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Motion of interfaces by the Allen-Cahn type equation with multiple-well potentials

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Abstract

We consider the singular limit of the Allen–Cahn type equation with a periodic nonlinear term. We obtain that several interfaces appear when the interface thickness parameter (denoted by ε) tends to 0. We also obtain that the interfaces move by the mean curvature flow with driving force term.

Keywords: Allen–Cahn equation; multiple-well potential; mean curvature flow; viscosity solution.

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1. Introduction

We consider the singular limit of the Allen–Cahn type equation of the form

$$u_t^\varepsilon - \Delta u^\varepsilon + \frac{1}{\varepsilon^2}(-\sin u^\varepsilon - \varepsilon a(1+\cos u^\varepsilon)) = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \tag{1.1}$$

where a is a given constant. A formal asymptotic analysis says that the internal transition layer of a solution of (1.1) approximates the evolving interface $\{\Gamma_t\}_{t\geq 0}$ under the mean curvature flow with a driving force of the form

$$V = -H + A \quad \text{on } \Gamma_t, \tag{1.2}$$

where V is the normal velocity, H is the mean curvature of Γ_t in the direction of the minus of the normal vector field, and A is a constant determined completely by the nonlinear term $-\sin u - \varepsilon a(1+\cos u)$. If the initial data $u^\varepsilon(x,0) = u_0(x) \in BUC(\mathbb{R}^N)$ satisfies

$$\sup_{\mathbb{R}^N} |u_0| \leq (2K_0+1)\pi$$

for some $K_0 \in \mathbb{N}$, the internal transition layers stay like as an *annual ring*. We shall see that all the internal transition layers approximates the motion of the interfaces moving by (1.2).

The Allen–Cahn equation

$$u_t^\varepsilon - \Delta u^\varepsilon + \frac{1}{\varepsilon^2}(W'(u^\varepsilon) - \varepsilon a) = 0, \tag{1.3}$$

where $W(t) = (t^2-1)^2/2$ is the double-well potential and a is a given constant, was introduced by Allen and Cahn in 1979. It is the L^2-gradient flow of the energy functional

$$E(u) = \int \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2}(W(u) - \varepsilon a u)\right). \tag{1.4}$$

The convergence of solutions of the Allen–Cahn equation to the mean curvature flow (1.2) has been proved in various setting, for example, by Chen(1992), Evans, Soner and Souganidis(1992). Let Γ_t be a solution of (1.2) and O_t be a region enclosed by Γ_t. If the initial data $u^\varepsilon(x,0)$ of a solution of (1.3) is positive in O_0 and negative in $\mathbb{R}^N \setminus (\Gamma_0 \cup O_0)$, then one expects that

$$u^\varepsilon(x,t) \to \begin{cases} \;\;1 & \text{for} \quad O_t, \\ -1 & \text{for} \quad \mathbb{R}^N \setminus (\Gamma_t \cup O_t) \end{cases} \tag{1.5}$$

locally uniformly as $\varepsilon \to 0$. Chen proved (1.5) if Γ_t is a solution of (1.2) in the classical sense. By using a level set formulation (see e.g. Giga, Y.(2006)) of the motion of Γ_t proposed by Chen, Giga and Goto(1991) or Evans and Spruck(1991), Evans, Soner and Souganidis proved (1.5) globally-in-time by interpreting Γ_t as the generalized solution of (1.2). The results on above are proved by the comparison principle and constructing a supersolution and a subsolution for the estimate of the convergence. Katsoulakis, Kossioris and Reitich(1995) extended this convergence result to the Neumann boundary condition in a convex domain. A new set theoretic approach to prove (1.5) is proposed by Barles and Souganidis(1998). Barles and Da Lio(2003) extended the set theoretic approach to the Neumann boundary problem without the convexity assumption of a domain. For an anisotropic version of the Allen–Cahn equation and the mean curvature flow equation, the convergence is studied by, for example, Elliott and Schätzle(1996, 1997), Elliott, Paolini and Schätzle(1996), Giga, Ohtsuka and Schätzle(to appear).

The equation (1.1) is the L^2-gradient flow of the energy functional of the form

$$F_\varepsilon(u) = \int \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2}(\cos u - \varepsilon a(u + \sin u)) \right).$$

We have the similar problem for the equation (1.1). The nonlinear term

$$f_\varepsilon(u) := -\sin u - \varepsilon a(1 + \cos u)$$

of (1.1) has a exactly three zeros $\pm\pi$ and some point $\alpha_\varepsilon \in (-\pi, \pi)$ in $[-\pi, \pi]$, which play the same role as the zero points of the nonlinear term of (1.3). Since f_ε is periodic, one expects that a solution of the equation (1.1) has several internal transition layers where a solution of (1.1) changes its value from $(2k-1)\pi$ to $(2k+1)\pi$. The aim of this paper is to prove for the solution u^ε of (1.1) and the generalized solution u of a level set equation of (1.2) with initial data $u^\varepsilon(x,0) = u(x,0) = u_0(x) \in BUC(\mathbb{R}^N)$, we have

$$u^\varepsilon(x,t) \to (2k+1)\pi$$

for $(x,t) \in \{(y,s) \in \mathbb{R}^N \times (0,T);\; u(y,s) \in (2\pi k, 2\pi(k+1))\}$ locally uniformly as $\varepsilon \to 0$.

We will discuss the existence and uniqueness of a solution for (1.1), the equation for the traveling wave solution of (1.1) and the level set equation of (1.2) in Section 2. In Section 3, we verify that the internal transition layer is generated in a very short time by using the strategy of §3 in Chen(1992). In Section 4, we shall give a uniform estimate of solutions of the traveling wave equation. By using this, we shall determine the constant A in (1.2) from the traveling wave equation. In Section 5, we construct a supersolution of (1.1) for the estimate of solutions u^ε by using a signed distance function from the generalized solution of (1.2). In Evans, Soner and Souganidis(1992), one can find a way to construct a supersolution which has a layer around an interface moving by (1.2). However, their method provides a way to construct a supersolution with single-height layer so that the value of it is, for example, π in a domain enclosed by some interface and $-\pi$ outside of the interfaces. For our problem, we need to construct a supersolution with the multiple-height layer. This is the one of characteristic difficulties. In Section 6, we shall prove a main result by using a properties proved in previous sections.

In Jerrard, R. L.(1997), the singular limit for the equation (1.1) with a nonlinear term $\varepsilon^{-1-\alpha} f_\varepsilon(\varepsilon^{-1+\alpha} u^\varepsilon)$ for $\alpha \in [0,1)$ instead of our nonlinear term is considered. He shows that a solution of this equation converges to a function which solves a level set equation of a some interface evolution equation.

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2. Preliminaries and main result

2.1. Allen–Cahn equations with multiple-well potentials

We consider the Allen–Cahn equation with the multiple-well potential of the form

$$u_t^\varepsilon - \Delta u^\varepsilon + \frac{1}{\varepsilon^2} f_\varepsilon(u^\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \times (0,T) \tag{2.1}$$

with the initial data

$$u^\varepsilon(\cdot,0) = u_0(\cdot) \in BUC(\mathbb{R}^N) \tag{2.2}$$

where f_ε is of the form

$$f_\varepsilon(u) = -\sin u - \varepsilon a(1+\cos u), \tag{2.3}$$

a is a constant, and $\varepsilon \in (0,1)$ is a small parameter satisfying $\varepsilon \ll 1$.

By straightforward calculation we obtain

$$f_\varepsilon(u) = -\sqrt{1+\varepsilon^2 a^2}\sin(u+\beta_\varepsilon) - \varepsilon a,$$

where $\beta_\varepsilon \in (0,\pi/2)$ satisfies

$$\cos\beta_\varepsilon = \frac{1}{\sqrt{1+\varepsilon^2 a^2}},\ \sin\beta_\varepsilon = \frac{\varepsilon a}{\sqrt{1+\varepsilon^2 a^2}}.$$

The function f_ε is periodic with $f_\varepsilon(u+2\pi) = f_\varepsilon(u)$ and has exactly three zeros $u = -\pi$, $u = -2\beta_\varepsilon =: \alpha_\varepsilon$ and $u = \pi$ in $[-\pi,\pi]$. Moreover we obtain from straightforward calculation,

$$f_\varepsilon'(\pm\pi) = 1,\ f_\varepsilon'(\alpha_\varepsilon) = -1.$$

The three zeros $\pm\pi$ and α_ε play the role of three zeros ± 1 and 0 in the case of Allen–Cahn equation, i.e., $f_\varepsilon(u) = 2u(u^2-1)$.

Remark 2.1. *In this paper we give the explicit form of nonlinear term f_ε by* (2.3)*. However, we note that the argument in this paper is valid for the periodic nonlinear term of the form*

$$f_\varepsilon(u) = f_0(u) + \varepsilon f_1(u),$$

where f_0 and f_1 are smooth functions satisfying

(i) *f_0 and f_1 are periodic with same period, for examples,*

$$f_0(u+2) = f_0(u), \ f_1(u+2) = f_1(u),$$

(ii) $f_0(\pm 1) = f_0(0) = 0$*,* $f_1(\pm 1) = 0$ $f_0 > 0$ *in* $(-1, 0)$ *and* $f_0 < 0$ *in* $(0, 1)$*,*

(iii) $f_0'(\pm 1) > 0$ *and* $f_0'(0) < 0$*,*

(iv) $\int_{-1}^{1} f_0(u)du = 0$*.*

We also remark that it is important that the periods of f_0 and f_1 are same. If the periods are different, then the driving force term of the interface evolution equation for each internal transition layers are various. We also need more complicated assumptions for f_0 and f_1 for such a situation.

In this paper we use the following notation

$$\lambda_j = \sup_{\varepsilon \in (0,1)} \sup_{\mathbb{R}} \left| \frac{d^j f_\varepsilon}{du^j} \right| \quad \text{for } j = 0, \ 1, \ 2. \tag{2.4}$$

Here we are interested in the case that $u_0 \in BUC(\mathbb{R}^N)$. Let $K_0 \in \mathbb{N}$ be a constant satisfying

$$\sup_{\mathbb{R}^N} |u_0| \le (2K_0 + 1)\pi. \tag{2.5}$$

We remark that we are interested in a situation such that several internal transition layers appear in a domain. If we would assume that $\sup_{\mathbb{R}^N} |u_0| \le \pi$, then only one internal transition layer which change the value of a solution of (2.1) from $-\pi$ to π appears so that it is essentially same as the Allen–Cahn equation case.

We here and hereafter consider viscosity solutions of (2.1) defined by Crandall, Ishii and Lions(1992). The comparison principle holds for viscosity solutions of (2.1).

Theorem 2.2. *Let u and v respectively be an upper and a lower semicontinuous sub- and supersolution of* (2.1) *in* $\mathbb{R}^N \times (0, T)$*. Moreover we assume that there exists a positive constant M satisfying $u \le M$ and $v \ge -M$. If $u(x, 0) \le v(x, 0)$ for $x \in \mathbb{R}$, then $u(x, t) \le v(x, t)$ for $(x, t) \in \mathbb{R}^N \times (0, T)$.*

We shall mention an idea of the proof in a few words. Set $\lambda_\varepsilon = \varepsilon^{-2}\lambda_1$, where λ_1 is a constant defined by (2.4). We have that the map $r \mapsto \lambda_\varepsilon + \varepsilon^{-2} f_\varepsilon(r)$ is monotone nondecreasing. We consider a function $\tilde{u}(x, t) := e^{-\lambda_\varepsilon t} u(x, t)$ and

$\tilde{v}(x,t) := e^{-\lambda_\varepsilon t} v(x,t)$. Then we have that $\tilde{u}$ and $\tilde{v}$ are a viscosity sub- and super-solutions of the equation of the form

$$\tilde{u}_t - \Delta \tilde{u} + e^{-\lambda_\varepsilon t}\left(\lambda_\varepsilon e^{\lambda_\varepsilon t}\tilde{u} + \frac{1}{\varepsilon^2} f_\varepsilon(e^{\lambda_\varepsilon t}\tilde{u})\right) = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \tag{2.6}$$

respectively. By applying the arguments as in Theorem 4.1 of Chen, Giga and Goto(1992) or §8 with the idea as in §5.D of Crandall, Ishii and Lions(1991), we have the conclusion of Theorem 2.2.

We also have the existence and uniqueness of a viscosity solution of (2.1).

Theorem 2.3. *For a given $u_0 \in BUC(\mathbb{R}^N)$, there exists a viscosity solution $u^\varepsilon \in C([0,\infty); BUC(\mathbb{R}^N))$ of (2.1) with initial data $u^\varepsilon(x,0) = u_0(x)$.*

We shall point out the idea of the proof of Theorem 2.3. The existence is established by the Perron's method due to H. Ishii. See Chen, Giga and Goto(1992) for the Perron's method and a construction of a solution. For the uniform continuity of u^ε, apply the method as in §5.D of Crandall, Ishii and Lions(1992) to (2.6).

It is convenient for our problem to consider the traveling wave solution of the form $u^\varepsilon = q(\varepsilon^{-1}x \cdot e - \varepsilon^{-2}ct)$ where $q\colon \mathbb{R} \to \mathbb{R}$. We have that q satisfies

$$-q'' - cq' + f_\varepsilon(q) = 0 \quad \text{in } \mathbb{R} \tag{2.7}$$

by straightforward calculation from (2.1). The constant c denotes the traveling speed of the internal transition layer of the solution of (2.1). Aronson and Weinberger(1978) show that there exists a unique pair (q,c) of a solution and a constant of (2.7) with boundary condition

$$q(\pm\infty) = \pm\pi.$$

In Section 4, we shall give an estimate of such a q, and the existence of the limit $\lim_{\varepsilon\to 0} \varepsilon^{-1}c$.

2.2. Level set equations for interfaces

A formal asymptotic analysis says that the internal transition layer of a solution of (2.1) approximates the motion of an interface $\{\Gamma_t\}_{t\geq 0}$ which moves by

$$V = -H + A \quad \text{on } \Gamma_t, \tag{2.8}$$

where V is the normal velocity of Γ_t, H is the mean curvature of Γ_t in the direction of the minus of the normal vector field, and A is a constant determined by $A = -\lim_{\varepsilon\to 0}\varepsilon^{-1}c$ of which the existence will be proved in Section 4.

We shall mention the relation of (2.1) and (2.8) globally-in-time whenever the interface Γ_t still appears. Therefore we introduce the level set formulation of (2.8) as in Chen, Giga and Goto(1991) or Evans and Spruck(1991). Let Γ_t be given by

$$\Gamma_t = \{y \in \mathbb{R}^N;\ u(y,t) = z\}$$

for some $z \in \mathbb{R}$. We obtain the level set equation of the form

$$u_t - |\nabla u| \left\{ \mathrm{div} \frac{\nabla u}{|\nabla u|} + A \right\} = 0 \quad \text{in } \mathbb{R}^N \times (0,T). \tag{2.9}$$

The comparison principle for viscosity solutions still holds for (2.9).

A general interface evolution equation including (2.9) is well studied, for example, by Chen, Giga and Goto(1991), Evans and Spruck(1991). Some result are obtained when a domain is bounded. However, we have the comparison principle, the invariance of under change of dependent variables, the existence and uniqueness of viscosity solutions for the uniform continuous and bounded initial data in a some domain which includes an unbounded case. See Giga(2006) more precise properties for (2.9).

2.3. Convergence result

We are now in the position to state our main result.

Theorem 2.4. *Let u^ε be a solution of* (2.1) *with $u^\varepsilon(x,0) = u_0(x) \in BUC(\mathbb{R}^N)$. Let u be a solution of* (2.9) *with $u(x,0) = u_0(x)$. Assume that u_0 satisfies* (2.5). *Then we have the followings for any $k \in [-K_0, K_0] \cap \mathbb{Z}$:*

(i) Assume that there exists $m_0 > 0$ such that $\{y \in \mathbb{R}^N;\ u(y,t) = 2\pi k - m\} \neq \emptyset$ for $t \in [0,T)$ provided that $m \in [0, m_0)$. Then, for any compact subset $K \subset \{(y,s) \in \mathbb{R}^N \times (0,T);\ u(x,t) < 2\pi k\}$, we have

$$\overline{\lim_{\varepsilon \to 0}} \sup_{(x,t) \in K} u^\varepsilon(x,t) \leq (2k-1)\pi.$$

(ii) Assume that there exists $m_0 > 0$ such that $\{y \in \mathbb{R}^N;\ u(y,t) = 2\pi k + m\} \neq \emptyset$ for $t \in [0,T)$ provided that $m \in [0, m_0)$. Then, for any compact subset $K \subset \{(y,s) \in \mathbb{R}^N \times (0,T);\ u(x,t) > 2\pi k\}$, we have

$$\underline{\lim_{\varepsilon \to 0}} \inf_{(x,t) \in K} u^\varepsilon(x,t) \geq (2k+1)\pi.$$

It is easy to see the following from Theorem 2.4;

Corollary 2.5. *Let u^ε be a solution of* (2.1) *with $u^\varepsilon(x,0) = u_0(x) \in BUC(\mathbb{R}^N)$. Let u be a solution of* (2.9) *with $u(x,0) = u_0(x)$. Assume that u_0 satisfies* (2.5). *Assume that $\Gamma_t^k \neq \emptyset$ and $\Gamma_t^{k+1} \neq \emptyset$ for $t \in [0,T)$. Then we have*

$$u^\varepsilon \to (2k+1) \text{ in } \{(y,s) \in \mathbb{R}^N \times (0,T);\ u(y,s) \in (2\pi k, 2\pi(k+1))\}$$

locally uniformly as $\varepsilon \to 0$.

The proof is given in Section 6.

The strategy of the proof is made up by 2 steps. The first step of the proof is to know a very short time behavior of the solution of (2.1) by using an idea of §3 as in Chen(1992), which is presented in Section 3. The second step is to construct a supersolution with multiple-height layer by using the traveling wave solution. However, the traveling wave solution provides single-height layer solution and there is no solution of (2.7) with a boundary condition which yields a multiple-height layer. To overcome this difficulty, we shall *pile* up several single-layer solution so that we construct a supersolution with a multiple-height layer. That is presented in Section 5. In this construction, we need a uniform estimate of the traveling wave solution with respect to ε, which is presented in Section 4.

3. Generation of interfaces

In this section we see a very short time behavior of the solution u^ε for (2.1). The aim of this section is to show that;

Theorem 3.1. *Let u^ε be a solution of* (2.1) *with $u^\varepsilon(x,0)=u_0(x)$. Assume that u_0 satisfies* (2.5). *Then, for any $b>0$ and $m>0$, there exist positive constants $\bar\varepsilon=\bar\varepsilon(b,m)$ and $\tau_0=\tau_0(b)$ such that, for any $k\in\mathbb{Z}\cap[-K_0,K_0]$,*

$$\begin{aligned} &u^\varepsilon(x,\tau_0\varepsilon^2|\log\varepsilon|)\geq(2k+1)\pi-b\varepsilon \quad \textit{for} \quad x\in\{y;\ u_0(y)\geq 2\pi k+m\},\\ &u^\varepsilon(x,\tau_0\varepsilon^2|\log\varepsilon|)\leq(2k-1)\pi+b\varepsilon \quad \textit{for} \quad x\in\{y;\ u_0(y)\leq 2\pi k-m\},\end{aligned}$$

provided that $\varepsilon\in(0,\bar\varepsilon)$.

In the following arguments, we shall mention only on the estimate from below. For the estimate from above, we consider the equation for $\bar u^\varepsilon := -u^\varepsilon$ of the form

$$\bar u^\varepsilon_t-\Delta\bar u^\varepsilon+\frac{1}{\varepsilon^2}g_\varepsilon(\bar u^\varepsilon)=0 \quad \text{in } \mathbb{R}^N\times(0,T) \tag{3.1}$$

with $g_\varepsilon(u)=-f_\varepsilon(-u)$ and $\bar u^\varepsilon(x,0)=-u_0$. The estimate of v^ε from below implies the estimate of u^ε from above.

We adjust the method as in §3 of Chen(1992) to our problem. Let $\zeta\colon\mathbb{R}\to\mathbb{R}$ be a smooth cut-off function satisfying

$$\zeta(u)=0 \quad \text{for } u\in(-\infty,\alpha_\varepsilon-\varepsilon/\lambda_1]\cup[\alpha_\varepsilon+3\varepsilon|\log\varepsilon|,+\infty), \tag{3.2}$$

$$\zeta(u)=1 \quad \text{for } u\in[\alpha_\varepsilon,\alpha_\varepsilon+2\varepsilon|\log\varepsilon|], \tag{3.3}$$

$$0\leq\zeta'(u)\leq 2\lambda_1/\varepsilon \quad \text{for } u\in(-\infty,\alpha_\varepsilon] \tag{3.4}$$

$$-2(\varepsilon|\log\varepsilon|)^{-1}\leq\zeta'(u)\leq 0 \quad \text{for } u\in[\alpha_\varepsilon,+\infty), \tag{3.5}$$

where λ_1 is the constant defined by (2.4). For $k\in\mathbb{Z}$, we define

$$\begin{aligned}\bar f_\varepsilon(s)=(1-\zeta(s-2\pi k))f_\varepsilon(s)+\zeta(s-2\pi k)\frac{\alpha_\varepsilon+2\pi k+\varepsilon|\log\varepsilon|-s}{|\log\varepsilon|}\\ \text{for } s\in[(2k-1)\pi,(2k+1)\pi].\end{aligned} \tag{3.6}$$

By the definition of $\bar f_\varepsilon$ we have

$$\bar f_\varepsilon=f_\varepsilon \quad \text{in } \bigcup_{k\in\mathbb{Z}}[\alpha_\varepsilon+2\pi k+3\varepsilon|\log\varepsilon|,\alpha_\varepsilon+2\pi(k+1)-\varepsilon/\lambda_1]. \tag{3.7}$$

Moreover we have that there exists $\varepsilon_0 > 0$ satisfying

$$\bar{f}_\varepsilon(u) \geq f_\varepsilon(u) \quad \text{for } u \in \mathbb{R} \text{ provided that } \varepsilon \in (0, \varepsilon_0). \tag{3.8}$$

In fact, for $u \in [-\pi, \pi]$, we have

$$\begin{aligned} \bar{f}_\varepsilon(u) =&(1 - \zeta(u))f_\varepsilon(u) + \zeta(u)\frac{\alpha_\varepsilon + \varepsilon|\log\varepsilon| - u}{|\log\varepsilon|} \\ =&f_\varepsilon(u) + \zeta(u)\left\{\varepsilon + (\alpha_\varepsilon - u)\left(\frac{1}{|\log\varepsilon|} + \frac{f_\varepsilon(\alpha_\varepsilon) - f_\varepsilon(u)}{\alpha_\varepsilon - u}\right)\right\}. \end{aligned} \tag{3.9}$$

Let $\varepsilon_0 \ll 1$ enough small so that we have

$$\begin{cases} f'_\varepsilon(u) \geq \dfrac{1}{2}f'_\varepsilon(0) \quad \text{for } u \in [\alpha_\varepsilon - \varepsilon/\lambda_1, \alpha_\varepsilon + 3\varepsilon|\log\varepsilon|], \\ \text{and } \ -\dfrac{1}{|\log\varepsilon|} \geq \dfrac{1}{2}f'_\varepsilon(0) \quad \text{provided that } \varepsilon \in (0, \varepsilon_0). \end{cases} \tag{3.10}$$

For $u \in [\alpha_\varepsilon - \varepsilon/\lambda_1, \alpha_\varepsilon]$, we have

$$(\alpha_\varepsilon - u)\left(\frac{1}{|\log\varepsilon|} + \frac{f_\varepsilon(\alpha_\varepsilon) - f_\varepsilon(u)}{\alpha_\varepsilon - u}\right) \geq (\alpha_\varepsilon - u)\frac{f_\varepsilon(\alpha_\varepsilon) - f_\varepsilon(u)}{\alpha_\varepsilon - u} \geq -\varepsilon.$$

By combining this and (3.9) we obtain $\bar{f}_\varepsilon \geq f_\varepsilon$. For $u \in [\alpha_\varepsilon, \alpha_\varepsilon + 3\varepsilon|\log\varepsilon|]$, we have

$$\frac{\alpha_\varepsilon + \varepsilon|\log\varepsilon| - u}{|\log\varepsilon|} \geq f_\varepsilon(u).$$

In fact, we have for $u \in [\alpha_\varepsilon, \alpha_\varepsilon + 3\varepsilon|\log\varepsilon|]$, there exists $\theta \in [\alpha_\varepsilon, u]$ satisfying $f_\varepsilon(u) = f_\varepsilon(\alpha_\varepsilon) + f'_\varepsilon(\theta)(u - \alpha_\varepsilon)$. Therefore we have for $u \in [\alpha_\varepsilon, \alpha_\varepsilon + 3\varepsilon|\log\varepsilon|]$,

$$f_\varepsilon(u) = f_\varepsilon(\alpha_\varepsilon) + f'_\varepsilon(\theta)(u - \alpha_\varepsilon) \leq \frac{1}{2}f'_\varepsilon(0)(u - \alpha_\varepsilon) < \frac{\alpha_\varepsilon + \varepsilon|\log\varepsilon| - u}{|\log\varepsilon|}.$$

by (3.10). This yields

$$\bar{f}_\varepsilon(u) \geq (1 - \zeta(u))f_\varepsilon(u) + \zeta(u)f_\varepsilon(u) = f_\varepsilon(u)$$

for $u \in [\alpha_\varepsilon, \alpha_\varepsilon + 3\varepsilon|\log\varepsilon|]$. We thus obtain $\bar{f}_\varepsilon \geq f_\varepsilon$ in $[-\pi, \pi]$. This implies $\bar{f}_\varepsilon \geq f_\varepsilon$ in $\mathbb{R}$ since $\bar{f}_\varepsilon$ is periodic whose length is 2π.

We here recall the comparison principle of ordinary differential equations, which is important tool in this section.

Lemma 3.2. *Let $D \subset \mathbb{R}$ be an interval. Let $F\colon D \to \mathbb{R}$ be a Lipschitz continuous function. Let $J \subset \mathbb{R}$ be an open interval, and u, $v\colon J \to D$ be functions satisfying*

$$\begin{aligned} &u(s_0) \leq v(s_0) \quad \textit{for some } s_0 \in J, \\ &u' \leq F(u),\ v' \geq G(v) \quad \textit{in } J. \end{aligned}$$

Then

$$u(s) \leq v(s) \quad \textit{for } s \in J \cap [s_0, \infty).$$

Proof. See (16,4) Lemma of Amann(1990). □

We prove the revised version of Lemma 3.1 as in Chen(1992) for our problem.

Lemma 3.3. *Assume that $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is a positive constant satisfying (3.10). Let $\bar{\omega} = \bar{\omega}(\xi, \tau)$ be a solution of*

$$\bar{\omega}_\tau + \bar{f}_\varepsilon(\bar{\omega}) = 0, \tag{3.11}$$

$$\bar{\omega}(\xi, 0) = \xi. \tag{3.12}$$

Then the followings hold.

(i) If $\xi \in (\alpha_\varepsilon + 2\pi k + \varepsilon|\log\varepsilon|, (2k+1)\pi)$, then $\bar{\omega}(\xi,\tau) \in (\xi, (2k+1)\pi)$. If $\xi \in ((2k-1)\pi, \alpha_\varepsilon + 2\pi k + \varepsilon|\log\varepsilon|)$, then $\bar{\omega}(\xi,\tau) \in ((2k-1)\pi, \xi)$.

(ii) There exists $\bar{\varepsilon} \in (0, \varepsilon_0)$ such that, for $b > 0$, there exists $\tau_0 = \tau_0(b)$ satisfying

$$\bar{\omega}(\xi,\tau) \geq 2\pi - b\varepsilon \quad \text{for } \xi \geq \alpha_\varepsilon + 3\varepsilon|\log\varepsilon|,\ \tau \geq \tau_0|\log\varepsilon|$$

provided that $\varepsilon \in (0, \bar{\varepsilon})$.

(iii) $\omega \in C^{2,1}(\mathbb{R} \times [0,\infty))$ and

$$\bar{\omega}_\xi(\xi,\tau) > 0 \quad \text{for } \tau \in [0,\infty).$$

(iv) For $\kappa > 0$, there exists $L = L(\kappa) > 0$ satisfying

$$\left|\frac{\bar{\omega}_{\xi\xi}(\xi,\tau)}{\bar{\omega}_\xi(\xi,\tau)}\right| \leq \frac{L}{\varepsilon} \quad \text{for } \tau \leq \kappa|\log\varepsilon|$$

provided that $\varepsilon \in (0, \bar{\varepsilon})$, where $\bar{\varepsilon}$ is as in (i).

We will give a detailed proof of Lemma 3.3 because it is necessary to clarify the dependence of ε for each constants since f_ε depends ε.

Proof. In the following arguments, we shall prove Lemma 3.3 only when $\xi \in [-\pi, \pi]$ since $\bar{\omega}(\xi,\tau) + 2\pi k = \bar{\omega}(\xi + 2\pi k, \tau)$. By the theory of ordinary differential equations we have $\bar{\omega} \in C^{2,1}(\mathbb{R} \times [0,\infty))$.

(i) We only prove the case that $\xi \in (\alpha_\varepsilon + \varepsilon|\log\varepsilon|, \pi)$ since a proof for the case $\xi \in (-\pi, \alpha_\varepsilon + \varepsilon|\log\varepsilon|)$ is similar.

Assume that there exists $\tau_1 > 0$ satisfying

$$\bar{\omega}(\xi, \tau_1) \geq \pi.$$

Set $\tau_2 = \inf\{\tau \in (0, \tau_1];\ \bar{\omega}(\xi,\tau) \geq \pi\}$. Since $\bar{\omega}(\xi, 0) = \xi < \pi$, we observe that $\tau_2 > 0$ and

$$\bar{\omega}(\xi,\tau) < \pi \quad \text{for } \tau < \tau_2, \tag{3.13}$$

$$\bar{\omega}(\xi,\tau_2) = \pi. \tag{3.14}$$

We have that $\bar{\omega}(\xi,\tau) \equiv \pi$ is a solution of (3.11) in some neighborhood J of τ_2 with initial data (3.14). By the uniqueness of a solution of (3.11) with initial data (3.14), we obtain

$$\bar{\omega}(\xi,\tau) \equiv \pi \quad \text{in } J,$$

which contradicts (3.13). We thereby obtain $\bar{\omega}(\xi,\tau) < \pi$. By similar arguments we obtain $\bar{\omega}(\xi,\tau) > \alpha_\varepsilon + \varepsilon|\log\varepsilon|$. We obtain

$$\bar{\omega}(\xi,\tau) \in (\alpha_\varepsilon + \varepsilon|\log\varepsilon|, \pi) \quad \text{for } \xi \in (\alpha_\varepsilon + \varepsilon|\log\varepsilon|, \pi) \text{ and } \tau > 0. \tag{3.15}$$

We next assume that there exists $\tau_1 > 0$ and $\tilde{\xi} \in (\alpha_\varepsilon + \varepsilon|\log\varepsilon|, \xi]$ satisfying $\bar{\omega}(\xi,\tau_1) = \tilde{\xi}$ and lead a contradiction. By (3.15) we obtain

$$\bar{f}_\varepsilon(\bar{\omega}(\xi,\tau)) < 0 \quad \text{for } \tau \in (0,\tau_1).$$

Therefore we obtain

$$\xi \geq \bar{\omega}(\xi,\tau_1) = \bar{\omega}(\xi,0) - \int_0^{\tau_1} \bar{f}_\varepsilon(\bar{\omega}(\xi,\sigma))d\sigma > \xi,$$

which is the contradiction.

(ii) We first remark that there exists σ_0, which is independent of ε, satisfying

$$g_\varepsilon(u) := f_\varepsilon(u) - \sigma_0(u-\alpha_\varepsilon)(u-\pi) < 0 \quad \text{for } u \in (\alpha_\varepsilon, \pi). \tag{3.16}$$

In fact, we obtain that, for example, $\sigma_0 = (2\pi)^{-1}$ is a desired constants by straightforward calculation.

Let $\xi \in (\alpha_\varepsilon + 3\varepsilon|\log\varepsilon|, \pi)$ and $\tilde{\omega} = \tilde{\omega}(\xi,\tau)$ be a function satisfying

$$\begin{cases} \tilde{\omega}_\tau + \sigma_0(\tilde{\omega}-\alpha_\varepsilon)(\tilde{\omega}-\pi) = 0 & \text{for } \tau > 0, \\ \tilde{\omega}(\xi,0) = \xi. \end{cases} \tag{3.17}$$

By similar argument as in (i) we obtain $\tilde{\omega}(\xi,\tau) \in (\xi,\pi)$ for $\xi \in (\alpha_\varepsilon,\pi)$. By (i) we also have that $\bar{\omega}(\xi,\tau) \in (\alpha_\varepsilon + 3\varepsilon|\log\varepsilon|, \pi)$. This and Lemma 3.2 yields that

$$\tilde{\omega}(\xi,\tau) \leq \bar{\omega}(\xi,\tau) \quad \text{for } \tau \geq 0 \text{ and } \xi \in (\alpha_\varepsilon + 3\varepsilon|\log\varepsilon|, \pi).$$

Since $\tilde{\omega}(\xi,\tau)$ is monotone increasing and $\tilde{\omega}(\xi,\tau) \geq \tilde{\omega}(\alpha_\varepsilon + 3\varepsilon|\log\varepsilon|, \tau)$, it suffices to obtain the estimate of $\hat{\tau}$ satisfying $\tilde{\omega}(\alpha_\varepsilon + 3\varepsilon|\log\varepsilon|, \hat{\tau}) = \pi - b\varepsilon$ from above.

By solving (3.17) we obtain that $\tilde{\omega}$ satisfies

$$-\sigma_0\tau + \tilde{C}_\varepsilon = -\frac{1}{\pi-\alpha_\varepsilon}\log\frac{\tilde{\omega}-\alpha_\varepsilon}{\pi-\tilde{\omega}}, \tag{3.18}$$

where

$$\tilde{C}_\varepsilon = -\frac{1}{\pi-\alpha_\varepsilon}\log\frac{3\varepsilon|\log\varepsilon|}{\pi-\alpha_\varepsilon-3\varepsilon|\log\varepsilon|}.$$

Let $\hat{\tau}$ be a constant satisfying $\tilde{\omega}(\alpha_\varepsilon + 3\varepsilon|\log\varepsilon|, \hat{\tau}) = \pi - b\varepsilon$. We obtain from a straightforward calculation

$$\hat{\tau} = \frac{1}{\sigma_0}\left(\tilde{C}_\varepsilon + \frac{1}{\pi-\alpha_\varepsilon}\log\frac{\pi-b\varepsilon-\alpha_\varepsilon}{b\varepsilon}\right).$$

Here we fix $\bar{\varepsilon} < \varepsilon_0$ satisfying

$$\bar{\varepsilon} < e^{-1}.$$

There exist numerical constants $\tilde{C}_1 > 0$ and $\tilde{C}_2 > 0$ satisfying

$$\begin{aligned} \tilde{C}_\varepsilon &\leq \tilde{C}_1 |\log \varepsilon|, \\ \frac{1}{\pi - \alpha_\varepsilon} \log \frac{\pi - b\varepsilon - \alpha_\varepsilon}{b\varepsilon} &\leq \tilde{C}_2 (1 + |\log b|) |\log \varepsilon|, \end{aligned}$$

provided that $\varepsilon \in (0, \bar{\varepsilon})$. Therefore we obtain

$$\hat{\tau} \leq \frac{1}{\sigma_0} (\tilde{C}_1 + \tilde{C}_2 (1 + |\log b|)) |\log \varepsilon|,$$

so $\tau_0(b) = (\tilde{C}_1 + \tilde{C}_2(1 + |\log b|))/\sigma_0$ is a desired constant.

(iii) By following the method of the proof of Lemma 3.1(ii) as in Chen(1992) we obtain

$$\bar{\omega}_\xi = \exp\left(- \int_0^\tau \bar{f}'_\varepsilon (\bar{\omega}(\xi, \sigma)) d\sigma \right) > 0 \quad \text{for } \tau \geq 0. \tag{3.19}$$

(iv) For $\xi \neq \pm\pi$, α_ε, we obtain

$$\begin{aligned} \bar{\omega}_\xi(\xi, \tau) &= \exp\left(\int_0^\tau \frac{\bar{f}'_\varepsilon(\bar{\omega}(\xi, \sigma))}{\bar{f}_\varepsilon(\bar{\omega}(\xi, \sigma))} \bar{\omega}_\tau(\xi, \sigma) d\sigma \right) \\ &= \exp\left(\int_0^\tau \frac{d}{d\sigma} (\log \bar{f}_\varepsilon(\bar{\omega}(\xi, \sigma))) d\sigma \right) \\ &= \frac{\bar{f}_\varepsilon(\bar{\omega}(\xi, \tau))}{\bar{f}_\varepsilon(\xi)}. \end{aligned}$$

Therefore we obtain from above,

$$\frac{\bar{\omega}_{\xi\xi}(\xi, \tau)}{\bar{\omega}_\xi(\xi, \tau)} = \frac{\bar{f}'_\varepsilon(\bar{\omega}(\xi, \tau)) - \bar{f}'_\varepsilon(\xi)}{\bar{f}_\varepsilon(\xi)} \quad \text{for } \xi \neq \pm\pi,\ \alpha_\varepsilon. \tag{3.20}$$

We remark that there exists a positive constant $\tilde{C}_3$ satisfying

$$\left| \frac{\bar{\omega}_{\xi\xi}(\pm\pi, \tau)}{\bar{\omega}_\xi(\pm\pi, \tau)} \right| \leq \tilde{C}_3 \quad \text{for } \tau \leq \kappa |\log \varepsilon|.$$

In fact we obtain

$$\frac{\bar{\omega}_{\xi\xi}(\pm\pi, \tau)}{\bar{\omega}_\xi(\pm\pi, \tau)} = \lim_{\xi \to \pm\pi} \frac{\bar{\omega}_{\xi\xi}(\xi, \tau)}{\bar{\omega}_\xi(\xi, \tau)} = \frac{\bar{f}''_\varepsilon(\bar{\omega}(\pm\pi, \tau))(\bar{\omega}_\xi(\pm\pi, \tau) - 1)}{\bar{f}''_\varepsilon(\pm\pi)}.$$

We have

$$\bar{f}''_\varepsilon(\bar{\omega}(\pm\pi, \tau)) = \bar{f}''_\varepsilon(\pm\pi) = f''_\varepsilon(\pm\pi) = -\varepsilon a.$$

From (3.19) we obtain

$$\bar{\omega}_\xi(\pm\pi, \tau) = \exp(-\tau) \in (0, 1].$$

Therefore we obtain

$$\left|\frac{\bar{\omega}_{\xi\xi}(\pm\pi,\tau)}{\bar{\omega}_{\xi}(\pm\pi,\tau)}\right| \le 2.$$

Moreover we remark that there exists $r_0 > 0$, which is independent of ε, satisfying

$$\begin{aligned} &\alpha_\varepsilon \le \bar{\omega}(\xi,\tau) \le \alpha_\varepsilon + 2\varepsilon|\log\varepsilon| \\ &\qquad \text{for } \xi \in (\alpha_\varepsilon + \varepsilon|\log\varepsilon| - r_0\varepsilon, \alpha_\varepsilon + \varepsilon|\log\varepsilon| + r_0\varepsilon),\ \tau \le \kappa|\log\varepsilon|. \end{aligned} \tag{3.21}$$

Here we only prove the second inequality because the proofs are symmetric. We may assume that $r_0 < |\log\varepsilon_0|$, which implies that $r_0 < |\log\varepsilon|$ since $\varepsilon < \varepsilon_0 < 1$. Let $\omega = \omega(\tau)$ be a solution of

$$\begin{aligned} &\omega_\tau + \frac{\alpha_\varepsilon + \varepsilon|\log\varepsilon| - \omega}{|\log\varepsilon|} = 0, \\ &\omega(0) = \alpha_\varepsilon + \varepsilon|\log\varepsilon| + r_0\varepsilon, \end{aligned}$$

where $r_0 > 0$ is a constant determined later. We obtain

$$\omega(\tau) = \alpha_\varepsilon + \varepsilon|\log\varepsilon| + r_0\varepsilon\exp\left(\frac{\tau}{|\log\varepsilon|}\right).$$

Fix $r_0 = \exp(-\kappa)$, which is determined so that r_0 satisfies $r_0 \le |\log\varepsilon|\exp(-\kappa)$. We have that $\omega(\tau) \le \alpha_\varepsilon + 2\varepsilon|\log\varepsilon|$ for $\tau \le \kappa|\log\varepsilon|$ and $\varepsilon \in (0,\varepsilon_0)$. This implies that ω satisfies

$$\omega_\tau + \bar{f}_\varepsilon(\omega) = 0 \quad \text{for } \tau \in (0, \kappa|\log\varepsilon|).$$

By Lemma 3.2 we obtain $\bar{\omega}(\tau,\xi) \le \omega(\tau)$ for $\xi \in (\alpha_\varepsilon + \varepsilon|\log\varepsilon|, \alpha_\varepsilon + \varepsilon|\log\varepsilon| + r_0\varepsilon)$ and $\tau \in (0, \kappa|\log\varepsilon|)$, which implies the desired conclusion.

We finish the proof of (iv). We observe that there exists $r_1 > 0$ satisfying

$$\left|\frac{\bar{\omega}_{\xi\xi}(\pm\pi,\tau)}{\bar{\omega}_{\xi}(\pm\pi,\tau)}\right| \le \tilde{C}_3 + 1 \quad \text{for } \tau \le \kappa|\log\varepsilon|,\ \xi \in (\pm\pi - r_1\varepsilon, \pm\pi + r_1\varepsilon), \tag{3.22}$$

and $\bar{r} = \min(r_0, r_1)$. By definition of $\bar{f}_\varepsilon$ there exists $\bar{c} > 0$ such that $\xi \in [-\pi, -\pi + \bar{r}\varepsilon) \cup (\alpha_\varepsilon + \varepsilon|\log\varepsilon| - \bar{r}\varepsilon, \alpha_\varepsilon + \varepsilon|\log\varepsilon| + \bar{r}\varepsilon) \cup (\pi - \bar{r}\varepsilon, \pi]$ if $\bar{f}_\varepsilon(\xi) < \bar{c}\varepsilon$. We divide a situation into 2 cases.

Case 1. Assume that ξ satisfies $\bar{f}_\varepsilon(\xi) \ge \bar{c}\varepsilon$. Then we obtain from (3.20),

$$\left|\frac{\bar{\omega}_{\xi\xi}(\xi,\tau)}{\bar{\omega}_{\xi}(\xi,\tau)}\right| = \left|\frac{\bar{f}'_\varepsilon(\bar{\omega}(\xi,\tau)) - \bar{f}'_\varepsilon(\xi)}{\bar{f}_\varepsilon(\xi)}\right| \le \frac{2\bar{\lambda}_1}{\bar{c}\varepsilon},$$

where $\bar{\lambda}_1 = \sup_{\varepsilon\in(0,\bar{\varepsilon})}\sup_{\mathbb{R}}|\bar{f}'_\varepsilon|$. We remark that

$$|\bar{f}'_\varepsilon| \le \lambda_1 + \frac{1}{|\log\varepsilon|} \le \lambda_1 + 1,$$

where λ_1 is a constant defined by (2.4). Therefore we obtain the desired conclusion in this case.

Case 2. Assume that ξ satisfies $\bar{f}_\varepsilon(\xi) < \bar{c}\varepsilon$. If $\xi \in [-\pi, -\pi + \bar{r}\varepsilon) \cup (\pi - \bar{r}\varepsilon, \pi]$, we obtain the desired conclusion by (3.22). If $\xi \in (\alpha_\varepsilon + \varepsilon|\log\varepsilon| - \bar{r}\varepsilon, \alpha_\varepsilon + \varepsilon|\log\varepsilon| + \bar{r}\varepsilon)$, then we obtain from (3.21) and (3.20)

$$\frac{\bar{\omega}_{\xi\xi}(\xi,\tau)}{\bar{\omega}_\xi(\xi,\tau)} = 0 \quad \text{for } \tau \le \kappa|\log\varepsilon|,$$

which includes the desired conclusion. □

We will give the estimate of solutions of (2.1) with *good* initial data.

Lemma 3.4. *Let $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ be smooth and satisfy*

$$\bar{C}_0 \ge \sup_{\mathbb{R}^N} |\nabla\varphi|^2 + \varepsilon \sup_{\mathbb{R}^N} |\Delta\varphi| < \infty \tag{3.23}$$

for some constant $\bar{C}_0$ independent of $\varepsilon \in (0,1)$, Let $\bar{\varepsilon}$ be as in Lemma 3.3. We have that:

(i) there exists a positive constant M such that

$$v(x,t) := \bar{\omega}\left(\varphi(x) - \frac{Mt}{\varepsilon}, \frac{t}{\varepsilon^2}\right)$$

is a subsolution of (2.1) *in $\mathbb{R}^N \times (0, \tau_0\varepsilon^2|\log\varepsilon|)$ provided that $\varepsilon \in (0,\bar{\varepsilon})$.*

(ii) for any $b > 0$, there exists a positive constant $M_0 = M_0(b)$ satisfying

$$v(x, \tau_0\varepsilon^2|\log\varepsilon|) \ge \pi - b\varepsilon \quad \textit{for } x \in \{y;\ \varphi(y) \ge \alpha_\varepsilon + M_0\varepsilon|\log\varepsilon|\}$$

provided that $\varepsilon \in (0,\bar{\varepsilon})$.

Proof. See the proof of Theorem 1 in §3 of Chen(1992). □

We are now in the position to prove Theorem 3.1.

Proof of Theorem 3.1. We first prove the case $k = 0$.

Let $m > 0$ and $x_0 \in \{y;\ u_0(y) > m\}$. By the uniform continuity of u_0 there exists $\delta > 0$ satisfying

$$\sup_{|x-y|<\delta} |u_0(x) - u_0(y)| < \frac{m}{2}. \tag{3.24}$$

We remark that $u_0 > m/2$ in $B_\delta(x_0) := \{x;\ |x - x_0| < \delta\}$. Here we define

$$\varphi(x) = -\frac{2(2K_0+1)\pi + m}{2\delta^2}|x - x_0|^2 + \frac{m}{2}.$$

We have

$$\varphi(x) \le u_0(x) \quad \text{for } x \in \mathbb{R}^N,$$

since $\varphi(x) \le m/2 \le u_0(x)$ for $x \in B_\delta(x_0)$ and $\varphi(x) \le -(2K_0+1)\pi \le u_0(x)$ for $x \in \mathbb{R}^N \setminus B_\delta(x_0)$. Moreover we have that there exists $r > 0$ satisfying

$$\varphi(x) > \frac{m}{4} \quad \text{for } x \in B_r(x_0). \tag{3.25}$$

Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

$$\theta(\sigma) = \begin{cases} \sigma & \sigma \geq -(2K_0+1)\pi, \\ -2(K_0+1)\pi & \sigma \leq -2(K_0+1)\pi, \end{cases}$$
$$\theta'(\sigma) \geq 0 \quad \text{for } \sigma \in \mathbb{R}.$$

We define

$$\bar{\varphi}(x) := \theta(\varphi(x)).$$

We then have

$$\bar{\varphi}(x) \leq u_0(x) \quad \text{for } x \in \mathbb{R}^N, \tag{3.26}$$
$$\bar{\varphi}(x) = \varphi(x) \quad \text{for } x \in B_\delta(x_0), \tag{3.27}$$
$$\sup_{\mathbb{R}^N} |\nabla \bar{\varphi}|^2 + \varepsilon \sup_{\mathbb{R}^N} |\Delta \bar{\varphi}| \leq \bar{C}_0 < +\infty \tag{3.28}$$

for some constant $\bar{C}_0$ independent of $\varepsilon \in (0,1)$. We now replace $\bar{\varepsilon} > 0$ smaller so that we have

$$\alpha_\varepsilon + M_0 \varepsilon |\log \varepsilon| \leq \frac{m}{4} \quad \text{provided that } \varepsilon \in (0, \bar{\varepsilon}). \tag{3.29}$$

We define

$$v(x,t) := \bar{\omega}\left(\bar{\varphi}(x) - \frac{M}{\varepsilon} t, \frac{t}{\varepsilon^2}\right),$$

where M is as in Lemma 3.4(ii). By (3.25), (3.27) and Lemma 3.4(ii) we obtain for $b > 0$,

$$v(x, \tau_0 \varepsilon^2 |\log \varepsilon|) \geq \pi - b\varepsilon \quad \text{for } x \in B_r(x_0) \text{ provided that } \varepsilon \in (0, \bar{\varepsilon}).$$

Since $v(x,0) = \bar{\varphi}(x) \leq u_0(x)$ for $x \in \mathbb{R}^N$, we obtain from Theorem 2.2,

$$u^\varepsilon(x, \tau_0 \varepsilon^2 |\log \varepsilon|) \geq v(x, \tau_0 \varepsilon^2 |\log \varepsilon|) \geq \pi - b\varepsilon \quad \text{for } x \in B_r(x_0)$$

provided that $\varepsilon \in (0, \bar{\varepsilon})$.

For a general $k \in \mathbb{Z} \cap [-K_0, K_0]$, let $x_0^k \in \{x;\ u_0(x) > 2\pi k + m\}$. We have $u_0(y) \geq 2\pi k + m/2$ for $y \in B_\delta(x_0^k)$. Define

$$\varphi_k(x) = -\frac{2(2K_0+1)\pi + m + 4\pi k}{2\delta^2} |x - x_0^k|^2 + \frac{m}{2} + 2\pi k,$$
$$\bar{\varphi}_k(x) = \theta(\varphi_k(x)).$$

We also obtain (3.26)–(3.28) for $\bar{\varphi}_k$ and there exists $r_k > 0$ satisfying

$$\bar{\varphi}_k(x) > 2\pi k + \frac{m}{4} \quad \text{for } x \in B_{r_k}(x_0^k). \tag{3.30}$$

Here we consider

$$v_k(x,t) = \bar{\omega}\left(\bar{\varphi}_k(x,t) - \frac{M}{\varepsilon} t, \frac{t}{\varepsilon^2}\right).$$

Since $\bar{\omega}(\xi, \tau) = \bar{\omega}(\xi - 2\pi k, \tau) + 2\pi k$ and (3.30) implies that $\bar{\varphi}_k - 2\pi k \geq \alpha_\varepsilon + M_0 \varepsilon |\log \varepsilon|$, we obtain

$$\begin{aligned} v_k(x,t) =& \bar{\omega}\left(\bar{\varphi}_k(x,t) - 2\pi k - \frac{M}{\varepsilon} t, \frac{t}{\varepsilon^2}\right) + 2\pi k \\ \geq& (2k+1)\pi - b\varepsilon \quad \text{for } x \in B_{r_k}(x_0^k). \end{aligned}$$

Therefore we obtain

$$u(x, \tau_0\varepsilon^2|\log\varepsilon|) \geq v_k(x, \tau_0\varepsilon^2|\log\varepsilon|) \geq (2k+1)\pi - b\varepsilon \quad \text{for } x \in B_{r_k}(x_0^k).$$

This yields the conclusion of Theorem 3.1. □

4. Uniform estimate of traveling waves

In the previous section we have the very short time behavior of a solution of (2.1). That result says that the solution of (2.1) becomes like as an initial data, which Evans, Soner and Souganidis(1992) introduced, in a very short time. Hence it is convenient to consider the traveling wave solution to construct a supersolution for an estimate as in Evans, Soner and Souganidis(1992). In this section we shall give an uniform estimate and some properties of the traveling wave solution.

The traveling wave solution of (2.1) is a solution of the form $q(\varepsilon^{-1}x\cdot e - \varepsilon^{-2}ct)$, where $e \in S^{n-1}$ and $c \in \mathbb{R}$. The function $q(s)$ satisfies the ordinary differential equation

$$q'' + cq' = f_\varepsilon(q) \quad \text{in } \mathbb{R}. \tag{4.1}$$

We are interested in a solution of (4.1) satisfying the boundary conditions

$$q(\pm\infty) = \pm\pi, \tag{4.2}$$
$$q(0) = \alpha_\varepsilon. \tag{4.3}$$

Aronson and Weinberger(1978) proved the existence and uniqueness of a pair (q, c) satisfying not only (4.1)–(4.2) but also

$$|q(s)| \leq \pi \quad \text{in } \mathbb{R} \tag{4.4}$$
$$q' > 0 \quad \text{in } \mathbb{R}, \tag{4.5}$$
$$q'(\pm\infty) = 0 \tag{4.6}$$

for $\varepsilon > 0$. Here and hereafter, we use the notation $(q_\varepsilon, c_\varepsilon)$ to clarify the dependence of ε.

For our problem, we are interested in the uniform bound of q_ε and the existence of the limit $\lim_{\varepsilon\to 0}\varepsilon^{-1}c_\varepsilon$.

Lemma 4.1. *Let q_ε be a solution of* (4.1) *with the conditions* (4.2)–(4.6)*. Then the followings hold.*

(i) There exists $\varepsilon_1 > 0$ such that, for any $R > 0$, we have

$$\inf_{\varepsilon\in(0,\varepsilon_1)} \inf_{[-R,R]} q_\varepsilon' > 0.$$

(ii) There exists a positive constants, $C_1 = C_1(\varepsilon_1)$ and $C_2 = C_2(\varepsilon_1)$ satisfying

$$|q_\varepsilon(s) - \pi| \leq C_1 \exp(-C_2 s) \quad \textit{for} \quad s > 0, \tag{4.7}$$
$$|q_\varepsilon(s) + \pi| \leq C_1 \exp(C_2 s) \quad \textit{for} \quad s < 0, \tag{4.8}$$
$$|q_\varepsilon'(s)|,\ |q_\varepsilon''(s)| \leq C_1 \exp(-C_2|s|) \quad \textit{for} \quad s \in \mathbb{R} \tag{4.9}$$

provided that $\varepsilon \in (0, \varepsilon_1)$, where ε_1 is as in (i).

(iii) There exists the limit $A := -\lim_{\varepsilon\to 0}\varepsilon^{-1}c_\varepsilon$.

Barles, Soner and Souganidis(1993) discuss the traveling waves for functions $u \mapsto f(x,t,u) - \varepsilon a$. However, they assume the existence of the limit $\lim_{\varepsilon\to 0}\varepsilon^{-1}c_\varepsilon$, and propose the example which fulfills their assumption. They proved that the pair (q,c) from a map $u \mapsto 2u(u^2-1) - \varepsilon a$ satisfies their assumption by using an explicit form of q or c. Therefore we need to prove the existence of the limit $\lim_{\varepsilon\to 0}\varepsilon^{-1}c_\varepsilon$. Here we shall prove it without using the explicit form of the solution q_ε or c_ε, which is one of the advantages over that of Barles, Soner and Souganidis(1993). Therefore it is easy to extend our proof to the case of the traveling waves for the function as in Remark 2.1.

Proof of Lemma 4.1. Let ε_1 satisfy $\varepsilon_1 < 1$ and

$$-\frac{\pi}{2} < \alpha_\varepsilon + \beta_\varepsilon < 0,\ -\frac{\pi}{2} < \alpha_\varepsilon < 0 \quad \text{for } \varepsilon \in (0,\varepsilon_1).$$

In the following arguments we shall replace ε_1 to smaller one later, at (4.14) and (4.22).

1. By multiplying q'_ε to (4.1) we obtain

$$q'_\varepsilon(s)q''_\varepsilon(s) + c_\varepsilon q'_\varepsilon(s)^2 = f_\varepsilon(q_\varepsilon(s))q'_\varepsilon(s). \tag{4.10}$$

By (4.2) and (4.6) we have

$$\begin{aligned}
&\int_{\mathbb{R}} q'_\varepsilon(\sigma)q''_\varepsilon(\sigma)d\sigma = q'_\varepsilon(\infty)^2 - q'_\varepsilon(-\infty)^2) = 0,\\
&\int_{\mathbb{R}} f_\varepsilon(q_\varepsilon(\sigma))d\sigma = \int_{-\pi}^{\pi} f_\varepsilon(u)du = -2\pi a\varepsilon.
\end{aligned}$$

Therefore we have from (4.10),

$$c_\varepsilon \int_{\mathbb{R}} q'_\varepsilon(\sigma)^2 d\sigma = -2\pi a\varepsilon. \tag{4.11}$$

Moreover, by integrating (4.10) in $(-\infty, s)$ we obtain

$$\frac{1}{2}q'_\varepsilon(s)^2 + c_\varepsilon \int_{-\infty}^{s} q'_\varepsilon(\sigma)^2 d\sigma = \int_{-\pi}^{q_\varepsilon(s)} f_\varepsilon(u)du. \tag{4.12}$$

By straightforward calculation we obtain from $q_\varepsilon \in (-\pi,\pi)$ and $\varepsilon \in (0,\varepsilon_1) \subset (0,1)$,

$$\begin{aligned}
\int_{-\pi}^{q_\varepsilon(s)} f_\varepsilon(u)du \le &\int_{-\pi}^{\alpha_\varepsilon} f_\varepsilon(u)du\\
=&\sqrt{1+\varepsilon^2a^2}(\cos(\alpha_\varepsilon+\beta_\varepsilon) - \cos(-\pi+\beta_\varepsilon)) - \varepsilon a(\alpha_\varepsilon + \pi)\\
\le&2\sqrt{1+\varepsilon_1^2a^2} + 2\pi\varepsilon_1|a| \le 2\sqrt{1+a^2} + 2\pi|a|.
\end{aligned}$$

Therefore we obtain

$$\frac{1}{2}q'_\varepsilon(s)^2 \le |c_\varepsilon| \int_{\mathbb{R}} q'_\varepsilon(\sigma)d\sigma + \int_{-\pi}^{\alpha_\varepsilon} f_\varepsilon(u)du \le 4\pi|a| + 2\sqrt{1+a^2},$$

which implies that there exists L_1 depending only on a satisfying

$$q_\varepsilon'(s) \le L_1 \quad \text{for } s \in \mathbb{R} \text{ provided that } \varepsilon \in (0, \varepsilon_1). \tag{4.13}$$

By integrating (4.10) in $(-\infty, 0)$ again we obtain

$$\begin{aligned}
\frac{1}{2} q_\varepsilon'(0)^2 =& - c_\varepsilon \int_{-\infty}^{0} q_\varepsilon'(\sigma)^2 d\sigma + \int_{-\pi}^{\alpha_\varepsilon} f_\varepsilon(u) du \\
\ge& - |c_\varepsilon| \int_{-\infty}^{\infty} q_\varepsilon'(\sigma)^2 d\sigma + \int_{-\pi}^{\alpha_\varepsilon} f_\varepsilon(u) du \\
=& - 2\pi |a| \varepsilon + \sqrt{1 + \varepsilon^2 a^2} (\cos(\alpha_\varepsilon + \beta_\varepsilon) - \cos(-\pi + \beta_\varepsilon)) - \varepsilon a (\alpha_\varepsilon + \pi).
\end{aligned}$$

Since α_ε, $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$, we replace ε_1 to smaller one so that we have

$$\begin{aligned}
-2\pi |a| \varepsilon + \sqrt{1 + \varepsilon^2 a^2} (\cos(\alpha_\varepsilon + \beta_\varepsilon) - \cos(-\pi + \beta_\varepsilon)) - \varepsilon a (\alpha_\varepsilon + \pi) \ge \frac{1}{2} \\
\text{provided that } \varepsilon \in (0, \varepsilon_1).
\end{aligned} \tag{4.14}$$

We obtain

$$q_\varepsilon'(0) \ge 1 \quad \text{provided that } \varepsilon \in (0, \varepsilon_1).$$

2. We verify (i) and that there exists $\sigma_0 > 0$ satisfying

$$\int_{\mathbb{R}} q_\varepsilon'(\sigma)^2 d\sigma \ge \sigma_0 > 0 \quad \text{provided that } \varepsilon \in (0, \varepsilon_1). \tag{4.15}$$

By Proposition 4.2 as in Aronson and Weinberger(1978) we have that $|c_\varepsilon| \le \sup_{\mathbb{R}} |f_\varepsilon'| \le \lambda_1$. Let $p_\varepsilon = q_\varepsilon'$. We have from (4.1), (4.5) and (4.6)

$$p_\varepsilon'' + c_\varepsilon p_\varepsilon' = f_\varepsilon'(q_\varepsilon) p_\varepsilon \quad \text{in } \mathbb{R}, \tag{4.16}$$

$$p_\varepsilon(0) \ge 1, \ p_\varepsilon(+\infty) = 0, \tag{4.17}$$

$$p_\varepsilon > 0 \quad \text{in } \mathbb{R}. \tag{4.18}$$

We give an estimate of $q_\varepsilon' = p_\varepsilon$ in $(0, +\infty)$ from below.

We first verify that there exists $\sigma_1 > 0$, which is independent of $\varepsilon \in (0, \varepsilon_1)$, such that $r_1(s) := \exp(-\sigma_1 s)$ satisfies

$$-r_1'' - c_\varepsilon r_1' + (1 + \lambda_1) r_1 \le 0 \quad \text{in } (0, \infty). \tag{4.19}$$

In fact, we have

$$-r_1'' - c_\varepsilon r_1' + (1 + \lambda_1) r_1 \le (-\sigma^2 + \lambda_1 \sigma_1 + 1 + \lambda_1) \exp(-s\sqrt{2 + \lambda_1}),$$

which yields the existence of $\sigma_1 > 0$ satisfying (4.19).

We demonstrate that $p_\varepsilon \ge r_1$ on $[0, \infty)$. If not, there exists $s_0 > 0$ satisfying $p_\varepsilon(s_0) - r_1(s_0) < 0$, $p_\varepsilon'(s_0) - r_1'(s_0) = 0$, and $p_\varepsilon''(s_0) - r_1''(s_0) \ge 0$ since $p_\varepsilon(0) \ge 1 = r_1(0)$ and $p_\varepsilon(\infty) = 0 = r_1(\infty)$. We obtain from (4.16), (4.19) and $r_1 > 0$,

$$\begin{aligned}
0 \ge& p_\varepsilon''(s_0) - r_1''(s_0) + c_\varepsilon (p_\varepsilon'(s_0) - r_1'(s_0)) - f_\varepsilon'(q_\varepsilon(s_0)) p_\varepsilon(s_0) + (1 + \lambda_1) r_1(s_0) \\
\ge& (\lambda_1 - f_\varepsilon'(q_\varepsilon(s_0))) p_\varepsilon(s_0) + r_1(s_0) > 0
\end{aligned}$$

which is the contradiction. By similar argument with $\bar{r}_1 := \exp(\sigma_0 s)$ we obtain $p_\varepsilon \geq \bar{r}_1$ on $(-\infty, 0]$. Consequently we obtain (i).

It is easy to get (4.15). We calculate that

$$\int_{\mathbb{R}} q_\varepsilon'(\sigma)^2 d\sigma \geq 2\int_0^\infty \exp(-2\sigma_1\sigma)d\sigma = \frac{1}{\sigma_1} =: \sigma_0.$$

Moreover, this yields that $c_\varepsilon \to 0$ as $\varepsilon \to 0$ by (4.11).

3. We verify that, for any $\mu > 0$, there exists $R_0 = R_0(\mu)$ satisfying

$$q(s) > \alpha_\varepsilon + \mu \quad \text{for} \quad s \geq R_0, \tag{4.20}$$

$$q(s) < \alpha_\varepsilon - \mu \quad \text{for} \quad s \leq -R_0 \tag{4.21}$$

provided that $\varepsilon \in (0, \varepsilon_1)$.

Since $q_\varepsilon'(s) \geq \exp(-\sigma_1|s|)$, we obtain for $s \in \mathbb{R}$,

$$|q_\varepsilon(s) - q_\varepsilon(0)| = \left|\int_0^s q_\varepsilon'(\sigma)d\sigma\right| \geq \int_0^{|s|} \exp(-\sigma_1\sigma)d\sigma = \frac{1}{\sigma_1}(1 - \exp(-\sigma_1|s|)),$$

which yields the existence of R_0 satisfying (4.20).

4. We are now in the position to show (4.7) and (4.8). We prove only (4.7) since the proofs are symmetric. By (4.4) it suffices to obtain the estimate of q_ε from below for $t \geq 0$.

Fix enough small $\mu > 0$. We replace ε_1 to smaller one so that we have

$$\begin{aligned} &|\alpha_\varepsilon| < \frac{\mu}{2},\ |\alpha_\varepsilon + \beta_\varepsilon| < \frac{\pi}{2} - \mu, \\ &f_\varepsilon\left(\frac{\mu}{2}\right) < -\frac{1}{2}\sin\frac{\mu}{2},\ f_\varepsilon\left(-\frac{\mu}{2}\right) > \frac{1}{2}\sin\frac{\mu}{2}, \end{aligned} \tag{4.22}$$

provided that $\varepsilon \in (0, \varepsilon_1)$. Then there exists $\nu > 0$ satisfying

$$f(u) < \nu(u - \pi) \quad \text{for } u \in \left(\frac{\mu}{2}, \pi\right) \tag{4.23}$$

$$f(u) > \nu(u - \pi) \quad \text{for } u \in \left(-\pi, -\frac{\mu}{2}\right). \tag{4.24}$$

In fact,

$$\nu = \frac{1}{2}\left(\pi - \frac{\mu}{2}\right)^{-1} \sin\frac{\mu}{2}$$

is the desired one by (4.22).

We now set $r_2 = \pi - 2\pi\exp(-\sigma_2(s - R_0))$, where $R_0 > 0$ is taken as in (4.20). We verify that there exists $\sigma_2 > 0$, which is independent of ε, satisfying

$$-r_2'' - c_\varepsilon r_2' + \nu(r_2 - \pi) \leq 0 \quad \text{in } (R_0, \infty). \tag{4.25}$$

By straightforward calculation we obtain

$$-r_2'' - c_\varepsilon r_2' + \nu(r_2 - \pi) = 2\pi(\sigma_2^2 - c_\varepsilon\sigma_2 - \nu)\exp(-\sigma_2(s - R_0)).$$

By (4.11) and (4.15) we have

$$\sigma_2^2 - c_\varepsilon\sigma_2 - \nu \leq \sigma_2^2 + \frac{2\pi|a|}{\sigma_0}\sigma_2 - \nu.$$

By solving $\sigma_2^2 + (2\pi|a|/\sigma_0)\sigma_2 - \nu < 0$, we observe the existence of a constant $\sigma_2 > 0$ satisfying (4.25).

By definition of r_2, we obtain

$$r_2(R_0) = -\pi \le q_\varepsilon(R_0),\ r_2(\infty) = \pi = q_\varepsilon(\infty).$$

Moreover we obtain from (4.23),

$$-q_\varepsilon'' - c_\varepsilon q_\varepsilon' + \nu(q_\varepsilon - \pi) \ge 0 \quad \text{in } (R_0, \infty),$$

which and (4.25) yield

$$q_\varepsilon \ge r_2 \quad \text{on } [R_0, \infty).$$

By (4.4) we also obtain

$$q_\varepsilon \ge -\pi \ge r_2 \quad \text{on } [0, R_0],$$

which implies (4.7).

We also obtain (4.8) by similar arguments with $\bar{r}_2(s) := -\pi + 2\pi \exp(\sigma_2(s + R_0))$.

5. We obtain (4.9).

We first give the estimate of q_ε' Since $q_\varepsilon' > 0$, it suffices to give an estimate of q_ε' from above. By properties of f_ε, there exists $R_1 > 0$ satisfying

$$f_\varepsilon'(q_\varepsilon(s)) \ge \frac{f_\varepsilon'(\pi)}{2} = \frac{f_\varepsilon'(-\pi)}{2} \quad \text{if } |s| \ge R_1.$$

For $s \ge 0$, we set $r_3(s) := L_1 \exp(-\sigma_3(s - R_1))$, where σ_3 is a positive constant satisfying

$$-r_3'' - c_\varepsilon r_3' + \frac{f_\varepsilon'(\pi)}{2} r_3 \ge 0 \quad \text{in } (R_1, \infty). \tag{4.26}$$

We now verify the existence of such σ_3. By straightforward calculation we obtain

$$-r_3'' - c_\varepsilon r_3' + \frac{f_\varepsilon'(\pi)}{2} r_3 \ge L_1 \left(-\sigma_3^2 - \frac{2\pi|a|}{\sigma_0}\sigma_3 + \frac{f_\varepsilon'(\pi)}{2} \right) \exp(-\sigma_3(s - R_1)).$$

By solving $-\sigma_3^2 - (2\pi|a|/\sigma_0)\sigma_3 + f_\varepsilon'(\pi)/2 > 0$, we observe the existence of a constant $\sigma_3 > 0$ satisfying (4.26), since $f_\varepsilon'(\pi) > 0$ and is independent of $\varepsilon \in (0, \varepsilon_1)$.

By the definition of r_3 we obtain

$$r_3(R_1) = L_1 \le q_\varepsilon'(0),\ r_3(\infty) = 0 = q_\varepsilon'(\infty)$$

and $p_\varepsilon = q_\varepsilon'$ satisfies

$$-p_\varepsilon'' - c_\varepsilon p_\varepsilon' + \frac{f_\varepsilon'(\pi)}{2} p_\varepsilon \le 0 \quad \text{in } (R_1, \infty),$$

we obtain $q_\varepsilon' \le r_3$ on $[R_1, \infty)$. Moreover we obtain

$$q_\varepsilon' \le L_1 \le r_3 \quad \text{on } [0, R_1],$$

which implies $q_\varepsilon' \le r_3$ on $[0, \infty)$.

It is easy to obtain the estimate of q''_ε since

$$|q''_\varepsilon(s)| \le |c_\varepsilon||q'_\varepsilon(s)| + |f_\varepsilon(q_\varepsilon(s))|$$

and $|f_\varepsilon(q_\varepsilon(s))| \le \lambda_1|q_\varepsilon(s)+\pi|$ for $s \le 0$ or $|f_\varepsilon(q_\varepsilon(s))| \le \lambda_1|q_\varepsilon(s)-\pi|$ for $s \ge 0$.

6. Finally, we verify the second property of (iii). By (4.11) it suffices to see that there exists $\lim_{\varepsilon\to 0}\int_{\mathbb{R}} q'_\varepsilon(\sigma)^2 d\sigma$. Let $q_0 \colon \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$\begin{aligned} -q''_0(s) - \sin q_0(s) &= 0 \quad \text{for } s \in \mathbb{R}, && (4.27)\\ q_0(\pm\infty) &= \pm\pi, && (4.28)\\ q_0(0) &= 0, && (4.29)\end{aligned}$$

i.e., the solution of (4.1) and boundary conditions (4.2)–(4.3) with $\varepsilon = 0$. By similar arguments as in §4 of Aronson and Weinberger(1978) and on above, such a function q_0 exists, is unique, and all of properties on above hold for q_0.

We verify that $|q_\varepsilon(s) - q_0(s)| + |q'_\varepsilon(s) - q'_0(s)| \to 0$ as $\varepsilon \to 0$. By integrating (4.1) and (4.27) on $[0, s]$ for $s > 0$ we obtain

$$q'_\varepsilon(s) - q'_0(s) = q'_\varepsilon(0) - q'_0(0) - c_\varepsilon \int_0^s q'_\varepsilon(\sigma)d\sigma + \int_0^s (f_\varepsilon(q_\varepsilon(\sigma)) - f_0(q_0(\sigma)))d\sigma,$$

where $f_0(u) = -\sin u$. Therefore we obtain

$$|q'_\varepsilon(s) - q'_0(s)| \le |q'_\varepsilon(0) - q'_0(0)| + |c_\varepsilon| \int_{\mathbb{R}} |q'_\varepsilon(\sigma)|d\sigma + \int_0^s |f_\varepsilon(q_\varepsilon(\sigma)) - f_0(q_0(\sigma))|d\sigma. \tag{4.30}$$

We give an estimate the first term of (4.30). By (4.12) and a similar calculation for (4.27) we obtain

$$\begin{aligned} q'_\varepsilon(0)^2 =& -2c_\varepsilon \int_{-\infty}^0 q'_\varepsilon(\sigma)^2 d\sigma + 2\int_{-\pi}^{\alpha_\varepsilon} f_\varepsilon(u)du \\ =& -2c_\varepsilon \int_{-\infty}^0 q'_\varepsilon(\sigma)^2 d\sigma + 4 - 2\varepsilon a(\alpha_\varepsilon + \pi), \\ q'_0(0)^2 =& 2\int_{-\pi}^0 f_0(u)du = 4. \end{aligned}$$

These yield that $\lim_{\varepsilon\to 0} q'_\varepsilon(0)$ exists and

$$\lim_{\varepsilon\to 0} q'_\varepsilon(0) = 2 = q'_0(0).$$

We next give an estimate the third term of (4.30). We obtain

$$\begin{aligned} |f_\varepsilon(q_\varepsilon(s)) - f_0(q_0(s))| =& |f_0(q_\varepsilon(s)) - f_0(q_0(s)) + \varepsilon f_1(q_\varepsilon(s))| \\ \le& |f_0(q_\varepsilon(s)) - f_0(q_0(s))| + \varepsilon|f_1(q_\varepsilon(s))|, \end{aligned}$$

where $f_1(u) = -a(1 + \cos u)$. Since $|f'_0(u)| = |-\cos u| \le 1$, we obtain

$$|f_0(q_\varepsilon(s)) - f_0(q_0(s))| \le |q_\varepsilon(s) - q_0(s)|.$$

Since $|f'_1(u)| = |a \sin u| \le |a|$ and $f'_1(\pi) = 0$, we obtain

$$|f_1(q_\varepsilon(s))| = |f_1(q_\varepsilon(s)) - f_1(\pi)| \le |a||q_\varepsilon(s) - \pi|.$$

By (4.7) we obtain

$$\int_0^s |f_1(q_\varepsilon(\sigma))|d\sigma \le |a| \int_0^s |q_\varepsilon(\sigma) - \pi| d\sigma \le |a| \int_0^\infty C_1 \exp(-C_2\sigma) d\sigma =: \Lambda_1.$$

We thus obtain

$$\int_0^s |f_\varepsilon(q_\varepsilon(\sigma)) - f_0(q_0(\sigma))| d\sigma \le \int_0^s |q_\varepsilon(\sigma) - q_0(\sigma)| d\sigma + \varepsilon\Lambda_1. \tag{4.31}$$

By combining (4.30) and (4.31) we obtain

$$|q_\varepsilon(s) - q_0(s)| + |q_\varepsilon'(s) - q_0'(s)| \le \Lambda_\varepsilon + \int_0^s (|q_\varepsilon(\sigma) - q_0(\sigma)| + |q_\varepsilon'(\sigma) - q_0'(\sigma)|) d\sigma,$$

where $\Lambda_\varepsilon = |q_\varepsilon'(0) - q_0'(0)| + |c_\varepsilon| \int_{\mathbb{R}} |q_\varepsilon'(\sigma)| d\sigma + \varepsilon\Lambda_1$. We remark that $\Lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. By Gronwall's inequality we obtain

$$|q_\varepsilon(s) - q_0(s)| + |q_\varepsilon'(s) - q_0'(s)| \le \Lambda_\varepsilon + \Lambda_\varepsilon \int_0^s \exp(s - \sigma) d\sigma \quad \text{for } s > 0,$$

which implies

$$\lim_{\varepsilon \to 0} (|q_\varepsilon(s) - q_0(s)| + |q_\varepsilon'(s) - q_0'(s)|) = 0 \quad \text{for } s > 0.$$

By similar argument we also obtain $q_\varepsilon(s) \to q_0(s)$ as $\varepsilon \to 0$ for $s < 0$.

Finally, we conclude (iii). Since $\lim_{\varepsilon\to 0} q_\varepsilon'(s) = q_0'(s)$ and (4.9) we obtain

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} q_\varepsilon'(\sigma)^2 d\sigma = \int_{\mathbb{R}} q_0'(\sigma)^2 d\sigma > 0.$$

Therefore we obtain

$$\lim_{\varepsilon \to 0} \frac{c_\varepsilon}{\varepsilon} = -\frac{2\pi a}{\int_{\mathbb{R}} q_0'(\sigma)^2 d\sigma} =: -A. \ \square$$

Remark 4.2. *Heuristically, one attempts to consider the traveling wave solution with multiple-heights, i.e., a solution of* (4.1) *with boundary conditions*

$$q_\varepsilon(-\infty) = -\pi \ , q_\varepsilon(\infty) = 3\pi, \tag{4.32}$$

to construct a supersolution for the estimate as in Theorem 2.4. Generally, however, there is no such a solution.

Let $a = 0$ and assume there exists a solution satisfying (4.1) *with the boundary condition* (4.32)*. We then have $c_\varepsilon = 0$ and so that $q = q_\varepsilon$ satisfies*

$$-q'' - \sin q = 0 \quad \textit{in } \mathbb{R}.$$

The boundary condition (4.32) *yields that there exists $s_0 \in \mathbb{R}$ satisfying $q(s_0) = \pi$. By integrating* (4.10) *in (s_0, s) for $s > s_0$ we obtain*

$$\frac{1}{2}(q'(s)^2 - q(s_0)^2) = \int_\pi^{q(s)} (-\sin u) du.$$

We remark that $q'(s_0)^2 > 0$ since, if not, then we have $q \equiv \pi$ in $\mathbb{R}$ from the uniqueness of a solution of (4.1)*, which contradicts* (4.32)*. This yields that*

$$q'(s)^2 \ge q'(s_0)^2 > 0$$

for $s > s_0$, since $\int_\pi^{q(s)} (-\sin u) du \ge 0$ for $s > s_0$. This is the contradiction to (4.32).

5. Supersolutions with multiple-height layer

In this section we construct a supersolution with multiple-height layers.

We introduce a signed distance function from an interface. Let $z \in \mathbb{R}$. Here and hereafter in this section, we assume that

$$\Gamma_t := \{x;\ u(x,t) = z\} \neq \emptyset \quad \text{provided that } t \in [0,T), \tag{5.1}$$

where u is a viscosity solution of (2.9), whose driving force term A is determined by Lemma 4.1 (iii), with initial data $u(x,0) = u_0$. We define the *signed distance function* from Γ_t with same sign as $u - z$ by

$$d(x,t) = \begin{cases} \inf\{|x-y|;\ u(y,t) = z\} & \text{if } (x,t) \text{ satisfies } u(x,t) \geq z, \\ -\inf\{|x-y|;\ u(y,t) = z\} & \text{otherwise.} \end{cases} \tag{5.2}$$

The following lemma lists some properties of the signed distance function as in Proposition 2.1 or Theorem 2.3 of Evans, Soner and Souganidis(1992).

Lemma 5.1. *Let u be a solution of* (2.9)*, whose driving force term A is defined by Lemma 4.1 (iii). For $z \in \mathbb{R}$, assume that* (5.1) *holds. Let $d(x,t)$ be a signed distance function from Γ_t defined by* (5.2)*. Then we obtain;*

(i) d is left continuous with respect to t, i.e.,

$$\lim_{x \to x_0, t \nearrow t_0} d(x,t) = d(x_0,t_0).$$

(ii) d is lower semicontinuous in $\{(x,t);\ d(x,t) > 0\}$ and satisfies

$$\left.\begin{array}{r} d_t - \Delta d - A|\nabla d| \geq 0, \\ |\nabla d| \geq 1,\ -|\nabla d| \geq -1 \end{array}\right\} \text{ in } \{(x,t);\ d(x,t) > 0\}$$

in the viscosity supersolution sense.

(iii) d is upper semicontinuous in $\{(x,t);\ d(x,t) < 0\}$ and satisfies

$$\left.\begin{array}{r} d_t - \Delta d - A|\nabla d| \leq 0, \\ |\nabla d| \leq 1,\ -|\nabla d| \leq -1 \end{array}\right\} \text{ in } \{(x,t);\ d(x,t) < 0\}$$

in the viscosity subsolution sense.

Proof. Apply the proof of Proposition 2.1 and Theorem 2.2 in Evans, Soner and Souganidis(1992) to prove (i), the first inequalities in (ii) and (iii). Here we shall prove only an estimate of $|\nabla d|$.

Let $(\hat{x},\hat{t}) \in \mathbb{R}^N \times (0,T)$ and $\phi \in C^2(\mathbb{R}^N \times (0,T))$. Assume that

$$d(x,t) - \phi(x,t) \geq d(\hat{x},\hat{t}) - \phi(\hat{x},\hat{t}) = 0 \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,T).$$

We now mention only the case $d(\hat{x},\hat{t}) > 0$ because the proofs are symmetric. Let $\hat{y} \in \{x;\ d(x,\hat{t}) = 0\}$ satisfy $d(\hat{x},\hat{t}) = |\hat{x} - \hat{y}| =: \hat{r}$. We demonstrate for $(\hat{x},\hat{t})$ satisfying $d(\hat{x},\hat{t}) > 0$,

$$\nabla\phi(\hat{x},\hat{t}) = \frac{\hat{x} - \hat{y}}{|\hat{x} - \hat{y}|}. \tag{5.3}$$

Let $\hat{p} = (\hat{x} - \hat{y})/|\hat{x} - \hat{y}|$ and $\Phi(s) = \phi(\hat{y} + s\hat{p}, \hat{t})$. We obtain for $h \in \mathbb{R}$,

$$\Phi(\hat{r} + h) - \Phi(\hat{r}) \leq d(\hat{y} + (\hat{r} + h)\hat{p}, \hat{t}) - d(\hat{y} + \hat{r}\hat{p}, \hat{t}) \leq \hat{r} + h - \hat{r} = h.$$

Therefore we obtain

$$\Phi'(\hat{r}) = \langle \nabla\phi(\hat{x}, \hat{t}), \hat{p} \rangle = 1, \tag{5.4}$$

in particular $|\nabla\varphi(\hat{x}, \hat{t})| \neq 0$. Moreover we obtain

$$\phi(x, \hat{t}) - \phi(\hat{x}, \hat{t}) \leq d(x, \hat{t}) - d(\hat{x}, \hat{t}) \leq |x - \hat{x}|.$$

By taking $x - \hat{x} = h\nabla\phi(\hat{x}, \hat{t})/|\nabla\phi(\hat{x}, \hat{t})|$ and sending $h \to 0$ we obtain

$$|\nabla\varphi(\hat{x}, \hat{t})| \leq 1.$$

This and (5.4) imply (5.3). □

We next recall the truncating function η as in Evans, Soner and Souganidis(1992). Let $\eta \colon \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

$$\eta(\sigma) = \begin{cases} \sigma - \delta & \sigma \geq \delta/2, \\ -\delta & \sigma \leq \delta/4, \end{cases} \tag{5.5}$$

$$0 \leq \eta'(\sigma) \leq C_\eta, \ |\eta''(\sigma)| \leq \frac{C_\eta}{\delta} \quad \text{for } \sigma \in \mathbb{R} \tag{5.6}$$

for some C_η. We remark that this function is convenience to construct a supersolution for estimate of u^ε from above. For the estimate from below, we have two ways. The first one is to apply the way to obtain an estimate of u^ε from above to the equations which $\bar{u}^\varepsilon = -u^\varepsilon$ or $\bar{u} = -u$ satisfy, i.e., (3.1) or

$$\bar{u}_t - |\nabla\bar{u}| \left\{ \operatorname{div}\frac{\nabla\bar{u}}{|\nabla\bar{u}|} - A \right\} = 0 \quad \text{in } \mathbb{R}^N \times (0, T) \tag{5.7}$$

with $\bar{u}^\varepsilon(x, 0) = \bar{u}(x, 0) = -u_0$. The second one is to construct a subsolution directly by using an another truncating function $\bar{\eta}$ satisfying

$$\bar{\eta}(\sigma) = \begin{cases} \sigma + \delta & \sigma \leq -\delta/2, \\ \delta & \sigma \geq -\delta/4, \end{cases}$$

$$0 \leq \bar{\eta}'(\sigma) \leq C_\eta, \ |\bar{\eta}''(\sigma)| \leq \frac{C_\eta}{\delta} \quad \text{for } \sigma \in \mathbb{R}.$$

Since the equations are isotropic, we obtain the estimate from below symmetrically. In this paper we shall construct only a supersolution. We next list the properties that the truncated distance function $\eta(d)$ satisfies.

Lemma 5.2. *Assume that* (5.1) *holds. Let* $w = \eta(d)$. *Then there exists a positive constant* C *satisfying*

$$\left.\begin{aligned} w_t - \mathit{\Delta} w - A|\nabla w| &\geq -\frac{C}{\delta}, \\ -|\nabla w| &\geq -C \end{aligned}\right\} \quad \text{in } \mathbb{R}^N \times (0, T),$$

in the viscosity supersolution sense. Moreover we obtain

$$\left.\begin{aligned} w_t - \mathit{\Delta} w - A|\nabla w| \geq 0, \\ |\nabla w| \geq 1, -|\nabla w| \geq -1 \end{aligned}\right\} \quad \text{in } \left\{ (x, t);\ d(x, t) > \frac{\delta}{2} \right\}$$

in the viscosity supersolution sense.

Proof. See the proof of Lemma 3.1 of Evans, Soner and Souganidis(1992). For more precise calculations to prove the estimate of $|\nabla w|$, see Lemma 3.4 of Giga, Ohtsuka and Schätzle(to appear), with $\gamma(p) = |p|$ and $\Lambda_\gamma = 1$, which are the notations of them. □

Here we construct a supersolution by the signed distance function and the traveling wave. Let $\delta > 0$ and $w(x,t) = \eta(d(x,t))$. For $b > 0$ and $\varepsilon > 0$, we define

$$\psi^{\varepsilon,b}(x,t) = q_\varepsilon\left(\frac{w(x,t) + \gamma_1 t + k_1 b}{\varepsilon}\right) + \varepsilon(\gamma_2 + k_2 b), \tag{5.8}$$

where q_ε is a solution of the ordinary differential equation (4.1)–(4.3) and γ_1, $\gamma_2 > 0$ are constants.

Proposition 5.3. *Assume that* (5.1) *holds. Assume that* $k_1 \in \mathbb{Z} \cap [0, K_1]$ $k_2 \in \mathbb{Z} \cap [0, K_2]$*, for some* K_1*,* $K_2 \in \mathbb{N}$*, respectively. For* $\delta > 0$*, there exist* $b_0 = b_0(\delta, K_1, K_2)$ $\gamma_1 = \gamma_1(\delta)$ *and* $\gamma_2 = \gamma_2(\delta)$ *such that, for* $b \in (0, b_0)$*, there exists* $\hat{\varepsilon} = \hat{\varepsilon}(\delta, b)$ *such that* $\psi = \psi^{\varepsilon,b}$ *satisfies*

$$\psi_t - \Delta\psi + \frac{1}{\varepsilon^2} f_\varepsilon(\psi) \geq \frac{\tilde{K}}{\varepsilon} + O(1) \quad \text{in } \mathbb{R}^N \times (0,T) \tag{5.9}$$

as $\varepsilon \to 0$ *in the viscosity supersolution sense provided that* $\varepsilon \in (0, \hat{\varepsilon})$*, where* $\tilde{K}$ *is a numerical constant.*

It is necessary to clarify the dependence of the parameters ε, b, k_1 or k_2 for the estimate of ψ. Therefore we give a detailed proof.

Proof. Let $(\hat{x}, \hat{t}) \in \mathbb{R}^N \times (0,T)$ and $\varphi \in C^2(\mathbb{R}^N \times (0,T))$ satisfy

$$\psi(x,t) - \varphi(x,t) \geq \psi(\hat{x}, \hat{t}) - \varphi(\hat{x}, \hat{t}) = 0 \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,T).$$

We take $\hat{\varepsilon}$ satisfying $\hat{\varepsilon} < \varepsilon_1$, which is as in Lemma 4.1, so that all of the estimates in Lemma 4.1 hold. In the following argument we shall replace $\hat{\varepsilon}$ to smaller one later, at (5.10), (5.14), and (5.15).

Since $q_\varepsilon' > 0$ for $\varepsilon > 0$, there exists $q_\varepsilon^{-1} \in C^\infty(\mathbb{R})$. Here we define

$$\tilde{\varphi}(x,t) = \varepsilon q_\varepsilon^{-1}(\varphi(x,t) - \varepsilon(\gamma_2 + k_2 b)) - \gamma_1 t - k_1 b.$$

Then we observe that $\tilde{\varphi} \in C^2(\mathbb{R}^N \times (0,T))$ and satisfies

$$\begin{cases} w(x,t) - \tilde{\varphi}(x,t) \geq w(\hat{x}, \hat{t}) - \tilde{\varphi}(\hat{x}, \hat{t}) = 0 \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,T), \\ \varphi(x,t) = q_\varepsilon\left(\dfrac{\tilde{\varphi}(x,t) + \gamma_1 t + k_1 b}{\varepsilon}\right) + \varepsilon(\gamma_2 + k_2 b). \end{cases}$$

By a straightforward calculation we obtain

$$\begin{aligned} \varphi_t &= \frac{q_\varepsilon'(h)}{\varepsilon}(\tilde{\varphi}_t + \gamma_1), \\ \nabla\varphi &= \frac{q_\varepsilon'(h)}{\varepsilon}\nabla\tilde{\varphi}, \\ \nabla^2\varphi &= \frac{q_\varepsilon''(h)}{\varepsilon^2}\nabla\tilde{\varphi} \otimes \nabla\tilde{\varphi} + \frac{q_\varepsilon'(h)}{\varepsilon}\nabla^2\tilde{\varphi}, \end{aligned}$$

where $h = h(x,t) = \varepsilon^{-1}(\tilde{\varphi}(x,t) + \gamma_1 t + k_1 b)$. Moreover we obtain

$$f_\varepsilon(\psi(\hat{x},\hat{t})) = f_\varepsilon(\varphi(\hat{x},\hat{t})) \geq f_\varepsilon(q_\varepsilon(\hat{h})) + \varepsilon(\gamma_2 + k_2 b) f'_\varepsilon(q'_\varepsilon(\hat{h})) - \varepsilon^2(\gamma_2 + k_2 b)^2 \lambda_2,$$

where $\hat{h} = h(\hat{x},\hat{t})$, and λ_2 is a constant defined by (2.4). Therefore we obtain

$$\varphi_t - \mathit{\Delta}\varphi + \frac{1}{\varepsilon^2} f_\varepsilon(\psi) \geq \varepsilon^{-2} I_0 + \varepsilon^{-1} I_1 - (\gamma_2 + k_2 b)^2 \lambda_2 \quad \text{at } (\hat{x},\hat{t}),$$

where

$$\begin{aligned} I_0 &= q'_\varepsilon(\hat{h})(|\nabla\tilde{\varphi}|^2 - 1), \\ I_1 &= q'_\varepsilon(\hat{h})(\gamma_1 + \tilde{\varphi}_t - \mathit{\Delta}\tilde{\varphi} - A|\nabla\tilde{\varphi}| + A|\nabla\tilde{\varphi}| + \varepsilon^{-1} c_\varepsilon) + (\gamma_2 + k_2 b) f'_\varepsilon(q_\varepsilon(\hat{h})). \end{aligned}$$

We divide a situation into 2 cases.

Case 1. Assume that $(\hat{x},\hat{t}) \in \{(x,t);\ d(x,t) > \delta/2\}$. By Lemma 5.2 we obtain

$$\left.\begin{array}{c} \tilde{\varphi}_t - \mathit{\Delta}\tilde{\varphi} - A|\nabla\tilde{\varphi}| \geq 0, \\ |\nabla\tilde{\varphi}| = 1 \end{array}\right\} \quad \text{at } (\hat{x},\hat{t}),$$

which implies

$$\begin{aligned} I_0 =& 0, \\ I_1 \geq& q'_\varepsilon(\hat{h})(\gamma_1 + A + \varepsilon^{-1} c_\varepsilon) + (\gamma_2 + k_2 b) f'_\varepsilon(q_\varepsilon(\hat{h})). \end{aligned}$$

We set

$$\gamma_1 = \frac{\delta}{4T}.$$

The reason why we set such a γ_1 will be clarify in Case 2, below. We replace $\hat{\varepsilon}$ to smaller one so that we have

$$|A + \varepsilon^{-1} c_\varepsilon| \leq \frac{\gamma_1}{2} = \frac{\delta}{8T} \quad \text{provided that } \varepsilon \in (0,\hat{\varepsilon}). \tag{5.10}$$

This yields

$$I_1 \geq \frac{q'_\varepsilon(\hat{h})\gamma_1}{2} + (\gamma_2 + k_2 b) f'_\varepsilon(q_\varepsilon(\hat{h})).$$

By a straightforward calculation we have $f'((2j+1)\pi) = 1 > 0$ for $j \in \mathbb{Z}$, in particular, the value is independent of ε. Therefore there exist $r_0 \in (0, 2\pi)$, which are independent of ε, satisfying

$$f'_\varepsilon((2j+1)\pi + r) \geq \frac{1}{2} f'_\varepsilon((2j+1)\pi) =: \nu_0 > 0 \quad \text{for } j \in \mathbb{Z} \text{ and } r \in (-r_0, r_0).$$

We remark that ν_0 is independent of $j \in \mathbb{Z}$. To apply Lemma 4.1 there exist $R = R(r_0, \hat{\varepsilon}) > 0$ and $\nu_1 = \nu_1(\hat{\varepsilon}, R) > 0$ satisfying

$$\left.\begin{array}{ll} q_\varepsilon(s) < -\pi + r_0 & \text{for } s < -R, \\ q_\varepsilon(s) > \pi - r_0 & \text{for } s > R, \\ q'_\varepsilon(s) \geq \nu_1 > 0 & \text{for } |s| \leq R \end{array}\right\} \text{provided that } \varepsilon \in (0,\hat{\varepsilon}).$$

Take b_0 small so that

$$k_2 b < \frac{\nu_1 \gamma_1}{8\lambda_1} \quad \text{provided that } k_2 \in [0, K_2] \text{ and } b \in (0, b_0), \tag{5.11}$$

where λ_1 is the constant defined by (2.4). We shall replace b_0 to smaller one later, at (5.13). We set

$$\gamma_2 = \frac{\nu_1\gamma_1}{8\lambda_1} = \frac{\nu_1\delta}{32\lambda_1 T}.$$

There exists $\bar{K} > 0$ satisfying

$$\varepsilon^{-2}I_0 + \varepsilon^{-1}I_1 \geq \frac{\bar{K}}{\varepsilon}. \tag{5.12}$$

We verify it. If $(\hat{x}, \hat{t})$ satisfies $\hat{h} > R$, then we obtain from $f_\varepsilon'(q_\varepsilon(\hat{h})) > \nu_0$ and $q_\varepsilon' > 0$,

$$I_1 \geq \gamma_2\nu_0.$$

If $(\hat{x}, \hat{t})$ satisfies $\hat{h} \leq R$, then we obtain from $k_2 b < \nu_1\gamma_1/(8\lambda_1) = \gamma_2$,

$$I_1 \geq \frac{\nu_1\gamma_1}{2} - 2\gamma_2\lambda_1 = \frac{\nu_1\gamma_1}{2} - \frac{2\nu_1\lambda_1\gamma_1}{8\lambda_1} = \frac{\nu_1\gamma_1}{4}.$$

Therefore we obtain (5.12) by setting $\bar{K} = \min(\gamma_2\nu_0, \nu_1\gamma_1/4)$, which implies

$$\varphi_t - \Delta\varphi + \frac{1}{\varepsilon^2}f_\varepsilon(\psi) \geq \frac{\bar{K}}{\varepsilon} - 4\gamma_2^2\lambda_2^2 \quad \text{at } (\hat{x}, \hat{t}).$$

Case 2. Assume that $(\hat{x}, \hat{t}) \in \{(x,t);\ d(x,t) \leq \delta/2\}$. Set γ_1 and γ_2 as above, i.e.,

$$\gamma_1 = \frac{\delta}{4T},\ \gamma_2 = \frac{\nu_1\gamma_1}{8\lambda_1}.$$

We replace b_0 to smaller one so that we have

$$k_1 b \leq \frac{\delta}{8T} \quad \text{provided that } k_1 \in [0, K_1] \text{ and } b \in (0, b_0). \tag{5.13}$$

This implies that

$$\hat{h} \leq \frac{1}{\varepsilon}\left(-\frac{\delta}{2} + \frac{\delta t}{4T} + k_1 b\right) \leq -\frac{\delta}{8\varepsilon} < 0.$$

We replace $\hat{\varepsilon} > 0$ to smaller one so that we have

$$\frac{\delta}{8\varepsilon} < -R \quad \text{provided that } \varepsilon \in (0, \hat{\varepsilon}), \tag{5.14}$$

which implies that $\hat{h} < -R$ provided that $\varepsilon \in (0, \hat{\varepsilon})$. By Lemma 5.2 we have

$$\left.\begin{aligned} \tilde{\varphi}_t - \Delta\tilde{\varphi} - A|\nabla\tilde{\varphi}| \geq -\frac{C}{\delta}, \\ |\nabla\tilde{\varphi}| \leq C \end{aligned}\right\} \text{ at } (\hat{x}, \hat{t}).$$

This implies for $\varepsilon < \hat{\varepsilon}$,

$$\begin{aligned} I_0 \geq& -|q_\varepsilon''(\hat{h})|(C^2+1), \\ I_1 \geq& q_\varepsilon'(\hat{h})\left(\frac{\gamma_1}{2} - \frac{C}{\delta} - A(C+1)\right) + (\gamma_2 + k_2 b)\nu_0 \\ \geq& \gamma_2\nu_0 - q_\varepsilon'(\hat{h})\left(\frac{C}{\delta} + A(C+1)\right). \end{aligned}$$

Therefore we obtain by Lemma 4.1

$$\begin{aligned}\varepsilon^{-2}I_0+\varepsilon^{-1}I_1 \geq & \left[-\frac{(C^2+1)}{\varepsilon^2}-\frac{1}{\varepsilon}\left(\frac{C}{\delta}+A(C+1)\right)\right]C_1\exp\left(-\frac{\delta C_2}{8\varepsilon}\right)+\frac{\gamma_2\nu_0}{\varepsilon}\\ =&\frac{1}{\varepsilon}\left[-\frac{\hat{C}}{\varepsilon}\exp\left(-\frac{\delta C_2}{8\varepsilon}\right)+\gamma_2\nu_0\right],\end{aligned}$$

where $\hat{C}$ is a positive numerical constant. Here we replace $\hat{\varepsilon}>0$ to smaller one to satisfy

$$-\frac{\hat{C}}{\varepsilon}\exp\left(-\frac{C_2\delta}{8\varepsilon}\right)\geq -\frac{\gamma_2\nu_0}{2} \quad \text{for } \varepsilon\in(0,\hat{\varepsilon}). \tag{5.15}$$

Therefore we obtain

$$\tilde{\varphi}_t-\Delta\tilde{\varphi}+\frac{1}{\varepsilon^2}f_\varepsilon(\psi)\geq\frac{\gamma_2\nu_0}{2\varepsilon}-4\gamma_2^2\lambda_2^2 \quad \text{at } (\hat{x},\hat{t}).$$

Let $\tilde{K}=\min(\bar{K},\gamma_2\nu_0/2)$. Then we obtain (5.9). $\square$

We shall construct a supersolution with a multiple-height layer around the interface. Let b_0 and $\hat{\varepsilon}$ be a constant determined from Proposition 5.3 with $K_1=K_2=2K_0+1$. For $\varepsilon\in(0,\hat{\varepsilon})$, $b\in(0,b_0)$ and $j=0,1,\ldots,2K_0$, we define

$$\psi_j^{\varepsilon,b}(x,t):=q_\varepsilon\left(\frac{w(x,t)+\gamma_1 t+(j+1)b}{\varepsilon}\right)+2\pi(K_0-j)+\varepsilon(\gamma_2+(j+1)\varepsilon). \tag{5.16}$$

We remark that $\psi_j^{\varepsilon,b}$ is a viscosity supersolution of (2.1) since $q(s):=q_\varepsilon(s)+2\pi(K_0-j)$ is still a solution of (4.1) with boundary condition $q(\pm\infty)=(2(K_0-j)\pm1)\pi$.

We construct a supersolution, which has twice height of a layer of $\psi_j^{\varepsilon,b}$, from $\psi_j^{\varepsilon,b}$ and $\psi_{j+1}^{\varepsilon,b}$. Let $\hat{\varepsilon}>0$ satisfy

$$q_\varepsilon\left(\frac{b}{4\varepsilon}\right)\geq\pi-\frac{b\varepsilon}{4},\ q_\varepsilon\left(-\frac{b}{4\varepsilon}\right)\leq-\pi+\frac{b\varepsilon}{4} \quad \text{for } \varepsilon\in(0,\hat{\varepsilon}) \tag{5.17}$$

in addition to the condition as in Proposition 5.3. We define

$$\tilde{\psi}_j^{\varepsilon,b}(x,t):=\begin{cases}\psi_j^{\varepsilon,b}(x,t) & \text{for } (x,t)\in U_j\\ \min\{\psi_j^{\varepsilon,b}(x,t),\psi_{j+1}^{\varepsilon,b}(x,t)\} & \text{for } (x,t)\in(\mathbb{R}^n\times[0,T))\setminus U_j,\end{cases} \tag{5.18}$$

where

$$U_j:=\{(x,t)\in\mathbb{R}^N\times[0,T);\ w(x,t)+\gamma_1 t>-(j+3/2)b\}.$$

We observe that $\tilde{\psi}_j^{\varepsilon,b}$ is a viscosity supersolution of (2.1). In fact, it is easy to see that $\tilde{\psi}_j^{\varepsilon,b}$ is a viscosity supersolution of (2.1) in $\mathbb{R}^N\times(0,T)\setminus\{(x,t);\ w(x,t)+\gamma_1 t=-(j+3/2)b\}$. If $(x,t)\in J_j:=\{(x,t);\ w(x,t)+\gamma_1 t\in(-(j+7/4)b,-(j+$

$5/4)b)\}$, we obtain

$$\begin{aligned}
\psi_j^{\varepsilon,b}(x,t) \leq & q_\varepsilon\left(-\frac{b}{4\varepsilon}\right)+2\pi(K_0-j)+\varepsilon(\gamma_2+(j+1)b)\\
\leq & (2(K_0-j)-1)\pi+\varepsilon\left(\gamma_2+\left(j+\frac{5}{4}\right)b\right),\\
\psi_{j+1}^{\varepsilon,b}(x,t) \geq & q_\varepsilon\left(\frac{b}{4\varepsilon}\right)+2\pi(K_0-(j+1))+\varepsilon(\gamma_2+(j+2)b)\\
\geq & (2(K_0-j)-1)\pi+\varepsilon\left(\gamma_2+\left(j+\frac{7}{4}\right)b\right),
\end{aligned}$$

which implies $\psi_j^{\varepsilon,b} \leq \psi_{j+1}^{\varepsilon,b}$ in J_j. Therefore we obtain $\tilde{\psi}_j^{\varepsilon,b} \equiv \psi_j^{\varepsilon,b}$ in J_j so that $\tilde{\psi}_j$ is a viscosity supersolution of (2.1) in a domain including $\{(x,t);\ w(x,t)+\gamma_1 t = -(j+3/2)b\}$. Here we summarize the more properties which $\tilde{\psi}_j$ satisfies:

Corollary 5.4. *Assume that* (5.1) *holds. Let* $\tilde{\psi}_j^{\varepsilon,b}$ *be a function defined by* (5.18). *Then the followings hold.*

(i) We have that $\tilde{\psi}_j^{\varepsilon,b}$ *is a viscosity supersolution of* (2.1).

(ii) We have

$$\tilde{\psi}_j^{\varepsilon,b}(x,t) = \psi_j^{\varepsilon,b}(x,t) \geq (2(K_0-j)+1)\pi+\varepsilon(\gamma_2+(j+3/4)b)$$

for $(x,t)\in\mathcal{D}^j$, *where*

$$\mathcal{D}^j := \{(x,t)\in\mathbb{R}^N\times[0,T);\ w(x,t)+\gamma_1 t \geq -(j+3/4)b\}.$$

(iii) We have

$$\tilde{\psi}_j^{\varepsilon,b}(x,t) = \psi_{j+1}^{\varepsilon,b}(x,t) \leq (2(K_0-j-1)-1)\pi+\varepsilon(\gamma_2+(j+7/4)b)$$

for $(x,t)\in\mathcal{O}^{j+1}$, *where*

$$\mathcal{O}^j := \{(x,t)\in\mathbb{R}^N\times[0,T);\ w(x,t)+\gamma_1 t \leq -(j+5/4)b\}.$$

Finally, we construct a supersolution which has multiple-height layers. For $k=0,\,1,\,\ldots,\,2K_0$, We define

$$v_k(x,t) := \begin{cases} \tilde{\psi}_0(x,t) & \text{for } (x,t)\in U_1,\\ \tilde{\psi}_j(x,t) & \text{for } (x,t)\in U_{j+1}\setminus U_j,\ j=1,\ 2,\ \ldots,\ k-2,\\ \tilde{\psi}_{k-1}(x,t) & \text{for } (x,t)\in(\mathbb{R}^N\times[0,T))\setminus U_{k-1}. \end{cases} \tag{5.19}$$

It is easy to see that v_k is a viscosity supersolution of (2.1). We list the properties of v_k.

Corollary 5.5. *Assume that* (5.1) *holds. Let* v_k *be a function defined on above. Then the followings hold.*

(i) We have v_k *is a viscosity supersolution of* (2.1).

(ii) We have

$$v_k(x,t) \geq (2K_0+1)\pi + \varepsilon(\gamma_2 + 3b/4) \quad \textit{for } (x,t) \in \mathcal{D}^0.$$

(iii) We have

$$v_k(x,t) \leq (2(K_0-k)-1)\pi + \varepsilon(\gamma_2 + (k+5/4)b) \quad \textit{for } (x,t) \in \mathcal{O}^k.$$

6. Approximation of the motion of interfaces

In this section we shall prove Theorem 2.4. We first prove the following.

Theorem 6.1. *Let u^ε be a solution of (2.1) with $u^\varepsilon(x,0) = u_0(x) \in BUC(\mathbb{R}^N)$. Let u be a solution of (2.9) with $u^\varepsilon(x,0) = u_0(x)$. Assume that there exists $m_0 > 0$ such that*

$$\{x;\ u(x,t) = 2\pi k - \mu\} \neq \emptyset$$

for $t \in [0,T)$ provided that $|\mu| \leq m_0$. Then the followings hold.

(i) For $m \in [0, m_0/2]$ and $k \in [-K_0, K_0] \cap \mathbb{Z}$, we have

$$\overline{\lim_{\varepsilon\to 0}}\, u^\varepsilon(x,t) \leq (2k-1)\pi$$

for $(x,t) \in \{(y,s) \in \mathbb{R}^N \times (0,T);\ u(x,t) \leq 2\pi k - m\}$.

(ii) For $m \in [0, m_0/2]$ and $k \in [-K_0, K_0] \cap \mathbb{Z}$, we have

$$\underline{\lim_{\varepsilon\to 0}}\, u^\varepsilon(x,t) \geq (2k+1)\pi$$

for $(x,t) \in \{(y,s) \in \mathbb{R}^N \times (0,T);\ u(x,t) \geq 2\pi k + m\}$.

Proof. We now prove only (i) since the proofs are symmetric.

Let $m \in (0, m_0/2)$ and set

$$D_t^m = \{x \in \mathbb{R}^N;\ u(x,t) \leq 2\pi k - m\}.$$

For $b \in (0, b_0)$, there exists $\bar\varepsilon > 0$ and $\tau_0 > 0$ satisfying

$$u^\varepsilon(x, \tau_0\varepsilon^2|\log\varepsilon|) \leq (2k-1)((2k-1)\pi + b\varepsilon)\chi_{D_0^m} + (2K_0+1)\pi\chi_{\mathbb{R}^N\setminus D_0^m}$$

by Theorem 3.1. Set

$$\Gamma_t^m = \{x \in \mathbb{R}^N;\ u(x,t) = 2\pi k - m\},$$

and define

$$d_m(x,t) = \left\{ \begin{array}{ll} -\inf\{|x-y|;\ y \in \Gamma_t^m\} & \text{if } (x,t) \text{ satisfies } x \in D_t^m, \\ \inf\{|x-y|;\ y \in \Gamma_t^m\} & \text{if } (x,t) \text{ satisfies } x \in \mathbb{R}^N \setminus D_t^m. \end{array} \right.$$

Let $\delta > 0$ be a constant satisfying

$$\sup_{|x-y|<\delta} |u_0(x) - u_0(y)| < \frac{m}{2}.$$

Take $b_0 > 0$ enough small so that b_0 satisfies not only the condition as in Proposition 5.3 but also satisfies

$$2(K_0+1)b_0 \leq \frac{3\delta}{4}. \tag{6.1}$$

Let $\eta = \eta_\delta$ be a function satisfying (5.5)–(5.6). Set $w(x,t) = \eta(d_{2m}(x,t))$. For $b \in (0, b_0)$, we set $\tilde{\varepsilon} = \min(\bar{\varepsilon}, \hat{\varepsilon})$, where $\bar{\varepsilon}$ is a constant as in Theorem 3.1 and $\hat{\varepsilon}$ is a constant satisfying all of the conditions as in Proposition 5.3 and (5.17). For $b \in (0, b_0)$ and $\varepsilon \in (0, \tilde{\varepsilon})$, we define $\psi_j^{\varepsilon,b}$ and $\tilde{\psi}_j^{\varepsilon,b}$ by (5.16) and (5.18), respectively. We consider a supersolution v_{K_0-k} of (2.1), which is defined by (5.19).

We first demonstrate

$$v_{K_0-k}(x,0) \geq ((2k-1)\pi + b\varepsilon)\chi_{D_0^m}(x) + (2K_0+1)\pi\chi_{\mathbb{R}^N \setminus D_0^m}(x), \tag{6.2}$$

where $\chi_U : \mathbb{R}^N \to \{0,1\}$ is the characteristic function of $U \subset \mathbb{R}^N$. Since

$$v_{K_0-k}(x,0) \geq (2k-1)\pi + \varepsilon(\gamma_2 + (j+1)b) > (2k-1)\pi + b\varepsilon,$$

it suffices to see that

$$v_{K_0-k}(x,0) \geq (2K_0+1)\pi \quad \text{for } x \in \mathbb{R}^N \setminus D_0^m.$$

To see this property, we first verify that

$$\{y \in \mathbb{R}^N;\ d_{2m}(y,0) < 2\delta\} \subset \{y \in \mathbb{R}^N;\ u_0(y) < 2\pi k - m\}. \tag{6.3}$$

Let $x \in \{y \in \mathbb{R}^N;\ d_{2m}(y,0) < 2\delta\}$. For $\mu > 0$, there exists $y \in D_0^{2m}$ such that

$$\begin{aligned} &d_{2m}(x,0) + \mu \geq |x-y|, \\ &u_0(y) \leq 2\pi k - 2m. \end{aligned}$$

Since $d_{2m}(x,0) < 2\delta$, we may assume that $y \in D_0^{2m}$ satisfies

$$|x-y| < 2\delta$$

by taking enough small $\mu > 0$. Let $z := (x+y)/2$. Then we obtain $|x-z| = |x-y|/2 < \delta$ and $|y-z| = |y-x|/2 < \delta$, so that we obtain

$$|u_0(x) - u_0(y)| \leq |u_0(x) - u_0(z)| + |u_0(z) - u_0(y)| < \frac{m}{2} + \frac{m}{2} = m.$$

Since $y \in D_0^{2m}$, we obtain

$$u_0(x) < 2\pi k - 2m + m = 2\pi k - m,$$

which yields (6.3).

The property (6.3) yields

$$\{y \in \mathbb{R}^N;\ d_{2m}(y,0) \geq 2\delta\} \supset \{y \in \mathbb{R}^N;\ u_0(y) \geq 2\pi k - m\},$$

which implies for $x \in \{y \in \mathbb{R}^N;\ u_0(y) \geq 2\pi k - m\}$,

$$w(x,0) + b > \delta \geq \frac{8}{3}(K_0+1)b_0 > \frac{b}{4}$$

by (6.1). Therefore we obtain $v_{K_0-k}(x,0)=\psi_0^{\varepsilon,b}(x,0)$ and

$$v_{K_0-k}(x,0)\geq q_\varepsilon\left(\frac{b}{4\varepsilon}\right)+2\pi K_0+\varepsilon(\gamma_2+b)>(2K_0+1)\pi$$

for $x\in\{y\in\mathbb{R}^N;\ u_0(y)\geq -m\}$ by definition of $\tilde{\varepsilon}$. We thus obtain (6.2).

We give an estimate of u^ε by using v_{K_0-k}. By (2.2) we obtain

$$u^\varepsilon(x,t+\tau_0\varepsilon^2|\log\varepsilon|)\leq v_{K_0-k}(x,t)\quad\text{for } (x,t)\in\mathbb{R}^N\times[0,T).$$

By Corollary 5.5 we obtain

$$u^\varepsilon(x,t+\tau_0\varepsilon^2|\log\varepsilon|)\leq(2k-1)\pi+\varepsilon(\gamma_2+(K_0-k+5/4)b)\quad\text{for } (x,t)\in\mathcal{O}^k$$

provided that $\varepsilon\in(0,\tilde{\varepsilon})$, where $\mathcal{O}^k$ is defined in Corollary 5.5(iii).

We now lead the conclusion. Let $(x_0,t_0)\in\{(y,s)\in\mathbb{R}^N\times(0,T);\ u(y,s)\leq 2\pi k-m\}$. There exists $b_0=b_0(m)$ such that, for $b\in(0,b_0)$, there exists $\tilde{\varepsilon}=\tilde{\varepsilon}(b)>0$, which are smaller than those we take on above, such that $t_0>\tau_0\varepsilon^2|\log\varepsilon|$ and $(x_0,t_0-\tau_0\varepsilon^2|\log\varepsilon|)\in\mathcal{O}^k$ provided that $b\in(0,b_0)$ and $\varepsilon\in(0,\tilde{\varepsilon})$. This implies

$$u^\varepsilon(x_0,t_0)\leq(2k-1)\pi+\varepsilon(\gamma_2+(K_0-k+5/4)b).$$

Since the choice of $(x_0,t_0)\in\{(y,s)\in\mathbb{R}^N\times(0,T);\ u(y,s)\leq 2\pi k-m\}$ is arbitrary, we obtain

$$\overline{\lim_{\varepsilon\to 0}}\,u^\varepsilon(x,t)\leq(2k-1)\pi\quad\text{for } (x,t)\in\{(y,s);\ u(y,s)\leq 2\pi k-m\}.\ \square$$

Here we remark that the choice of $\varepsilon>0$ for the estimate of $\overline{\lim}_{\varepsilon\to 0}\,u^\varepsilon(x,t)$ is independent of (x,t) if there exists $\tilde{\tau}>0$ satisfying $t\geq\tilde{\tau}$.

Proof of Theorem 2.4. Let Ω be a compact subset satisfying

$$\Omega\subset\{(y,s)\in\mathbb{R}^N\times(0,T);u(x,t)<2\pi k\}.$$

Then there exists $\nu>0$ and $m>0$ satisfying

$$\begin{aligned}&t\geq\nu>0,\\&u(x,t)\leq 2\pi k-m\quad\text{on }\Omega.\end{aligned}$$

Therefore we obtain the conclusion. $\square$

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