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PARTIAL DATA FOR THE CALDERÓN PROBLEM IN TWO DIMENSIONS

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Abstract. We show in two dimensions that measuring Dirichlet data for the conductivity equation on an open subset of the boundary and, roughly speaking, Neumann data in slightly larger set than the complement uniquely determines the conductivity on a simply connected domain. The proof is reduced to show a similar result for the Schrödinger equation. Using Carleman estimates with degenerate weights we construct appropriate complex geometrical optics solutions to prove the results.

1. Introduction

The Electrical Impedance Tomography (EIT) inverse problem consists in determining the electrical conductivity of a body by making voltage and current measurements at the boundary of the body. Substantial progress has been made on this problem since Calderón's pioneer contribution [7]. The inverse problem is also known as Calderón's problem. This problem can be reduced to studying the Dirichlet-to-Neumann (DN) map associated to the Schrödinger equation. A key ingredient in several of the results is the construction of complex geometrical optics for the Schrödinger equation (see [23] for a recent survey). Using this method in dimension $n \geq 3$ for the conductivity equation the first global uniqueness result for C^2 conductivities was proven in [20] and the regularity was improved to having $3/2$ derivatives in [3] and [18]. More singular conormal conductivities were considered in [11]. These results were also proven by showing a corresponding result for the Schrödinger equation.

In two dimensions the first global uniqueness result for Calderón's problem for full data was in [17] for conductivities having two derivatives, and this was improved to Lipschitz conductivities in [4] and for merely L^∞ conductivities in [2]. However, the corresponding result for the Schrödinger equation was not known until the recent breakthrough [5]. As for the uniqueness in determining two coefficicents, see [8].

Much less is known if the DN map is only measured on part of the boundary. We only review here the results where no a-priori information is assumed on the bounded potential. In dimensions $n \geq 3$ a global result is shown in [6] where partial measurements of the DN map are assumed. It is shown in [6] that for C^2 conductivities if we measure the DN map restricted to, roughly speaking, a slightly larger than the half of the boundary, then one can determine uniquely the potential. The proof relies on a Carleman estimate with an exponential weight with a linear phase. The Carleman estimate can also be used to

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construct complex geometrical optics solutions for the Schrödinger equation. In [15] the regularity assumption on the conductivity was relaxed to $C^{3/2+\epsilon}, \epsilon > 0$. Stability estimates for the uniqueness result of [6] were given in [12]. Stability estimates for the magnetic Schrödinger operator with partial data in the setting of [6] can be found in [22].

In [14], the result in [6] was generalized to show that by all possible pairs of Dirichlet data on an arbitrary open subset Γ_+ of the boundary and Neumann data on a slightly larger subboundary than $\partial\Omega \setminus \Gamma_+$, one can uniquely determine the potential. The case of the magnetic Schrödinger equation was considered in [9] and improvement on the regularity of the coefficients can be found in [16].

In this paper we show a result similar to [14] in two dimensions by constructing complex geometrical optics solutions with degenerate weights. We note that in two dimensions the problem is formally determined while in dimension three or higher is overdetermined. We now state the main result more precisely.

Let $\Omega \subset \mathbf{R}^2$ be a simply connected bounded domain with smooth boundary. The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current the potential $u \in H^1(\Omega)$ with given boundary voltage potential $f \in H^{\frac{1}{2}}(\partial\Omega)$ is a solution of the Dirichlet problem

$$\begin{aligned} \operatorname{div}(\gamma\nabla u) &= 0 \text{ in } \Omega, \\ u\big|_{\partial\Omega} &= f. \end{aligned} \tag{1.1}$$

The Dirichlet to Neumann (DN) map, or voltage to current map, is given by

$$\Lambda_\gamma(f) = \gamma\frac{\partial u}{\partial\nu}\Big|_{\partial\Omega}, \tag{1.2}$$

where ν denotes the unit outer normal to $\partial\Omega$. This problem can be reduced to studying the set of Cauchy data for the Schrödinger equation with the potential q given by:

$$q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}. \tag{1.3}$$

$$C_q = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega} \right) \mid (\Delta - q)u = 0 \text{ on } \Omega,\ u \in H^1(\Omega) \right\}. \tag{1.4}$$

We have $C_q \subset H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$.

By using a conformal map, thanks to the Kellog-Warchawski theorem (see e.g. p 42 [19]), without loss of generality we assume that $\Omega = \{x \in \mathbf{R}^2 | \, |x| < 1\}$.

Let $\Gamma_- = \{(\cos\theta, \sin\theta) | \theta \in (-\theta_0, \theta_0)\}$ be a connected subdomain in $\partial\Omega$ and $\theta_0 \in (0, \pi]$, $\widehat{x}_\pm$ the boundary of Γ_-: $\partial\Gamma_- = \{\widehat{x}_\pm\}$. Denote $\Gamma_+ = S^1 \setminus \Gamma_-$. Let $\epsilon > 0$ be a small number such that $\theta_0 + \epsilon \in (0, \pi]$. Denote by $\Gamma_{-,\epsilon} = \{(\cos\theta, \sin\theta) | \theta \in (-\theta_0 - \epsilon, \theta_0 + \epsilon)\}$ and by $\widehat{x}_{\pm,\epsilon}$ the endpoints of $\Gamma_{-,\epsilon}$.

We have

Theorem 1.1. *Let $q_j \in C^{1+\epsilon}(\overline{\Omega})$, $j = 1, 2$. Consider the following sets of partial Cauchy data:*

$$\mathcal{C}_{q_j} = \left\{ \left(u|_{\Gamma_+}, \frac{\partial u}{\partial \nu}\Big|_{\Gamma_{-,\epsilon}} \right) \mid (\Delta - q_j)u = 0 \ \text{on}\ \Omega,\ u \in H^1(\Omega) \right\}, \quad j = 1, 2. \tag{1.5}$$

Assume

$$\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$$

with some $\epsilon > 0$. *Then*

$$q_1 = q_2.$$

As a direct consequence of Theorem 1.1 we have

Corollary 1.1. *Let* $\gamma_j \in C^{3+\epsilon}(\overline{\Omega})$, $j = 1, 2$, *be strictly positive. Assume that* $\gamma_1 = \gamma_2$ *on* $\partial\Omega$ *and*

$$\Lambda_{\gamma_1} u = \Lambda_{\gamma_2} u \quad \text{on} \quad \Gamma_{-,\epsilon} \text{ for all } u \in H^{\frac{1}{2}}(\Gamma_+).$$

Then $\gamma_1 = \gamma_2$.

The proof of Theorem 1.1 uses Carleman estimates for the Laplacian with degenerate limiting Carleman weights. The results of [6] and [14] use complex geometrical optics solutions (CGO) of the form

$$u = e^{\tau(\varphi + \sqrt{-1}\psi)}(a + r), \tag{1.6}$$

where $\nabla\varphi \cdot \nabla\psi = 0, |\nabla\varphi|^2 = |\nabla\psi|^2$ and ϕ is a limiting Carleman weight and a is smooth and non-vanishing and $\|r\|_{L^2(\Omega)} = O(\frac{1}{\tau})$, $\|r\|_{H^1(\Omega)} = O(1)$. Examples of limiting Carleman weights are the linear phase $\varphi(x) = x \cdot \omega, \omega \in S^{n-1}$, used in [6], and the non-linear phase $\varphi(x) = \ln|x - x_0|$, where $x_0 \in \mathbf{R}^n \setminus \overline{\Omega}$ which was used in [14]. For a complete characterization of possible local Carleman weights in the Euclidean space and more general manifolds see [10].

In two dimensions the limiting Carleman weights are harmonic functions so that there is a larger class of complex geometrical optics solutions. This freedom was used in [24] to determine inclusions for a large class of systems in two dimensions. In particular, one can use the harmonic function $\phi = z^n$ as limiting Carleman weight, assuming that 0 is outside the domain.

In this paper we construct complex geometrical optics solutions of the form

$$u = e^{\tau(\varphi + \sqrt{-1}\psi)}(a + r)\sqrt{-1} + u_r \tag{1.7}$$

where u_r is a "reflected" term to guarantee that the solution vanishes in particular subsets of the boundary, φ is a harmonic function having a finite number of non-degenerate critical points in Ω, and ψ is the corresponding conjugate harmonic function. However we need to modify the form with ϕ harmonic but having non-degenerate critical points. Solutions as in (1.6) with degenerate harmonic functions were also used in [5] but here the phase function needs to satisfy further restrictions in order to use them for the partial data problem. Another complication is that the correction term r and the reflected term u_r do not have the same asymptotic behavior in τ as in [14] because of the degeneration of the phase so that one needs to further decompose these terms and analyze their asymptotic behavior in τ. See section 3 for more details. In section 2 we prove a general Carleman estimate for degenerate weights. Finally in section 4 we prove Theorem 1.1.

2. **Carleman estimates with degenerate weights**

Throughout the paper we use the following notations:

Notations $i = \sqrt{-1}$, $x_1, x_2, \xi_1, \xi_2 \in \mathbf{R}$, $z = x_1 + ix_2$,$\zeta = \xi_1 + i\xi_2$, $\frac{\partial}{\partial z} = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$, $H^{1,\tau}(\Omega)$ denotes the space $H^1(\Omega)$ with norm $\|v\|^2_{H^{1,\tau}(\Omega)} = \|v\|^2_{H^1(\Omega)} + \tau^2\|v\|^2_{L^2(\Omega)}$. The tangential derivative on the boundary is given by $\partial_\tau = \nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2}$, with $\nu = (\nu_1, \nu_2)$ the unit outer normal to $\partial\Omega$, $B(\widehat{x}, \delta) = \{x \in \mathbf{R}^2 | |x - \widehat{x}| \le \delta\}$, $f(x) : \mathbf{R}^2 \to \mathbf{R}^1$, f'' is the Hessian matrix with entries $\frac{\partial^2 f}{\partial x_i \partial x_j}$.

Let $\Phi(z) = \varphi_1(x_1, x_2) + i\varphi_2(x_1, x_2)$ be a holomorphic function in Ω:

$$\frac{\partial \Phi(z)}{\partial \overline{z}} = 0 \quad \text{in } \Omega, \quad \Phi \in C^2(\overline{\Omega}). \tag{2.1}$$

Denote by $\mathcal{H}$ the set of critical points of a function Φ

$$\mathcal{H} = \left\{ z \in \overline{\Omega} | \frac{\partial \Phi}{\partial z}(z) = 0 \right\}.$$

Assume that Φ has no critical points at the boundary and nondegenerate critical points in the interior;

$$\mathcal{H} \cap \partial\Omega = \{\emptyset\}, \quad \Phi''(z) \ne 0 \quad \forall z \in \mathcal{H}. \tag{2.2}$$

Then Φ we have only a finite number of critical points:

$$\operatorname{card} \mathcal{H} < \infty. \tag{2.3}$$

Denote $\frac{\partial \Phi}{\partial z}(z) = \psi_1(x_1, x_2) + i\psi_2(x_1, x_2)$.

We will prove Carleman estimates for the conjugated operator

$$\Delta_\tau = e^{\tau\Phi}\Delta e^{-\tau\Phi}. \tag{2.4}$$

We will use the factorization

(2.5)
$$e^{\tau\varphi_1}\Delta e^{-\tau\varphi_1}\widetilde{v} = \left(4\frac{\partial}{\partial z} - 2\tau\frac{\partial \Phi}{\partial z}\right)\left(4\frac{\partial}{\partial \overline{z}} - 2\tau\frac{\partial \overline{\Phi}}{\partial \overline{z}}\right)\widetilde{v} = \left(4\frac{\partial}{\partial \overline{z}} - 2\tau\frac{\partial \overline{\Phi}}{\partial \overline{z}}\right)\left(4\frac{\partial}{\partial z} - 2\tau\frac{\partial \Phi}{\partial z}\right)\widetilde{v}$$

and prove Carleman estimates first for every term in the factorization.

Proposition 2.1. *Let Φ satisfy (2.1) and (2.2). Let $\widetilde{f} \in L^2(\Omega)$ and $\widetilde{v}$ be solution to the problem*

$$2\frac{\partial \widetilde{v}}{\partial z} + \tau\frac{\partial \Phi}{\partial z}\widetilde{v} = \widetilde{f} \quad \text{in } \Omega \tag{2.6}$$

or $\widetilde{v}$ be solution to the problem

$$2\frac{\partial \widetilde{v}}{\partial \overline{z}} + \tau\frac{\partial \overline{\Phi}}{\partial \overline{z}}\widetilde{v} = \widetilde{f} \quad \text{in } \Omega. \tag{2.7}$$

In the case (2.6) we have

$$\left\| \left(\frac{\partial}{\partial x_1} + i\psi_2\tau \right) \widetilde{v} \right\|^2_{L^2(\Omega)} + \tau \int_{\partial\Omega} (\nabla\varphi_1, \nu)|\widetilde{v}|^2 d\sigma$$

$$+Re \int_{\partial\Omega} i \left(\left(\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \widetilde{v} \right) \overline{\widetilde{v}} d\sigma + \left\| \left(-i\frac{\partial}{\partial x_2} + \tau\psi_1 \right) \widetilde{v} \right\|^2_{L^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\Omega)} \tag{2.8}$$

and $\widetilde{v}$ *solves (2.7) we have*

$$\left\| \left(\frac{\partial}{\partial x_1} - i\psi_2\tau \right) \widetilde{v} \right\|^2_{L^2(\Omega)} + \tau \int_{\partial\Omega} (\nabla\varphi_1, \nu)|\widetilde{v}|^2 d\sigma + Re \int_{\partial\Omega} i \left(\left(-\nu_2 \frac{\partial}{\partial x_1} + \nu_1 \frac{\partial}{\partial x_2} \right) \widetilde{v} \right) \overline{\widetilde{v}} d\sigma$$

$$+ \left\| \left(i\frac{\partial}{\partial x_2} + \psi_1\tau \right) \widetilde{v} \right\|^2_{L^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\Omega)}. \tag{2.9}$$

Proof. We prove the statement of the proposition first for the equation $2\frac{\partial\widetilde{v}}{\partial z} + \tau\frac{\partial\Phi}{\partial z}\widetilde{v} = \widetilde{f}$. Since $\frac{\partial}{\partial z} + \tau\frac{\partial\Phi}{\partial z} = (\frac{\partial}{\partial x_1} + i\psi_2\tau) + (\frac{\partial}{i\partial x_2} + \psi_1\tau)$, taking the L^2- norm of the right and left hand sides of (2.6) we have

$$\left\| \left(\frac{\partial}{\partial x_1} + i\psi_2\tau \right) \widetilde{v} \right\|^2_{L^2(\Omega)} + 2\mathrm{Re} \left(\left(\frac{\partial}{\partial x_1} + i\psi_2\tau \right) \widetilde{v}, \left(-i\frac{\partial}{\partial x_2} + \psi_1\tau \right) \widetilde{v} \right)_{L^2(\Omega)}$$
$$+ \left\| \left(-i\frac{\partial}{\partial x_2} + \psi_1\tau \right) \widetilde{v} \right\|^2_{L^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\Omega)}.$$

Since we take the commutator to have $[(\frac{\partial}{\partial x_1} + i\psi_2\tau), (\frac{\partial}{i\partial x_2} + \psi_1\tau)] \equiv 0$, we obtain

$$\left\| \left(\frac{\partial}{\partial x_1} + i\psi_2\tau \right) \widetilde{v} \right\|^2_{L^2(\Omega)} + \left(\left(\frac{\partial}{\partial x_1} + i\psi_2\tau \right) \widetilde{v}, \overline{(-i\nu_2\widetilde{v})} \right)_{L^2(\partial\Omega)} + \left(\overline{\nu_1\widetilde{v}}, \left(-i\frac{\partial}{\partial x_2} + \psi_1\tau \right) \widetilde{v} \right)_{L^2(\partial\Omega)}$$
$$+ \left\| \left(-i\frac{\partial}{\partial x_2} + \psi_1\tau \right) \widetilde{v} \right\|^2_{L^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\Omega)}.$$

This equality implies

$$\left\| \left(\frac{\partial}{\partial x_1} + i\psi_2\tau \right) \widetilde{v} \right\|^2_{L^2(\Omega)} + \tau \int_{\partial\Omega} (\psi_1\nu_1 - \psi_2\nu_2)|\widetilde{v}|^2 d\sigma + \int_{\partial\Omega} i \left(\left(\nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2} \right) \widetilde{v} \right) \overline{\widetilde{v}} d\sigma$$
$$+ \left\| \left(-i\frac{\partial}{\partial x_2} + \psi_1\tau \right) \widetilde{v} \right\|^2_{L^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\Omega)}.$$

Finally by (2.1) we observe that $\psi_1 = \frac{1}{2}(\frac{\partial\varphi_1}{\partial x_1} + \frac{\partial\varphi_2}{\partial x_2}) = \frac{\partial\varphi_1}{\partial x_1}$ and $\psi_2 = \frac{1}{2}(\frac{\partial\varphi_2}{\partial x_1} - \frac{\partial\varphi_1}{\partial x_2}) = -\frac{\partial\varphi_1}{\partial x_2}$. Therefore from the above equality (2.8) follows immediately.

Now we prove the statement of the theorem first for the equation (2.7). Since $\frac{\partial}{\partial \overline{z}} + \tau\frac{\partial\overline{\Phi}}{\partial \overline{z}} = (\frac{\partial}{\partial x_1} - i\psi_2\tau) + (-\frac{\partial}{i\partial x_2} + \psi_1\tau)$, taking the L^2- norm of the right and left hand sides of (2.7) we have

$$\left\|\left(\frac{\partial}{\partial x_1}-i\psi_2\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}+2\mathrm{Re}\left(\left(\frac{\partial}{\partial x_1}-i\psi_2\tau\right)\widetilde{v},\left(i\frac{\partial}{\partial x_2}+\psi_1\tau\right)\widetilde{v}\right)_{L^2(\Omega)}$$
$$+\left\|\left(i\frac{\partial}{\partial x_2}+\psi_1\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}=\|\widetilde{f}\|^2_{L^2(\Omega)}.$$

Since $[(\frac{\partial}{\partial x_1}-i\psi_2\tau),(-\frac{\partial}{i\partial x_2}+\psi_1\tau)]\equiv 0$, we obtain

$$\left\|\left(\frac{\partial}{\partial x_1}-i\psi_2\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}+\left(\left(\frac{\partial}{\partial x_1}-i\psi_2\tau\right)\widetilde{v},\overline{(i\nu_2\widetilde{v})}\right)_{L^2(\partial\Omega)}+\left(\overline{\nu_1\widetilde{v}},\left(i\frac{\partial}{\partial x_2}+\psi_1\tau\right)\widetilde{v}\right)_{L^2(\partial\Omega)}$$
$$+\left\|\left(i\frac{\partial}{\partial x_2}+\psi_1\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}=\|\widetilde{f}\|^2_{L^2(\Omega)}.$$

This equality implies

$$\left\|\left(\frac{\partial}{\partial x_1}-i\psi_2\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}+\tau\int_{\partial\Omega}(\psi_1\nu_1-\psi_2\nu_2)|\widetilde{v}|^2d\sigma+\int_{\partial\Omega}i\left(\left(-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2}\right)\widetilde{v}\right)\overline{\widetilde{v}}d\sigma$$
$$+\left\|\left(i\frac{\partial}{\partial x_2}+\psi_1\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}=\|\widetilde{f}\|^2_{L^2(\Omega)}.$$

Finally we observe that $\psi_1=\frac{1}{2}(\frac{\partial\varphi_1}{\partial x_1}+\frac{\partial\varphi_2}{\partial x_2})=\frac{\partial\varphi_1}{\partial x_1}$ and $\psi_2=\frac{1}{2}(\frac{\partial\varphi_2}{\partial x_1}-\frac{\partial\varphi_1}{\partial x_2})=-\frac{\partial\varphi_1}{\partial x_2}$. Thus (2.9) follows immediately from the above equality (2.9), finishing the proof of the proposition. □

Let u solve

$$\Delta u=f\quad\text{in }\Omega,\quad u|_{\partial\Omega}=0. \tag{2.10}$$

Denote

$$\partial\Omega_+=\{(x_1,x_2)\in\partial\Omega|(\nabla\varphi_1,\nu)>0\}$$

and

$$\partial\Omega_-=\{(x_1,x_2)\in\partial\Omega|(\nabla\varphi_1,\nu)<0\}.$$

The main result of this section is the following Carleman estimate with degenerate weights.

Theorem 2.1. *Suppose that Φ satisfies (2.1), (2.2). Let $f\in L^2(\Omega)$ and a solution to (2.10) with $u\in H^1(\Omega)$ be a real valued function. Then there is a positive constant $C>0$ such that:*

$$\frac{1}{C_5}\left(\tau\|ue^{\tau\varphi_1}\|^2_{L^2(\Omega)}+\|ue^{\tau\varphi_1}\|^2_{H^1(\Omega)}+\tau^2\left\|\left|\frac{\partial\Phi}{\partial z}\right|ue^{\tau\varphi_1}\right\|^2_{L^2(\Omega)}\right)-\tau\int_{\partial\Omega_-}(\nu,\nabla\varphi_1)\left|\frac{\partial u}{\partial\nu}\right|^2e^{2\tau\varphi_1}d\sigma$$
$$\le C\left(\|fe^{s\varphi_1}\|^2_{L^2(\Omega)}+\tau\int_{\partial\Omega_+}(\nu,\nabla\varphi_1)\left|\frac{\partial u}{\partial\nu}\right|^2e^{2\tau\varphi_1}d\sigma\right). \tag{2.11}$$

Proof. As indicated earlier we can take Ω to be the unit ball. Denote $\widetilde{v} = ue^{\tau\varphi_1}$. Observe that $\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}$ and $\varphi_1(x_1, x_2) = \frac{1}{2}(\Phi(z) + \overline{\Phi(z)})$. Therefore

$$e^{\tau\varphi_1}\Delta e^{-\tau\varphi_1}\widetilde{v} = \left(4\frac{\partial}{\partial z} - 2\tau\frac{\partial\Phi}{\partial z}\right)\left(4\frac{\partial}{\partial \overline{z}} - 2\tau\frac{\partial\overline{\Phi}}{\partial \overline{z}}\right)\widetilde{v} = \left(4\frac{\partial}{\partial \overline{z}} - 2\tau\frac{\partial\overline{\Phi}}{\partial \overline{z}}\right)\left(4\frac{\partial}{\partial z} - 2\tau\frac{\partial\Phi}{\partial z}\right)\widetilde{v} = fe^{\tau\varphi_1}.$$

Denote $\widetilde{w}_1 = (4\frac{\partial}{\partial \overline{z}} - 2\tau\frac{\partial\overline{\Phi}}{\partial \overline{z}})\widetilde{v}$, $\widetilde{w}_2 = (4\frac{\partial}{\partial z} - 2\tau\frac{\partial\Phi}{\partial z})\widetilde{v}$ and $\frac{\partial\Phi}{\partial z} = \psi_1(x_1, x_2) + i\psi_2(x_1, x_2)$. Thanks to the boundary condition (2.10), we have

$$\widetilde{w}_1|_{\partial\Omega} = 4\partial_{\overline{z}}\widetilde{v}|_{\partial\Omega} = 2(\nu_1 + i\nu_2)\frac{\partial\widetilde{v}}{\partial\nu}|_{\partial\Omega},\ \ \widetilde{w}_2|_{\partial\Omega} = 4\partial_z\widetilde{v}|_{\partial\Omega} = 2(\nu_1 - i\nu_2)\frac{\partial\widetilde{v}}{\partial\nu}|_{\partial\Omega}.$$

By Proposition 2.1

$$\left\|\left(\frac{\partial}{\partial x_1} - i\psi_2\tau\right)\widetilde{w}_1\right\|^2_{L^2(\Omega)} - \tau\int_{\partial\Omega}(\nabla\varphi_1, \nu)\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma + \mathrm{Re}\int_{\partial\Omega} i\left(\left(\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2}\right)\widetilde{w}_1\right)\overline{\widetilde{w}_1}d\sigma$$
$$+\left\|\left(i\frac{\partial}{\partial x_2} + \psi_1\tau\right)\widetilde{w}_1\right\|^2_{L^2(\Omega)} = \frac{1}{4}\|fe^{s\varphi_1}\|^2_{L^2(\Omega)}$$

and

$$\left\|\left(\frac{\partial}{\partial x_1} + i\psi_2\tau\right)\widetilde{w}_2\right\|^2_{L^2(\Omega)} - \tau\int_{\partial\Omega}(\nabla\varphi_1, \nu)\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma + \mathrm{Re}\int_{\partial\Omega} i\left(\left(-\nu_2\frac{\partial}{\partial x_1} + \nu_1\frac{\partial}{\partial x_2}\right)\widetilde{w}_2\right)\overline{\widetilde{w}_2}d\sigma$$
$$+\left\|\left(i\frac{\partial}{\partial x_2} - \psi_1\tau\right)\widetilde{w}_2\right\|^2_{L^2(\Omega)} = \frac{1}{4}\|fe^{s\varphi_1}\|^2_{L^2(\Omega)}.$$

Let us simplify the integral $\mathrm{Re}\, i\int_{\partial\Omega}\left(\left(\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2}\right)\widetilde{w}_1\right)\overline{\widetilde{w}_1}d\sigma$. We recall that $\widetilde{v} = ue^{\tau\varphi_1}$ and $\widetilde{w}_1 = 2(\nu_1 + i\nu_2)\frac{\partial\widetilde{v}}{\partial\nu} = 2(\nu_1 + i\nu_2)\frac{\partial u}{\partial\nu}e^{\tau\varphi_1}$. Denote $A + iB = (\nu_1 + i\nu_2)$. Thus

$$\mathrm{Re}\int_{\partial\Omega} i\left(\left(\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2}\right)\widetilde{w}_1\right)\overline{\widetilde{w}_1}d\sigma =$$
$$\mathrm{Re}\int_{\partial\Omega} 4i\left(\left(\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2}\right)\left[(A+iB)\frac{\partial u}{\partial\nu}e^{\tau\varphi_1}\right]\right)(A - iB)\frac{\partial u}{\partial\nu}e^{\tau\varphi_1}d\sigma =$$
$$\mathrm{Re}\int_{\partial\Omega} 4i\left[\left(\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2}\right)(A+iB)\right]\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2(A - iB)d\sigma +$$
$$\mathrm{Re}\int_{\partial\Omega} 2i(A^2 + B^2)\left(\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2}\right)\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma =$$
$$4\int_{\partial\Omega}\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma.$$

Let us simplify the integral $\mathrm{Re}\int_{\partial\Omega} i\left(\left(-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2}\right)\widetilde{w}_2\right)\overline{\widetilde{w}_2}d\sigma$. We recall that $\widetilde{v}=ue^{\tau\varphi_1}$ and $\widetilde{w}_2=2(\nu_1-i\nu_2)\frac{\partial\widetilde{v}}{\partial\nu}=2(\nu_1-i\nu_2)\frac{\partial u}{\partial\nu}e^{\tau\varphi_1}$. We conclude

$$\begin{aligned}
&\mathrm{Re}\int_{\partial\Omega} i\left(\left(-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2}\right)\widetilde{w}_2\right)\overline{\widetilde{w}_2}d\sigma = \\
&\mathrm{Re}\int_{\partial\Omega} 4i\left(\left(-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2}\right)\left[(A-iB)\frac{\partial u}{\partial\nu}e^{\tau\varphi_1}\right]\right)(A+iB)\frac{\partial u}{\partial\nu}e^{\tau\varphi_1}d\sigma = \\
&\mathrm{Re}\int_{\partial\Omega} 4i\left[\left(-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2}\right)(A-iB)\right]\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2(A+iB)d\sigma - \\
&\mathrm{Re}\int_{\partial\Omega} 2i(A^2+B^2)\left(\nu_2\frac{\partial}{\partial x_1}-\nu_1\frac{\partial}{\partial x_2}\right)\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma = \\
&\int_{\partial\Omega} 4\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma.
\end{aligned}\tag{2.12}$$

Using the above formulae we obtain

$$\begin{aligned}
&\left\|\left(\frac{\partial}{\partial x_1}+i\psi_2\tau\right)\widetilde{w}_2\right\|^2_{L^2(\Omega)}+\left\|\left(i\frac{\partial}{\partial x_2}-\psi_1\tau\right)\widetilde{w}_2\right\|^2_{L^2(\Omega)}-2\tau\int_{\partial\Omega}(\nu,\nabla\varphi_1)\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma \\
&+\left\|\left(\frac{\partial}{\partial x_1}-i\psi_2\tau\right)\widetilde{w}_1\right\|^2_{L^2(\Omega)}+\left\|\left(i\frac{\partial}{\partial x_2}+\psi_1\tau\right)\widetilde{w}_1\right\|^2_{L^2(\Omega)} \\
&+4\int_{\partial\Omega}\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma=\frac{1}{2}\|fe^{s\varphi_1}\|^2_{L^2(\Omega)}.
\end{aligned}\tag{2.13}$$

Let a function $\widetilde{\psi}_k$ satisfy

$$\frac{\partial\widetilde{\psi}_1}{\partial x_1}=\psi_2,\quad \frac{\partial\widetilde{\psi}_2}{\partial x_2}=\psi_1\quad \text{in }\Omega.$$

We can rewrite equality (2.13) in the form

$$\begin{aligned}
&\left\|\frac{\partial}{\partial x_1}(e^{i\widetilde{\psi}_1\tau}\widetilde{w}_2)\right\|^2_{L^2(\Omega)}+\left\|\frac{\partial}{\partial x_2}(e^{-i\widetilde{\psi}_2\tau}\widetilde{w}_2)\right\|^2_{L^2(\Omega)}-2\tau\int_{\partial\Omega}(\nu,\nabla\varphi_1)\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma \\
&+\left\|\frac{\partial}{\partial x_1}(e^{-i\widetilde{\psi}_1\tau}\widetilde{w}_1)\right\|^2_{L^2(\Omega)}+\left\|\frac{\partial}{\partial x_2}(e^{i\widetilde{\psi}_2\tau}\widetilde{w}_1)\right\|^2_{L^2(\Omega)} \\
&+4\int_{\partial\Omega}\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma=\frac{1}{2}\|fe^{s\varphi_1}\|^2_{L^2(\Omega)}.
\end{aligned}\tag{2.14}$$

Observe that there exists some positive constant $C > 0$, independent of τ such that

$$\frac{1}{C}(\|\widetilde{w}_1\|^2_{L^2(\Omega)} + \|\widetilde{w}_2\|^2_{L^2(\Omega)}) \le \frac{1}{2}\left\|\frac{\partial}{\partial x_1}(e^{i\widetilde{\psi}_2\tau}\widetilde{w}_2)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\frac{\partial}{\partial x_2}(e^{i\widetilde{\psi}_1\tau}\widetilde{w}_2)\right\|^2_{L^2(\Omega)} - \tau\int_{\partial\Omega_-}(\nu,\nabla\varphi_1)\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma$$

$$+\frac{1}{2}\left\|\frac{\partial}{\partial x_1}(e^{-i\widetilde{\psi}_1\tau}\widetilde{w}_1)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\frac{\partial}{\partial x_2}(e^{i\widetilde{\psi}_2\tau}\widetilde{w}_1)\right\|^2_{L^2(\Omega)}. \tag{2.15}$$

Since $\widetilde{v}$ is the real-valued function we have

$$\left\|2\frac{\partial\widetilde{v}}{\partial x_1} + \tau\psi_1\widetilde{v}\right\|^2_{L^2(\Omega)} + \left\|2\frac{\partial\widetilde{v}}{\partial x_2} - \tau\psi_2\widetilde{v}\right\|^2_{L^2(\Omega)} \le C_0(\|\widetilde{w}_1\|^2_{L^2(\Omega)} + \|\widetilde{w}_2\|^2_{L^2(\Omega)}).$$

Therefore

$$4\left\|\frac{\partial\widetilde{v}}{\partial x_1}\right\|^2_{L^2(\Omega)} - 2\tau\int_\Omega\left(\frac{\partial\psi_1}{\partial x_1} - \frac{\partial\psi_2}{\partial x_2}\right)\widetilde{v}^2 dx$$

$$+\|\tau\psi_1\widetilde{v}\|^2_{L^2(\Omega)} + 4\left\|\frac{\partial\widetilde{v}}{\partial x_2}\right\|^2_{L^2(\Omega)} + \|\tau\psi_2\widetilde{v}\|^2_{L^2(\Omega)} \le C_1(\|\widetilde{w}_1\|^2_{L^2(\Omega)} + \|\widetilde{w}_2\|^2_{L^2(\Omega)}). \tag{2.16}$$

By the Cauchy-Riemann equations the second integral is zero.

Now since by assumption (2.2) the function Φ has zeros of at most order one, we have

$$\tau\|\widetilde{v}\|^2_{L^2(\Omega)} \le C\left(\|\widetilde{v}\|^2_{H^1(\Omega)} + \tau^2\left\|\left|\frac{\partial\Phi}{\partial z}\right|\widetilde{v}\right\|^2_{L^2(\Omega)}\right). \tag{2.17}$$

By (2.16) and (2.17)

$$\tau\|\widetilde{v}\|^2_{L^2(\Omega)} + \|\widetilde{v}\|^2_{H^1(\Omega)} + \tau^2\left\|\left|\frac{\partial\Phi}{\partial z}\right|\widetilde{v}\right\|^2_{L^2(\Omega)} \le C_1(\|\widetilde{w}_1\|^2_{L^2(\Omega)} + \|\widetilde{w}_2\|^2_{L^2(\Omega)}). \tag{2.18}$$

Using (2.18), we obtain from (2.14) and (2.15)

$$\frac{1}{C_5}\left(\tau\|\widetilde{v}\|^2_{L^2(\Omega)} + \|\widetilde{v}\|^2_{H^1(\Omega)} + \tau^2\left\|\left|\frac{\partial\Phi}{\partial z}\right|\widetilde{v}\right\|^2_{L^2(\Omega)}\right) - \tau\int_{\partial\Omega}(\nu,\nabla\varphi_1)\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma$$

$$+\int_{\partial\Omega}8\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma \le \|fe^{s\varphi_1}\|^2_{L^2(\Omega)} - \tau\int_{\partial\Omega_-}(x_1\nu_1 - x_2\nu_2)\left|\frac{\partial\widetilde{v}}{\partial\nu}\right|^2 d\sigma$$

concluding the proof of the theorem. □

We note that in the theorem we can add a zeroth order term to the Laplacian and the estimate is valid for large enough τ.

As usual the Carleman estimate implies existence of solutions of the solution for the Schrödinger equation satisfying estimates with appropriate weights.

Consider the following problem

$$\Delta u + q_0 u = f \quad \text{in } \Omega, \quad u|_{\partial\Omega_-} = 0. \tag{2.19}$$

Proposition 2.2. *Let $q_0 \in L^\infty(\Omega)$. There exists $\tau_0 > 0$ such that for all $\tau > \tau_0$ there exists a solution to problem (2.19) such that*

$$\|ue^{\tau\varphi_1}\|_{L^2(\Omega)} \le C\|fe^{\tau\varphi_1}\|_{L^2(\Omega)}/\sqrt{\tau}. \tag{2.20}$$

Proof. Let us introduce the space

$$H = \left\{ v \in H^1_0(\Omega) | \, \Delta v + q_0 v \in L^2(\Omega), \frac{\partial v}{\partial \nu}|_{\partial\Omega_+} = 0 \right\}$$

with the scalar product

$$(v_1, v_2)_H = \int_\Omega e^{-2\tau\varphi_1}(\Delta v_1 + q_0 v_1)(\Delta v_2 + q_0 v_2) dx.$$

By Proposition 2.1 H is a Hilbert space. Consider the linear functional on $H : v \to \int_\Omega v f dx$. By (2.11) this is a continuous linear functional with the norm estimated by a constant $C\|fe^{\tau\varphi_1}\|_{L^2(\Omega)}/\sqrt{\tau}$. Therefore by the Riesz theorem there exists an element $\widehat{v} \in H$ so that

$$\int_\Omega v f dx = \int_\Omega e^{-2\tau\varphi_1}(\Delta\widehat{v} + q_0\widehat{v})(\Delta v + q_0 v) dx.$$

Then, as a solution to (2.19), we take the function $u = e^{-2\tau\varphi_1}(\Delta\widehat{v} + q_0\widehat{v})$. □

3. Complex geometrical optics solutions with degenerate weights

In this section we construct the complex geometrical optics which we will use.

We first observe that we can put the set Γ_- and $\mathcal{S}$ in a more convenient position on the boundary of the unit ball and slightly deform the ball itself.

Namely

$$\Omega \subset B(0,1), \quad \Gamma_- \subset S^1, \quad \mathcal{S} = \partial\Omega \setminus \Gamma_{-,\epsilon} \subset S^1. \tag{3.1}$$

Let $\ell_+ \in \Gamma_+$ be a piece of $\partial\Omega$ between points $\hat{x}_+$ and $\hat{x}_{+,\epsilon}$ and $\ell_- \in \Gamma_+$ be a piece of $\partial\Omega$ between points $\hat{x}_-$ and $\hat{x}_{-,\epsilon}$. Then

$$\ell_\pm \subset B(0,1). \tag{3.2}$$

We construct CGO solutions of the Schrödinger equation $\Delta + q_1$, with q_1 satisfying the conditions of Theorem 1.1.

$$L_1 u = \Delta u + q_1 u = 0 \quad \text{in } \Omega. \tag{3.3}$$

Let $\Phi(z)$ be a holomorphic function satisfying (2.1) and (2.2). Let us fix small positive constants ϵ, ϵ' and consider two domains:

$$\partial\Omega_{-,-\epsilon} = \{x \in \partial\Omega | (\nabla\varphi_1, \nu) < -\epsilon\}, \quad \partial\Omega_{+,\epsilon'} = \{x \in \partial\Omega | (\nabla\varphi_1, \nu) > \epsilon'\}. \tag{3.4}$$

Suppose that

$$\overline{\Gamma}_- \subset \partial\Omega_{-,-\epsilon}, \tag{3.5}$$

and endpoints in $B(0,1)$ such that

$$\overline{\mathcal{S}} \subset \partial\Omega_{+,\epsilon'}. \tag{3.6}$$

We will construct solutions to (3.3) of the form

$$u_1(x) = e^{\tau\Phi(z)}a(z) - \chi_1(x)e^{\tau\Phi(\frac{1}{\bar z})}a\left(\frac{1}{\bar z}\right) + e^{\tau\varphi_1}u_{11} + e^{\tau\varphi_1}u_{12}, \quad u_1|_{\Gamma_-} = 0. \tag{3.7}$$

We explain in the next subsections the different phase functions φ_1 and the amplitudes $a(z)$ in (3.7). The function Φ and ϕ_1 satisfy (2.1) and (2.2). Moreover we derive the behavior for large τ of the different pieces of the CGO solutions.

3.1. **The amplitude $a(z)$ and the function χ_1.** The amplitude $a(z)$ has the following properties:

$$a \in C^2(\overline{\Omega}), \quad \frac{\partial a}{\partial \bar z} \equiv 0, \quad a(z) \neq 0 \text{ on } \overline{\Omega}.$$

Next we construct the cut-off function $\chi_1(x)$.

By (3.1), (3.2) there exists a neighborhood $\mathcal{O}_1$ of the set Γ_- such that $\widetilde{\varphi}_1(x) = \operatorname{Re}\Phi(\frac{1}{\bar z})$ is a harmonic function satisfying

$$\widetilde{\varphi}_1(x) < \varphi(x), \quad \forall x \in \Omega \cap \mathcal{O}_1, \tag{3.8}$$

$$\partial\Omega \cap \mathcal{O}_1 \subset \partial\Omega_{-,-\frac{\epsilon}{2}}, \tag{3.9}$$

$$\operatorname{supp} \nabla\chi_1 \subset\subset B(0,1) \cap \mathcal{O}_1. \tag{3.10}$$

Consider the following integral

$$J(\tau) = \int_\Omega \chi_1 r(x) e^{\tau\Phi(\frac{1}{\bar z}) - \tau\overline{\Phi(z)}}dx.$$

We have

Proposition 3.1. *Let $r \in C^{1+\epsilon}(\overline{\Omega})$ for some positive ϵ. Then*

$$J(\tau) = o\left(\frac{1}{\tau}\right).$$

Proof. Observe that the function χ_1 can be chosen in such a way that

$$\partial_{\bar z}\left(\Phi\left(\frac{1}{\bar z}\right) - \overline{\Phi(z)}\right)|_{\text{supp}\chi_1} \neq 0. \tag{3.11}$$

Assume that for some point from $\partial\Omega_{-,-\epsilon}$ we have

$$\partial_{\bar z}\left(\Phi\left(\frac{1}{\bar z}\right) - \overline{\Phi(z)}\right)|_{\text{supp}\chi_1} = 0,$$

and the above equality is equivalent to

$$\operatorname{Re}(\Phi'(z)z) = 0.$$

This equality and the Cauchy-Riemmann equations imply that at this point $\frac{\partial\varphi}{\partial\nu} = 0$ which is a contradiction. Since it suffices to choose $\operatorname{supp}\chi_1$ close to Γ_-, the proof of (3.11) is completed.

Therefore

$$J(\tau)=\int_\Omega \chi_1 r(x)e^{\tau\Phi(\frac{1}{\overline{z}})-\tau\overline{\Phi(z)}}dx=\frac{1}{\tau}\int_\Omega \chi_1 r(x)\frac{1}{\partial_{\overline{z}}(\Phi(\frac{1}{\overline{z}})-\overline{\Phi(z)})}\partial_{\overline{z}}e^{\tau\Phi(\frac{1}{\overline{z}})-\tau\overline{\Phi(z)}}dx.$$

Integrating by parts we have:

$$J(\tau)=-\frac{1}{\tau}\int_\Omega \partial_{\overline{z}}(\chi_1 r(x))\frac{1}{\partial_{\overline{z}}(\Phi(\frac{1}{\overline{z}})-\overline{\Phi(z)})}e^{\tau\Phi(\frac{1}{\overline{z}})-\tau\overline{\Phi(z)}}dx$$

$$+\frac{1}{2\tau}\int_{\partial\Omega} \chi_1 r(x)\frac{1}{\partial_{\overline{z}}(\Phi(\frac{1}{\overline{z}})-\overline{\Phi(z)})}(\nu_1+i\nu_2)e^{\tau\Phi(\frac{1}{\overline{z}})-\tau\overline{\Phi(z)}}d\sigma=J_1+J_2.$$

Observe that on $\partial\Omega$

$$e^{\tau\Phi(\frac{1}{\overline{z}})-\tau\overline{\Phi(z)}}=e^{2\tau i\mathrm{Im}\,\Phi(z)}.$$

Using stationary phase, taking into account that $\partial_\nu \mathrm{Re}\Phi=\partial_\tau \mathrm{Im}\Phi\ne 0$ on $\mathrm{supp}\chi_1\cap\partial\Omega$, we obtain

$$J_2=o\left(\frac{1}{\tau}\right).$$

Next we observe that since $r\in C^{1+\epsilon}(\overline{\Omega})$ we have

$$J_1=o\left(\frac{1}{\tau}\right).$$

The proof of the proposition is finished. ☐

3.2. **Construction of u_{11}.** The function $e^{\tau\Phi(z)}a(z)-\chi_1(x)e^{\tau\Phi(\frac{1}{\overline{z}})}a(\frac{1}{\overline{z}})$ does not satisfy (3.3). We construct the next term in the asymptotic expansion- the function u_{11}. Before we start the construction of this term we need several Propositions.

Let us introduce the operators:

$$\partial_{\overline{z}}^{-1}g=\frac{1}{2\pi i}\int_\Omega\frac{g(\zeta)}{\zeta-z}d\zeta\wedge d\overline{\zeta}=-\frac{1}{\pi}\int_\Omega\frac{g(\zeta)}{\zeta-z}d\xi_1d\xi_2,$$

$$\partial_{z}^{-1}g=-\frac{1}{2\pi i}\overline{\int_\Omega\frac{\overline{g}(\zeta)}{\zeta-z}d\zeta\wedge d\overline{\zeta}}=-\frac{1}{\pi}\int_\Omega\frac{g(\zeta)}{\overline{\zeta}-\overline{z}}d\xi_1d\xi_2. \tag{3.12}$$

Then we know (e.g., [25] p. 56):

Proposition 3.2. *Let $m\ge 0$ be an integer number, $\alpha\in(0,1)$. The operators $\partial_{\overline{z}}^{-1},\partial_z^{-1}\in\mathcal{L}(C^{m+\alpha}(\overline{\Omega}),C^{m+\alpha+1}(\overline{\Omega}))$.*

Here and henceforth $\mathcal{L}(X,Y)$ denotes the Banach space of all bounded linear operators from a Banach space X to another Banach space Y.

We define two other operators:

$$R_\Phi g=e^{\tau(\overline{\Phi(z)}-\Phi(z))}\partial_{\overline{z}}^{-1}(ge^{\tau(\Phi(z)-\overline{\Phi(z)})}),\quad \widetilde{R}_\Phi g=e^{\tau(\overline{\Phi(z)}-\Phi(z))}\partial_{z}^{-1}(ge^{\tau(\Phi(z)-\overline{\Phi(z)})}). \tag{3.13}$$

Proposition 3.3. *Let $g \in C^\epsilon(\overline{\Omega})$ for some positive ϵ. The function $R_\Phi g$ is a solution to*

$$\partial_{\overline{z}} R_\Phi g - \tau \overline{\frac{\partial \Phi(z)}{\partial z}} R_\Phi g = g \quad in\ \Omega. \tag{3.14}$$

The function $\widetilde{R}_\Phi g$ solves

$$\partial_z \widetilde{R}_\Phi g + \tau \frac{\partial \Phi(z)}{\partial z} \widetilde{R}_\Phi g = g \quad in\ \Omega. \tag{3.15}$$

Proof. The proof is by direct computations:

$$\partial_z \widetilde{R}_\Phi g + \tau \frac{\partial \Phi(z)}{\partial z} \widetilde{R}_\Phi g = \partial_z (e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_z^{-1} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})}))$$
$$+ \tau \frac{\partial \Phi(z)}{\partial z} (e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_{\overline{z}}^{-1} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})})) =$$
$$-\tau \frac{\partial \Phi(z)}{\partial z} (e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_z^{-1} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})})) + (e^{\tau(\overline{\Phi(z)} - \Phi(z))} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})}))$$
$$+ \tau \frac{\partial \Phi(z)}{\partial z} (e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_{\overline{z}}^{-1} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})})) = g.$$

□

Denote

$$\mathcal{O}_\epsilon = \{x \in \Omega | dist(x, \partial\Omega) \le \epsilon\}.$$

Proposition 3.4. *Let $g \in C^1(\Omega), g|_{\mathcal{O}_\epsilon} \equiv 0, g(x) \neq 0$ for all $x \in \mathcal{H}$. Then*

$$|R_\Phi g(x)| + |\widetilde{R}_\Phi g(x)| \le C \max_{x \in \mathcal{H}} |g(x)|/\tau \tag{3.16}$$

for all $x \in \mathcal{O}_{\epsilon/2}$. If $g \in C^2(\overline{\Omega})$ and $g|_{\mathcal{H}} = 0$ then

$$|R_\Phi g(x)| + |\widetilde{R}_\Phi g(x)| \le C/\tau^2 \tag{3.17}$$

for all $x \in \mathcal{O}_{\epsilon/2}$.

Proof. Observe that $e^{\tau(\Phi(z) - \overline{\Phi(z)})} = e^{2i\tau \mathrm{Im}\Phi(z)}$. By the Cauchy-Riemann equations, the sets of the critical points of $\Phi(z)$ and $\mathrm{Im}\Phi(z)$ are exactly the same. Therefore by our assumptions the Hessian of $\mathrm{Im}\Phi(z)$ is nondegenerate at each point of $\mathcal{H}$. It suffices to show that

$$\left| \int_\Omega \frac{g(\zeta)}{z - \zeta} d\zeta \wedge d\overline{\zeta} \right| \le C \max_{x \in \mathcal{H}} |g(x)|/\tau \quad \text{and} \quad \left| \int_\Omega \frac{g(\zeta)}{z - \zeta} d\zeta \wedge d\overline{\zeta} \right| \le C/\tau^2.$$

We observe that for any $z = x_1 + ix_2 \in \mathcal{O}_{\frac{\epsilon}{2}}$ function $\frac{g(\zeta)}{z-\zeta}$ is smooth compactly supported function of the variable ζ. The statement of the proposition follows from the standard stationary phase argument (see e.g. [13]). □

Denote

$$r(z) = \Pi_{k=1}^{\ell} (z - z_k) \ \text{ where } \mathcal{H} = \{z_1, \dots, z_\ell\}. \tag{3.18}$$

Proposition 3.5. *Let $g \in C^1(\overline{\Omega}), g|_{\mathcal{O}_\epsilon} \equiv 0$. Then for each $\delta \in (0,1)$ there exists a constant $C(\delta)$ such that*

$$\|\widetilde{R}_\Phi (\overline{r(z)} g)\|_{L^2(\Omega)} \le C(\delta) \|g\|_{C^1(\overline{\Omega})}/\tau^{1-\delta}, \quad \|R_\Phi (r(z) g)\|_{L^2(\Omega)} \le C(\delta) \|g\|_{C^1(\overline{\Omega})}/\tau^{1-\delta}. \tag{3.19}$$

Proof. Denote $v = \widetilde{R}_\Phi(\overline{r(z)}g)$. By Proposition 3.4

$$\|v\|_{L^2(\mathcal{O}_{\epsilon/2})} \le C/\tau. \tag{3.20}$$

Then by Proposition 3.3

$$\frac{\partial v}{\partial z} + \tau \frac{\partial \Phi}{\partial z} v = \overline{r(z)} g \quad \text{in } \Omega.$$

There exists a function p such that

$$-\frac{\partial p}{\partial \overline{z}} + \tau \frac{\overline{\partial \Phi(z)}}{\partial z} p = v \quad \text{in } \Omega$$

and there exists a constant $C > 0$ independent of τ such that

$$\|p\|_{L^2(\Omega)} \le C\|v\|_{L^2(\Omega)}. \tag{3.21}$$

Let χ be a nonnegative function such that $\chi \equiv 0$ on $\mathcal{O}_{\frac{\epsilon}{16}}$ and $\chi \equiv 1$ on $\Omega \setminus \mathcal{O}_{\frac{\epsilon}{8}}$. Setting $\widetilde{p} = \chi p$, using $g|_{\mathcal{O}_\epsilon} \equiv 0$, we have that

$$\int_\Omega \overline{r(z)} g \overline{p} dx = \int_{\Omega \setminus \mathcal{O}_\epsilon} \overline{r(z)} g \overline{p} dx = \int_\Omega \overline{r(z)} g \overline{\widetilde{p}} dx$$

and

$$-\frac{\partial \widetilde{p}}{\partial \overline{z}} + \tau \frac{\overline{\partial \Phi(z)}}{\partial z} \widetilde{p} = \chi v - p \frac{\partial \chi}{\partial \overline{z}} \quad \text{in } \Omega. \tag{3.22}$$

Then

$$\|\chi^{\frac{1}{2}} v\|^2_{L^2(\Omega)} = \int_\Omega \overline{r(z)} g \overline{p} dx + \int_\Omega p \frac{\partial \chi}{\partial \overline{z}} \overline{v} dx. \tag{3.23}$$

Note that

$$\|\widetilde{p}\|_{H^1(\Omega)} \le C\tau \|p\|_{L^2(\Omega)} \le C\tau \|v\|_{L^2(\Omega)}, \quad \int_\Omega \overline{r(z)} g \overline{p} dx = \int_\Omega g \overline{r(z) \widetilde{p}} dx. \tag{3.24}$$

Taking the scalar product of (3.22) and $\frac{\overline{r(z)}}{\partial_z \Phi(z)} g$ we obtain

$$\int_\Omega \frac{\overline{r(z)}}{\partial_z \Phi(z)} g \overline{\left(-\frac{\partial \widetilde{p}}{\partial \overline{z}} + \tau \frac{\overline{\partial \Phi(z)}}{\partial z} \widetilde{p} \right)} dx = \int_\Omega \frac{\overline{r(z)}}{\partial_z \Phi(z)} g \overline{\left(\chi v - p \frac{\partial \chi}{\partial \overline{z}} \right)} dx,$$

$$\tau \int_\Omega g \overline{r(z) \widetilde{p}} dx = \int_\Omega \frac{\overline{r(z)}}{\partial_z \Phi(z)} g \overline{\left(\chi v + p \frac{\partial \chi}{\partial \overline{z}} \right)} dx - \int_\Omega \frac{\partial}{\partial z} \left(\frac{\overline{r(z)}}{\partial_z \Phi(z)} g \right) \overline{\widetilde{p}} dx.$$

By (3.24) and the Sobolev embedding theorem, for each $\epsilon \in (0, \frac{1}{2})$ we have

$$\left| \int_\Omega \frac{\partial}{\partial z} \left(\frac{\overline{r(z)}}{\partial_z \Phi(z)} g \right) \overline{\widetilde{p}} dx \right| \le \left| \int_\Omega \frac{\overline{r(z)} \partial_z^2 \Phi(z)}{(\partial_z \Phi(z))^2} g \overline{\widetilde{p}} dx \right| + \left| \int_\Omega \frac{\overline{r(z)}}{\partial_z \Phi(z)} \frac{\partial g}{\partial z} \overline{\widetilde{p}} dx \right|$$

$$\le C\|g\|_{C^1(\overline{\Omega})} \left\| \frac{1}{\partial_z \Phi(z)} \right\|_{L^{2-\epsilon}(\Omega)} \|\widetilde{p}\|_{L^{\frac{2-\epsilon}{1-\epsilon}}(\Omega)} \le C\|\widetilde{p}\|_{H^{\delta_3(\epsilon)}(\Omega)} \le C\tau^{\delta_4} \|v\|_{L^2(\Omega)}. \tag{3.25}$$

Here we choose $\delta_3(\epsilon) > 0$ such that $\delta_3(\epsilon) \to +0$ as $\epsilon \to +0$ and $H^{\delta_3(\epsilon)}(\Omega) \subset L^{\frac{2-\epsilon}{1-\epsilon}}(\Omega)$. Therefore

$$\left|\int_\Omega gr(z)\overline{\widetilde{p}}dx\right| \le C\tau^{-1+\delta_4}\|v\|_{L^2(\Omega)} \quad \text{as } \delta_4 \to +0. \tag{3.26}$$

By (3.20)

$$\left|\int_\Omega p\frac{\partial\chi}{\partial\overline{z}}\overline{v}dx\right| \le C\|p\|_{L^2(\Omega)}\|v\|_{L^2(\mathcal{O}_{\frac{\epsilon}{8}})} \le C\|p\|_{L^2(\Omega)}/\tau. \tag{3.27}$$

By (3.21), (3.26) and (3.27) we obtain from (3.23)

$$\|v\|^2_{L^2(\Omega)} \le C(\tau^{-1+\delta_4}\|v\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}/\tau) \le C\tau^{-1+\delta_4}\|v\|_{L^2(\Omega)}.$$

In the last estimate we used (3.21). □

We construct the function u_{11} in the form $u_{11} = (u_{11,1}+u_{11,2})$ where the functions $u_{11,k}$ are defined in the following way: Let $e_i \in C^\infty(\overline{\Omega})$, $e_1 + e_2 \equiv 1$, e_2 is zero in some neighborhood of $\mathcal{H}$ and e_1 is zero in a neighborhood of $\partial\Omega$. The second term u_{11} in the asymptotic (3.7), is constructed to satisfy

$$\Delta u_{11} + 4\tau\frac{\partial\Phi(z)}{\partial z}\partial_{\overline{z}}u_{11} = aq_1 + o\left(\frac{1}{\tau}\right) \quad \text{in } \Omega. \tag{3.28}$$

Let $m_1(z), m_2(z), m_3(z)$ be polynomials satisfying

$$(\partial_{\overline{z}}^{-1}(aq_1) - m_1(z))|_{\mathcal{H}} = 0.$$

$$m_2(z)|_{\mathcal{H}} = 0, \quad (\partial_z(\partial_{\overline{z}}^{-1}(aq_1) - m_1(z)) - m_2(z))|_{\mathcal{H}} = 0.$$

$$m_3(z)|_{\mathcal{H}} = \partial_z m_3(z)|_{\mathcal{H}} = 0, \quad \partial_z^2(\partial_{\overline{z}}^{-1}(aq_1) - m_1(z) - m_2(z) - m_3(z))|_{\mathcal{H}} = 0.$$

The equation for u_{11} can be transformed into

$$4\partial_z u_{11} + 4\tau\frac{\partial\Phi(z)}{\partial z}u_{11} = \partial_{\overline{z}}^{-1}(aq_1) - \sum_{k=1}^3 m_k(z) + o\left(\frac{1}{\tau}\right).$$

Then

$$4\partial_z u_{11,1} + 4\tau\frac{\partial\Phi(z)}{\partial z}u_{11,1} = e_1\left(\partial_{\overline{z}}^{-1}(aq_1) - \sum_{k=1}^3 m_k(z)\right)$$

and we define $u_{11,1}$ as

$$u_{11,1}(x) = \frac{1}{4}\widetilde{R}_\Phi\left(e_1(\partial_{\overline{z}}^{-1}(aq_1) - \sum_{k=1}^3 m_k(z))\right) \tag{3.29}$$

and we define $u_{11,2}$ as

$$u_{11,2}(x) = \frac{1}{4}e_2\left(\partial_{\overline{z}}^{-1}(aq_1) - \sum_{k=1}^3 m_k(z)\right)/(\tau\partial_z\Phi(z)). \tag{3.30}$$

Since by the assumption e_2 vanishes near the zeros of Φ, the function $u_{11,2}$ is smooth.

We will apply Proposition 3.5 to the function $u_{11,1}$ to get the asymptotic behavior in τ. In order to do that we need to represent the function

$$\mathcal{G}_1 = e_1\left(\partial_{\overline{z}}^{-1}(aq_1) - \sum_{k=1}^{3} m_k(z)\right), \tag{3.31}$$

in the form

$$\mathcal{G}_1 = \overline{r(z)}g(x),$$

where g is some function from $C^1(\overline{\Omega})$. This is an equivalent representation of the function $m = \partial_{\overline{z}}^{-1}(aq_1) - \sum_{k=1}^{3} m_k(z)$ in the form

$$m = \overline{r(z)}g_1, \quad g_1 \in C^1(\overline{\Omega}).$$

We remind that the polynomial $r(z)$ is given by (3.18). Denote as $p = \partial_{\overline{z}}^{-1}(aq_1)$. Let x_j be a critical point of the function $Im\Phi$ and $z_j \in \mathcal{H}$ (see (3.18)). By Taylor's formula $p(x) = p(z_j) + p_1(z - z_j) + p_2(\overline{z} - \overline{z}_j) + p_{11}(z - z_j)^2 + p_{12}(z - z_j)(\overline{z} - \overline{z}_j) + p_{22}(\overline{z} - \overline{z}_j)^2 + q(z, \overline{z})$. Then $m = p_2(\overline{z} - \overline{z}_j) + p_{22}(\overline{z} - \overline{z}_j)^2 + p_{12}(z - z_j)(\overline{z} - \overline{z}_j) + q(z, \overline{z})$ and we set $g_1 = (p_2(\overline{z} - \overline{z}_j) + p_{22}(\overline{z} - \overline{z}_j)^2 + p_{12}(z - z_j)(\overline{z} - \overline{z}_j) + q(z, \overline{z}))/\overline{r(z)}$. Let us show that $g_1 \in C^1(\overline{\Omega})$. Obviously $(p_2(\overline{z} - \overline{z}_j) + p_{22}(\overline{z} - \overline{z}_j)^2 + p_{12}(z - z_j)(\overline{z} - \overline{z}_j))/\overline{r(z)}$ is a smooth function and $\widetilde{q}(z, \overline{z}) = q(z, \overline{z})/\overline{r(z)}$ is C^1 outside of $z = 0$. Continue the function $\widetilde{q}$ by zero on $z = 0$. Since $q = o(|z|^3)$ the partial derivatives of this function at zero vanishes.

By Proposition 3.5

$$\|u_{11,1}\|_{L^2(\Omega)} \le C(\delta)/\tau^{1-\delta} \quad \forall \delta \in (0, 1). \tag{3.32}$$

3.3. **Construction of u_{12}.** We will define u_{12} as a solution to the inhomogeneous problem

$$\Delta(u_{12}e^{\tau\varphi_1}) + q_1 u_{12} e^{\tau\varphi_1} \tag{3.33}$$

$$= (q_1 u_{11} + \Delta u_{11,2})e^{\tau\Phi} - L_1\left(\chi_1 e^{\tau\Phi(\frac{1}{\overline{z}})} a\left(\frac{1}{\overline{z}}\right)\right) \quad \text{in } \Omega,$$

$$u_{12}|_{\Gamma_-} = 0. \tag{3.34}$$

This can be done since

$$\|q_1 u_{11} + \Delta u_{11,2}\|_{L^2(\Omega)} \le C(\delta)/\tau^{1-\delta} \quad \forall \delta \in (0, 1)$$

and by (3.8), (3.10)

$$\left\| L_1\left(\chi_1 e^{\tau\Phi(\frac{1}{\overline{z}})} a\left(\frac{1}{\overline{z}}\right)\right) e^{-\tau\varphi_1} \right\|_{L^2(\Omega)} = o\left(\frac{1}{\tau^2}\right).$$

By Proposition 2.2 there exists a solution to (3.33) satisfying

$$\|u_{12}\|_{L^2(\Omega)} \le C/\tau^{\frac{3}{2}-\delta}, \quad \forall \delta \in (0, 1). \tag{3.35}$$

3.4. **Replacing Φ by $-\overline{\Phi}$.** Now we construct CGO solutions for the potential q_2 satisfying the conditions of the Theorem 1.1 but with Φ replaced by $-\overline{\Phi}$ and the solution vanishes on $\mathcal{S}$.

This is very similar to what we have already done.

Consider the Schrödinger equation

$$L_2 v = \Delta v + q_2 v = 0 \quad \text{in } \Omega. \tag{3.36}$$

We will construct solutions to (3.36) of the form

$$v_1(x) = e^{-\tau\overline{\Phi(z)}}\overline{b(z)} - \chi_1(x)e^{-\tau\overline{\Phi(\frac{1}{\overline{z}})}}\overline{b\left(\frac{1}{\overline{z}}\right)} + e^{-\tau\varphi_1}v_{11} + e^{-\tau\varphi_1}v_{12}, \quad v_1|_{\mathcal{S}} = 0. \tag{3.37}$$

The construction of v_1 repeats the corresponding steps of the construction of u_1. In fact the only difference is that the parameter τ is negative or in terms of the weight function we use $-\varphi_1$ instead of φ_1. We provide the details for the sake of completeness. The amplitude $b(z)$ has the following properties:

$$b \in C^2(\overline{\Omega}), \quad \frac{\partial b}{\partial \overline{z}} \equiv 0, \quad b(z) \neq 0 \text{ in } \overline{\Omega}.$$

Next we construct the cut-off function $\chi_2(x)$ with $\operatorname{supp}\chi_2 \in \mathcal{O}_2$ where $\mathcal{O}_2$ is a neighborhood of $\mathcal{S}$, and

$$\widetilde{\varphi}_1(x) > \varphi(x), \qquad \forall x \in \Omega \cap \mathcal{O}_2, \tag{3.38}$$

$$\partial\Omega \cap \mathcal{O}_2 \subset \partial\Omega_{+,\frac{\epsilon'}{2}}, \tag{3.39}$$

$$\operatorname{supp} \nabla\chi_2 \subset\subset B(0,1) \cap \mathcal{O}_2, \tag{3.40}$$

$$\operatorname{supp}\chi_2 \cap \operatorname{supp}\chi_1 = \emptyset. \tag{3.41}$$

Consider the following integral

$$\widetilde{J}(\tau) = \int_\Omega \chi_2 r(x) e^{-\tau\overline{\Phi(\frac{1}{\overline{z}})} + \tau\Phi(z)} dx.$$

Similarly to Proposition 3.1 we have

Proposition 3.6. *Let $r \in C^{1+\epsilon}(\overline{\Omega})$ for some positive ϵ. Then*

$$\widetilde{J}(\tau) = o\left(\frac{1}{\tau}\right).$$

Now we construct v_{11} Let $e_i \in C^\infty(\overline{\Omega})$, $e_1(x) + e_2(x) \equiv 1$, e_2 is zero on some neighborhood of $\mathcal{H}$ and e_1 is zero on some neighborhood of $\partial\Omega$. Then

$$\Delta v_{11} - 4\tau\overline{\frac{\partial\Phi(z)}{\partial z}}\partial_z v_{11} = \overline{b}q_2 + o\left(\frac{1}{\tau}\right).$$

Let $\widetilde{m}_1(\overline{z}), \widetilde{m}_2(\overline{z}), \widetilde{m}_3(\overline{z})$ be polynomials satisfying

$$(\partial_z^{-1}(\overline{b}q_2) - \widetilde{m}_1(\overline{z}))|_{\mathcal{H}} = 0,$$

$$\widetilde{m}_2(\overline{z})|_{\mathcal{H}} = 0, \quad (\partial_{\overline{z}}(\partial_z^{-1}(\overline{b}q_2) - \widetilde{m}_1(\overline{z})) - \widetilde{m}_2(\overline{z}))|_{\mathcal{H}} = 0$$

and

$$\widetilde{m}_3(\overline{z})|_{\mathcal{H}} = \partial_{\overline{z}}\widetilde{m}_3(\overline{z})|_{\mathcal{H}} = 0, \quad \partial^2_{\overline{z}}(\partial_z^{-1}(\bar{b}q_2) - \widetilde{m}_1(\overline{z}) - \widetilde{m}_2(\overline{z}) - \widetilde{m}_3(\overline{z}))|_{\mathcal{H}} = 0.$$

The equation for v_{11} can be transformed into

$$4\partial_{\overline{z}} v_{11} - 4\tau\overline{\frac{\partial\Phi(z)}{\partial z}} v_{11} = \left(\partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^{3}\widetilde{m}_k(\overline{z})\right) + o\left(\frac{1}{\tau}\right).$$

Then

$$4\partial_{\overline{z}} v_{11,1} - 4\tau\overline{\frac{\partial\Phi(z)}{\partial z}} v_{11,1} = e_1\left(\partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^{3}\widetilde{m}_k(\overline{z})\right)$$

and we take $v_{11,1}$ as

$$v_{11,1} = \frac{1}{4}R_\Phi\left(e_1\left(\partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^{3}\widetilde{m}_k(\overline{z})\right)\right) \tag{3.42}$$

and we take $v_{11,2}$ as

$$v_{11,2} = \frac{1}{4}e_2\left(\partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^{3}\widetilde{m}_k(\overline{z})\right) / \left(\tau\overline{\frac{\partial\Phi}{\partial z}}\right). \tag{3.43}$$

Thanks to our assumption on the function e_2, this function is smooth. Let us show that we can apply Proposition 3.4 to the function $v_{11,1}$. In order to do that we need to represent the function

$$\mathcal{G}_2 = e_1\left(\partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^{3}\widetilde{m}_k(\overline{z})\right), \tag{3.44}$$

in the form

$$\mathcal{G}_2 = zg(x),$$

where g is some function from $C^1(\overline{\Omega})$. This is an equivalent representation of the function $m = \partial_z^{-1}(\bar{b}q_2) - \sum_{k=1}^{3}\widetilde{m}_k(\overline{z})$ in the form

$$m = r(z)g_1, \quad g_1 \in C^1(\overline{\Omega}).$$

Denote as $p = \partial_z^{-1}(\bar{b}q_2)$. Let x_j be a critical point of the function $Im\Phi$ and z_j be an arbitrary critical point of the function Φ. By Taylor's formula $p(x) = p(x_j)+p_1(x_j)(z-z_j)+p_2(x_j)(\overline{z}-\overline{z}_j) + p_{11}(z-z_j)^2 + p_{12}(z-z_j)(\overline{z}-\overline{z}_j) + p_{22}(\overline{z}-\overline{z}_j)^2 + q(z,\overline{z})$. Then $m = p_1(x_j)(z-z_j) + p_{11}(z-z_j)^2 + p_{12}(z-z_j)(\overline{z}-z_j) + q(z,\overline{z})$ and we set $g_1 = (p_1(x_j)(z-z_j) + p_{11}(z-z_j)^2 + p_{12}(z-z_j)(\overline{z}-\overline{z}_j) + q(z,\overline{z}))/r(z)$. Let us show that $g_1 \in C^1(\overline{\Omega})$. Obviously $(p_1(z-z_j) + p_{11}(z-z_j)^2 + p_{12}(z-z_j)(\overline{z}-\overline{z}_j))/r(z)$ is a smooth function and $\widetilde{q}(z,\overline{z}) = q(z,\overline{z})/r(z)$ is C^1 outside of $z=0$. Continue the function $\widetilde{q}$ by zero on $z=0$. Since $q = o(|z|^3)$ the partial derivatives of this function at zero vanishes.

By Proposition 3.4

$$\|v_{11,2}\|_{L^2(\Omega)} + \|v_{11,1}\|_{L^2(\Omega)} \le C(\delta)/\tau^{1-\delta}, \quad \forall\delta \in (0,1). \tag{3.45}$$

Let v_{12} be a solution to the problem

$$\Delta(v_{12}e^{-\tau\varphi_1}) + q_2v_{12}e^{-\tau\varphi_1} \tag{3.46}$$

$$= (q_2 v_{11} + \Delta v_{11,2}) e^{-\tau\Phi} + L_2\left(\chi_2 e^{-\tau\overline{\Phi(\frac{1}{\bar z})}}\overline{b\left(\frac{1}{\bar z}\right)}\right) \quad \text{in } \Omega$$

and

(3.47) $$v_{12}|_{\mathcal{S}} = 0.$$

Then since

$$\|q_2 v_{11} + \Delta v_{11,2}\|_{L^2(\Omega)} \le C(\delta)/\tau^{1-\delta}, \quad \forall \delta \in (0,1)$$

and by (3.40)

$$\left\| L_2\left(\chi_2 e^{-\tau\overline{\Phi(\frac{1}{\bar z})}}\overline{b\left(\frac{1}{\bar z}\right)}\right) e^{\tau\varphi_1}\right\|_{L^2(\Omega)} = o\left(\frac{1}{\tau^2}\right),$$

by Proposition 2.2 there exists a solution to problem (3.46) such that

(3.48) $$\|v_{12}\|_{L^2(\Omega)} \le C(\delta)/\tau^{\frac{3}{2}-\delta}, \quad \forall \delta \in (0,1).$$

4. **Proof of the theorem**

Proposition 4.1. *Suppose that Φ satisfies (2.1),(2.2), (3.5) and (3.6). Let $\{x_1, \dots, x_\ell\}$ be the set of critical points of the function $Im\Phi$. Then for any potentials $q_1, q_2 \in C^\ell(\Omega), \ell > 1$ with the same DN maps and for any holomorphic functions a and b, we have*

$$\sum_{k=1}^{\ell} \frac{2\pi (q a \bar b)(x_k)}{\tau (det\, Im\Phi'')(x_k)} = 0, \quad q = q_1 - q_2.$$

Proof. Let u_1 be a solution to (3.3) and satisfy (3.37), and u_2 be a solution to the following equation

$$\Delta u_2 + q_2 u_2 = 0 \quad \text{in } \Omega, \quad u_2|_{\partial\Omega} = u_1, \quad \nabla u_2|_{\Gamma_{-,-\epsilon}} = \nabla u_1.$$

Denoting $u = u_1 - u_2$ we obtain

(4.1) $$\Delta u + q_2 u = -q u_1 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\Gamma_{-,-\epsilon}} = 0.$$

We multiply (4.1) by v and integrate over Ω. By (3.35) and (3.48), we have

$$0 = \int_\Omega q u_1 v dx = \int_\Omega q(a\bar b + \bar b u_{11} + a v_{11}) e^{\tau(\Phi(z) - \overline{\Phi(z)})} dx$$

$$+ \int_\Omega \left(q\chi_1(x) e^{\tau\Phi(\frac{1}{\bar z})} a\left(\frac{1}{\bar z}\right) \bar b e^{-\tau\overline{\Phi(z)}} + q\chi_1(x) e^{-\tau\overline{\Phi(\frac{1}{\bar z})}}\overline{b\left(\frac{1}{\bar z}\right)} a e^{\tau\Phi(z)}\right.$$

(4.2) $$\left. + q\chi_1(x) e^{\tau\Phi(\frac{1}{\bar z})} a\left(\frac{1}{\bar z}\right) \chi_2(x) e^{-\tau\overline{\Phi(\frac{1}{\bar z})}}\overline{b\left(\frac{1}{\bar z}\right)}\right) dx + o\left(\frac{1}{\tau}\right).$$

By Propositions 3.1 and 3.6

$$\int_\Omega \left(q\chi_1(x) e^{\tau\Phi(\frac{1}{\bar z})} a\left(\frac{1}{\bar z}\right) \bar b e^{-\tau\overline{\Phi(z)}} + q\chi_2(x) e^{-\tau\overline{\Phi(\frac{1}{\bar z})}}\overline{b\left(\frac{1}{\bar z}\right)} a e^{\tau\Phi(z)}\right) dx = o\left(\frac{1}{\tau}\right).$$

By (3.41)

$$\int_\Omega q\chi_1(x)e^{\tau\Phi(\frac{1}{\overline{z}})}a\left(\frac{1}{\overline{z}}\right)\chi_2(x)e^{-\tau\overline{\Phi(\frac{1}{\overline{z}})}}\overline{b\left(\frac{1}{\overline{z}}\right)}dx=0.$$

Therefore we can rewrite (4.2) as

$$\sum_{k=1}^{\ell}\frac{2\pi(q a\overline{b})(x_k)}{\tau(\det \mathrm{Im}\Phi'')(x_k)}+\int_\Omega q(\overline{b}u_{11}+av_{11})e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx+o\left(\frac{1}{\tau}\right)=0. \tag{4.3}$$

By (3.30), (3.43) and the fact that

$$\int_\Omega \overline{b}qu_{11,2}e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx=$$

$$\frac{1}{4\tau}\int_\Omega \overline{b}q\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1)-\sum_{k=1}^3 m_k(z))}{\tau\partial_z\Phi(z)}e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx=o\left(\frac{1}{\tau}\right), \tag{4.4}$$

and the fact that

$$\int_\Omega aqv_{11,2}e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx=$$

$$\frac{1}{4\tau}\int_\Omega aq\frac{e_2(\partial_{z}^{-1}(\overline{b}q_2)-\sum_{k=1}^3 \widetilde{m}_k(\overline{z}))}{\tau\overline{\partial_z\Phi(z)}}e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx=o\left(\frac{1}{\tau}\right), \tag{4.5}$$

which follows from the stationary phase $e_2|_{\mathcal{H}}=0$ we obtain

$$\sum_{k=1}^{\ell}\frac{2\pi(q a\overline{b})(x_k)}{\tau(\det \mathrm{Im}\Phi'')(x_k)}+\int_\Omega q(\overline{b}u_{11,1}+av_{11,1})e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx+o\left(\frac{1}{\tau}\right)=0. \tag{4.6}$$

By (3.13), (3.43) and (3.29)

$$0=\sum_{k=1}^{\ell}\frac{2\pi(q a\overline{b})(x_k)}{\tau(\det \mathrm{Im}\Phi'')(x_k)}+\int_\Omega q(\overline{b}\widetilde{R}_\Phi\mathcal{G}_1+aR_\Phi\mathcal{G}_2)e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx+o\left(\frac{1}{\tau}\right)=$$

$$\sum_{k=1}^{\ell}\frac{2\pi(q a\overline{b})(x_k)}{\tau(\det \mathrm{Im}\Phi'')(x_k)}-\int_\Omega((\partial_z^{-1}(q\overline{b}))\mathcal{G}_1+(\partial_{\overline{z}}^{-1}(qa))\mathcal{G}_2)e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx+o\left(\frac{1}{\tau}\right)=$$

$$\sum_{k=1}^{\ell}\frac{2\pi(q a\overline{b})(x_k)}{\tau(\det \mathrm{Im}\Phi'')(x_k)}+o\left(\frac{1}{\tau}\right). \tag{4.7}$$

We remind the definitions of the functions $\mathcal{G}_1$ and $\mathcal{G}_2$ introduced in (3.31) and (3.44).

In order to get rid of the integral $\int_\Omega((\partial_z^{-1}(q\overline{b}))\mathcal{G}_1+(\partial_{\overline{z}}^{-1}(qa))\mathcal{G}_2)e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx$, we used the stationary phase lemma (see e.g. Theorem 7.7.5 [13]) and the fact that $\mathcal{G}_1|_{\mathcal{H}}=\mathcal{G}_2|_{\mathcal{H}}=0$. Passing to the limit in this equality as $\tau\to+\infty$ we obtain $\sum_{k=1}^{\ell}\frac{2\pi(qa\overline{b})(x_k)}{\tau(\det \mathrm{Im}\Phi'')(x_k)}=0$. □

The Proposition 4.1 plays the key role in the proof of the Theorem 1.1. In order to be able to use this proposition we need to prove the existence of the weight function Φ. The following proposition will allow us to construct this function.

Let $\mathcal{G}_\epsilon$ be a non-empty open subset of the boundary $\partial\Omega$: the union of the segment between $\widehat{x}_+$ and $\widehat{x}_{+,\epsilon}$ and the segment between $\widehat{x}_{-,\epsilon}$ and $\widehat{x}_-$.

Consider the Cauchy problem for the Laplace operator

$$\Delta\psi = 0 \quad \text{in } \Omega, \quad \left(\psi, \frac{\partial\psi}{\partial\nu}\right)|_{\partial\Omega\setminus\mathcal{G}_\epsilon} = (a(x), b(x)). \tag{4.8}$$

The following proposition establishes the solvability of (4.8) for a dense set of Cauchy data.

Proposition 4.2. *There exist a set $\mathcal{O} \subset C^2(\overline{\partial\Omega\setminus\mathcal{G}_\epsilon}) \times C^1(\overline{\partial\Omega\setminus\mathcal{G}_\epsilon})$ such that for each $(a,b) \in \mathcal{O}$ problem (4.8) has at least one solution $\psi \in C^2(\overline{\Omega})$ and $\overline{\mathcal{O}} = C^2(\overline{\partial\Omega\setminus\mathcal{G}_\epsilon}) \times C^1(\overline{\partial\Omega\setminus\mathcal{G}_\epsilon})$.*

Proof. First we observe that without the loss of generality we may assume that $a \equiv 0$. Consider the following extremal problem

$$J(\psi) = \left\|\frac{\partial\psi}{\partial\nu} - b\right\|^2_{H^2(\partial\Omega\setminus\mathcal{G}_\epsilon)} + \epsilon\|\psi\|^2_{H^2(\partial\Omega)} + \frac{1}{\epsilon}\left\|\Delta^2\psi\right\|^2_{L^2(\Omega)} \to \inf, \tag{4.9}$$

$$\psi \in \mathcal{X}. \tag{4.10}$$

Here $\mathcal{X} = \left\{\delta(x) | \delta \in H^2(\Omega), \Delta^2\delta \in L^2(\Omega), \Delta\delta|_{\partial\Omega} = \delta|_{\partial\Omega\setminus\mathcal{G}_\epsilon} = 0, \delta|_{\partial\Omega} \in H^2(\partial\Omega), \frac{\partial\psi}{\partial\nu} \in H^2(\partial\Omega\setminus\mathcal{G}_\epsilon)\right\}$.

For each $\epsilon > 0$ there exists a unique solution to (4.9) which we denote as $\widehat{\psi}_\epsilon$. By the Fermat theorem (see e.g. [1] p. 155) we have

$$J'(\widehat{\psi}_\epsilon)[\delta] = 0, \quad \forall\delta \in \mathcal{X}.$$

Here $\mathcal{X} = \left\{\delta(x) | \delta \in H^2(\Omega), \Delta^2\delta \in L^2(\Omega), \Delta\delta|_{\partial\Omega} = \delta|_{\partial\Omega\setminus\mathcal{G}_\epsilon} = 0, \delta|_{\partial\Omega} \in H^2(\partial\Omega), \frac{\partial\psi}{\partial\nu} \in H^2(\partial\Omega\setminus\mathcal{G}_\epsilon)\right\}$.
This equality can be written in the form

$$\left(\frac{\partial\widehat{\psi}_\epsilon}{\partial\nu} - b, \frac{\partial\delta}{\partial\nu}\right)_{H^2(\partial\Omega\setminus\mathcal{G}_\epsilon)} + \epsilon(\widehat{\psi}_\epsilon, \delta)_{H^2(\partial\Omega)} + \frac{1}{\epsilon}(\Delta^2\widehat{\psi}_\epsilon, \Delta^2\delta)_{L^2(\Omega)} = 0.$$

This equality implies that the sequence $\{\frac{\partial\widehat{\psi}_\epsilon}{\partial\nu}\}$ is bounded in $H^2(\partial\Omega\setminus\mathcal{G}_\epsilon)$, the sequence $\{\sqrt{\epsilon}\widehat{\psi}_\epsilon\}$ converges to zero in $H^2(\partial\Omega)$ and $\left\{\frac{1}{\sqrt{\epsilon}}\Delta^2\widehat{\psi}_\epsilon\right\}$ is bounded in $L^2(\Omega)$.

Therefore there exist $q \in H^2(\partial\Omega\setminus\mathcal{G}_\epsilon)$ and $p \in L^2(\Omega)$ such that

$$\frac{\partial\widehat{\psi}_{\epsilon_k}}{\partial\nu} - b \rightharpoonup q \quad \text{weakly in } H^2(\partial\Omega\setminus\mathcal{G}_\epsilon) \tag{4.11}$$

and

$$\left(q, \frac{\partial\delta}{\partial\nu}\right)_{H^2(\partial\Omega\setminus\mathcal{G}_\epsilon)} + (p, \Delta^2\delta)_{L^2(\Omega)} = 0. \tag{4.12}$$

Next we claim that

$$\Delta p = 0 \quad \text{in } \Omega \tag{4.13}$$

in the sense of distributions. Suppose that (4.13) is already proved. This implies

$$(p, \Delta^2\delta)_{L^2(\Omega)} = 0 \quad \forall\delta \in H^4(\Omega), \Delta\delta|_{\partial\Omega} = \frac{\partial\Delta\delta}{\partial\nu}|_{\partial\Omega} = 0.$$

This equality and (4.12) imply that

$$\left(q, \frac{\partial \delta}{\partial \nu}\right)_{H^2(\partial\Omega\setminus\mathcal{G}_\epsilon)} = 0 \quad \forall \delta \in H^4(\Omega), \Delta\delta|_{\partial\Omega} = \frac{\partial \Delta\delta}{\partial \nu}|_{\partial\Omega} = 0. \tag{4.14}$$

Then using the trace theorem we conclude that $q = 0$ and (4.11) implies that

$$\frac{\partial \widehat{\psi}_{\epsilon_k}}{\partial \nu} - b \rightharpoonup 0 \quad \text{weakly in } H^2(\partial\Omega \setminus \mathcal{G}_\epsilon).$$

By the Sobolev embedding theorem

$$\frac{\partial \widehat{\psi}_{\epsilon_k}}{\partial \nu} - b \to 0 \quad \text{in } C^2(\partial\Omega \setminus \mathcal{G}_\epsilon).$$

Therefore the sequence $\{\widehat{\psi}_{\epsilon_k} - \widetilde{\psi}_{\epsilon_k}\}$, with

$$\Delta\widetilde{\psi}_{\epsilon_k} = \Delta\widehat{\psi}_{\epsilon_k} \quad \text{in } \Omega, \ \ \widetilde{\psi}_{\epsilon_k}|_{\partial\Omega} = 0$$

represents the desired approximation for solution of the Cauchy problem (4.8).

Now we prove (4.13). Let $\widetilde{x}$ be an arbitrary point in Ω and let $\widetilde{\chi}$ be a smooth function such that it is zero in some neighborhood of $\partial\Omega \setminus \mathcal{G}_\epsilon$ and the set $\mathcal{B} = \{x \in \Omega | \widetilde{\chi}(x) = 1\}$ contains an open connected subset $\mathcal{F}$ such that $\widetilde{x} \in \mathcal{F}$ and $\mathcal{G}_\epsilon \cap \overline{\mathcal{F}}$ is an open set in $\partial\Omega$. By (4.12)

$$0 = (p, \Delta^2(\widetilde{\chi}\delta))_{L^2(\Omega)} = (\widetilde{\chi}p, \Delta^2\delta)_{L^2(\Omega)} + (p, [\Delta^2, \widetilde{\chi}]\delta)_{L^2(\Omega)}.$$

That is,

$$(\widetilde{\chi}p, \Delta^2\delta)_{L^2(\Omega)} + ([\Delta^2, \widetilde{\chi}]^* p, \delta)_{L^2(\Omega)} = 0. \tag{4.15}$$

This equality implies that $\widetilde{\chi}p \in H^1(\Omega)$.

Next we take another smooth cut off function $\widetilde{\chi}_1$ such that $\operatorname{supp}\widetilde{\chi}_1 \subset \mathcal{B}$. A neighborhood of $\widetilde{x}$ belongs to $\mathcal{B}_1 = \{x | \widetilde{\chi}_1 = 1\}$, the interior of $\mathcal{B}_1$ is connected, and $\text{Int } \mathcal{B}_1 \cap \mathcal{G}_\epsilon$ contains an open subset $\mathcal{O}$ in $\partial\Omega$. Similarly to (4.16) we have

$$(\widetilde{\chi}_1 p, \Delta^2\delta)_{L^2(\Omega)} + ([\Delta^2, \widetilde{\chi}_1]^* p, \delta)_{L^2(\Omega)} = 0. \tag{4.16}$$

This equality implies that $\widetilde{\chi}_1 p \in H^2(\Omega)$. Let ω be a domain such that $\omega \cap \Omega = \emptyset$, $\partial\omega \cap \partial\Omega \subset \mathcal{O}$ contains a set open in $\partial\Omega$.

We extend p on ω by zero. Then

$$(\Delta(\widetilde{\chi}_1 p), \Delta\delta)_{L^2(\Omega\cup\omega)} + ([\Delta^2, \widetilde{\chi}]^* p, \delta)_{L^2(\Omega\cup\omega)} = 0.$$

Hence

$$\Delta^2(\widetilde{\chi}_1 p) = 0 \quad \text{in } \text{Int } \mathcal{B}_1 \cup \omega, \quad p|_\omega = 0.$$

By Holmgren's theorem $\Delta(\widetilde{\chi}_1 p)|_{\text{Int } \mathcal{B}_1} = 0$, that is, $(\Delta p)(\widetilde{x}) = 0$. □

Completion of the proof of Theorem 1.1. It suffices to prove that $q(0) = 0$. We take $\mathcal{G}_\epsilon$ in the previous proposition to be the union of the segment between $\widehat{x}_+$ and $\widehat{x}_{+,\epsilon}$ and the segment between $\widehat{x}_{-,\epsilon}$ and $\widehat{x}_-$.

We will show that $q_1(0) = q_2(0)$. By obvious changes of the argument below we can prove that $q_1(x) = q_2(x)$ for any point $x \in \Omega$.

Suppose for some Cauchy data the problem (4.8) is solved. Next, since Ω is simply connected, we construct a function φ such that the function $\Phi(z) = \varphi(x) + i\psi(x)$ is holomorphic in Ω. Consider the function $\widetilde{\Phi}(z) = z^2\Phi(z)$. Observe that $Im\widetilde{\Phi} = (x_1^2 - x_2^2)\psi(x) + 2x_1x_2\varphi(x)$. In particular by (4.8) and the Cauchy-Riemann equations, we have

$$Im\widetilde{\Phi}|_{\partial\Omega\setminus\mathcal{G}_\epsilon} = (x_1^2 - x_2^2)a(x) + 2x_1x_2c(x), \quad \frac{\partial c(x)}{\partial\tau} = b(x).$$

Since we can choose a, b from a dense set in $C^1(\overline{\partial\Omega\setminus\mathcal{G}_\epsilon})$ and the tangential derivatives of $(x_1^2 - x_2^2)$ and x_1x_2 are not equal zero simultaneously we can choose a, b such that

$$\frac{\partial \mathrm{Im}\widetilde{\Phi}}{\partial\tau}|_{\overline{\Gamma}_-} = \frac{\partial \mathrm{Re}\widetilde{\Phi}}{\partial\nu}|_{\overline{\Gamma}_-} < 0, \frac{\partial \mathrm{Im}\widetilde{\Phi}}{\partial\tau}|_{\overline{\partial\Omega\setminus\Gamma_{-,\epsilon}}} = \frac{\partial \mathrm{Re}\widetilde{\Phi}}{\partial\nu}|_{\overline{\partial\Omega\setminus\Gamma_{-,\epsilon}}} > 0. \tag{4.17}$$

Obviously the function $\widetilde{\Phi}$ has a critical point at zero. We may assume that $\partial_z^2\widetilde{\Phi}(0) \neq 0$. Really if $\Phi(0) \neq 0$ then $\partial_z^2\widetilde{\Phi}(0) = 2\Phi(0)$. If $\Phi(0) = 0$ we modify this function by adding a small real number: $\Phi(z) + \epsilon$. Obviously we will have (4.17).

A general function $\widetilde{\Phi}$ may have a degenerate critical points. In order to avoid them, we approximate the function $\widetilde{\Phi}$ in $C^1(\overline{\Omega})$ by a sequence of holomorphic functions $\{\widetilde{\Phi}_k\}_{k=1}^\infty$ such that

$$\widetilde{\Phi}_k \to \widetilde{\Phi} \quad \text{in } C^1(\overline{\Omega}), \ \frac{\partial \mathrm{Re}\widetilde{\Phi}_k}{\partial\nu}|_{\overline{\Gamma}_-} < 0 \ \frac{\partial \mathrm{Re}\widetilde{\Phi}_k}{\partial\nu}|_{\overline{\partial\Omega\setminus\Gamma_{-,\epsilon}}} > 0, \tag{4.18}$$

$$\mathcal{H}_k = \{z|\partial_z\Phi_k(z) = 0\}, \ card\mathcal{H}_k < \infty, \ H_k \cap \partial\Omega = \{\emptyset\}, \ \partial_z^2\widetilde{\Phi}_k(z_\ell) \neq 0, \ \forall z_\ell \in \mathcal{H}_k. \tag{4.19}$$

Let us show that such a sequence exists. For any $\epsilon_1 \in (0,1)$ we consider a function $\widetilde{\Phi}(z/(1+\epsilon_1))$. Obviously

$$\widetilde{\Phi}(\cdot/(1+\epsilon_1)) \to \widetilde{\Phi} \quad \text{in } C^1(\overline{\Omega}), \quad \text{as } \epsilon_1 \to +0.$$

Each function $\widetilde{\Phi}(z/(1+\epsilon_1))$ is holomorphic in $B(0, 1+\epsilon_1)$ and in $B(0,1)$ it can be approximated by a polynomial. Let $\epsilon_1 \in (0,1)$ be an arbitrary but fixed. Consider the sequence of such polynomials. Let $p(z) = \sum_{k=0}^\kappa c_k z^k$ be a polynomial from this sequence. Consider the polynomial $p'(z) = \sum_{k=1}^\kappa kc_k z^{k-1} = \Pi_{k=1}^\ell (z - \widehat{z}_k)^{s(k)}$. Here we assume $\widehat{z}_j \neq \widehat{z}_k$ for $k \neq j$. Let us construct an approximation of the polynomial $p(z)$ by a sequence of polynomials of the order κ. We do the construction in the following way. First pick up all $s(k)$ such that $s(k) \geq 2$. Denote the set of such indices as $\mathcal{U}$. Let $\widehat{k} \in \mathcal{U}$. Consider the sequences $\{\widehat{z}_{k,\ell_1,\epsilon_2}\}, \dots, \{\widehat{z}_{k,\ell_{s(\widehat{k})},\epsilon_2}\}$ such that

$$\widehat{z}_{k,\ell_i,\epsilon_2} \to \widehat{z}_k \quad \text{as } \epsilon_2 \to +0, \ \forall \ell_i \in \{\ell_1, \dots, \ell_{s(\widehat{k})}\},$$
$$\widehat{z}_{k,\ell_i,\epsilon_2} \neq \widehat{z}_{k,\ell_j,\epsilon_2}, \quad 1 \leq k \leq \kappa, \text{ if } \ell_i \neq \ell_j.$$

The polynomial

$$p'_{\epsilon_2}(z) = \Pi_{k=1}^\ell \Pi_{i=1}^{s(k)} (z - \widehat{z}_{k,i,\epsilon_2})$$

does not have any zeros of order greater then one. By the construction we have

$$p'_{\epsilon_2}(z) = \sum_{k=1}^\kappa kc_{k,\epsilon_2} z^{k-1}$$

satisfying

$$c_{k,\epsilon_2} \to c_k, \quad \forall k \in \{1, \dots, \kappa\}.$$

This means that the sequence of polynomials $p_{\epsilon_2}(z) = \sum_{k=0}^{\kappa} c_{k,\epsilon_2} z^k, c_{0,\epsilon_2} = c_0$ converges to $p(z)$ in $C^1(\Omega)$ and for small ϵ_2 these polynomials do not have critical points. Let us fix some sufficiently large $\widehat{k}$ and consider $k > \widehat{k}$. Then $card\,\mathcal{H}_{k_1} = card\,\mathcal{H}_{k_2}$ for all $k_1 > \widehat{k}$ and $k_2 > \widehat{k}$. Let $card\,\mathcal{H}_k = \ell$ and points $z_1 = \widetilde{x}_{1,1} + i\widetilde{x}_{2,1}, \dots, z_\ell = \widetilde{x}_{1,\ell} + i\widetilde{x}_{2,\ell}$ represent all critical points of the function $\widetilde{\Phi}_k(z) = \varphi_k(z) + i\psi_k(z)$.

Then by Proposition 4.1 we have

$$\sum_{j=1}^{\ell} \frac{q(\widetilde{x}_j)}{|\det \psi_k''(\widetilde{x}_j)|^{\frac{1}{2}}} = 0, \quad \widetilde{x}_j = (\widetilde{x}_{1,j}, \widetilde{x}_{2,j}).$$

Let $\widehat{j} \in \{1, \dots, \ell\}$. Consider the polynomial

$$p(z) = p_1(z) + ip_2(z) = \frac{d_1}{2} \frac{\Pi_{k\ne\widehat{j}}(z - z_k)^3}{\Pi_{k\ne\widehat{j}}(z_{\widehat{j}} - z_k)^3}(z - z_{\widehat{j}})^2 + d\frac{\Pi_{k\ne\widehat{j}}(z - z_k)^3}{\Pi_{k\ne\widehat{j}}(z_{\widehat{j}} - z_k)^3}(z - z_{\widehat{j}}).$$

Then

$$\partial_z^2 p(z_{\widehat{j}}) = d_1 \in \mathbb{C},\ \partial_z^2 p(z_{\widehat{j}}) = d \in \mathbb{C}, \tag{4.20}$$

$$p(z_j) = \partial_z p(z_j) = \partial_z^2 p(z_j) = 0\ j \in \{1, \dots, \ell\} \setminus \{\widehat{j}\}. \tag{4.21}$$

Consider the function $\widetilde{\Phi}_k(z) + \epsilon p(z)$. For small ϵ the set of critical points of this function consists exactly of ℓ points,which we denote as $z_j(\epsilon)$ ($\widetilde{x}_j(\epsilon) = (\mathrm{Re} z_j(\epsilon), \mathrm{Im} z_j(\epsilon))$). These critical points have the following properties:

$$z_j(0) = z_j, \frac{\partial z_j(\epsilon)}{\partial \epsilon}|_{\epsilon=0} = 0, \quad j \ne \widehat{j}, \quad \frac{\partial z_{\widehat{j}}(\epsilon)}{\partial \epsilon}|_{\epsilon=0} = \frac{d}{\partial_z^2 \Phi_j(z_{\widehat{j}})}. \tag{4.22}$$

In fact, there exists $\epsilon_0 > 0$ such that

$$z_j = z_j(\epsilon), \quad \forall \epsilon \in (-\epsilon_0, \epsilon_0), \quad j \ne \widehat{j}.$$

Then by Proposition 4.1 we have

$$J(\epsilon) = \sum_{j=1}^{\ell} \frac{q(\widetilde{x}_j(\epsilon))}{|\det(\psi_k + \epsilon p)''(\widetilde{x}_j(\epsilon))|^{\frac{1}{2}}} = 0.$$

Taking the derivative of the function $J(\epsilon)$ at zero, we have:

$$\begin{aligned} &\frac{1}{|\overline{\partial_z^2 \widetilde{\Phi}_k(\widetilde{x}_{\widehat{j}})}|^2} \frac{q_{x_1}(\widetilde{x}_{\widehat{j}}(0))\mathrm{Re}(d\overline{\partial_z^2 \widetilde{\Phi}_k(\widetilde{x}_{\widehat{j}})}) + q_{x_2}(\widetilde{x}_{\widehat{j}}(0))\mathrm{Im}(d\overline{\partial_z^2 \widetilde{\Phi}_k(\widetilde{x}_{\widehat{j}})})}{|\det(\psi_k)''(x_{\widehat{j}}(0))|^{\frac{1}{2}}} \\ &-\frac{1}{2}\sum_{j=1}^{\ell}\Big(\frac{q(\widetilde{x}_j(0))(-2\psi_{kx_1x_2}(\widetilde{x}_j(0))\mathrm{Im} p_{x_1x_2}(\widetilde{x}_j(0)) - 2\psi_{kx_1x_1}(\widetilde{x}_j(0))\mathrm{Im} p_{x_1x_1}(\widetilde{x}_j(0))}{|\det \psi_k''(\widetilde{x}_j(0))|^{\frac{3}{2}}} \\ &-\frac{1}{2}\frac{q(\widetilde{x}_j(0))(\partial_{x_1}(\det \psi_k''(\widetilde{x}_j(0))\mathrm{Re}(d\overline{\partial_z^2 \widetilde{\Phi}_k(\widetilde{x}_{\widehat{j}})} + \partial_{x_2}(\det \psi_k''(\widetilde{x}_{\widehat{j}}(0))\mathrm{Im}(d\overline{\partial_z^2 \widetilde{\Phi}_k(\widetilde{x}_{\widehat{j}})}}{|\overline{\partial_z^2 \widetilde{\Phi}_k(\widetilde{x}_{\widehat{j}})}|^2 |\det \psi_k''(\widetilde{x}_{\widehat{j}}(0))|^{\frac{3}{2}}}\Big) = 0. \end{aligned} \tag{4.23}$$

The first and third terms of (4.23) are independent of $\mathrm{Im}p_{x_1x_2}(\widetilde{x}_j(0))$ and $\mathrm{Im}p_{x_1x_1}(\widetilde{x}_j(0))$. Consequently

$$\frac{1}{2}\sum_{j=1}^{\ell}\frac{q(\widetilde{x}_j(0))(-2\psi_{kx_1x_2}(\widetilde{x}_j(0))\mathrm{Im}p_{x_1x_2}(\widetilde{x}_j(0))-2\psi_{kx_1x_1}(x_j(0))\mathrm{Im}p_{x_1x_1}(\widetilde{x}_j(0))}{|\mathrm{det}\psi''_k(\widetilde{x}_j(0))|^{\frac{3}{2}}}=0.$$

This formula and (4.22) imply that $q(\widetilde{x_j}(0)) = 0$. Since by (4.18) and (4.19) the set $\mathcal{H}_k$ converges to the set of critical points of $\widetilde{\Phi}$ and 0 belongs to the set of critical points of $\widetilde{\Phi}$, we have $q(0) = 0$. ■

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