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The Gel'fand-Levitan theory for one-dimensional hyperbolic systems with impulsive inputs

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The Gel'fand-Levitan theory for one-dimensional hyperbolic systems with impulsive inputs

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Abstract

We consider a wave equation with damping coefficient

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + p_1(x)\frac{\partial u}{\partial t}(x,t) + p_2(x)\frac{\partial u}{\partial x}(x,t),\ 0<x<1, -T<t<T,$$

$$u(x,0)=0, \quad \frac{\partial u}{\partial t}(x,0)=\delta(x), \quad 0 \le x \le 1,$$

$$\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(1,t)=0, \quad -T \le t \le T$$

where $T \ge 2$, the complex-valued functions $p_1, p_2 \in C^1[0,1]$ and $\delta(x)$ is the Dirac delta function. We discuss an inverse problem of determining simultaneously the coefficients $p_1(x)$ and $p_2(x)$, $0 \le x \le 1$ from observation data $u(0,t), -T \le t \le T$. We prove a reconstruction formula for $p_1(x)$ and $p_2(x)$ from $u(0,t)$ by establishing an intrinsic relation with the inverse spectral theory.

1 Introduction

In the present paper, we consider the following initial-boundary value problem:

$$\begin{cases} L_p u(x,t) = 0, \ \ 0<x<1, -T<t<T, \\ u(x,0)=0, \quad \dfrac{\partial u}{\partial t}(x,0)=\delta(x), \quad 0 \le x \le 1, \\ \dfrac{\partial u}{\partial x}(0,t)=\dfrac{\partial u}{\partial x}(1,t)=0, \quad -T \le t \le T \end{cases} \tag{1.1}$$

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where the operator L_p is defined by

$$L_p = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - p_1(x)\frac{\partial}{\partial t} - p_2(x)\frac{\partial}{\partial x},$$

$T \geq 2$, $p_1, p_2 \in C^1[0,1]$ are complex-valued and $\delta(x)$ is the Dirac delta function.

Problem (1.1) can describe, for example, small vibrations of a string with damping term and a telegraph equation. Our main object here is to determine the operator L_p, i.e., to simultaneously determine the coefficients $p_1(x)$, $p_2(x)$ on $[0,1]$, from boundary data at one end $u(0,t)$, $-T \leq t \leq T$. This problem was solved by A. S. Blagoveshchenskii (see [1, 2]) when $p_1 \equiv 0$, but for general p_1 it is difficult and has been open for a long time.

Replacing the boundary condition in (1.1) by $u(0,t) = u(1,t) = 0$, we can similarly discuss an inverse problem of determining p_1 and p_2 by $\frac{\partial u}{\partial x}(0,t)$, $-T \leq t \leq T$, but we here do not give details. For an interval I, let $H^{-1}(I)$ be the dual space of the Soblev space $H_0^1(I)$, $(H^1(I))'$ be the dual space of the Soblev space $H^1(I)$, and $\mathcal{D}'(I)$ denotes the distribution space on open set I. Moreover $\left(H^1(0,1)\right)'$ denotes the dual space of $H^1(0,1)$, which is not a space of distributions in $(0,1)$. Henceforth $\overline{\Omega}$ denotes the closure of a set. Since $\delta \in \left(H^1(0,1)\right)'$, by means of the Soblev embedding $C[0,1] \subset H^1(0,1)$, we can apply the general theory for abstract evolution equations of the second order (e.g., Lions and Magenes [6]). More precisely we can follow the proof of Theorem 9.3 (p.288) in [6] to the equation with damping term $p_1\frac{\partial}{\partial t}$ with $p_1 \in C^1[0,1]$, and prove that there exists a unique solution

$$u \in C\left([-T,T]; L^2(0,1)\right) \cap C^1\left([-T,T]; \left(H^1(0,1)\right)'\right).$$

Moreover we can prove more detailed regularity.

Proposition 1.

$$\frac{\partial u}{\partial t} \in C([-T,T]; (H^1(0,1))') \cap C\big([0,1]; H^{-1}(-T,T)\big).$$

We are ready to state our main result, which gives a reconstructive formula and requires us to solve a Fredholm equation of the second kind.

Theorem 1. *Let $p_1, p_2 \in C^1[0,1]$, $T \geq 2$ and let us set*

$$v(t) = u(0,t), \quad -T \leq t \leq T,$$

where u is the solution to (1.1). We denote the derivatives of $v(x+y), v(x-y), v(-x+y)$ and $v(-x-y)$ with respect to x in the sense of $\mathcal{D}'((0,1)^2)$ by $\frac{\partial}{\partial x}$, and define

$$V_{11}(x,y) = \frac{1}{4}\left(\frac{\partial v(x+y)}{\partial x} + \frac{\partial v(x-y)}{\partial x} - \frac{\partial v(-x+y)}{\partial x} - \frac{\partial v(-x-y)}{\partial x}\right) - \delta(x-y), \tag{1.2}$$

$$V_{12}(x,y) = \frac{1}{4}\left(\frac{\partial v(x+y)}{\partial x} - \frac{\partial v(x-y)}{\partial x} - \frac{\partial v(-x+y)}{\partial x} + \frac{\partial v(-x-y)}{\partial x}\right), \tag{1.3}$$

$$V_{21}(x,y) = \frac{1}{4}\left(-\frac{\partial v(x+y)}{\partial x} - \frac{\partial v(x-y)}{\partial x} - \frac{\partial v(-x+y)}{\partial x} - \frac{\partial v(-x-y)}{\partial x}\right), \tag{1.4}$$

$$V_{22}(x,y) = \frac{1}{4}\left(-\frac{\partial v(x+y)}{\partial x} + \frac{\partial v(x-y)}{\partial x} - \frac{\partial v(-x+y)}{\partial x} + \frac{\partial v(-x-y)}{\partial x}\right) - \delta(x-y). \tag{1.5}$$

Then $V(x,y) := (V_{ij}(x,y))_{1\le i,j\le 2}$ *are in* $\left(C^1(\overline{\Omega})\right)^4$ *and* $\left(C^1\left(\overline{(0,1)^2\backslash\Omega}\right)\right)^4$, *where* $\Omega = \{(x,y) \in (0,1)^2 : 0 < y < x < 1\}$. *Moreover, there exists a unique solution* $M = (M_{ij})_{1\le i,j\le 2} \in \left(C^1\left(\overline{\Omega}\right)\right)^4$ *to*

$$V(x,y) + M(x,y) + \int_0^x M(x,\tau)V(\tau,y)\mathrm{d}\tau = 0,\ (x,y) \in \overline{\Omega}. \tag{1.6}$$

The solution M *to equation (1.6) satisfies that for* $0 \le x \le 1$

$$2(M_{12} - M_{21})(x,x) = -p_1(x)\cosh\left(\int_0^x p_1(s)\mathrm{d}s\right) + p_2(x)\sinh\left(\int_0^x p_1(s)\mathrm{d}s\right), \tag{1.7}$$

$$2(M_{11} - M_{22})(x,x) = -p_1(x)\sinh\left(\int_0^x p_1(s)\mathrm{d}s\right) + p_2(x)\cosh\left(\int_0^x p_1(s)\mathrm{d}s\right). \tag{1.8}$$

It is easy to conclude from Theorem 1 the following

Corollary. *If* $\widetilde{u}(0,t) \equiv u(0,t), -T \le t \le T$ *where* $\widetilde{u}$ *is the solution to (1.1) with the coefficients* $\widetilde{p}_1, \widetilde{p}_2$*, then* $\widetilde{p}_1 \equiv p_1, \widetilde{p}_2 \equiv p_2$ *on* $[0,1]$.

Theorem 1 gives a reconstruction scheme for $p_1(x), p_2(x)$ from $u(0,t)$:
(i) Set $v(t) = u(0,t),\ -T \le t \le T$.
(ii) Find $V_{ij}, 1 \le i,j \le 2$ by (1.2)–(1.5).
(iii) Solve (1.6) with respect to $M_{ij}, 1 \le i,j \le 2$.
(iv) Solve (1.7) and (1.8) with respect to p_1 and p_2.

Remark. The nonlinear equations (1.7) and (1.8) can be converted to an initial value problem of a system of ordinary differential equations, and so p_1, p_2 are uniquely solvable.

In Romanov [11] and Ramm [10], the inverse problem of determining the potential $q(x)$ from boundary observation for the one-dimensional wave equation $u_{tt} = u_{xx} - q(x)u$ with impulse input has been considered. In [11], Chapter 2, Section 4, a reconstruction formula is established, which is an analogue to the Gel'fand-Levitan theory for a Sturm-Liouville problem (e.g., Gel'fand and Levitan [3], Levitan [4] and Marchenko [7]). In [11], this problem is reduced to an integral equation of Volterra type by the characteristic method. On the other hand, in our case this kind of treatment seems impossible, so that the analogue of the Gel'fand-Levitan theory to a nonstationary case has been unsolved for the telegraph equation. However, we can represent the solution of the direct problem in terms of the root vectors of the nonsymmetric differential operator corresponding to problem (1.1). For similar nonselfadjoint operators related with damped hyperbolic equations, we refer readers to Shubov [13, 14] and Shubov, Martin, Dauer and Belinskiy [15].

In order to solve the inverse problem we establish an intrinsic relation with the inverse spectral theory (see Ning and Yamamoto [9]). The relevant inverse spectral theory has been studied further in Ning [8]. Recently, an identification problem which is close to the inverse problem considered in the present paper has been studied in Trooshin and Yamamoto [17]. Here we treat the inverse problem only in one dimensional case. One can formulate similar inverse problems for general dimensions. For example, for the multidimensional case, we refer to Romanov and Yamamoto [12]. For the case when the boundary conditions are nonhomogeneous, we refer to Ye and Si [18]. However, the inverse problems with inner observation data in $(0,1)$ (e.g. $u(\frac{1}{2},t)$) or with general initial boundary conditions are still open.

The rest of the paper is devoted to the proof of Theorem 1. The proof relies on the Gel'fand-Levitan theory for the corresponding nonsymmetric first-order system in $x \in (0,1)$, which is a reconstruction scheme of coefficients by the spectral characteristics. The relevant Gel'fand-Levitan theory is established in Ning and Yamamoto [9], and thanks to the Dirac delta input, we can extract the spectral characteristics from the data $u(0,t)$, $-T \le t \le T$ in terms of the eigenfunction expansion of $u(x,t)$. In Section 2 we present known results on spectral properties and the inverse problem related with the hyperbolic equation in (1.1), and in Section 3 we establish the representation of the solution to (1.1) by means of the eigenfunctions and prove Proposition 1. In Section 4 we complete the proof of Theorem 1.

2 Inverse spectral problem

Setting

$$\Phi = \begin{pmatrix} \Phi^{(1)}(x,t) \\ \Phi^{(2)}(x,t) \end{pmatrix} := \begin{pmatrix} \dfrac{\partial u}{\partial t}(x,t) \\ \dfrac{\partial u}{\partial x}(x,t) \end{pmatrix}, \tag{2.1}$$

we can formally rewrite (1.1) as follows:

$$\begin{cases} \dfrac{\partial \Phi}{\partial t}(x,t) = B\dfrac{\partial \Phi}{\partial x}(x,t) + P(x)\Phi(x,t), \quad 0<x<1, -T<t<T, \\ \Phi(x,0) = \begin{pmatrix} \delta(x) \\ 0 \end{pmatrix}, \quad 0 \le x \le 1, \\ \Phi^{(2)}(0,t) = \Phi^{(2)}(1,t) = 0, \quad -T \le t \le T \end{cases} \tag{2.2}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ P(x) = \begin{pmatrix} p_1(x) & p_2(x) \\ 0 & 0 \end{pmatrix}.$$

We mainly consider (2.2). Throughout this paper, let $(L^2(0,1))^2$, $(H^1(0,1))^2$ denote the product spaces of the complex-valued Lebesgue space $L^2(0,1)$ and the complex-valued Sobolev space $H^1(0,1)$ respectively. By $(\cdot,\cdot)$ we denote the scalar product in $(L^2(0,1))^2$:

$$(f,g) = \int_0^1 f^T(x)\overline{g(x)}\mathrm{d}x = \int_0^1 \left(f^{(1)}(x)\overline{g^{(1)}(x)} + f^{(2)}(x)\overline{g^{(2)}(x)}\right)\mathrm{d}x$$

for $f = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix} \in (L^2(0,1))^2$, $g = \begin{pmatrix} g^{(1)} \\ g^{(2)} \end{pmatrix} \in (L^2(0,1))^2$. Here and henceforth $\overline{c}$ denotes the complex conjugate of $c \in \mathbb{C}$ and $\cdot^T$ denotes the transpose of a vector or matrix under consideration. Now we set

$$\mathcal{A}_P\varphi(x) = \left(B\frac{\partial}{\partial x} + P(x)\right)\varphi(x),$$

and define a differential operator A_P in $(L^2(0,1))^2$ by $A_P\varphi = \mathcal{A}_P\varphi$ for

$$\varphi \in D(A_P) \equiv \left\{ \varphi = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} \in \left(H^1(0,1)\right)^2 : \varphi^{(2)}(0) = \varphi^{(2)}(1) = 0 \right\}.$$

We can directly verify that the adjoint operator A_P^* of A_P in $(L^2(0,1))^2$ is given by

$$\begin{cases} (A_P^*\varphi^*)(x) = -B\dfrac{\mathrm{d}\varphi^*}{\mathrm{d}x}(x) + \overline{P^T(x)}\varphi^*(x), \quad \varphi^* \in D(A_P^*),\ 0 < x < 1, \\ D(A_P^*) = \left\{ \varphi^* = \begin{pmatrix} \varphi^{*(1)} \\ \varphi^{*(2)} \end{pmatrix} \in \left(H^1(0,1)\right)^2 : \varphi^{*(2)}(0) = \varphi^{*(2)}(1) = 0 \right\} \end{cases}$$

and that $\mathcal{A}_P^* = -\mathcal{A}_{-\overline{P^T}}$.

For the spectrum $\sigma(A_P)$ of the operator A_P, we have

Proposition 2.1.
(i) There exist $N_1 \in \mathbb{N}$ and $\Sigma_1, \Sigma_2 \subset \sigma(A_P)$ such that $\sigma(A_P) = \Sigma_1 \bigcup \Sigma_2$, $\Sigma_1 \bigcap \Sigma_2 = \emptyset$ and the following properties hold:

(1) Σ_1 *consists of* $2N_1 - 1$ *eigenvalues including algebraic multiplicities and*

$$\Sigma_1 \subset \left\{ \lambda \in \mathbb{C} : \left| \mathrm{Im}\left(\lambda - \frac{1}{2}\int_0^1 p_1(s)\mathrm{d}s \right) \right| \le (N_1 - \frac{1}{2})\pi \right\}.$$

(2) Σ_2 *consists of eigenvalues with algebraic multiplicity 1 and, with a suitable numbering* $\{\lambda_n\}_{n\in\mathbb{Z}}$ *of* $\sigma(A_P)$*, the eigenvalues have an asymptotic behaviour*

$$\lambda_n = \frac{1}{2}\int_0^1 p_1(s)\mathrm{d}s + n\pi\sqrt{-1} + O\left(\frac{1}{|n|}\right) \tag{2.3}$$

as $|n| \to \infty$.
(ii) The set of all the root vectors of A_P *is a Riesz basis in* $(L^2(0,1))^2$.

Here by the root vector $\phi \neq 0$ for an eigenvalue λ of A_P, we mean that $(A_P - \lambda)^m\phi = 0$ with some $m \in \mathbb{N}$. We call $\dim\{\phi : (A_P - \lambda)^m\phi = 0 \text{ with some } m \in \mathbb{N}\}$ and $\dim\{\phi : (A_P - \lambda)\phi = 0\}$ the algebraic multiplicity and the geometric multiplicity of λ respectively. For the proof of Proposition 2.1, see Theorem 1.1 in Trooshin and Yamamoto [16].

We say that an eigenvalue λ is *simple* if the algebraic multiplicity of λ is 1. Henceforth, for the convenience of notations, we put the spectrum $\sigma(A_P) = \Sigma_1 \bigcup \Sigma_2$ by a suitable renumbering as follows:

$$\begin{aligned} \Sigma_1 &= \left\{ \lambda^i \in \sigma(A_P) : m_i \ge 2, 1 \le i \le N \right\}, \\ \Sigma_2 &= \left\{ \lambda_n \in \sigma(A_P) : \lambda_n \text{ is simple}, n \in \mathbb{Z} \right\}, \end{aligned}$$

where m_i denotes the algebraic multiplicity of λ^i.

Remark. The spectrum may only consists of simple eigenvalues and then Σ_1 does not appear.

Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Here and henceforth let $\varphi_n = \varphi_n(x)$ be the eigenvector of A_P for λ_n such that $\varphi_n(0) = e_1$ and $\varphi_n^* = \varphi_n^*(x)$ be the eigenvector of A_P^* for $\overline{\lambda_n}$ such that $\varphi_n^*(0) = e_1$, $n \in \mathbb{Z}$.

Proposition 2.2 ([9]). *We can uniquely construct constants* $\alpha_j^i,\ 1 \le j \le m_i - 1, 1 \le i \le N$ *and root vectors* $\left\{\varphi_j^i\right\}_{1\le j\le m_i}$ *of* A_P *for* λ^i *and* $\left\{\varphi_j^{i*}\right\}_{1\le j\le m_i}$ *of* A_P^* *for* $\overline{\lambda^i}$ *(*$1 \le i \le N$*) satisfying*
(i)

$$\begin{cases} (\mathcal{A}_P - \lambda^i)\varphi_1^i = 0,\ (\mathcal{A}_P - \lambda^i)\varphi_j^i = \varphi_{j-1}^i, \quad 2 \le j \le m_i, \\ \varphi_j^i(0) = e_1,\ \varphi_j^i \in D(A_P), \quad 1 \le j \le m_i \end{cases}$$

and

$$\begin{cases} (\mathcal{A}_P^* - \overline{\lambda^i})\varphi_{m_i}^{i*} = 0, (\mathcal{A}_P^* - \overline{\lambda^i})\varphi_j^{i*} = \varphi_{j+1}^{i*}, \quad 1 \le j \le m_i - 1, \\ \varphi_{m_i}^{i*}(0) = e_1,\ \varphi_j^{i*}(0) = \overline{\alpha_j^i} e_1, \quad 1 \le j \le m_i - 1, \\ \varphi_j^{i*} \in D(A_P^*),\ 1 \le j \le m_i. \end{cases}$$

(ii)

$$\left(\varphi_j^i, \varphi_n^*\right) = 0, \quad \left(\varphi_n, \varphi_j^{i*}\right) = 0, \quad \textit{for } 1 \le j \le m_i,\ 1 \le i \le N,\ n \in \mathbb{Z}.$$

(iii)

$$\left(\varphi_j^i, \varphi_l^{k*}\right) = 0 \ \text{ if } i \ne k \text{ or } j \ne l,\ 1 \le j \le m_i,\ 1 \le l \le m_k, \quad 1 \le i, k \le N,$$

and

$$\left(\varphi_j^i, \varphi_j^{i*}\right) = \left(\varphi_{m_i}^i, \varphi_{m_i}^{i*}\right), \quad \textit{for } 1 \le j \le m_i,\ 1 \le i \le N. \tag{2.4}$$

As for the construct of α_j^i, see Appendix in [9].

We set $\rho^i = \left(\varphi_{m_i}^i, \varphi_{m_i}^{i*}\right),\ 1 \le i \le N$, $\rho_n = (\varphi_n, \varphi_n^*)$, $n \in \mathbb{Z}$ and $\boldsymbol{\alpha}^i = (\alpha_1^i, \cdots, \alpha_{m_i-1}^i),\ \ 1 \le i \le N$. By (2.4), we have

$$\left(\varphi_j^i, \varphi_j^{i*}\right) = \rho^i, \quad \text{for } 1 \le j \le m_i.$$

It can be proved that (see [9])

$$\rho^i \ne 0,\ 1 \le i \le N; \quad \rho_n \ne 0,\ n \in \mathbb{Z}. \tag{2.5}$$

Then we define the *spectral characteristics* of A_P by

$$S(P) := \left\{\lambda^i, m_i, \rho^i, \boldsymbol{\alpha}^i\right\}_{1\le i\le N} \bigcup \left\{\lambda_n, \rho_n\right\}_{n\in\mathbb{Z}}.$$

In terms of the spectral characteristics, we can prove the Parseval equality.

Proposition 2.3 ([9]). *Let* $f, g \in (L^2(0,1))^2$.
(i) (*the Parseval equality with respect to* A_P)

$$(f, g) = \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\left(f, \varphi_j^{i*}\right)\left(\varphi_j^i, g\right)}{\rho^i} + \sum_{n\in\mathbb{Z}} \frac{\left(f, \varphi_n^*\right)\left(\varphi_n, g\right)}{\rho_n}.$$

(ii) (*expansion*)

$$f = \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\left(f, \varphi_j^{i*}\right)}{\rho^i}\varphi_j^i + \sum_{n\in\mathbb{Z}} \frac{\left(f, \varphi_n^*\right)}{\rho_n}\varphi_n,$$

$$g=\sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{(g,\varphi_j^i)}{\overline{\rho^i}}\varphi_j^{i*}+\sum_{n\in\mathbb{Z}}\frac{(g,\varphi_n)}{\overline{\rho_n}}\varphi_n^*,$$

where both series are convergent in $(L^2(0,1))^2$.

For $\lambda\in\mathbb{C}$, set

$$S(x,\lambda)=\begin{pmatrix}\cosh(\lambda x)\\ \sinh(\lambda x)\end{pmatrix}, \tag{2.6}$$

$$S^*(x,\overline{\lambda})=\begin{pmatrix}\cosh(\overline{\lambda}x)\\ -\sinh(\overline{\lambda}x)\end{pmatrix}. \tag{2.7}$$

Then

$$\begin{cases}(\mathcal{A}_0-\lambda)\,S=0,\\ S(0,\lambda)=e_1,\end{cases} \tag{2.8}$$

$$\begin{cases}\left(\mathcal{A}_0^*-\overline{\lambda}\right)S^*=0,\\ S^*(0,\overline{\lambda})=e_1\end{cases} \tag{2.9}$$

and $\left(S(\cdot,\lambda),S^*(\cdot,\overline{\lambda})\right)=1$. For $n\in\mathbb{Z}$, let $\mu_n\in\sigma(A_0)$ and let us denote $S_n(x)=S(x,\mu_n)$, $S_n^*(x)=S(x,\overline{\mu_n})$. Here a short calculation shows that μ_n is simple and equal to $n\pi\sqrt{-1}$.

Let $S_{(j)}(x,\lambda)$ and $S^*_{(j)}(x,\overline{\lambda})$, $1\le j\le m_i$ satisfy the following initial value problems respectively:

$$\begin{cases}(\mathcal{A}_0-\lambda)\,S_{(1)}=0,\ (\mathcal{A}_0-\lambda)\,S_{(j)}=S_{(j-1)},\ 2\le j\le m_i,\\ S_{(j)}(0,\lambda)=e_1,\ 1\le j\le m_i,\end{cases} \tag{2.10}$$

$$\begin{cases}\left(\mathcal{A}_0^*-\overline{\lambda}\right)S^*_{(m_i)}=0,\ \left(\mathcal{A}_0^*-\overline{\lambda}\right)S^*_{(j)}=S^*_{(j+1)},\ 1\le j\le m_i-1,\\ S^*_{(m_i)}(0,\overline{\lambda})=e_1,\ S^*_{(j)}(0,\overline{\lambda})=\overline{\alpha_j^i}e_1,\ 1\le j\le m_i-1.\end{cases} \tag{2.11}$$

Then, we can directly find the solutions of (2.10) and (2.11) possess the following forms:

$$S_{(j)}(x,\lambda)=\begin{pmatrix}\displaystyle\sum_{k=0}^{j-1}\frac{x^k}{k!}\theta_k(x,\lambda)\\ \displaystyle\sum_{k=0}^{j-1}\frac{x^k}{k!}\theta_{k+1}(x,\lambda)\end{pmatrix}, \tag{2.12}$$

$$S^*_{(j)}(x,\overline{\lambda})=\begin{pmatrix}\displaystyle\sum_{k=j}^{m_i}\overline{\alpha_k^i}\frac{x^{k-j}}{(k-j)!}\theta_{k-j}(x,\overline{\lambda})\\ \displaystyle-\sum_{k=j}^{m_i}\overline{\alpha_k^i}\frac{x^{k-j}}{(k-j)!}\theta_{k+1-j}(x,\overline{\lambda})\end{pmatrix}, \tag{2.13}$$

where $\alpha^i_{m_i} = 1$,

$$\theta_k(x,\lambda) = \begin{cases} \cosh(\lambda x), \ k \text{ even} \\ \sinh(\lambda x), \ k \text{ odd} \end{cases}. \tag{2.14}$$

Put

$$C^*(x,\overline{\lambda}) = \int_0^x S^*(t,\overline{\lambda})\mathrm{d}t, \quad C^*_{(j)}(x,\overline{\lambda}) = \int_0^x S^*_{(j)}(t,\overline{\lambda})\mathrm{d}t, \tag{2.15}$$

$$C(y,\lambda) = \int_0^y S(t,\lambda)\mathrm{d}t, \quad C_{(j)}(y,\lambda) = \int_0^y S_{(j)}(t,\lambda)\mathrm{d}t, \tag{2.16}$$

and

$$\begin{aligned} f(x,y) &= \sum_{i=1}^N \sum_{j=1}^{m_i} \frac{\overline{C^*_{(j)}(x,\overline{\lambda^i})} C^T_{(j)}(y,\lambda^i)}{\rho^i} \\ &+ \sum_{n\in\mathbb{Z}} \left\{ \frac{\overline{C^*(x,\overline{\lambda_n})} C^T(y,\lambda_n)}{\rho_n} - \overline{C^*(x,\overline{\mu_n})} C^T(y,\mu_n) \right\}. \end{aligned} \tag{2.17}$$

We further put

$$F(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y). \tag{2.18}$$

Proposition 2.4 ([9]).
(i) The series in (2.17) is convergent absolutely and uniformly in $[0,1]^2$.
(ii) $f \in \left(C[0,1]^2\right)^4$ *and* $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y} \in \left(C^1(\overline{\Omega})\right)^4, \in \left(C^1\left(\overline{(0,1)^2 \backslash \Omega}\right)\right)^4$.

In [9] the following two theorems have been proved.

Theorem 2.1 (Uniqueness). *The spectral characteristics* $S(P)$ *uniquely determines* P.

Theorem 2.2 (Reconstruction).
Let $P = \begin{pmatrix} p_1 & p_2 \\ 0 & 0 \end{pmatrix} \in \left(C^1[0,1]\right)^4$, $S(P) = \left\{\lambda^i, m_i, \rho^i, \boldsymbol{\alpha}^i\right\}_{1\le i\le N} \bigcup \{\lambda_n, \rho_n\}_{n\in\mathbb{Z}}$ *be the spectral characteristics of* A_P *and let* $F(x,y)$ *be given by* (2.17) *and* (2.18). *Then there exists* $M \in \left(C^1\left(\overline{\Omega}\right)\right)^4$ *such that*

$$F(x,y) + M(x,y) + \int_0^x M(x,\tau)F(\tau,y)\mathrm{d}\tau = 0, \ (x,y) \in \overline{\Omega}. \tag{2.19}$$

Moreover, for $0 \le x \le 1$ *we have*

$$2(M_{12} - M_{21})(x,x) = -p_1(x)\cosh\left(\int_0^x p_1(s)\mathrm{d}s\right) + p_2(x)\sinh\left(\int_0^x p_1(s)\mathrm{d}s\right),$$

$$2(M_{11}-M_{22})(x,x) = -p_1(x)\sinh\left(\int_0^x p_1(s)\mathrm{d}s\right)+p_2(x)\cosh\left(\int_0^x p_1(s)\mathrm{d}s\right).$$

Equation (2.19) corresponds to the Gel'fand-Levitan equation for the Sturm-Liouville problem (e. g., [3], [4], [7]).

3 Direct Problem

In this section, we give the representation of $u(x,t)$ by means of the eigenfunctions constructed in Proposition 2.2. First we show

Lemma 3.1.

$$\left\|\sum_{n\in\mathbb{Z}}\gamma_n\exp(\lambda_n t)\varphi_n^{(1)}\right\|_{C([-T,T];(H^1(0,1))')} \le C_1\sup_{n\in\mathbb{Z}}|\gamma_n|.$$

Here and henceforth $\varphi_n^{(1)}$ means the first component of φ_n and C_j denotes generic constants which are independent of $n\in\mathbb{Z}$ and special choices of γ_n, $\zeta\in H^1(0,1)$, $\omega\in H_0^1(-T,T)$.

Proof. Set $C_2 = \sup_{n\in\mathbb{Z}}|\gamma_n|$. For some fixed suitably large $N_2\in\mathbb{N}$, if $|n|\ge N_2$, then by (2.3) there exists a constant $C_1>0$, such that $|\lambda_n|\ge C_1|n|$ and $|\lambda_n - p_k(x)|\ge C_1|n|$, $k=1,2$ for any $x\in[0,1]$ since $p_k\in C^1[0,1]$.

Then when $|n|\ge N_2$, since $\varphi_n^{(2)}(0)=\varphi_n^{(2)}(1)=0$ and

$$\frac{\mathrm{d}\varphi_n^{(2)}(x)}{\mathrm{d}x} + p_1(x)\varphi_n^{(1)}(x) + p_2(x)\varphi_n^{(2)}(x) = \lambda_n\varphi_n^{(1)}(x),$$

by integration by parts we have for any $\zeta\in H^1(0,1)$

$$\begin{aligned}&\left|\int_0^1 \varphi_n^{(1)}(x)\overline{\zeta(x)}dx\right|\\ \le\ &\left|\int_0^1 \frac{p_2(x)\varphi_n^{(2)}(x)\overline{\zeta(x)}}{\lambda_n - p_1(x)}dx\right| + \left|\int_0^1 \frac{\varphi_n^{(2)}(x)}{(\lambda_n-p_1(x))^2}\left((\lambda_n - p_1(x))\frac{\mathrm{d}\overline{\zeta(x)}}{\mathrm{d}x} + \overline{\zeta(x)}\frac{\mathrm{d}p_1(x)}{\mathrm{d}x}\right)\mathrm{d}x\right|.\end{aligned}$$

Moreover, noting that

$$\frac{1}{\lambda_n - p_1(x)} = \frac{1}{\lambda_n} + \frac{p_1(x)}{\lambda_n(\lambda_n - p_1(x))}$$

and by means of the transformation formula ([9]) we can prove that $\sup_{n\in\mathbb{Z}}\|\varphi_n^{(2)}\|_{C[0,1]}<\infty$, we see by (2.3) that

$$\begin{aligned}&\left|\int_0^1 \varphi_n^{(1)}(x)\overline{\zeta(x)}dx\right|\\ \le\ &\frac{C_3}{|n|}\left\{\left|\int_0^1 \varphi_n^{(2)}(x)p_2(x)\overline{\zeta(x)}dx\right| + \left|\int_0^1 \varphi_n^{(2)}(x)\frac{d\overline{\zeta(x)}}{dx}dx\right|\right\} + \frac{C_3}{n^2}\|\zeta\|_{H^1(0,1)}.\end{aligned}$$

Therefore by (2.3) and the Schwarz inequality, we have

$$\begin{aligned}
&\left|\int_0^1\left(\sum_{|n|\ge N_2}\gamma_n\exp(\lambda_n t)\varphi_n^{(1)}(x)\right)\overline{\zeta(x)}dx\right| \\
\le\ & \sum_{|n|\ge N_2}|\gamma_n||\exp(\lambda_n t)|\left|\int_0^1\varphi_n^{(1)}(x)\overline{\zeta(x)}\mathrm{d}x\right| \\
\le\ & C_2C_3\sum_{|n|\ge N_2}\left\{\frac{1}{|n|}\left\{\left|\int_0^1\varphi_n^{(2)}(x)p_2(x)\overline{\zeta(x)}dx\right|+\left|\int_0^1\varphi_n^{(2)}(x)\frac{d\overline{\zeta(x)}}{dx}dx\right|\right\}+\frac{1}{n^2}\|\zeta\|_{H^1(0,1)}\right\} \\
\le\ & C_2C_3\left(\sum_{|n|\ge N_2}\frac{1}{n^2}\right)^{\frac12}\left\{\sum_{|n|\ge N_2}\left(\left|\int_0^1\varphi_n^{(2)}(x)p_2(x)\overline{\zeta(x)}dx\right|^2+\left|\int_0^1\varphi_n^{(2)}(x)\frac{d\overline{\zeta(x)}}{dx}dx\right|^2\right)\right\}^{\frac12} \\
&+C_2C_3\|\zeta\|_{H^1(0,1)}, \qquad -T\le t\le T.
\end{aligned}$$

On the other hand, by Proposition 2.1 (ii) we see that for any $\varrho\in L^2(0,1)$

$$\sum_{|n|\ge N_2}\left|(\varphi_n^{(2)},\varrho)\right|^2\le C_4\|\varrho\|^2_{L^2(0,1)}, \tag{3.1}$$

where the constant $C_4>0$ is independent of ϱ.

Therefore

$$\begin{aligned}
&\sup_{\|\zeta\|_{H^1(0,1)}=1}\left|\int_0^1\left(\sum_{|n|\ge N_2}\gamma_n\exp(\lambda_n t)\varphi_n^{(1)}(x)\right)\overline{\zeta(x)}dx\right| \\
\le\ & C_2C_3C_4\left(\|p_2\zeta\|_{L^2(0,1)}+\left\|\frac{d\zeta}{dx}\right\|_{L^2(0,1)}\right)+C_2C_3\|\zeta\|_{H^1(0,1)}\le C_5C_2.
\end{aligned}$$

Thus the proof of the lemma is complete. □

Proposition 1 follows directly from

Lemma 3.2. *Let φ_i^{j*}, ρ^i, λ^i, φ_k^i, α_j^i, φ_n^*, ρ_n, λ_n, φ_n be defined in Proposition 2.2. Then for the solution u to (1.1), we have*

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t)=&\sum_{i=1}^N\sum_{j=1}^{m_i}\frac{\alpha_j^i}{\rho^i}\left(\exp(\lambda^i t)\sum_{k=1}^j\frac{t^{j-k}}{(j-k)!}\varphi_k^{i(1)}(x)\right) \\
&+\sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\exp(\lambda_n t)\varphi_n^{(1)}(x)
\end{aligned}\tag{3.2}$$

in $C([-T,T];(H^1(0,1))')\cap C([0,1];H^{-1}(-T,T))$.

Proof. Let us consider (2.2) for $\Phi(\cdot,0)=a\in\mathcal{D}(A_P^\ell)$ with sufficiently large $\ell\in\mathbb{N}$. Then, by the

Fourier method or the separation of variables, we have

$$\Phi(x,t) = \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{(a,\varphi_j^{i*})}{\rho^i}\left(\exp(\lambda^i t)\sum_{k=1}^{j}\frac{t^{j-k}}{(j-k)!}\varphi_k^i(x)\right)$$

$$+\sum_{n\in\mathbb{Z}} \frac{(a,\varphi_n^*)}{\rho_n}\exp(\lambda_n t)\varphi_n(x)$$

in $(C([-T,T];L^2(0,1)))^2$ in terms of Proposition 2.2 and (2.3). In particular,

$$\frac{\partial u}{\partial t}(x,t) = \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{(a,\varphi_j^{i*})}{\rho^i}\left(\exp(\lambda^i t)\sum_{k=1}^{j}\frac{t^{j-k}}{(j-k)!}\varphi_k^{i(1)}(x)\right)$$

$$+\sum_{n\in\mathbb{Z}} \frac{(a,\varphi_n^*)}{\rho_n}\exp(\lambda_n t)\varphi_n^{(1)}(x)$$

in $C([-T,T];L^2(0,1))$, where

$$\begin{cases} (L_p u)(x,t) = 0, \qquad 0<x<1,\ -T<t<T, \\ \left(u(x,0), \dfrac{\partial u}{\partial t}(x,0)\right)^T = a(x), \quad 0<x<1, \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0, \quad -T<t<T, \end{cases}$$

provided that $a \in \mathcal{D}(A_p^\ell)$. We apply a usual density argument. That is, choosing an approximating sequence $a_m \in \mathcal{D}(A_p^\ell)$, $m \in \mathbb{N}$ such that

$$\lim_{m\to\infty} \|a_m - (\delta(x),0)\|_{L^2(0,1)\times(H^1(0,1))'} = 0. \tag{3.3}$$

Then

$$\lim_{m\to\infty} \frac{(a_m,\varphi_n^*)}{\rho_n} = \frac{1}{\rho_n}, \qquad n\in\mathbb{Z}$$

by Proposition 2.2 (ii). Hence Lemma 3.1 yields

$$\lim_{m\to\infty}\sum_{n\in\mathbb{Z}} \frac{(a_m,\varphi_n^*)}{\rho_n}\exp(\lambda_n t)\varphi_n^{(1)} = \sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\exp(\lambda_n t)\varphi_n^{(1)}$$

in $C([-T,T];(H^1(0,1))')$. On the other hand, by (3.3) and Theorem 9.3 (p.288) in Lions and Magenes [6], it follows that the first components of $\Phi(a_m)$, $m\in\mathbb{N}$ tend to $\Phi(\delta(x),0)$ in the space $C([-T,T];(H^1(0,1))')$. Therefore, applying Proposition 2.2 again, we obtain that (3.2) converges in $C([-T,T];(H^1(0,1))')$ for the solution to (1.1).

Next we prove that (3.2) holds also in $C([0,1];H^{-1}(-T,T))$ for the solution u to (1.1). Let $\omega \in H_0^1(-T,T)$ be arbitrary. Noting that $\sup_{n\in\mathbb{Z}}\|\varphi_n^{(1)}\|_{C[0,1]} < \infty$ (see [9]), we obtain by integration by parts and the Schwarz inequality that

$$\begin{aligned}
&\left|\int_{-T}^{T}\sum_{|n|\ge N_2}\frac{1}{\rho_n}\exp(\lambda_n t)\varphi_n^{(1)}(x)\omega(t)\mathrm{d}t\right| \\
\le\ & \sum_{|n|\ge N_2}\left|\frac{1}{\rho_n\lambda_n}\int_{-T}^{T}\exp(\lambda_n t)\varphi_n^{(1)}(x)\frac{\mathrm{d}\omega}{\mathrm{d}t}(t)\mathrm{d}t\right| \\
\le\ & C_6\sum_{|n|\ge N_2}\frac{1}{|\lambda_n|}\left|\int_{-T}^{T}\exp(\lambda_n t)\frac{\mathrm{d}\omega}{\mathrm{d}t}(t)\mathrm{d}t\right| \\
\le\ & C_6\left(\sum_{|n|\ge N_2}\frac{1}{|\lambda_n|^2}\right)^{\frac{1}{2}}\left(\sum_{|n|\ge N_2}\left|\int_{-T}^{T}\exp(\lambda_n t)\frac{\mathrm{d}\omega}{\mathrm{d}t}(t)\mathrm{d}t\right|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.4}$$

Next we claim that

$$\sum_{|n|\ge N_2}\left|\int_{-T}^{T}\exp(\lambda_n t)\beta(t)\mathrm{d}t\right|^2 \le C_7\|\beta\|^2_{L^2(-T,T)} \quad \text{for any } \beta\in L^2(-T,T). \tag{3.5}$$

We set $c_0 = \frac{1}{2}\int_0^1 p_1(s)\mathrm{d}s$. Then, by (2.3), we have $\lambda_n = n\pi\sqrt{-1}+c_0+O\left(\frac{1}{|n|}\right)$ as $|n|\to\infty$. By the trigonometric series, we have for any $\eta\in L^2(-1,1)$

$$\sum_{n\in\mathbb{Z}}\left|\int_{-1}^{1}\exp((n\pi\sqrt{-1}+c_0)t)\eta(t)dt\right|^2 = \frac{2}{\pi}\|\exp(c_0\cdot)\eta(\cdot)\|^2_{L^2(-1,1)} \le C_8\|\eta\|^2_{L^2(-1,1)}. \tag{3.6}$$

Let $[T]$ denote the maximum integer not greater than T, and let $\widetilde{\beta}\in L^2(-[T]-1,[T]+1)$ be the 0-extension of $\beta\in L^2(-T,T)$. Then by (3.6) we have for any $\beta\in L^2(-T,T)$

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}\left|\int_{-T}^{T}\exp((n\pi\sqrt{-1}+c_0)t)\beta(t)\mathrm{d}t\right|^2 = \sum_{n\in\mathbb{Z}}\left|\int_{-[T]-1}^{[T]+1}\exp((n\pi\sqrt{-1}+c_0)t)\widetilde{\beta}(t)\mathrm{d}t\right|^2 \\
&= \sum_{n\in\mathbb{Z}}\left|([T]+1)\int_{-1}^{1}\exp\left((n\pi\sqrt{-1}+c_0)([T]+1)s\right)\widetilde{\beta}(([T]+1)s)\mathrm{d}s\right|^2 \\
&\le ([T]+1)^2C_8\|\widetilde{\beta}\left([T]+1\right)\cdot\|^2_{L^2(-1,1)} \\
&= ([T]+1)C_8\|\beta\|^2_{L^2(-T,T)} \equiv C_9\|\beta\|^2_{L^2(-T,T)}.
\end{aligned} \tag{3.7}$$

On the other hand, by (2.3) we have

$$\begin{aligned}
&\left|\int_{-T}^{T} \exp(\lambda_n t)\beta(t)dt\right| \\
\le\ & \left|\int_{-T}^{T} \exp((n\pi\sqrt{-1}+c_0)t)\beta(t)dt\right| + \left|\int_{-T}^{T} \{\exp(\lambda_n t) - \exp((n\pi\sqrt{-1}+c_0)t)\}\beta(t)dt\right| \\
\le\ & \left|\int_{-T}^{T} \exp((n\pi\sqrt{-1}+c_0)t)\beta(t)dt\right| + \frac{C_{10}}{|n|}\int_{-T}^{T} |\beta(t)|dt, \qquad |n| \ge N_2
\end{aligned}$$

and so

$$\begin{aligned}
&\left|\int_{-T}^{T} \exp(\lambda_n t)\beta(t)dt\right|^2 \\
\le\ & 2\left|\int_{-T}^{T} \exp((n\pi\sqrt{-1}+c_0)t)\beta(t)dt\right|^2 + \frac{C_{11}}{n^2}\|\beta\|^2_{L^2(-T,T)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{|n|\ge N_2}\left|\int_{-T}^{T} \exp(\lambda_n t)\beta(t)dt\right|^2 \\
\le\ & 2\sum_{|n|\ge N_2}\left|\int_{-T}^{T} \exp((n\pi\sqrt{-1}+c_0)t)\beta(t)dt\right|^2 + C_{11}\sum_{|n|\ge N_2}\frac{1}{n^2}\|\beta\|^2_{L^2(-T,T)}.
\end{aligned}$$

Thus by (3.7) the proof of (3.5) is complete. Then by (3.4) and (3.5) we have

$$\begin{aligned}
&\left\|\sum_{|n|\ge N_2}\frac{1}{\rho_n}\exp(\lambda_n t)\varphi_n^{(1)}(x)\right\|_{C([0,1];H^{-1}(-T,T))} \\
=\ & \sup_{\|\omega\|_{H_0^1(-T,T)}=1}\left|\int_{-T}^{T}\sum_{|n|\ge N_2}\frac{1}{\rho_n}\exp(\lambda_n t)\varphi_n^{(1)}(x)\omega(t)\mathrm{d}t\right| \\
\le\ & C_{12}\left(\sum_{|n|\ge N_2}\frac{1}{|n|^2}\right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

Thus the proof of Lemma 3.2 is complete.

4 Proof of Theorem 1

We divide the proof of Theorem 1 into three steps. In the first and the second steps we prove $V = F$, where V is defined in Section 1 and F is defined in Section 2. Thus the proof of Theorem

1 except for the unique solvability of integral equation (1.6) follows from Proposition 2.4 and Theorem 2.2. Finally, the proof of the unique solvability of integral equation (1.6) will given in the third step.

First step: By Proposition 2.4, the series (2.17) is convergent in $\mathcal{D}'((0,1)^2)$: the distribution of $(x,y) \in (0,1)^2$. Therefore, we can differentiate termwise (2.17) in the sense of distributions to obtain (note that the derivative of a continuously differentiable function in the sense of distributions coincides with its derivative in the usual sense)

$$\begin{aligned} F(x,y) &= \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{\overline{S^*_{(j)}(x,\overline{\lambda^i})}S^T_{(j)}(y,\lambda^i)}{\rho^i} \\ &\quad + \sum_{n\in\mathbb{Z}} \left\{ \frac{\overline{S^*(x,\overline{\lambda_n})}S^T(y,\lambda_n)}{\rho_n} - \overline{S^*(x,\overline{\mu_n})}S^T(y,\mu_n) \right\}. \end{aligned} \tag{4.1}$$

Particularly, by (2.6), (2.7), (2.12) and (2.13), we have

$$\begin{aligned} & F_{11}(x,y) \\ = & \sum_{i=1}^{N}\sum_{j=1}^{m_i} \frac{1}{\rho^i} \left\{ \sum_{k=j}^{m_i} \alpha^i_k \frac{x^{k-j}}{(k-j)!}\theta_{k-j}(x,\lambda^i) \sum_{l=0}^{j-1} \frac{y^l}{l!}\theta_l(y,\lambda^i) \right\} \\ & + \sum_{n\in\mathbb{Z}} \frac{1}{\rho_n} \cosh(\lambda_n x)\cosh(\lambda_n y) - \sum_{n\in\mathbb{Z}} \cosh(\mu_n x)\cosh(\mu_n y). \end{aligned} \tag{4.2}$$

To calculate $F_{11}(x,y)$ we first prove a lemma.

Lemma 4.1. *It holds that in* $\mathcal{D}'((0,1)^2)$

$$\frac{\partial^2 \min\{x,y\}}{\partial x \partial y} = \delta(x-y).$$

Proof. Let $\phi \in C_0^\infty((0,1)^2)$. We have to verify

$$LHS := \int_0^1\int_0^1 \frac{\partial^2\phi(x,y)}{\partial x\partial y}\min\{x,y\}\mathrm{d}x\mathrm{d}y = \int_0^1 \phi(x,x)\mathrm{d}x.$$

Integrating by parts, we have

$$\begin{aligned} & LHS \\ = & \int_0^1 \left(\int_0^x \frac{\partial^2\phi(x,y)}{\partial x\partial y}\min\{x,y\}\mathrm{d}y \right)\mathrm{d}x + \int_0^1 \left(\int_0^y \frac{\partial^2\phi(x,y)}{\partial x\partial y}\min\{x,y\}\mathrm{d}x \right)\mathrm{d}y \\ = & \int_0^1 x\frac{\partial\phi(x,x)}{\partial x}\mathrm{d}x - \int_0^1 \left(\int_0^x \frac{\partial\phi(x,y)}{\partial x}\mathrm{d}y \right)\mathrm{d}x \\ & + \int_0^1 y\frac{\partial\phi(y,y)}{\partial y}\mathrm{d}y - \int_0^1 \left(\int_0^y \frac{\partial\phi(x,y)}{\partial y}\mathrm{d}x \right)\mathrm{d}y. \end{aligned}$$

Since $\dfrac{\mathrm{d}}{\mathrm{d}x}\phi(x,x) = \dfrac{\partial \phi}{\partial x}(x,x) + \dfrac{\partial \phi}{\partial y}(x,x)$ and $\displaystyle\int_0^1 y\frac{\partial \phi(y,y)}{\partial y}\mathrm{d}y = \int_0^1 x\frac{\partial \phi(x,x)}{\partial y}\mathrm{d}x$, by integration by parts we obtain

$$\int_0^1 x\frac{\partial \phi(x,x)}{\partial x}\mathrm{d}x + \int_0^1 y\frac{\partial \phi(y,y)}{\partial y}\mathrm{d}y = \int_0^1 x\frac{\mathrm{d}}{\mathrm{d}x}\phi(x,x)\mathrm{d}x = -\int_0^1 \phi(x,x)\mathrm{d}x.$$

Now we change orders of integrations, so that

$$-\int_0^1 \left(\int_0^x \frac{\partial \phi(x,y)}{\partial x}\mathrm{d}y\right)\mathrm{d}x = -\int_0^1 \left(\int_y^1 \frac{\partial \phi(x,y)}{\partial x}\mathrm{d}x\right)\mathrm{d}y = \int_0^1 \phi(y,y)\mathrm{d}y$$

and

$$-\int_0^1 \left(\int_0^y \frac{\partial \phi(x,y)}{\partial y}\mathrm{d}x\right)\mathrm{d}x = \int_0^1 \phi(x,x)\mathrm{d}x.$$

Therefore we see that $LHS = \displaystyle\int_0^1 \phi(x,x)\mathrm{d}x$. The proof is complete. □

It follows from the Parseval equality with respect to A_0 that (see also Lemma 4.4 in [9])

$$\sum_{n\in\mathbb{Z}} \overline{C^*(x,\overline{\mu_n})}C^T(y,\mu_n) = \min\{x,y\}E, \tag{4.3}$$

where E denotes the 2×2 unit matrix. In the sense of $\mathcal{D}'((0,1)^2)$, we differentiate termwise the left hand side of (4.3) and apply Lemma 4.1, so that we obtain

$$\sum_{n\in\mathbb{Z}} \overline{S^*(x,\overline{\mu_n})}S^T(y,\mu_n) = \delta(x-y)E.$$

In particular, we have in $\mathcal{D}'((0,1)^2)$

$$\sum_{n\in\mathbb{Z}} \cosh(\mu_n x)\cosh(\mu_n y) = \delta(x-y)$$

and

$$F_{11}(x,y) = G_{11}(x,y) - \delta(x-y), \tag{4.4}$$

where

$$\begin{aligned} G_{11}(x,y) \quad &= \quad \sum_{i=1}^N\sum_{j=1}^{m_i}\frac{1}{\rho^i}\left\{\sum_{k=j}^{m_i}\alpha_k^i\frac{x^{k-j}}{(k-j)!}\theta_{k-j}(x,\lambda^i)\sum_{l=0}^{j-1}\frac{y^l}{l!}\theta_l(y,\lambda^i)\right\} \\ &\qquad + \sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\cosh(\lambda_n x)\cosh(\lambda_n y). \end{aligned} \tag{4.5}$$

Second step:

We first prove $V_{11} = F_{11}$. By (3.2) and Lemma 3.2 we have

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{\partial u}{\partial t}(0,t) = \sum_{i=1}^N\sum_{j=1}^{m_i}\frac{\alpha_j^i}{\rho^i}\left(\exp(\lambda^i t)\sum_{k=1}^j\frac{t^{j-k}}{(j-k)!}\right) + \sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\exp(\lambda_n t)$$

in $H^{-1}(-T,T) \subset \mathcal{D}'(-T,T)$. Here and henceforth $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}$ denote differentiations in the sense of distributions under consideration. Since $v(0) = u(0,0) = 0$, we have in $\mathcal{D}'(-T,T)$

$$v(t) = \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{\alpha_j^i}{\rho^i}\left(\sum_{k=1}^{j}\int_0^t \exp(\lambda^i\tau)\frac{\tau^{j-k}}{(j-k)!}\mathrm{d}\tau\right) + \sum_{n\in\mathbb{Z}}\frac{\exp(\lambda_n t)-1}{\rho_n\lambda_n}.$$

Then we obtain for $0 \le x, y \le 1$

$$v(x+y) = \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{\alpha_j^i}{\rho^i}\left(\sum_{k=1}^{j}\int_0^{x+y} \exp(\lambda^i\tau)\frac{\tau^{j-k}}{(j-k)!}\mathrm{d}\tau\right) + \sum_{n\in\mathbb{Z}}\frac{\exp(\lambda_n(x+y))-1}{\rho_n\lambda_n} \tag{4.6}$$

in $\mathcal{D}'((0,1)^2)$ as a function of x and y, and we can differentiate termwise the right hand side of (4.6) in $\mathcal{D}'((0,1)^2)$ to obtain

$$\frac{\partial v(x+y)}{\partial x} = \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{\alpha_j^i}{\rho^i}\left(\exp\left(\lambda^i(x+y)\right)\sum_{k=1}^{j}\frac{(x+y)^{j-k}}{(j-k)!}\right) + \sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\exp\left(\lambda_n(x+y)\right) \tag{4.7}$$

Similarly we can obtain that in $\mathcal{D}'((0,1)^2)$

$$\frac{\partial v(x-y)}{\partial x} = \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{\alpha_j^i}{\rho^i}\left(\exp\left(\lambda^i(x-y)\right)\sum_{k=1}^{j}\frac{(x-y)^{j-k}}{(j-k)!}\right) + \sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\exp\left(\lambda_n(x-y)\right), \tag{4.8}$$

$$-\frac{\partial v(-x+y)}{\partial x} = \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{\alpha_j^i}{\rho^i}\left(\exp\left(\lambda^i(-x+y)\right)\sum_{k=1}^{j}\frac{(-x+y)^{j-k}}{(j-k)!}\right) + \sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\exp\left(\lambda_n(-x+y)\right) \tag{4.9}$$

and

$$-\frac{\partial v(-x-y)}{\partial x} = \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{\alpha_j^i}{\rho^i}\left(\exp\left(\lambda^i(-x-y)\right)\sum_{k=1}^{j}\frac{(-x-y)^{j-k}}{(j-k)!}\right) + \sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\exp\left(\lambda_n(-x-y)\right). \tag{4.10}$$

From (4.7)–(4.10), we have

$$\begin{aligned}
&\frac{\partial v(x+y)}{\partial x} + \frac{\partial v(x-y)}{\partial x} - \frac{\partial v(-x+y)}{\partial x} - \frac{\partial v(-x-y)}{\partial x} \\
= &\sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{\alpha_j^i}{\rho^i}\left(\exp\left(\lambda^i(x+y)\right)\sum_{k=1}^{j}\frac{(x+y)^{j-k}}{(j-k)!} + \exp\left(\lambda^i(x-y)\right)\sum_{k=1}^{j}\frac{(x-y)^{j-k}}{(j-k)!}\right) \\
&\quad + \sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\left(\exp\Big(\lambda_n(x+y)\Big) + \exp\Big(\lambda_n(x-y)\Big)\right) \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{\alpha_j^i}{\rho^i}\left(\exp\left(\lambda^i(-x+y)\right)\sum_{k=1}^{j}\frac{(-x+y)^{j-k}}{(j-k)!} + \exp\left(\lambda^i(-x-y)\right)\sum_{k=1}^{j}\frac{(-x-y)^{j-k}}{(j-k)!}\right) \\
&\quad + \sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\left(\exp\Big(\lambda_n(-x+y)\Big) + \exp\Big(\lambda_n(-x-y)\Big)\right).
\end{aligned} \tag{4.11}$$

On the other hand, since by (2.14) $\theta_k(x,\lambda) = \frac{1}{2}\big(\exp(\lambda x) + (-1)^k \exp(-\lambda x)\big)$, we see by (4.5) that

$$
\begin{aligned}
& G_{11}(x,y) \\
= \; & \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\left\{\sum_{k=j}^{m_i}\alpha_k^i\left(\exp(\lambda^i x)\frac{x^{k-j}}{(k-j)!} + \exp(-\lambda^i x)\frac{(-x)^{k-j}}{(k-j)!}\right)\right\} \\
& \qquad \times\left\{\sum_{l=0}^{j-1}\left(\exp(\lambda^i y)\frac{y^l}{l!} + \exp(-\lambda^i y)\frac{(-y)^l}{l!}\right)\right\} \\
& + \frac{1}{4}\sum_{n\in\mathbb{Z}}\frac{1}{\rho_n}\Bigg\{\exp\Big(\lambda_n(x+y)\Big) + \exp\Big(\lambda_n(x-y)\Big) \\
& \qquad + \exp\Big(\lambda_n(-x+y)\Big) + \exp\Big(\lambda_n(-x-y)\Big)\Bigg\}.
\end{aligned}
\tag{4.12}
$$

Therefore, by (4.4), (4.11) and (4.12), we see that $V_{11} = F_{11}$ holds if and only if

$$
\begin{aligned}
& \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{1}{\rho^i}\left\{\sum_{k=j}^{m_i}\alpha_k^i\left(\exp(\lambda^i x)\frac{x^{k-j}}{(k-j)!} + \exp(-\lambda^i x)\frac{(-x)^{k-j}}{(k-j)!}\right)\right\} \\
& \qquad \times\left\{\sum_{l=0}^{j-1}\left(\exp(\lambda^i y)\frac{y^l}{l!} + \exp(-\lambda^i y)\frac{(-y)^l}{l!}\right)\right\} \\
= & \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{\alpha_j^i}{\rho^i}\left(\exp\left(\lambda^i(x+y)\right)\sum_{k=1}^{j}\frac{(x+y)^{j-k}}{(j-k)!} + \exp\left(\lambda^i(x-y)\right)\sum_{k=1}^{j}\frac{(x-y)^{j-k}}{(j-k)!}\right) \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{\alpha_j^i}{\rho^i}\left(\exp\left(\lambda^i(-x+y)\right)\sum_{k=1}^{j}\frac{(-x+y)^{j-k}}{(j-k)!} + \exp\left(\lambda^i(-x-y)\right)\sum_{k=1}^{j}\frac{(-x-y)^{j-k}}{(j-k)!}\right).
\end{aligned}
$$

It is equivalent to show that for any $1 \leq i \leq N$

$$
\begin{aligned}
I_1 := \sum_{j=1}^{m_i} \alpha_j^i & \left\{ \sum_{k=1}^{j} \left(\exp(\lambda^i x) \frac{x^{j-k}}{(j-k)!} + \exp(-\lambda^i x) \frac{(-x)^{j-k}}{(j-k)!} \right) \right\} \\
& \times \left\{ \sum_{l=0}^{k-1} \left(\exp(\lambda^i y) \frac{y^l}{l!} + \exp(-\lambda^i y) \frac{(-y)^l}{l!} \right) \right\} \\
= \sum_{j=1}^{m_i} \alpha_j^i & \left(\exp\left(\lambda^i (x+y)\right) \sum_{k=1}^{j} \frac{(x+y)^{j-k}}{(j-k)!} + \exp\left(\lambda^i (x-y)\right) \sum_{k=1}^{j} \frac{(x-y)^{j-k}}{(j-k)!} \right) \\
+ \sum_{j=1}^{m_i} \alpha_j^i & \left(\exp\left(\lambda^i (-x+y)\right) \sum_{k=1}^{j} \frac{(-x+y)^{j-k}}{(j-k)!} + \exp\left(\lambda^i (-x-y)\right) \sum_{k=1}^{j} \frac{(-x-y)^{j-k}}{(j-k)!} \right) \\
=: I_2. &
\end{aligned}
$$

Since first changing the order of summation on j, k and then exchanging k with j, we have

$$
\begin{aligned}
& \sum_{j=1}^{m_i} \left\{ \sum_{k=j}^{m_i} \alpha_k^i \left(\exp(\lambda^i x) \frac{x^{k-j}}{(k-j)!} + \exp(-\lambda^i x) \frac{(-x)^{k-j}}{(k-j)!} \right) \right\} \\
& \qquad \times \left\{ \sum_{l=0}^{j-1} \left(\exp(\lambda^i y) \frac{y^l}{l!} + \exp(-\lambda^i y) \frac{(-y)^l}{l!} \right) \right\} \\
= & \sum_{j=1}^{m_i} \alpha_j^i \left\{ \sum_{k=1}^{j} \left(\exp(\lambda^i x) \frac{x^{j-k}}{(j-k)!} + \exp(-\lambda^i x) \frac{(-x)^{j-k}}{(j-k)!} \right) \right\} \\
& \qquad \times \left\{ \sum_{l=0}^{k-1} \left(\exp(\lambda^i y) \frac{y^l}{l!} + \exp(-\lambda^i y) \frac{(-y)^l}{l!} \right) \right\}.
\end{aligned}
$$

On one hand, we have

$$\sum_{k=1}^{j}\left(\exp(\lambda^i x)\frac{x^{j-k}}{(j-k)!}+\exp(-\lambda^i x)\frac{(-x)^{j-k}}{(j-k)!}\right)$$

$$\times\sum_{l=0}^{k-1}\left(\exp(\lambda^i y)\frac{y^l}{l!}+\exp(-\lambda^i y)\frac{(-y)^l}{l!}\right)$$

$$= \sum_{k=1}^{j}\sum_{l=0}^{k-1}\left(\exp\left(\lambda^i(x+y)\right)\frac{x^{j-k}}{(j-k)!}\frac{y^l}{l!}+\exp\left(\lambda^i(x-y)\right)\frac{x^{j-k}}{(j-k)!}\frac{(-y)^l}{l!}\right)$$

$$+\sum_{k=1}^{j}\sum_{l=0}^{k-1}\left(\exp\left(\lambda^i(-x+y)\right)\frac{(-x)^{j-k}}{(j-k)!}\frac{y^l}{l!}+\exp\left(\lambda^i(-x-y)\right)\frac{(-x)^{j-k}}{(j-k)!}\frac{(-y)^l}{l!}\right).$$

On the other hand, for any $a, b \in \mathbb{R}$ we have

$$\sum_{k=1}^{j}\frac{(a+b)^{j-k}}{(j-k)!}=\sum_{k=1}^{j}\sum_{l=0}^{j-k}\frac{a^{j-k-l}}{(j-k-l)!}\frac{b^l}{l!}$$

$$= \sum_{s=1}^{j}\sum_{\substack{k+l=s\\ k,l\geq 0}}\frac{a^{j-k-l}}{(j-k-l)!}\frac{b^l}{l!}=\sum_{s=1}^{j}\frac{a^{j-s}}{(j-s)!}\sum_{l=0}^{s-1}\frac{b^l}{l!}.$$

Then, comparing the coefficients of $\exp\left(\lambda^i(\cdot)\right)$ in I_1 and I_2, we see that $V_{11} = F_{11}$. Similarly we can prove $V_{12} = F_{12}$, $V_{21} = F_{21}$ and $V_{22} = F_{22}$.

Third step: Now we prove that (1.6) is uniquely solvable. Note that it follows from Proposition 2.4 that $V = F \in \left(C^1(\overline{\Omega})\right)^4, \in \left(C^1\left(\overline{(0,1)^2\backslash\Omega}\right)\right)^4$. For fixed $x \in (0,1]$ the integral equation (1.6) is a Fredholm equation of the second kind with respect to $M(x,y)$. By Fredholm's alternative theorem, for the solvability of (1.6), it is sufficient to show that for fixed $x \in (0,1]$ ($x = 0$ is trivial) the corresponding homogeneous equation has only the trivial solution, which is equivalent to show the following:

$$\begin{cases} \text{the } 1\times 2 \text{ vector-valued continuous function } \mathsf{m}(y) \text{ satisfying} \\ \mathsf{m}(y)+\displaystyle\int_0^x \mathsf{m}(\tau)F(\tau,y)\mathrm{d}\tau = 0, \quad 0\leq y\leq x\leq 1 \\ \text{is nothing but the zero vector.}\end{cases} \tag{4.13}$$

First we point out that in fact $\mathsf{m}(y) \in (C^1[0,x])^2$ by the continuous differentiability of $F(\tau,y)$. Let $\mathsf{n} \in (C^1[0,x])^2$ be arbitrary 2×1 vector-valued function. Next we set

$$\mathcal{J}[\mathsf{m},\mathsf{n}] = \int_0^x \mathsf{m}(y)\mathsf{n}(y)\mathrm{d}y + \int_0^x\int_0^x \mathsf{m}(\tau)F(\tau,y)\mathsf{n}(y)\mathrm{d}\tau\mathrm{d}y.$$

For $\mathcal{J}[\mathsf{m},\mathsf{n}]$, we show

Lemma 4.2. *It holds that* $\mathcal{J}[\mathsf{m},\mathsf{n}] = \displaystyle\sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{G_j^{i*}(\mathsf{m})G_j^i(\mathsf{n})}{\rho^i}+\sum_{n\in\mathbb{Z}}\frac{G_n^*(\mathsf{m})G_n(\mathsf{n})}{\rho_n}$, *where*

$$G_j^{i*}(\mathsf{m})=\int_0^x \mathsf{m}(\tau)\overline{S_{(j)}^*(\tau,\overline{\lambda^i})}\mathrm{d}\tau,\quad G_j^i(\mathsf{n})=\int_0^x S_{(j)}^T(y,\lambda^i)\mathsf{n}(y)\mathrm{d}y,$$

$$G_n^*(\mathsf{m})=\int_0^x \mathsf{m}(\tau)\overline{S^*(\tau,\overline{\lambda_n})}\mathrm{d}\tau,\quad G_n(\mathsf{n})=\int_0^x S^T(y,\lambda_n)\mathsf{n}(y)\mathrm{d}y.$$

Proof. First we should note that, from the definition of $F(\cdot,\cdot)$ and Proposition 2.4, it follows that

$$\begin{aligned}\mathcal{J}[\mathsf{m},\mathsf{n}] &= \int_0^x \mathsf{m}(y)\mathsf{n}(y)\mathrm{d}y+\sum_{i=1}^N\sum_{j=1}^{m_i}\int_0^x\int_0^x \mathsf{m}(\tau)\frac{\overline{S_{(j)}^*(\tau,\overline{\lambda^i})}S_{(j)}^T(y,\lambda^i)}{\rho^i}\mathsf{n}(y)\mathrm{d}\tau\mathrm{d}y\\ &+ \sum_{n\in\mathbb{Z}}\int_0^x\int_0^x \mathsf{m}(\tau)\left\{\frac{\overline{S^*(\tau,\overline{\lambda_n})}S^T(y,\lambda_n)}{\rho_n}-\overline{S^*(\tau,\overline{\mu_n})}S^T(y,\mu_n)\right\}\mathsf{n}(y)\mathrm{d}\tau\mathrm{d}y\end{aligned}$$

Then the Parseval equality with respect to A_0 completes the proof of Lemma 4.2. □

Lemma 4.3. *If there exists a* 1×2 *vector-valued function* $\mathsf{m}\in(C^1[0,x])^2$ *such that for any* 2×1 *vector-valued function* $\mathsf{n}\in(C^1[0,x])^2$

$$\sum_{i=1}^N\sum_{j=1}^{m_i}\frac{G_j^{i*}(\mathsf{m})G_j^i(\mathsf{n})}{\rho^i}+\sum_{n\in\mathbb{Z}}\frac{G_n^*(\mathsf{m})G_n(\mathsf{n})}{\rho_n}=0,$$

then $\mathsf{m}\equiv 0$.

Proof. First by the transformation formulae (see Lemma 4.1 in [9] with $\mu=\nu=0$)

$$S_{(j)}^*(\cdot,\overline{\lambda^i})=X(-\overline{P^T},0,0)\varphi_j^{i*}(\cdot),\quad S_{(j)}(\cdot,\lambda^i)=X(P,0,0)\varphi_j^i(\cdot),$$

$$S_n^*(\cdot,\overline{\lambda_n})=X(-\overline{P^T},0,0)\varphi_n^*(\cdot),\quad S_n(\cdot,\lambda_n)=X(P,0,0)\varphi_n(\cdot),$$

it follows from change of the order of integrals that

$$G_j^{i*}(\mathsf{m})=\int_0^x\left\{\mathsf{m}(\tau)\overline{R(-\overline{P^T},0)(\tau)}+\int_\tau^x \mathsf{m}(t)\overline{K(-\overline{P^T},0,0)(t,\tau)}\mathrm{d}t\right\}\overline{\varphi_j^{i*}(\tau)}\mathrm{d}\tau,$$

$$G_j^i(\mathsf{n})=\int_0^x\left(\varphi_j^i(y)\right)^T\left\{R(P,0)(y)\mathsf{n}(y)+\int_y^x K^T(P,0,0)(t,y)\mathsf{n}(t)\mathrm{d}t\right\}\mathrm{d}y,$$

$$G_n^*(\mathsf{m})=\int_0^x\left\{\mathsf{m}(\tau)\overline{R(-\overline{P^T},0)(\tau)}+\int_\tau^x \mathsf{m}(t)\overline{K(-\overline{P^T},0,0)(t,\tau)}\mathrm{d}t\right\}\overline{\varphi_n^*(\tau)}\mathrm{d}\tau,$$

$$G_n(\mathsf{n})=\int_0^x\left(\varphi_n(y)\right)^T\left\{R(P,0)(y)\mathsf{n}(y)+\int_y^x K^T(P,0,0)(t,y)\mathsf{n}(t)\mathrm{d}t\right\}\mathrm{d}y.$$

Therefore, the Parseval equality with respect to A_P yields

$$\sum_{i=1}^{N}\sum_{j=1}^{m_i}\frac{G_j^{i*}(\mathsf{m})G_j^i(\mathsf{n})}{\rho^i}+\sum_{n\in\mathbb{Z}}\frac{G_n^*(\mathsf{m})G_n(\mathsf{n})}{\rho_n}$$

$$= \int_0^x\left\{\mathsf{m}(y)\overline{R(-\overline{P^T},0)(y)}+\int_y^x \mathsf{m}(t)\overline{K(-\overline{P^T},0,0)(t,y)}\mathrm{d}t\right\}$$

$$\times\left\{R(P,0)(y)\mathsf{n}(y)+\int_y^x K^T(P,0,0)(t,y)\mathsf{n}(t)\mathrm{d}t\right\}\mathrm{d}y$$

$$= \int_0^x\left\{\mathsf{m}(y)+\int_y^x \mathsf{m}(t)\overline{K(-\overline{P^T},0,0)(t,y)}R(P,0)(y)\mathrm{d}t\right\}$$

$$\times\left\{\mathsf{n}(y)+\int_y^x R(0,P)(y)K^T(P,0,0)(t,y)\mathsf{n}(t)\mathrm{d}t\right\}\mathrm{d}y.$$

The last identity follows from $\overline{R(-\overline{P^T},0)(\cdot)}=R(0,P)(\cdot)=R^{-1}(P,0)(\cdot)$.
By the assumption we see that

$$\mathsf{m}(y)+\int_y^x \mathsf{m}(t)\overline{K(-\overline{P^T},0,-\overline{\mu})(t,y)}R(P,0)(y)\mathrm{d}t=0,$$

which is a Volterra integral equation with a continuous kernel, and therefore $\mathsf{m}(y)\equiv 0$. □

(4.13) follows easily from Lemma 4.2 and Lemma 4.3. Consequently (1.6) admits a unique solution. Moreover, this solution is of $\left(C^1(\overline{\Omega})\right)^4$ (for the proof see, e.g., Levitan and Sargsjan [5]). Thus the proof of Theorem 1 is complete. □

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