $i \in \{0, \dots, d\}$, we consider a Stratonovich stochastic differential equation

$$\begin{cases} dX^i(t,x) = V_i(X^i(t,x)) \circ dB^i_t, \\ X(0,x) = x. \end{cases}$$

For $s \geq 0$, let us define a operator $Q^{(i)}_s$ on $L_\infty(\mathbf{R}^N)$ by

$$Q^{(i)}_s(g)(x) := E[g(X^i(s,x))],\ x \in \mathbf{R}^N.$$

We set

$$f_1 := \frac{1}{24},\ f_2 := -\frac{16}{15} \text{ and } f_3 := \frac{81}{40}.$$

For $\theta \in \{1,2,3\}$ and $t \geq 0$, let

$$\tilde{Q}^{[\theta]}_{(t)} := \frac{1}{2}\left(\left(\overset{\curvearrowright}{\prod_{i=0}^{d}} Q^{(i)}_{t/\theta}\right)^\theta + \left(\overset{\curvearrowleft}{\prod_{i=0}^{d}} Q^{(i)}_{t/\theta}\right)^\theta\right).$$

Also, define operators

$$Q_{(n)} := f_3\left(\tilde{Q}^{[3]}_{(T/n)}\right)^n + f_2\left(\tilde{Q}^{[2]}_{(T/n)}\right)^n + f_1\left(\tilde{Q}^{[1]}_{(T/n)}\right)^n$$

and

$$Q_{(n,1)} := f_3\tilde{Q}^{[3]}_{(T/n)} + f_2\tilde{Q}^{[2]}_{(T/n)} + f_1\tilde{Q}^{[1]}_{(T/n)}.$$

Theorem 3.2. *There exists a constant $C > 0$ such that*

$$\left\|(P_T - Q_{(n)})g\right\|_\infty \leq \frac{C}{n^6}\|g\|_{54(d+1)}$$

for any $g \in C^\infty_b(\mathbf{R}^N)$ and $n \in \mathbf{N}$.

Proof. Let $s := T/n$. Then

$$\begin{aligned}
(Q_{(n)} - P_T)g(x) &= \sum_{\theta\in\{1,2,3\}} f_\theta\left(\left(\tilde{Q}^{[\theta]}_{(s)}\right)^n - P_T\right)g(x) \\
&= \sum_{\theta\in\{1,2,3\}} f_\theta \sum_{k=0}^{n-1} P_{ks}\left(\tilde{Q}^{[\theta]}_{(s)} - P_s\right)P_{(n-k-1)s}g(x) \\
&\quad + \sum_{\theta\in\{1,2,3\}} f_\theta \sum_{k=1}^{n-1}\sum_{l=0}^{k-1} P_{ls}\left(\tilde{Q}^{[\theta]}_{(s)} - P_s\right)P_{(k-l-1)s}\left(\tilde{Q}^{[\theta]}_{(s)} - P_s\right)P_{(n-k-1)s}g(x) \\
&\quad + \sum_{\theta\in\{1,2,3\}} f_\theta \sum_{k=1}^{n-1}\sum_{l=1}^{k-1}\sum_{m=1}^{l-1} (\tilde{Q}^{[\theta]}_{(s)})^m\left(\tilde{Q}^{[\theta]}_{(s)} - P_s\right)P_{(l-m-1)s}\left(\tilde{Q}^{[\theta]}_{(s)} - P_s\right) \\
&\quad\quad \times P_{(k-l-1)s}\left(\tilde{Q}^{[\theta]}_{(s)} - P_s\right)P_{(n-k-1)s}g(x).
\end{aligned}$$

The first term on the right-hand side of this equality becomes

$$\sum_{k=0}^{n-1} P_{ks}\left(Q_{(n,1)} - P_s\right)P_{(n-k-1)s}g(x).$$

By (2.4) and Proposition 3.1, there exists a constant $C_1 > 0$ such that

$$\left\| \sum_{k=0}^{n-1} P_{ks} \left(Q_{(n,1)} - P_s \right) P_{(n-k-1)s} g \right\|_\infty \leq \frac{C_1}{n^6} \|g\|_{42(d+1)}$$

for any $n \in \mathbf{N}$. Similarly as the third term on the right-hand side, there exists a constant $C_2 > 0$ such that

$$\begin{aligned} \Bigg\| \sum_{\theta \in \{1,2,3\}} f_\theta \sum_{k=1}^{n-1} \sum_{l=1}^{k-1} \sum_{m=1}^{l-1} (\tilde{Q}_{(s)}^{[\theta]})^m \left(\tilde{Q}_{(s)}^{[\theta]} - P_s \right) P_{(l-m-1)s} \left(\tilde{Q}_{(s)}^{[\theta]} - P_s \right) \\ \times P_{(k-l-1)s} \left(\tilde{Q}_{(s)}^{[\theta]} - P_s \right) P_{(n-k-1)s} g \Bigg\|_\infty \leq \frac{C_2}{n^6} \|g\|_{54(d+1)} \end{aligned}$$

by (2.1) and Proposition 3.1. Finally we consider the second term on the right-hand side. By (2.1), (2.2) and Proposition 3.1, there exists a constant $C_3 > 0$ such that

$$\begin{aligned} & \Bigg\| \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} P_{ls} \left(\tilde{Q}_{(s)}^{[2]} - P_s \right) P_{(k-l-1)s} \left(\tilde{Q}_{(s)}^{[2]} - P_s \right) P_{(n-k-1)s} g \\ & - \frac{1}{16} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} P_{ls} \left(\tilde{Q}_{(s)}^{[1]} - P_s \right) P_{(k-l-1)s} \left(\tilde{Q}_{(s)}^{[1]} - P_s \right) P_{(n-k-1)s} g \Bigg\|_\infty \\ = & \Bigg\| \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} P_{ls} \left(\tilde{Q}_{(s)}^{[2]} - \frac{1}{4} \tilde{Q}_{(s)}^{[1]} + \frac{3}{4} P_s \right) P_{(k-l-1)s} \left(\tilde{Q}_{(s)}^{[2]} - P_s \right) P_{(n-k-1)s} g \\ & + \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} P_{ls} \left(\tilde{Q}_{(s)}^{[1]} - P_s \right) P_{(k-l-1)s} \left(\tilde{Q}_{(s)}^{[2]} - \frac{1}{4} \tilde{Q}_{(s)}^{[1]} + \frac{3}{4} P_s \right) P_{(n-k-1)s} g \Bigg\|_\infty \\ \leq & \frac{C_3}{n^6} \|g\|_{32(d+1)}. \end{aligned}$$

Similarly by (2.1), (2.3) and Proposition 3.1, there exists a constant $C_4 > 0$ such that

$$\begin{aligned} & \Bigg\| \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} P_{ls} \left(\tilde{Q}_{(s)}^{[3]} - P_s \right) P_{(k-l-1)s} \left(\tilde{Q}_{(s)}^{[3]} - P_s \right) P_{(n-k-1)s} g \\ & - \frac{1}{81} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} P_{ls} \left(\tilde{Q}_{(s)}^{[1]} - P_s \right) P_{(k-l-1)s} \left(\tilde{Q}_{(s)}^{[1]} - P_s \right) P_{(n-k-1)s} g \Bigg\|_\infty \\ \leq & \frac{C_4}{n^6} \|g\|_{48(d+1)}. \end{aligned}$$

Hence there exists a constant $C_5 > 0$ such that

$$\left\| \sum_{\theta \in \{1,2,3\}} f_\theta \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} P_{ls} \left(\tilde{Q}_{(s)}^{[\theta]} - P_s \right) P_{(k-l-1)s} \left(\tilde{Q}_{(s)}^{[\theta]} - P_s \right) P_{(n-k-1)s} g \right\|_\infty \leq \frac{C_5}{n^6} \|g\|_{48(d+1)}.$$

Then we have our assertion. ■

4. Implementation of the approximation operators

We regard a vector space $C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$ as a noncommutative algebra with multiplication given by conposition and follow the notation of Section 2.

For a vector field $W \in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$, let $y(t,x)$ be the solution of the ordinary differential equation

$$\begin{cases} \dfrac{d}{dt}y(t,x) = W(y(t,x)), \\ y(0,x) = x. \end{cases} \tag{4.1}$$

We define $\mathrm{Exp}(W)(x) := y(1,x)$.

Let $s := T/n$ and $\theta \in \{1,2,3\}$. Let $\{\Lambda_k\}_{k\in\{1,\dots,n\}}$ and $\{Z_k\}_{k\in\{1,\dots,\theta n\}}$ be independent random variables, where each Λ_k is Bernoulli random variable and $Z_k = (Z_k^i)_{i\in\{1,\dots,d\}}$ is a standard d-dimensional normal random variable. Let $x_0 \in \mathbf{R}^N$, $\tilde{Z}_{k,\theta}^{0,n} := s/\theta$ and $\tilde{Z}_{k,\theta}^{i,n} := \sqrt{s/\theta}Z_k^i$ for $i \in \{1,\dots,d\}$. Then we inductively define $\{X_k^{[\theta],n}\}_{k\in\{0,\dots,n\}}$ by

$$X_0^{[\theta],n} := x_0,$$
$$X_{k+1}^{[\theta],n} := \begin{cases} \left(\overset{\curvearrowright}{\prod_{i=0}^{d}}\mathrm{Exp}\left(\tilde{Z}_{k,\theta}^{i,n}V_i\right)\right)^\theta \left(X_k^{[\theta],n}\right) & \text{if } \Lambda_k = +1, \\ \left(\overset{\curvearrowleft}{\prod_{i=0}^{d}}\mathrm{Exp}\left(\tilde{Z}_{k,\theta}^{i,n}V_i\right)\right)^\theta \left(X_k^{[\theta],n}\right) & \text{if } \Lambda_k = -1. \end{cases}$$

Then by a routine computation we obtain:

Proposition 4.1. *For $g \in C_b^\infty(\mathbf{R}^N)$ and $\theta \in \{1,2,3\}$,*

$$\left(\tilde{Q}_{(T/n)}^{[\theta]}\right)^n g(x_0) = E\left[g\left(X_n^{[\theta],n}\right)\right].$$

Remark 4.1. To calculate $E\left[g\left(X_n^{[\theta],n}\right)\right]$ numerically using Proposition 4.1, we need to approximate an integral over a finite-dimensional space. Such numerical integrations are rapidly performed using the quasi-Monte-Carlo method [13]. From Proposition 4.1, it appears we need a $3(d+1)n$-dimensional uniform random variable. However, we can calculate $Q_n g(x)$ by $3dn+1$-dimensional integrations if we implement n-dimensional Bernoulli random variables by a one-dimensional uniform random variable.

Remark 4.2. Let $\breve{Q}_t^{(0)}$, $\breve{Q}_t^{(d+1)} := Q_{t/2}^{(0)}$ and $\breve{Q}_t^{(i)} := Q_t^{(i)}$ for $i \in \{1, \dots, d\}$. Now define

$$\begin{aligned}
\breve{Q}_{(t)}^{[1]} &:= \frac{1}{2}\left(\prod_{i=0}^{\overset{\curvearrowright}{d+1}} \breve{Q}_t^{(i)} + \prod_{i=0}^{\overset{\curvearrowleft}{d+1}} \breve{Q}_t^{(i)}\right) \\
&= \frac{1}{2} Q_{t/2}^{(0)} \left(\prod_{i=1}^{\overset{\curvearrowright}{d}} Q_t^{(i)} + \prod_{i=1}^{\overset{\curvearrowleft}{d}} Q_t^{(i)}\right) Q_{t/2}^{(0)}, \\
\breve{Q}_{(t)}^{[2]} &:= \frac{1}{2}\left(\left(\prod_{i=0}^{\overset{\curvearrowright}{d+1}} \breve{Q}_{t/2}^{(i)}\right)^2 + \left(\prod_{i=0}^{\overset{\curvearrowleft}{d+1}} \breve{Q}_{t/2}^{(i)}\right)^2\right) \\
&= \frac{1}{2} Q_{t/4}^{(0)} \left(\prod_{i=1}^{\overset{\curvearrowright}{d}} Q_{t/2}^{(i)} \prod_{i=0}^{\overset{\curvearrowright}{d}} Q_{t/2}^{(i)} + \prod_{i=0}^{\overset{\curvearrowleft}{d}} Q_{t/2}^{(i)} \prod_{i=1}^{\overset{\curvearrowleft}{d}} Q_{t/2}^{(i)}\right) Q_{t/4}^{(0)},
\end{aligned}$$

$$\begin{aligned}
\breve{Q}_{(t)}^{[3]} &:= \frac{1}{2}\left(\left(\prod_{i=0}^{\overset{\curvearrowright}{d+1}} \breve{Q}_{t/3}^{(i)}\right)^3 + \left(\prod_{i=0}^{\overset{\curvearrowleft}{d+1}} \breve{Q}_{t/3}^{(i)}\right)^3\right) \\
&= \frac{1}{2} Q_{t/6}^{(0)} \left(\prod_{i=1}^{\overset{\curvearrowright}{d}} Q_{t/3}^{(i)} \prod_{i=0}^{\overset{\curvearrowright}{d}} Q_{t/3}^{(i)} \prod_{i=0}^{\overset{\curvearrowright}{d}} Q_{t/3}^{(i)} + \prod_{i=0}^{\overset{\curvearrowleft}{d}} Q_{t/3}^{(i)} \prod_{i=0}^{\overset{\curvearrowleft}{d}} Q_{t/3}^{(i)} \prod_{i=1}^{\overset{\curvearrowleft}{d}} Q_{t/3}^{(i)}\right) Q_{t/6}^{(0)}, \\
\breve{Q}_{(n)} &:= f_3\left(\breve{Q}_{(T/n)}^{[3]}\right)^n + f_2\left(\breve{Q}_{(T/n)}^{[2]}\right)^n + f_1\left(\breve{Q}_{(T/n)}^{[1]}\right)^n, \\
\breve{Q}'_{(n)} &:= \frac{4}{3}\left(\breve{Q}_{(T/n)}^{[2]}\right)^n - \frac{1}{3}\left(\breve{Q}_{(T/n)}^{[1]}\right)^n.
\end{aligned}$$

Then similarly as before

$$\left\|(\breve{Q}_{(n)} - P_T)g\right\|_\infty \le \frac{C}{n^6}\|g\|_{54(d+1)+18}$$

and

$$\left\|(\breve{Q}'_{(n)} - P_T)g\right\|_\infty \le \frac{C}{n^4}\|g\|_{24(d+1)+12}.$$

Using $\{\breve{Q}_{(n)}\}_{n\in\mathbf{N}}$ or $\{\breve{Q}'_{(n)}\}_{n\in\mathbf{N}}$ instead of $\{Q_{(n)}\}_{n\in\mathbf{N}}$ in order to approximate P_T may be better from a practical point of view.

Remark 4.3. For the purpose of constructing the approximate operators, a good approximate solution to the ODE (4.1) will suffice. For example, we could use a 13-th order Runge-Kutta scheme [1].

Remark 4.4. One could also show the convergence of the algorithm when $g \in L_\infty(\mathbf{R}^N)$ and the vector fields of the SDEs satisfy a condition that is weaker than the Hörmander condition [7, 8, 9].

Remark 4.5. We also showed the convergence of the algorithms when the SDEs are jump-type in [3].

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