

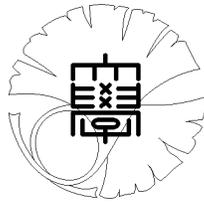
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polyhedral sound-hard  
obstacles with a single incoming wave**

by

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# Uniqueness in determining polyhedral sound-hard obstacles with a single incoming wave

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## Abstract

We consider the inverse acoustic scattering problem of determining a sound-hard obstacle by far field measurements. It is proved that a polyhedral scatterer in  $\mathbb{R}^n$ ,  $n \geq 2$ , consisting of finitely many solid polyhedra, is uniquely determined by a single incoming plane wave.

## 1 Introduction

Let  $D$  be a compact subset of  $\mathbb{R}^n$ ,  $n \geq 2$ , and assume that a time harmonic plane wave  $u^{in}(x) = \exp(ikd \cdot x)$ ,  $x \in \mathbb{R}^n$ , is incident on the scatterer  $D$ . Here  $k > 0$  is the wave number, which is kept fixed throughout the paper, and  $d \in \mathbb{S}^{n-1}$  is the incident direction. In the case of a sound-hard scatterer  $D$ , the total field  $u$  which is the sum of  $u^{in}$  and the scattered field  $u^{sc}$  satisfies the following exterior boundary value problem in  $D^c := \mathbb{R}^n \setminus D$ :

$$\left. \begin{aligned} \Delta u + k^2 u &= 0 \quad \text{in } D^c, \quad \partial_\nu u = 0 \quad \text{on } \partial D, \\ \lim_{r \rightarrow \infty} r^{(n-1)/2} (\partial_r u^{sc} - iku^{sc}) &= 0, \quad r = |x|. \end{aligned} \right\} \quad (1.1)$$

Here  $\nu$  is the unit normal to  $\partial D$  pointing to  $D^c$ , and the relation in the second line of (1.1) is the Sommerfeld radiation condition which holds uniformly in all directions  $\hat{x} = x/|x| \in \mathbb{S}^{n-1}$  as  $|x| \rightarrow \infty$ . This condition implies that the asymptotic behaviour at infinity of the scattered field  $u^{sc}$  is governed by the relation

$$u^{sc}(x) = r^{(1-n)/2} \exp(ikr) \{u_\infty(\hat{x}) + O(r^{-1})\}, \quad r \rightarrow \infty, \quad (1.2)$$

holding uniformly in all directions  $\hat{x} \in \mathbb{S}^{n-1}$ ; see, e.g., [3], [11]. The function  $u_\infty$  defined by (1.2) on  $\mathbb{S}^{n-1}$  is called the far field pattern of  $u^{sc}$ . In the case of a sound-soft scatterer  $D$ , the boundary condition on  $\partial D$  in (1.1) is replaced by  $u = 0$ .

The inverse acoustic scattering problem consists in determining a (sound-soft or sound-hard) scatterer by its far field pattern  $u_\infty$  for one or several incident directions, and its uniqueness presents important and challenging open problems since many years; see, e.g., [3], [4], [7]. Uniqueness results with a minimal number of incident waves have recently been obtained within the class of polygonal and polyhedral scatterers.

**Definition 1** *A compact set  $D \subset \mathbb{R}^n$  is called a polyhedral obstacle if  $D$  is the union of finitely many convex polyhedra and its exterior  $D^c$  is connected. We shall say that a compact set  $D \subset \mathbb{R}^n$  with connected exterior  $D^c$  is a polyhedral scatterer if  $D$  is the union of a polyhedral obstacle and finitely many cells, where a cell is defined as the closure of an open connected subset of an  $(n - 1)$ -dimensional hyperplane.*

Note that a polyhedral obstacle coincides with the closure of its interior, whereas a polyhedral scatterer may also contain  $(n - 1)$ -dimensional components (e.g., screens).

It was proved in [1] that any sound-soft polyhedral scatterer is uniquely determined by the far field pattern of a single incident wave. The method in [1] is based on a careful study of the nodal set  $\mathcal{N} = \{x : u(x) = 0\}$  of the direct solution in  $D^c$ , the reflection principle for the Helmholtz equation, and the construction of a path in  $D^c$  connecting a point on  $\partial D$  to infinity and intersecting  $\mathcal{N}$  suitably.

The approach of [1] was considerably simplified in [8] to obtain a shorter proof in the sound-soft case, together with the result that any sound-hard polyhedral scatterer is uniquely determined by the far field patterns for  $n$  linearly independent incident directions, generalizing previous work in the 2D case [2]. Moreover, these uniqueness results can be extended to scatterers with impedance and mixed type (Dirichlet/Neumann) boundary conditions [9], [10].

What about uniqueness in the inverse Neumann problem with only one incident direction? The counter-examples of [9] show that, in general, a polyhedral scatterer cannot be uniquely reconstructed using less than  $n$  incident waves. On the other hand, it was shown in [6] and [10] that one incident wave is enough to recover a polygonal obstacle (in the sense of Definition 1 for  $n = 2$ ). The aim of this paper is to prove the higher dimensional analogue of that result.

**Theorem.** *For fixed  $k > 0$  and  $d \in \mathbb{S}^{n-1}$ , a polyhedral obstacle  $D \subset \mathbb{R}^n$  ( $n \geq 2$ ) is uniquely determined by the far field pattern  $u_\infty$ .*

For the proof of our uniqueness result, the notion of a Neumann plane is of importance (cf. also [8]).

**Definition 2** *Let  $\Pi \subset \mathbb{R}^n$  be an  $(n - 1)$ -dimensional hyperplane. A non-void open connected component  $\pi$  of  $\Pi \cap D^c$  such that  $\partial_\nu u = 0$  in  $\pi$  (where  $u$  is the solution of (1.1)) is called a Neumann plane of  $u$ .*

Note that in contrast to a Dirichlet plane in the sound-soft case, which is always a bounded set (see [1], [8]), a Neumann plane may be unbounded. As a key preliminary result, we

will show in Section 2 that there are at most finitely many unbounded Neumann planes of  $u$  (Lemma 2). To carry out the proof of the theorem in Section 3, we modify the path and reflection arguments of [1] and [8] in that we construct a path to infinity avoiding the unbounded Neumann planes. Note that our approach here differs essentially from that in the 2D case [6], [10], which is mainly based on the finiteness of the set of bounded Neumann planes (lines) and is difficult to extend to higher dimensions.

## 2 Preliminaries

Let  $D \subset \mathbb{R}^n$  be a polyhedral obstacle, and let  $u \in H_{loc}^1(D^c)$  be the unique solution of the direct problem (1.1); see, e.g., [5, Chap. 3.4]. Note that the solution  $u$  to the homogeneous Helmholtz equation is real-analytic in  $D^c$ . We first collect some properties of the Neumann planes of  $u$ , which will be needed in the sequel.

Let  $\pi \subset \Pi$  be a Neumann plane of  $u$ , where  $\Pi \subset \mathbb{R}^n$  is an  $(n - 1)$ -dimensional hyperplane. Then its boundary  $\partial\pi$  is always a subset of  $\partial D$  (but may be void) since otherwise the zero set of  $\partial_\nu u$  on  $\Pi \cap D^c$  could be extended.

A Neumann plane  $\pi \subset \Pi$  may be bounded or unbounded, but  $\Pi$  can contain at most one unbounded Neumann plane for  $n \geq 3$  and at most two Neumann planes (lines) if  $n = 2$ . This is clear for  $n = 2$  since a connected open set in  $\mathbb{R}$  is a (bounded or unbounded) interval, and there cannot lie more than two unbounded intervals on an infinite straight line. Here and in the following, we refer to [12, Chaps. 2.9, 2.10] for the properties of connected sets. For  $n \geq 3$ , assume there are two different unbounded Neumann planes  $\pi_1, \pi_2$  on  $\Pi$ . However, outside a sufficiently large ball  $B \subset \Pi$ , there is no continuous curve on  $\Pi$  connecting points from  $\pi_1$  and  $\pi_2$  since it would intersect  $\partial D$ . Note that  $\partial D$  is bounded and  $B^c$  is connected.

Moreover, a hyperplane  $\Pi$  contains at most finitely many bounded Neumann planes. Note that  $\overline{D^c}$  is the union of  $Q^c$  and finitely many bounded convex polyhedra, where  $Q$  is a sufficiently large closed cube. Hence  $\overline{D^c}$  is the union of finitely many (possibly unbounded) convex polyhedra. Moreover, the intersection of the interior of such a polyhedron with  $\Pi$  is either void or an  $(n - 1)$ -dimensional open convex polyhedron. Thus the intersection of  $\Pi$  with  $D^c$  can only have finitely many connected components.

**Lemma 1** *The normal to an unbounded Neumann plane is always orthogonal to the wave vector  $d$  of the incident wave  $u^{in}$ .*

The proof is analogous to that of Lemma 9 in [2], using the fact that  $\lim_{r \rightarrow \infty} |\nabla u^{sc}| = 0$ ; see also Lemma 2 in [8].

For  $n = 2$ , it was proved in [10, Cor. 2.16] that all unbounded Neumann lines must lie on one infinite straight line, so that there exist at most two unbounded Neumann lines of  $u$ . Using Lemma 1, we can prove the following weaker version of this result which is valid in any dimensions and is sufficient for our purposes.

**Lemma 2** *There is at most a finite number of unbounded Neumann planes of  $u$ , say  $\pi_j \subset \Pi_j$ ,  $j = 1, \dots, N$ , where  $\Pi_j$  are  $(n-1)$ -dimensional hyperplanes. Of course, the set of unbounded Neumann planes may be void,  $N = 0$ .*

Note that the hyperplanes  $\Pi_j$  must be mutually different for  $n \geq 3$ , whereas this need not be the case for  $n = 2$ .

*Proof of Lemma 2.* Assume there exists a sequence of (different) hyperplanes  $\{\Pi_j : j \in \mathbb{N}\}$  such that there is (at least) one unbounded Neumann plane  $\pi_j \subset \Pi_j$  for each  $j$ . We first show that the convex hull  $\mathcal{D}$  of the polyhedral obstacle  $D$  must be symmetric with respect to each  $\Pi \in \{\Pi_j\}$ .

Let  $R$  denote the reflection with respect to  $\Pi$ . If  $\mathcal{D}$  were not symmetric, i.e.  $\mathcal{D} \neq R(\mathcal{D})$ , there would exist a vertex  $P \in R(\mathcal{D})$  such that  $P \in \mathcal{D}^c$ . Applying even reflection to the solution  $u$  of (1.1) and using the fact that at least  $n$  faces (cells) of  $D$  meet at the vertex  $R(P)$  of  $\mathcal{D}$  (and  $D$ ), we obtain  $n$  unbounded Neumann planes passing  $P$  and having linearly independent normal vectors. Note that  $P \in \mathcal{D}^c \subset D^c$  is a vertex of the convex set  $R(\mathcal{D})$ , and  $u$  is even symmetric with respect to  $\Pi$  and analytic in  $D^c$ . This contradicts Lemma 1 since there do not exist more than  $n - 1$  linearly independent vectors orthogonal to  $x_n$ -direction.

Finally, we observe that  $\mathcal{D}$  cannot be symmetric with respect to infinitely many hyperplanes since the number of vertices of  $\mathcal{D}$  is finite. This contradiction finishes the proof of the lemma.  $\square$

### 3 Proof of Theorem

#### 3.1 The case $N = 0$

To prove the theorem in this case (where no unbounded Neumann plane of  $u$  exists), we employ path and reflection arguments due to [1] and later modified in [8]. Here we follow [8] in spirit, but present a shorter version.

*Step 1: existence of a Neumann plane*

Assume contrarily that there is another polyhedral obstacle  $D_1 \neq D$  such that the far fields of  $u$  and  $u_1$  (the solution of problem (1.1) for  $D_1$ ) coincide on  $\mathbb{S}^{n-1}$ . The following arguments are standard, and we refer to [6], [8] for the details. We have

$$u_1 = u \quad \text{in the unbounded connected component } \Omega \text{ of } \mathbb{R}^n \setminus (D \cup D_1). \quad (3.1)$$

Furthermore, since  $D^c$  and  $D_1^c$  are connected, we obtain  $\partial\Omega \not\subset D \cup D_1$  and can assume without loss of generality that

$$S := (\partial D_1 \setminus D) \cap \partial\Omega \neq \emptyset. \quad (3.2)$$

By (3.1) and (3.2), there is a cell  $F \subset S$  such that  $\partial_\nu u = 0$  on  $F$ , and denoting by  $\Pi$  the hyperplane containing  $F$  and by  $\text{int}(F)$  the interior of the set  $F$ , we find a Neumann plane  $\pi$  of  $u$  such that  $\text{int}(F) \subset \pi \subset \Pi$  which must be bounded by our assumption.

*Step 2: path argument*

Choose a point  $P \in \text{int}(F)$  and a continuous and injective path  $\gamma(t)$ ,  $t \geq 0$ , starting at  $P = \gamma(0)$  and leading to infinity in the connected set  $\Omega$ . In fact, if  $B$  is a sufficiently large ball (centered at the origin), we can first connect  $P$  with some point  $Q \in B^c \cap \Omega$  by finitely many segments parallel to the coordinate axes, and  $Q$  may be connected to infinity e.g. by a ray parallel to  $x_n$ -direction. Note that the set  $\gamma := \{\gamma(t) = t \geq 0\}$  is homeomorphic to  $[0, \infty)$ .

Let  $\mathcal{M}$  be the set of intersection points of  $\gamma$  with all (bounded) Neumann planes of  $u$ . By Step 1,  $\mathcal{M} \neq \emptyset$ . Moreover,  $\mathcal{M}$  is bounded since there is no bounded Neumann plane outside a sufficiently large ball  $B$ . (A Neumann plane  $\pi$ , with  $\pi \cap B^c \neq \emptyset$  and  $B$  large, must be unbounded since  $\partial D$  is bounded.)

By Lemma 2 in [8],  $\mathcal{M}$  is also closed, hence compact. Thus there exists  $t_0 \geq 0$  such that no Neumann plane of  $u$  can intersect  $\gamma(t)$  for  $t > t_0$ . Let  $\pi_0 \subset \Pi_0$  be a Neumann plane passing  $\gamma(t_0)$ , where  $\Pi_0$  is an  $(n-1)$ -dimensional hyperplane.

*Step 3: reflection argument and final contradiction*

We now apply the reflection argument of [1, Lemma 3.7] to prove the existence of a Neumann plane  $\pi'$  intersecting  $\gamma(t)$  at some  $t' > t_0$  which is a contradiction.

Let  $R$  denote the reflection with respect to the plane  $\Pi_0$ , and choose  $x^+ = \gamma(t_0 + \varepsilon)$  for  $\varepsilon > 0$  sufficiently small and  $x^- = R(x^+)$ . Let  $G^\pm$  be the connected component of  $D^c \setminus \pi_0$  containing  $x^\pm$ , and denote by  $E^\pm$  the connected component of  $G^\pm \cap R(G^\mp)$  containing  $x^\pm$ . We set  $E = E^+ \cup \pi_0 \cup E^-$ . Note that  $E$  is a connected open set whose boundary consists of cells of  $\partial D$  and  $R(\partial D)$ .

Then, by the (even) reflection principle for the Helmholtz equation in  $D^c$ , we obtain that  $u$  is even symmetric in  $E$  (with respect to  $\Pi_0$ ), so that  $\partial_\nu u = 0$  on  $\partial E$  and  $E \cap \Pi_0$ . Moreover,  $E$  is bounded since otherwise  $\Pi_0$  would contain an unbounded Neumann plane.

Hence,  $\gamma(t)$  must intersect  $\partial E$  at some  $t' > t_0$ , so that there exists a Neumann plane  $\pi'$  passing  $\gamma(t')$ .  $\square$

**Remark 1** *The decisive step in the above argument is the boundedness of the set  $E$ . Here this is ensured by the fact that  $\Pi_0$  does not contain an unbounded Neumann plane.*

*Another possibility to prove boundedness of  $E$  is to use a bounded connected component  $G^-$  of  $D^c \setminus \pi_0$  with  $\partial_\nu u = 0$  on  $\partial G^-$ , in which case  $\Pi_0$  may contain an unbounded Neumann plane. We will employ a version of this argument to prove existence of a bounded Neumann plane in the case  $N \geq 1$ .*

## 3.2 The case $N \geq 1$

We now assume that there is at least one unbounded Neumann plane of  $u$ .

**Definition 3** *Let  $\pi$  be a Neumann plane of  $u$ . We write  $\pi \in \mathcal{P}_0$  if  $\pi$  is either an unbounded Neumann plane ( $\pi_j \subset \Pi_j$ ,  $j = 1, \dots, N$ ) or a bounded Neumann plane lying on one of the*

hyperplanes  $\Pi_j$  ( $j = 1, \dots, N$ ). Otherwise, if  $\pi$  is bounded and not contained in one of the hyperplanes  $\Pi_j$ , we write  $\pi \in \mathcal{P}_1$ .

To prove the theorem for  $N \geq 1$  by contradiction along the lines of Section 3.1, we will modify the path and reflection arguments appropriately. More precisely, we construct a path  $\gamma(t)$ ,  $t \geq 0$ , which starts at a Neumann plane of  $u$  and leads to infinity avoiding the finitely many (cf. Section 2) Neumann planes of  $\mathcal{P}_0$  for all  $t > 0$ . Then we prove existence of a bounded Neumann plane intersecting  $\gamma(t)$  at some  $t^* > 0$  by using the reflection argument. Finally, the path and reflection arguments of Section 3.2 are applied again to obtain a contradiction to the existence of a “last” intersection point of  $\{\gamma(t) : t \geq t^*\}$  with the Neumann planes of  $\mathcal{P}_1$ .

*Step 1.* By Lemma 2 we can assume that  $\mathcal{P}_0 = \{\pi_1, \dots, \pi_{N+M}\}$ , where  $M \geq 0$  and  $\pi_j$ ,  $j > N$ , are bounded Neumann planes lying on the hyperplanes  $\Pi_k$ ,  $k = 1, \dots, N$ . We introduce the open set (cf. (3.1))

$$\Sigma := \Omega \setminus \bigcup_{j=1}^{N+M} \pi_j, \quad (3.3)$$

which has only finitely many bounded and unbounded connected components, but at least one unbounded connected component. In fact,  $\bar{\Omega}$  is the union of finitely many (possibly unbounded) convex polyhedra, and by the hyperplanes  $\Pi_1, \dots, \Pi_N$  each of these convex polyhedra is cut into a finite number of polyhedral connected components. Recall that each  $\pi_j$  ( $j = 1, \dots, N+M$ ), i.e. any Neumann plane on  $\Pi_1, \dots, \Pi_N$ , extends to the boundary of  $\Omega$  and/or to infinity.

Let  $\Omega_1, \dots, \Omega_l$  be the bounded components of  $\Sigma$  if there is any, and the case  $l = 0$  is not excluded. We remove the bounded components (which may block the exit to infinity) from  $\Sigma$  by setting

$$\Sigma_1 := \Sigma \setminus \bigcup_{j=1}^l \bar{\Omega}_j \quad (3.4)$$

and observe that  $\partial\Sigma_1$  consists of cells lying on  $\partial\Omega$  and the Neumann planes from  $\mathcal{P}_0$ . The following lemma is crucial for the path and reflection arguments of the next step.

**Lemma 3** *There is a cell  $F^* \subset \partial\Sigma_1$  with the following properties.*

- (i)  $\text{int}(F^*)$  is contained in a Neumann plane  $\pi_0$  (which may be unbounded),
- (ii)  $F^*$  lies on the boundary of a bounded connected component, say  $G^-$ , of  $\mathbb{R}^n \setminus (\bar{\Sigma}_1 \cup D)$  such that  $\partial_\nu u = 0$  on  $\partial G^-$ ,
- (iii)  $F^*$  belongs to the boundary of some (unbounded) connected component  $\Omega^*$  of  $\Sigma_1$ .

*Proof of Lemma 3.* Let first  $l \geq 1$  and set  $\mathcal{K} = \bigcup_{j=1}^l \bar{\Omega}_j$ . Note that  $\bar{\Omega}$ ,  $\bar{\Sigma}_1$  and  $\mathcal{K}$  are unions of finitely many convex polyhedra (bounded ones in case of  $\mathcal{K}$ ), and by (3.3), (3.4) we have

$$\bar{\Omega} = \bar{\Sigma} = \mathcal{K} \cup \bar{\Sigma}_1, \quad \text{int}(\mathcal{K}) \cap \text{int}(\bar{\Sigma}_1) = \emptyset. \quad (3.5)$$

The connectedness of  $\Omega$  and (3.5) imply

$$S^* := \partial\bar{\Sigma}_1 \cap \partial\mathcal{K} \cap \Omega \neq \emptyset, \quad (3.6)$$

and since  $\partial\bar{\Sigma}_1$  and  $\partial\mathcal{K}$  consist of cells, there exists a cell  $F$  with

$$F \subset S^* \subset \partial\bar{\Sigma}_1 \subset \partial\Sigma_1. \quad (3.7)$$

Moreover, there is a subcell  $F' \subset F$  such that

$$F' \subset \partial\Omega_{j^*} \quad \text{for some } j^* \leq l. \quad (3.8)$$

To verify (3.8), we note that each  $\bar{\Omega}_j$  is the union of finitely many convex polyhedra, and the intersection of the boundary of such a polyhedron with  $F$  is either a cell or a convex set of dimension  $\leq n - 2$  (possibly void). However, the latter case cannot occur for each of those convex polyhedra since its intersection with  $F$  is nowhere dense in  $F$ .

From (3.6)–(3.8) we now observe that the cell  $F'$  satisfies (i) and (ii). In particular, it is contained in a Neumann plane of  $\mathcal{P}_0$ , and we can choose as  $G^-$  the bounded component  $\Omega_{j^*}$ .

Let now  $l = 0$ , i.e.  $\Sigma_1 = \Sigma$ . As in Step 1 in Section 3.1 we can choose a cell  $F \subset S \subset \partial\Omega$ , which additionally belongs to the boundary of a bounded connected component of  $D^c \setminus \partial D_1$ , say  $G^-$ , for which  $\partial_\nu u = 0$  on  $\partial G^-$ . This follows immediately from (3.1) and (3.2), provided there exists a bounded connected component of  $D^c \setminus (\partial D_1 \cap \partial\Omega)$ . If the latter set is unbounded and connected, we obtain  $\partial_\nu u = 0$  on  $\partial(D^c \setminus \partial D_1)$ . Note that we have  $D^c \setminus \partial D_1 \subset D^c \setminus (\partial D_1 \cap \partial\Omega)$  and thus  $u = u_1$  in  $D^c \setminus \partial D_1$ . Therefore, since  $\partial(D^c \setminus \partial D_1) \subset \partial D \cup \partial D_1$ ,  $\partial_\nu u = 0$  on  $\partial D$  and  $\partial_\nu u_1 = 0$  on  $\partial D_1$ , we obtain  $\partial_\nu u = 0$  on  $\partial(D^c \setminus \partial D_1)$ . Furthermore, by (3.2), and since  $D$  and  $D_1$  are assumed to be polyhedral obstacles, there is a bounded connected component of  $D^c \setminus \partial D_1$ , so that we have  $\partial_\nu u = 0$  on its boundary.

Therefore, in any case, there exists a cell  $F$  satisfying (i) and (ii). Finally, since  $\Sigma_1$  (where  $\Sigma_1 = \Sigma$  for  $l = 0$ ) only consists of finitely many (unbounded) polyhedral components, we can choose a subcell  $F^* \subset F$  which also satisfies (iii); see the proof of (3.8).  $\square$

*Step 2.* Now we select a (continuous and injective) path  $\gamma(t)$ ,  $t \geq 0$ , starting at some point  $P^* = \gamma(0) \in \text{int}(F^*)$  and leading to infinity in the connected set  $\Omega^*$  (from Lemma 3).

Indeed, for a sufficiently large ball  $B$  centred at the origin, we first connect  $P^*$  to some point  $Q^* \in B^c \cap \Omega^*$  by finitely many segments parallel to the coordinate axes. Then, by Lemma 1 and because there is no bounded Neumann plane outside  $B$ , we may connect  $Q^*$  to infinity by a ray parallel to  $x_n$ -direction.

Next we show that  $\gamma$  must intersect another Neumann plane  $\pi$  at some point  $P = \gamma(t^*)$ ,  $t^* > 0$ . For this we apply the reflection argument of Section 3.1 to  $u$  with respect to the hyperplane  $\Pi_0 \supset F^*$ , where  $F^*$  is the cell from Lemma 3.

In particular, we use the bounded component  $G^-$  from Lemma 3 and choose  $G^+$  as the connected component of  $D^c \setminus \pi_0$  containing  $\gamma(\varepsilon)$  for sufficiently small  $\varepsilon > 0$  (cf. Section 3.1, Step 3). Thus the existence of such a Neumann plane  $\pi$  is guaranteed (cf. also Remark 1), and by our construction of the path  $\gamma$ ,  $\pi$  must belong to  $\mathcal{P}_1$ .

Denoting by  $\mathcal{M}^*$  the set of intersection points of  $\gamma^* = \{\gamma(t) : t \geq t^*\}$  with all Neumann planes of  $\mathcal{P}_1$ , we can repeat the arguments of Section 3.1 to show that  $\mathcal{M}^*$  is compact. Hence, there exists a “last” intersection point of  $\gamma^*$  with the (bounded) Neumann planes from  $\mathcal{P}_1$ .

*Step 3.* Performing another reflection step, we obtain a final contradiction as in Section 3.1 since  $\gamma^*$  can only intersect Neumann planes from  $\mathcal{P}_1$ . This finishes the proof of the theorem.

□

**Remark 2** *The theorem is not true for polyhedral scatterers in the sense of Definition 1 as counter-examples in [9] show. In fact, the results of Liu and Zou [8] on the uniqueness of polyhedral scatterers in the inverse Neumann problem with  $n$  linearly independent incident directions cannot be improved in general. We also note that the arguments of Section 3.1 are sufficient to prove these uniqueness results with  $n$  incident waves since the corresponding (simultaneous) Neumann planes are always bounded.*

**Remark 3** *Combining our approach with some of the arguments in [9], [10], the theorem can be extended to polyhedral obstacles with mixed type and impedance boundary conditions.*

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