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A CARLEMAN ESTIMATE FOR THE LINEAR SHALLOW SHELL EQUATION AND AN INVERSE SOURCE PROBLEM

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dedicated to the 70th birthday of Professor Li Ta-tsien

ABSTRACT. We consider an elastic bi-dimensional body whose reference configuration is a shallow shell. We establish a Carleman estimate for the linear shallow shell equation and apply it to prove a conditional stability for an inverse problem of determining external source terms by observations of displacement in a neighbourhood of the boundary over a time interval.

§1. Introduction.

The problems of controllability or observability for thin bi-dimensional bodies as membranes or plates have been discussed for many years and we refer to Lagnese and Lions [17], Lions [22], Russell [25] as early works. Here we do not intend a complete list of works and we list some of important papers on the controllability for shell and related problems; Cagnol, Lasiecka, Lebiedzik and Zolésio [3], Geymonat, Loreti and Valente [6], [7], Komornik [16], Lasiecka [18], Lasiecka and Marchand [19], Lasiecka, Triggiani and Valente [20], Telega and Bielski [26], Valente [27]. In

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this paper we consider the case of an elastic body that occupies a two-dimensional domain slightly curved and we establish a Carleman estimate. A Carleman estimate yields observability inequalities by an argument in Cheng, Isakov, Yamamoto and Zhou [4], Kazemi and Klibanov [14]. From a geometrical point of view, this is of great interest because a general shell can be approximated through a juxtaposition of shallow shells. From a theoretical point of view, the approach retained here which is based on Carleman estimates, is powerful, while by the multipliers technique, we were not able to obtain the controllability of a Koiter shell by a boundary action without a "shallowness" restriction (Miara and Valente [23]). In the static case, let us now briefly recall the equilibrium equations of a shallow shell with middle surface S, thickness 2ε and curvature $\varepsilon\theta$ (this expression of the curvature has been rigorously justified in Ciarlet and Miara [5]). More precisely, let $\Omega \subset \mathbb{R}^2$ be a bounded connected domain with Lipschitz boundary $\partial\Omega$, and let a point in Ω be denoted by $x = (x_1, x_2)$, let θ: $\overline{\Omega} \to \mathbb{R}$, $\theta \in C^3(\overline{\Omega})$. Let $\partial_j = \frac{\partial}{\partial x_j}$ and $\partial_{ij} = \partial_i\partial_j = \frac{\partial^2}{\partial x_i \partial x_j}$. Then the middle surface of the shell is therefore given by the set $S = \{(x_1, x_2, \varepsilon\theta(x_1, x_2)); (x_1, x_2) \in \overline{\Omega}\}$ and the shell with thickness 2ε occupies the domain $\{(x_1, x_2, \varepsilon\theta(x_1, x_2)) + x_3\mathbf{a}(x_1, x_2); (x_1, x_2) \in \overline{\Omega}, -\varepsilon \le x_3 \le \varepsilon\}$, where $\mathbf{a}(x_1, x_2)$ is the unit outward normal vector to the middle surface S at the point (x_1, x_2). Hence, if we denote by Θ^ε the mapping

$$\Theta^\varepsilon(x_1, x_2, x_3) = \big(x_1, x_2, \varepsilon\theta(x_1, x_2)\big) + x_3\mathbf{a}(x_1, x_2), \quad (x_1, x_2) \in \overline{\Omega},\ -\varepsilon \le x_3 \le \varepsilon,$$

then the reference configuration of the shell is $\Theta^\varepsilon\big(\Omega \times (-\varepsilon, \varepsilon)\big)$. Subjected to applied volume forces $\mathbf{F} = (F_1, F_2, F_3) \in \{L^2(\Omega)\}^3$ (for simplicity, no surface forces are taken into account in this presentation), a shell which is clamped on its lateral surface, undergoes a scaled Kirchhoff-Love displacement field of the form $\Big(u_1 -$

$x_3\partial_1 u_3, u_2 - x_3\partial_2 u_3, u_3\Big)$. For the precise meaning of the scaling, see Ciarlet and Miara [5]. In this section, Latin exponents and indices take their values in the set $\{1,2,3\}$, Greek indices take their values in the set $\{1,2\}$, the Einstein convention for repeated exponents and indices is used. These notations are used especially for the compatibility with the notations in Ciarlet and Miara [5], Miara and Valente [23]. [24]. Throughout this paper, bold face letters represent vectors.

The three-dimensional vector–valued function $\mathbf{u} = (u_1, u_2, u_3) : \overline{\Omega} \longrightarrow \mathbb{R}^3$ describes the displacement of the middle surface of the shell and solves a boundary value problem for shallow shell equations: Find $\mathbf{u} = (u_1, u_2, u_3) \in \{H_0^1(\Omega)\}^2 \times H_0^2(\Omega)$ such that

$$\begin{cases} -\displaystyle\int_\Omega m_{\alpha\beta}(u_3)\partial_{\alpha\beta}v_3 d\Omega + \int_\Omega n^\theta_{\alpha\beta}(\mathbf{u})(\partial_\alpha\theta)\partial_\beta v_3 d\Omega = \int_\Omega v_3\left(\int_{-1}^1 (f_3 + y_3\partial_\alpha f_\alpha)dy_3\right)d\Omega, \\ \qquad\qquad \forall v_3 \in H_0^2(\Omega), \\ \displaystyle\int_\Omega n^\theta_{\alpha\beta}(\mathbf{u})\partial_\beta v_\alpha d\Omega = \int_\Omega v_\alpha\left(\int_{-1}^1 f_\alpha dy_3\right)d\Omega, \quad \forall v_\alpha \in H_0^1(\Omega). \end{cases} \tag{1.1}$$

For a Saint Venant-Kirchhoff isotropic material with Lamé coefficients $\widetilde{\lambda}$ and $\widetilde{\mu}$, the constitutive law reads:

$$\begin{cases} m_{\alpha\beta}(u_3) := -\Big(\dfrac{4\widetilde{\lambda}\widetilde{\mu}}{3(\widetilde{\lambda}+2\widetilde{\mu})}(\Delta u_3)\delta_{\alpha\beta} + \dfrac{4}{3}\widetilde{\mu}\partial_{\alpha\beta}u_3\Big), \\ n^\theta_{\alpha\beta}(\mathbf{u}) = \dfrac{4\widetilde{\lambda}\widetilde{\mu}}{\widetilde{\lambda}+2\widetilde{\mu}}e^\theta_{\rho\rho}(\mathbf{u})\delta_{\alpha\beta} + 4\widetilde{\mu}e^\theta_{\alpha\beta}(\mathbf{u}), \\ e^\theta_{\alpha\beta}(\mathbf{u}) = \dfrac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) + \dfrac{1}{2}((\partial_\alpha\theta)\partial_\beta u_3 + (\partial_\beta\theta)\partial_\alpha u_3). \end{cases} \tag{1.2}$$

The outline of the contents of the paper is as follows: In the next section we rewrite the shallow shell equations in a more appropriate form to deal with the evolution problem, in section 3 we establish a Carleman estimate for the evolution problem and finally in section 4 we solve the inverse problem.

§2. The evolution problem of the shallow shell equation.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$, $x = (x_1, x_2)$, $\partial_j = \frac{\partial}{\partial x_j}$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_{jk} = \partial_j\partial_k = \frac{\partial^2}{\partial x_j \partial x_k}$, and let θ: $\overline{\Omega} \to \mathbb{R}$ be given and sufficiently smooth,

$$\mathbf{u}(x_1, x_2, t) = (u_1(x,t), u_2(x,t), u_3(x,t)),$$

$$\widetilde{\mathbf{u}}(x_1, x_2, t) = (u_1(x,t), u_2(x,t)) \quad : \overline{\Omega} \to \mathbb{R}^2.$$

Then, considering the force of inertia in (1.1) and (1.2), we can describe an evolution problem for **shallow shell equation:**

$$\rho\partial_t^2\widetilde{\mathbf{u}} - \mu\triangle\widetilde{\mathbf{u}} - (\lambda+\mu)\nabla\mathrm{div}\,\widetilde{\mathbf{u}} - (\mathrm{div}\,\widetilde{\mathbf{u}})\nabla\lambda - \left(\nabla\widetilde{\mathbf{u}} + (\nabla\widetilde{\mathbf{u}})^T\right)\nabla\mu$$
$$- \widetilde{\mathbf{G}}(D^2u_3, \nabla u_3) = \widetilde{\mathbf{F}}, \tag{2.1}$$

and

$$\rho\partial_t^2 u_3 + \frac{\lambda+2\mu}{3}\triangle^2 u_3 + \left(\frac{4}{3}\nabla\mu + \frac{2}{3}\nabla\lambda\right)\cdot\nabla\left(\triangle u_3\right)$$
$$+ \left(\mu\triangle\widetilde{\mathbf{u}} + (\lambda+\mu)\nabla\mathrm{div}\,\widetilde{\mathbf{u}}\right)\cdot\nabla\theta + G_3(\nabla\widetilde{\mathbf{u}}, D^2u_3, \nabla u_3) = F_3 \text{ in } Q \equiv \Omega\times(0,T). \tag{2.2}$$

Here

$$\widetilde{\mathbf{G}}(D^2u_3, \nabla u_3) = \sum_{k=1}^{2}\left\{\nabla\left(\lambda(\partial_k\theta)(\partial_k u_3)\right) + \partial_k\left\{\mu(\partial_k u_3)(\nabla\theta) + \mu(\partial_k\theta)(\nabla u_3)\right\}\right\},$$

$$G_3(\nabla\widetilde{\mathbf{u}}, D^2u_3, \nabla u_3) = \left\{(\mathrm{div}\,\widetilde{\mathbf{u}})\nabla\lambda + (\nabla\widetilde{\mathbf{u}} + (\nabla\widetilde{\mathbf{u}})^T)\nabla\mu + \widetilde{\mathbf{G}}(D^2u_3, \nabla u_3)\right\}\cdot\nabla\theta$$
$$+\frac{1}{3}(\triangle\lambda)\left(\triangle u_3\right) + \sum_{j,k=1}^{2}\frac{2}{3}\left(\partial_{jk}\mu\right)\left(\partial_{jk}u_3\right) + \sum_{j,k=1}^{2} n_{jk}^{\theta}\partial_{jk}\theta,$$

$$\widetilde{\mathbf{F}} = (F_1, F_2)$$

and

$$\lambda = 4\widetilde{\lambda}\widetilde{\mu}/(\widetilde{\lambda} + 2\widetilde{\mu}), \quad \mu = 2\widetilde{\mu},$$

$$\triangle = \partial_1^2 + \partial_2^2, \quad \nabla = (\partial_1, \partial_2), \quad \nabla_{x,t} = (\partial_1, \partial_2, \partial_t),$$

$$\mathrm{div}\,\widetilde{\mathbf{u}} = \partial_1 u_1 + \partial_2 u_2, \quad \mathrm{rot}\,\widetilde{\mathbf{u}} = \partial_1 u_2 - \partial_2 u_1,$$

$$\nabla\widetilde{\mathbf{u}} = \begin{pmatrix} \partial_1 u_1 & \partial_1 u_2 \\ \partial_2 u_1 & \partial_2 u_2 \end{pmatrix}, \quad D^2 u_3 = (\partial_1^2 u_3, \partial_2^2 u_3, \partial_1\partial_2 u_3)$$

and $(\nabla\widetilde{\mathbf{u}})^T$ is the transpose matrix of $\nabla\widetilde{\mathbf{u}}$.

Henceforth we set

$$L_e\mathbf{u} = \rho\partial_t^2\widetilde{\mathbf{u}} - \mu\triangle\widetilde{\mathbf{u}} - (\lambda+\mu)\nabla\mathrm{div}\,\widetilde{\mathbf{u}} - (\mathrm{div}\,\widetilde{\mathbf{u}})\nabla\lambda - \left(\nabla\widetilde{\mathbf{u}} + (\nabla\widetilde{\mathbf{u}})^T\right)\nabla\mu$$
$$- \widetilde{\mathbf{G}}(D^2u_3, \nabla u_3)$$

and

$$L_p\mathbf{u} = \rho\partial_t^2 u_3 + \frac{\lambda+2\mu}{3}\triangle^2 u_3 + \left(\frac{4}{3}\nabla\mu + \frac{2}{3}\nabla\lambda\right)\cdot\nabla(\triangle u_3)$$
$$+ (\mu\triangle\widetilde{\mathbf{u}} + (\lambda+\mu)\nabla\mathrm{div}\,\widetilde{\mathbf{u}})\cdot\nabla\theta + G_3(\nabla\widetilde{\mathbf{u}}, D^2u_3, \nabla u_3) \text{ in } Q.$$

§3. Carleman estimate.

In this section, we establish a Carleman estimate for the shallow shell equation.

We assume that $\rho = \rho(x)$, $\lambda = \lambda(x)$ and $\mu = \mu(x)$ are in $C^2(\overline{\Omega})$ and positive in $\overline{\Omega}$. We set $t_0 = T/2$,

$$\varphi(x,t) = e^{\gamma(|x-x_0|^2 - \nu|t-t_0|^2)}, \tag{3.1}$$

γ and ν are positive constants, $x_0 = (x_0^1, x_0^2) \in \mathbb{R}^2\setminus\overline{\Omega}$. We set

$$Lu_3 = \rho\partial_t^2 u_3 + \frac{\lambda+2\mu}{3}\triangle^2 u_3 + \left(\frac{4}{3}\nabla\mu + \frac{2}{3}\nabla\lambda\right)\cdot\nabla(\triangle u_3).$$

First we present

Lemma 1. *We assume that ρ, μ and λ are in $C^2(\overline{\Omega})$ and positive on $\overline{\Omega}$ and that*

$$\left(\nabla\log\left(\sqrt{\frac{3\rho}{\lambda+2\mu}}\right)(x)\cdot(x-x_0)\right) > -2, \quad x\in\overline{\Omega}. \tag{3.2}$$

Let $\nu > 0$ be arbitrarily fixed in (3.1). Then there exists a number $\gamma_0 > 0$ such that for arbitrary $\gamma \ge \gamma_0$, we can choose $s_0 \ge 0$ satisfying: there exists a constant $C > 0$ such that

$$\int_Q \Bigg\{ s|\nabla \partial_t v|^2 + s|\nabla \triangle v|^2 + s^3|\partial_t v|^2 + s^3|\triangle v|^2 + s\sum_{j,k=1}^{2} |\partial_j\partial_k v|^2 + s^4|\nabla v|^2 + s^6|v|^2 + \sum_{j,k,\ell=1}^{2} |\partial_j\partial_k\partial_\ell v|^2 \Bigg\} e^{2s\varphi} dxdt \le C\int_Q |Lv|^2 e^{2s\varphi} dxdt$$

for all $s > s_0$ and every real-valued $v \in L^2(0,T;H_0^4(\Omega))\cap H^1(0,T;H_0^3(\Omega))\cap H^2(0,T;H_0^1(\Omega))$ satisfying $v(\cdot,0) = \partial_t v(\cdot,0) = v(\cdot,T) = \partial_t v(\cdot,T) = 0$, provided that the right hand side of what is finite. The constants s_0 and C continuously depend on T, γ, $\|\rho\|_{C^1(\overline{\Omega})}$, $\|\lambda\|_{C^1(\overline{\Omega})}$, $\|\mu\|_{C^1(\overline{\Omega})}$, and γ_0 continuously depends on T, $\|\rho\|_{C^1(\overline{\Omega})}$, $\|\lambda\|_{C^1(\overline{\Omega})}$, $\|\mu\|_{C^1(\overline{\Omega})}$.

We note that if λ and μ are positive constants, then condition (3.2) is automatically satisfied.

Proof of Lemma 1. Except for the term $\sum_{j,k,\ell=1}^2 |\partial_j\partial_k\partial_\ell v|^2 e^{2s\varphi}$, by Yuan and Yamamoto [28], all the terms on the left hand side is proved to be estimated by the right hand side. We have to estimate $\partial_j\partial_k\partial_\ell v$. We have

$$\triangle((\partial_j v)e^{s\varphi}) = \triangle(\partial_j v)e^{s\varphi} + 2s\nabla(\partial_j v)\cdot(\nabla\varphi)e^{s\varphi} + (\partial_j v)(s\triangle\varphi + s^2|\nabla\varphi|^2)e^{s\varphi},$$

and so

$$|\triangle((\partial_j v)e^{s\varphi})|^2 \le C|\triangle(\partial_j v)|^2 e^{2s\varphi} + Cs^2|\nabla(\partial_j v)|^2 e^{2s\varphi} + Cs^4|\nabla v|^2 e^{2s\varphi}. \tag{3.3}$$

Similarly we have

$$|\triangle(ve^{s\varphi})|^2 \le C|\triangle v|^2 e^{2s\varphi} + Cs^2|\nabla v|^2 e^{2s\varphi} + Cs^4|v|^2 e^{2s\varphi}.$$

Lemma 1 yields

$$s^2\int_Q |\triangle(ve^{s\varphi})|^2 dxdt \le C\int_Q |Lv|^2 e^{2s\varphi} dxdt.$$

Here the a priori estimate for the Dirichlet problem for $\triangle$ implies

$$\sum_{j,k=1}^2 \int_Q s^2 |\partial_j\partial_k(ve^{s\varphi})|^2 dxdt \le C\int_Q |Lv|^2 e^{2s\varphi} dxdt.$$

Since

$$\partial_j\partial_k(ve^{s\varphi}) = (\partial_j\partial_k v)e^{s\varphi} + se^{s\varphi}\{(\partial_j\varphi)\partial_k v + (\partial_k\varphi)\partial_j v\}$$

$$+\{s(\partial_j\partial_k\varphi) + s^2(\partial_j\varphi)(\partial_k\varphi)\}ve^{2s\varphi},$$

we have

$$\sum_{j,k=1}^2 \int_Q s^2|\partial_j\partial_k v|^2 e^{2s\varphi} dxdt \le C\int_Q |Lv|^2 e^{2s\varphi} dxdt \tag{3.4}$$

in terms of Lemma 1. Hence by (3.3) and Lemma 1, we have

$$\int_Q |\triangle((\partial_j v)e^{s\varphi})|^2 dxdt$$

$$\le C\int_Q (|\nabla(\triangle v)|^2 + s^2\sum_{j,k=1}^2 |\partial_j\partial_k v|^2 + s^4|\nabla v|^2)e^{2s\varphi} dxdt \le C\int_Q |Lv|^2 e^{2s\varphi} dxdt.$$

Hence the a priori estimate for the Dirichlet problem for Δ yields

$$\sum_{j,k,\ell=1}^2 \int_Q |\partial_j\partial_k((\partial_\ell v)e^{s\varphi})|^2 dxdt \le C\int_Q |Lv|^2 e^{2s\varphi} dxdt,$$

that is,

$$\sum_{j,k,\ell=1}^2 \int_Q |\partial_j\partial_k\partial_\ell v|^2 e^{2s\varphi} dxdt$$

$$\le C\int_Q |Lv|^2 e^{2s\varphi} dxdt + C\int_Q \left(s^2\sum_{j,k} |\partial_j\partial_k v|^2 + s^4|\nabla v|^2\right) e^{2s\varphi} dxdt.$$

In terms of (3.4), we obtain

$$\sum_{j,k,\ell=1}^{2}\int_Q |\partial_j\partial_k\partial_\ell v|^2 e^{2s\varphi}dxdt \le C\int_Q |Lv|^2 e^{2s\varphi}dxdt. \tag{3.5}$$

Thus the proof of Lemma 1 is completed.

Next we present a Carleman estimate for the two-dimensional isotropic Lamé system. We set $\widetilde{\mathbf{u}} = (u_1, u_2)$ and

$$L_0\widetilde{\mathbf{u}} = \rho\partial_t^2\widetilde{\mathbf{u}} - \mu\triangle\widetilde{\mathbf{u}} - (\lambda+\mu)\nabla\mathrm{div}\,\widetilde{\mathbf{u}} - (\mathrm{div}\,\widetilde{\mathbf{u}})\nabla\lambda - (\nabla\widetilde{\mathbf{u}} + (\nabla\widetilde{\mathbf{u}})^T)\nabla\mu.$$

Let $x_0 \notin \overline{\Omega}$. Set

$$D = \sqrt{\sup_{x\in\overline{\Omega}}|x-x_0|^2 - \inf_{x\in\overline{\Omega}}|x-x_0|^2}.$$

We introduce conditions on a function a:

$$\begin{cases} \|a\|_{C^3(\overline{\Omega})} \le M_0, \\ a(x) \ge \theta_1 > 0, \quad \dfrac{(x-x_0)\cdot\nabla a(x)}{2a(x)} \le 1-\theta_0, \quad x\in\overline{\Omega}, \end{cases} \tag{3.6}$$

where the constants $M_0 > 0$, $0 < \theta_0 < 1$, $\theta_1 > 0$ are given. We fix a positive constant ν such that

$$\nu + \frac{M_0 D}{\sqrt{\theta_1}}\sqrt{\nu} \le \theta_0\theta_1, \qquad \theta_1\inf_{x\in\overline{\Omega}}|x-x_0|^2 - \nu\sup_{x\in\overline{\Omega}}|x-x_0|^2 > 0. \tag{3.7}$$

Here we note that such $\nu > 0$ exists because $x_0 \notin \overline{\Omega}$.

Then

Lemma 2. *We assume that $\rho,\lambda,\mu \in C^3(\overline{\Omega})$, $\rho,\lambda,\mu > 0$ on $\overline{\Omega}$, and that (3.2) holds and $\frac{\mu}{\rho}$ and $\frac{\lambda+2\mu}{\rho}$ satisfy (3.6). Let $\varphi(x,t)$ be defined by (3.1), $\gamma > 0$ be sufficiently large, and $\nu > 0$ satisfy (3.7). Then*

$$\int_Q (s|\nabla_{x,t}rot\,\widetilde{\mathbf{u}}|^2 + s|\nabla_{x,t}div\,\widetilde{\mathbf{u}}|^2 + s|\nabla_{x,t}\widetilde{\mathbf{u}}|^2 + s^3|rot\,\widetilde{\mathbf{u}}|^2 + s^3|div\,\widetilde{\mathbf{u}}|^2 + s^3\,|\widetilde{\mathbf{u}}|^2)e^{2s\varphi}dxdt$$
$$\le C\int_Q \left(|L_0\widetilde{\mathbf{u}}|^2 + |\nabla(L_0\widetilde{\mathbf{u}})|^2\right)e^{2s\varphi}dxdt$$

for $s \ge s_0$ *and* $\widetilde{\mathbf{u}} \in \{H_0^3(Q)\}^2$.

The proof of the lemma is found in Imanuvilov and Yamamoto [10], [11], [12] for example.

Now we proceed to the derivation of the Carleman estimate for the shallow shell equation. Since $\triangle u_1 = \partial_1(\mathrm{div}\,\widetilde{\mathbf{u}}) - \partial_2 \mathrm{rot}\,\widetilde{\mathbf{u}}$ and $\triangle u_2 = \partial_2(\mathrm{div}\,\widetilde{\mathbf{u}}) + \partial_1 \mathrm{rot}\,\widetilde{\mathbf{u}}$, we have

$$|\triangle \widetilde{\mathbf{u}}|^2 \le C(|\nabla(\mathrm{div}\,\widetilde{\mathbf{u}})|^2 + |\nabla(\mathrm{rot}\,\widetilde{\mathbf{u}})|^2).$$

Hence Lemma 2 yields

$$\int_Q s|\triangle \widetilde{\mathbf{u}}|^2 e^{2s\varphi} dxdt \le C \int_Q (|L_0 \widetilde{\mathbf{u}}|^2 + |\nabla(L_0 \widetilde{\mathbf{u}})|^2) e^{2s\varphi} dxdt. \tag{3.8}$$

On the other hand, since

$$\begin{aligned} \partial_t^2 \widetilde{\mathbf{u}} = \frac{1}{\rho} L_0 \widetilde{\mathbf{u}} + \frac{\mu}{\rho} \triangle \widetilde{\mathbf{u}} + \frac{\lambda+\mu}{\rho} \nabla(\mathrm{div}\,\widetilde{\mathbf{u}}) \\ +(\mathrm{div}\,\widetilde{\mathbf{u}}) \frac{\nabla \lambda}{\rho} + (\nabla \widetilde{\mathbf{u}} + (\nabla \widetilde{\mathbf{u}})^T) \frac{\nabla \mu}{\rho}, \end{aligned}$$

we obtain

$$\begin{aligned} \int_Q |\partial_t^2 \widetilde{\mathbf{u}}|^2 e^{2s\varphi} dxdt \le C \int_Q |L_0 \widetilde{\mathbf{u}}|^2 e^{2s\varphi} dxdt \\ +C \int_Q (|\triangle \widetilde{\mathbf{u}}|^2 + |\nabla(\mathrm{div}\,\widetilde{\mathbf{u}})|^2 + |\nabla \widetilde{\mathbf{u}}|^2) e^{2s\varphi} dxdt. \end{aligned}$$

Therefore by Lemma 2 and (3.8), we have

$$\int_Q |\partial_t^2 \widetilde{\mathbf{u}}|^2 e^{2s\varphi} dxdt \le C \int_Q (|L_0 \widetilde{\mathbf{u}}|^2 + |\nabla(L_0 \widetilde{\mathbf{u}})|^2) e^{2s\varphi} dxdt. \tag{3.9}$$

Applying Lemma 2 and (3.9), we obtain

$$\begin{aligned} &\int_Q (s|\nabla \mathrm{rot}\,\widetilde{\mathbf{u}}|^2 + s|\nabla \mathrm{div}\,\widetilde{\mathbf{u}}|^2 + s|\nabla \widetilde{\mathbf{u}}|^2 + s|\partial_t\,(\mathrm{rot}\,\widetilde{\mathbf{u}})\,|^2 + s|\partial_t\,(\mathrm{div}\,\widetilde{\mathbf{u}})\,|^2 + s|\partial_t \widetilde{\mathbf{u}}|^2 \\ &+s^3|\mathrm{rot}\,\widetilde{\mathbf{u}}|^2 + s^3|\mathrm{div}\,\widetilde{\mathbf{u}}|^2 + s^3|\widetilde{\mathbf{u}}|^2 + s|\triangle \widetilde{\mathbf{u}}|^2 + |\partial_t^2 \widetilde{\mathbf{u}}|^2) e^{2s\varphi} dxdt \\ &\le C \int_Q \left\{ \left(\sum_{|\alpha| \le 2} |\partial_x^\alpha u_3|^2 + |\widetilde{\mathbf{F}}|^2 \right) + \sum_{|\alpha| \le 3} |\partial_x^\alpha u_3|^2 + |\nabla \widetilde{\mathbf{F}}|^2 \right\} e^{2s\varphi} dxdt \\ &\le C \int_Q \left(\sum_{|\alpha| \le 3} |\partial_x^\alpha u_3|^2 + |\widetilde{\mathbf{F}}|^2 + |\nabla \widetilde{\mathbf{F}}|^2 \right) e^{2s\varphi} dxdt. \end{aligned} \tag{3.10}$$

Applying Lemma 1, (3.4) and (3.5) to (2.2), we have

$$\int_Q \Bigg\{ s|\nabla\partial_t u_3|^2 + s|\nabla\triangle u_3|^2 + s^3|\partial_t u_3|^2 + s^3|\triangle u_3|^2$$
$$+s^2\sum_{j,k=1}^{2} |\partial_j\partial_k u_3|^2 + s^4|\nabla u_3|^2 + s^6|u_3|^2 + \sum_{|\alpha|=3} |\partial_x^\alpha u_3|^2 \Bigg\} e^{2s\varphi}dxdt$$
$$\le C\int_Q (|\triangle\widetilde{\mathbf{u}}|^2 + |\nabla\mathrm{div}\,\widetilde{\mathbf{u}}|^2 + |\nabla\widetilde{\mathbf{u}}|^2 + |\widetilde{\mathbf{u}}|^2)e^{2s\varphi}dxdt + C\int_Q |F_3|^2 e^{2s\varphi}dxdt. \tag{3.11}$$

Here we absorbed $\sum_{j,k=1}^{2}|\partial_j\partial_k u_3|^2$ and $|\nabla u_3|^2$ into the left hand side. Substitute (3.11) into the right hand side of (3.10), we obtain

$$\int_Q (s|\nabla\mathrm{rot}\,\widetilde{\mathbf{u}}|^2 + s|\nabla\mathrm{div}\,\widetilde{\mathbf{u}}|^2 + s|\nabla\widetilde{\mathbf{u}}|^2 + s|\partial_t\left(\mathrm{rot}\,\widetilde{\mathbf{u}}\right)|^2 + s|\partial_t\left(\mathrm{div}\,\widetilde{\mathbf{u}}\right)|^2 + s|\partial_t\widetilde{\mathbf{u}}|^2$$
$$+s^3|\mathrm{rot}\,\widetilde{\mathbf{u}}|^2 + s^3|\mathrm{div}\,\widetilde{\mathbf{u}}|^2 + s^3|\widetilde{\mathbf{u}}|^2 + s|\triangle\widetilde{\mathbf{u}}|^2 + |\partial_t^2\widetilde{\mathbf{u}}|^2)e^{2s\varphi}dxdt$$
$$\le C\int_Q (|\triangle\widetilde{\mathbf{u}}|^2 + |\nabla\mathrm{div}\,\widetilde{\mathbf{u}}|^2 + |\nabla\widetilde{\mathbf{u}}|^2 + |\widetilde{\mathbf{u}}|^2)e^{2s\varphi}dxdt$$
$$+C\int_Q (|\widetilde{\mathbf{F}}|^2 + |\nabla\widetilde{\mathbf{F}}|^2 + |F_3|^2)e^{2s\varphi}dxdt.$$

Absorbing the first term on the right hand side into the left hand side, we reach

$$\int_Q (s|\nabla\mathrm{rot}\,\widetilde{\mathbf{u}}|^2 + s|\nabla\mathrm{div}\,\widetilde{\mathbf{u}}|^2 + s|\nabla\widetilde{\mathbf{u}}|^2 + s|\partial_t\left(\mathrm{rot}\,\widetilde{\mathbf{u}}\right)|^2 + s|\partial_t\left(\mathrm{div}\,\widetilde{\mathbf{u}}\right)|^2 + s|\partial_t\widetilde{\mathbf{u}}|^2$$
$$+s^3|\mathrm{rot}\,\widetilde{\mathbf{u}}|^2 + s^3|\mathrm{div}\,\widetilde{\mathbf{u}}|^2 + s^3|\widetilde{\mathbf{u}}|^2 + s|\triangle\widetilde{\mathbf{u}}|^2 + |\partial_t^2\widetilde{\mathbf{u}}|^2)e^{2s\varphi}dxdt$$
$$\le C\int_Q (|\widetilde{\mathbf{F}}|^2 + |\nabla\widetilde{\mathbf{F}}|^2 + |F_3|^2)e^{2s\varphi}dxdt. \tag{3.12}$$

Again the application of (3.12) in (3.11) yields

$$\int_Q \Bigg\{ s|\nabla\partial_t u_3|^2 + s|\nabla\triangle u_3|^2 + s^3|\partial_t u_3|^2 + s^3|\triangle u_3|^2$$
$$+s^2\sum_{j,k=1}^{2} |\partial_j\partial_k u_3|^2 + s^4|\nabla u_3|^2 + s^6|u_3|^2 + \sum_{|\alpha|=3} |\partial_x^\alpha u_3|^2 \Bigg\} e^{2s\varphi}dxdt$$
$$\le C\int_Q (\frac{1}{s}|\widetilde{\mathbf{F}}|^2 + \frac{1}{s}|\nabla\widetilde{\mathbf{F}}|^2 + |F_3|^2)e^{2s\varphi}dxdt. \tag{3.13}$$

Thus we proved

Theorem 1 (Carleman estimate). *We assume that* ρ, λ, $\mu \in C^3\left(\overline{\Omega}\right)$, $\rho(x) > 0$, $\lambda(x) > 0$, $\mu(x) > 0$ *for all* $x \in \overline{\Omega}$, *and that (3.2) holds,* $\frac{\mu}{\rho}$ *and* $\frac{\lambda+2\mu}{\rho}$ *satisfy (3.6). Let* $\varphi(x,t)$ *be defined by (3.1) and* $\nu > 0$ *satisfy (3.7). Then there exists* $\gamma_0 > 0$ *such that for any* $\gamma > \gamma_0$, *we can choose* $s_0 = s_0(\gamma) > 0$ *and* $C = C(s_0, \gamma_0, M_0, \theta_0, \theta_1, \nu, \Omega, T, x_0, \omega) > 0$ *such that*

$$\begin{aligned}
&\int_Q \Bigg\{ s|\nabla_{x,t} rot\, \widetilde{\mathbf{u}}|^2 + s|\nabla_{x,t} div\, \widetilde{\mathbf{u}}|^2 + s|\nabla_{x,t}\widetilde{\mathbf{u}}|^2 \\
&+s^3|rot\, \widetilde{\mathbf{u}}|^2 + s^3|div\, \widetilde{\mathbf{u}}|^2 + s^3|\widetilde{\mathbf{u}}|^2 + s|\triangle\widetilde{\mathbf{u}}|^2 + |\partial_t^2\widetilde{\mathbf{u}}|^2 \Bigg\} e^{2s\varphi} dxdt \\
&+ \int_Q \Bigg\{ s|\nabla\partial_t u_3|^2 + s|\nabla\triangle u_3|^2 + s^3|\partial_t u_3|^2 + s^3|\triangle u_3|^2 \\
&+s^2 \sum_{j,k=1}^{2} |\partial_j\partial_k u_3|^2 + s^4|\nabla u_3|^2 + s^6|u_3|^2 + \sum_{|\alpha|=3} |\partial_x^\alpha u_3|^2 \Bigg\} e^{2s\varphi} dxdt \\
\le& C\int_Q (|L_e\mathbf{u}|^2 + |\nabla(L_e\mathbf{u})|^2 + |L_p\mathbf{u}|^2) e^{2s\varphi} dxdt.
\end{aligned}$$

for all $s \ge s_0$, $\widetilde{\mathbf{u}} \in \{H_0^3(Q)\}^2$ *and* $u_3 \in H_0^4(Q)$.

This is a Carleman estiamate for the linear shallow shell equation, which is novel to the authors' best knowledge. As for Carleman estimates and applications to the observability and inverse problems for nonstationary Lamé system and a plate equation, see Cheng, Isakov, Yamamoto and Zhou [4], Imanuvilov, Isakov and Yamamoto [9], Imanuvilov and Yamamoto [10], [11], [12], Isakov [13].

§4. Inverse source problem.

We consider

$$L_e\mathbf{u} = \widetilde{\mathbf{F}} \tag{4.1}$$

$$L_p\mathbf{u} = F_3 \quad \text{in } Q \equiv \Omega \times (0,T), \tag{4.2}$$

$$\mathbf{u} = \partial_t\mathbf{u} = 0 \qquad \text{on } \Omega \times \{t_0\}, \tag{4.3}$$

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{4.4}$$

where $t_0 = \frac{T}{2}$, $\widetilde{\mathbf{F}} = (F_1, F_2)$, and $\widetilde{\mathbf{u}} = (u_1, u_2)$ and u_3 depends on $(x,t) \in Q$.

Inverse source problem. We assume that

$$\widetilde{\mathbf{F}}(x,t) = \begin{pmatrix} F_1(x,t) \\ F_2(x,t) \end{pmatrix} = \begin{pmatrix} f_1(x)R_1(x,t) \\ f_2(x)R_2(x,t) \end{pmatrix}, \quad F_3(x,t) = f_3(x)R_3(x,t). \tag{4.5}$$

Let an observation subdomain $\omega \subset \Omega$ satisfy $\partial\Omega \subset \partial\omega$ and $T > 0$ be suitably given. Then determine an x-dependent component $(f_1(x),\ f_2(x),\ f_3(x))$, $x \in \Omega$ of an exterior force (F_1, F_2, F_3) from the observations of

$$\mathbf{u}(x,t) = (\widetilde{\mathbf{u}}(x,t), u_3(x,t)), \quad (x,t) \in Q_\omega \equiv \omega \times (0,T).$$

The condition $\partial\omega \supset \partial\Omega$ means that $\omega \subset \Omega$ is a neighbourhood of $\partial\Omega$. We can relax the condition $\partial\omega \supset \partial\Omega$, but we cannot choose an arbitrary subdomain ω, because the equation in $\widetilde{\mathbf{u}}$ is hyperbolic, so that we need some geometric condition on ω (e.g., [10]). This condition is related with the pseudo-convexity which is necessary for proving a Carleman estimate (e.g., Hörmander [8]).

We are ready to state the main result for the inverse source problem.

Theorem 2. *We assumne that $\rho,\ \lambda,\ \mu \in C^3\left(\overline{\Omega}\right)$, $\rho(x) > 0$, $\lambda(x) > 0$, $\mu(x) > 0$ for all $x \in \overline{\Omega}$, and that (3.2) holds, and $\frac{\mu}{\rho}$ and $\frac{\lambda+2\mu}{\rho}$ satisfy (3.6). Let $\nu > 0$ satisfy (3.7). Furthermore we assume that*

$$T > \frac{2D}{\sqrt{\nu}}, \tag{4.6}$$

and there exist constants $M_1, r_0 > 0$ such that

$$\|R_j\|_{W^{3,\infty}(Q)} \le M_1, \quad |R_j(x,t_0)| \ge r_0, \quad x \in \overline{\Omega}, \quad j = 1,2,3. \tag{4.7}$$

Moreover, we assume that $\widetilde{\mathbf{u}} \in \{W^{5,\infty}(Q)\}^2$, $u_3 \in W^{6,\infty}(Q)$, $f_j \in C^1(\overline{\Omega})$, $j = 1, 2$, $f_3 \in C(\overline{\Omega})$ *satisfy (4.1) - (4.5) and*

$$\|\widetilde{\mathbf{u}}\|_{\{W^{4,\infty}(Q)\}^2} \le M_2, \qquad \|u_3\|_{W^{3,\infty}(Q)} \le M_2. \tag{4.8}$$

Then there exist constants $\kappa = \kappa(M_0, M_1, M_2, \theta_0, \theta_1, \Omega, T, x_0, \rho, \lambda, \mu) \in (0,1)$ *and* $C_0 = C_0(M_0, M_1, M_2, \theta_0, \theta_1, \Omega, T, x_0, \rho, \lambda, \mu) > 0$ *such that*

$$\|f_1\|_{H^1(\Omega)} + \|f_2\|_{H^1(\Omega)} + \|f_3\|_{L^2(\Omega)} \le C_0 \left(\|\widetilde{\mathbf{u}}\|_{\{H^4(Q_\omega)\}^2} + \|u_3\|_{H^6(Q_\omega)}\right)^\kappa. \tag{4.9}$$

This kind of inverse problems was considered firstly in Bukhgeim and Klibanov [2], whose method is based on Carleman estimates. See also Bellassoued and Yamamoto [1], Klibanov [15] and the references therein. Here we do not give more detailed references on inverse problems by Carleman estimates.

Proof. The proof is adapted from e.g., [10]. By (4.6) and the definition (3.1) of φ, we have

$$\varphi(x, t_0) \ge d \ge 1, \quad 0 < \varphi(x, 0) = \varphi(x, T) < d, \quad x \in \overline{\Omega}, \tag{4.10}$$

where $d = \exp(\gamma \inf_{x\in\overline{\Omega}} |x - x_0|^2)$. Therefore, for any given small $\eta \in \left(0, d - \sup_{x\in\overline{\Omega}} \varphi(x,T)\right)$, we can choose a sufficiently small $\delta = \delta(\eta) > 0$, such that

$$0 < \varphi(x,t) \le d - \eta, \quad (x,t) \in \overline{\Omega} \times ([0, 2\delta] \cup [T - 2\delta, T]). \tag{4.11}$$

In order to apply Theorem 1, we introduce two cut-off functions χ_1 and χ_2 satisfying $\chi_1 \in C^\infty(\mathbb{R})$, $\chi_2 \in C_0^\infty(\Omega)$, $0 \le \chi_1(t) \le 1$ for $t \in \mathbb{R}$, $0 \le \chi_2(x) \le 1$ for $x \in \overline{\Omega}$,

$$\chi_1(t) = \begin{cases} 0, & t \in [0, \delta] \cup [T - \delta, T], \\ 1, & t \in [2\delta, T - 2\delta], \end{cases} \tag{4.12}$$

and $\chi_2(x) = 1$ for $x \in \overline{\Omega \backslash \omega}$.

We set $\mathbf{w} = (\widetilde{\mathbf{w}}, w_3)$, $\widetilde{\mathbf{w}} = (w_1, w_2) = \chi_1 \partial_t^2 \widetilde{\mathbf{u}} \in \{W^{3,\infty}(Q)\}^2$ and $w_3 = \chi_1 \partial_t^2 u_3 \in W^{4,\infty}(Q)$. Then, by (4.1) and (4.2), we have

$$L_e \mathbf{w} = \chi_1 \left(\partial_t^2 \widetilde{\mathbf{F}}\right) + 2\rho \left(\partial_t \chi_1\right) \left(\partial_t^3 \widetilde{\mathbf{u}}\right) + \rho \left(\partial_t^2 \chi_1\right) \left(\partial_t^2 \widetilde{\mathbf{u}}\right), \tag{4.13}$$

$$L_p \mathbf{w} = \chi_1 \left(\partial_t^2 F_3\right) + 2\rho \left(\partial_t \chi_1\right) \left(\partial_t^3 u_3\right) + \rho \left(\partial_t^2 \chi_1\right) \left(\partial_t^2 u_3\right) \text{ in } Q \tag{4.14}$$

and

$$\mathbf{w} = 0, \qquad \text{on } \partial\Omega \times (0, T). \tag{4.15}$$

Moreover, we set $\mathbf{v} = (\widetilde{\mathbf{v}}, v_3) = \chi_2(\widetilde{\mathbf{w}}, w_3)$. Then, by definition of χ_1 and χ_2, (4.13) and (4.14), we have $\widetilde{\mathbf{v}} \in \{H_0^3(Q)\}^2$ and $v_3 \in H_0^4(Q)$,

$$L_e \mathbf{v} = \chi_1 \left(\partial_t^2 \widetilde{\mathbf{F}}\right) + 2\rho \left(\partial_t \chi_1\right) \left(\partial_t^3 \widetilde{\mathbf{u}}\right) + \rho \left(\partial_t^2 \chi_1\right) \left(\partial_t^2 \widetilde{\mathbf{u}}\right) - L_e \left((1 - \chi_2)\mathbf{w}\right), \tag{4.16}$$

$$L_p \mathbf{v} = \chi_1 \left(\partial_t^2 F_3\right) + 2\rho \left(\partial_t \chi_1\right) \left(\partial_t^3 u_3\right) + \rho \left(\partial_t^2 \chi_1\right) \left(\partial_t^2 u_3\right) - L_p \left((1 - \chi_2)\mathbf{w}\right) \text{ in } Q. \tag{4.17}$$

Here we have used

$$\widetilde{\mathbf{v}} = \widetilde{\mathbf{w}} - (1 - \chi_2)\widetilde{\mathbf{w}}, \quad v_3 = w_3 - (1 - \chi_2) w_3 \text{ in } Q. \tag{4.18}$$

Furthermore, by (4.16), we have

$$\begin{aligned}\nabla(L_e \mathbf{v}) &= \chi_1 \nabla \left(\partial_t^2 \widetilde{\mathbf{F}}\right) + 2 \left(\partial_t \chi_1\right) \nabla \left(\rho \left(\partial_t^3 \widetilde{\mathbf{u}}\right)\right) + \left(\partial_t^2 \chi_1\right) \nabla \left(\rho \left(\partial_t^2 \widetilde{\mathbf{u}}\right)\right) \\ &\quad - \nabla \left\{L_e((1 - \chi_2)\mathbf{w})\right\} \text{ in } Q.\end{aligned} \tag{4.19}$$

Applying Theorem 1, we obtain

$$\begin{aligned}&\int_Q \left(s^3 \left|\widetilde{\mathbf{v}}\right|^2 + s \left|\nabla_{x,t} \widetilde{\mathbf{v}}\right|^2 + s \left|\triangle \widetilde{\mathbf{v}}\right|^2 + s^6 \left|v_3\right|^2 + s^3 \left|\nabla_{x,t} v_3\right|^2\right) e^{2s\varphi} dxdt \\ &\le C_1 \int_Q (|L_e \mathbf{v}|^2 + |\nabla(L_e \mathbf{v})|^2 + |L_p \mathbf{v}|^2) e^{2s\varphi} dxdt\end{aligned} \tag{4.20}$$

for all large $s > 0$. Here and henceforth, C_k denote generic positive constants which may depend on s_0, s_1, s_2, γ, M_0, M_1, M_2, θ_0, θ_1, D, d, Ω, T, ω, χ_1, χ_2, δ, η, but are independent of s. By the definition of χ_1 and χ_2, we have

$$(1-\chi_2(x))\widetilde{\mathbf{w}}(x,t)=0,\ (1-\chi_2(x))w_3(x,t)=0 \text{ for } (x,t)\in\overline{(\Omega\setminus\omega)\times(0,T)} \quad (4.21)$$

and $(\partial_t\chi_2)(t) = \left(\partial_t^2\chi_2\right)(t) = 0$ for $t \in (0,\delta)\cup(2\delta,T-2\delta)\cup(T-\delta,T)$. Therefore, by (4.5), (4.7), (4.8), (4.11) and (4.16), we have

$$\begin{aligned}
&\int_Q |L_e\mathbf{v}|^2 e^{2s\varphi}dxdt \le C_2\int_Q |\chi_1\partial_t^2\widetilde{\mathbf{F}}|^2 e^{2s\varphi}dxdt + C_3\int_{Q_\omega}|L_e((1-\chi_2)\mathbf{w})|^2 e^{2s\varphi}dxdt\\
&+C_4\left(\int_\delta^{2\delta}+\int_{T-2\delta}^{T-\delta}\right)\int_\Omega(|2\rho(\partial_t\chi_1)(\partial_t^3\widetilde{\mathbf{u}})|^2+|\rho(\partial_t^2\chi_1)(\partial_t^2\widetilde{\mathbf{u}})|^2)e^{2s\varphi}dxdt\\
&\le C_5\left\{\int_Q(|f_1|^2+|f_2|^2)e^{2s\varphi}dxdt+e^{2s(d-\eta)}+e^{2s\Phi}\Theta\right\}
\end{aligned}$$

for all $s > 0$, where $\Phi = \sup_{(x,t)\in\overline{Q_\omega}}\varphi(x,t) \ge 1$ and

$$\Theta = \|\widetilde{\mathbf{u}}\|^2_{\{H^4(Q_\omega)\}^2} + \|u_3\|^2_{H^6(Q_\omega)}. \quad (4.22)$$

Similarly, by (4.5), (4.7), (4.8), (4.11), (4.17) and (4.19), we have

$$\begin{aligned}
&\int_Q(|L_p\mathbf{v}|^2+|\nabla(L_e\mathbf{v})|^2)e^{2s\varphi}dxdt\\
\le& C_6\left\{\int_Q\left(\sum_{k=1}^2\left(|f_k|^2+|\nabla f_k|^2\right)+|f_3|^2\right)e^{2s\varphi}dxdt+e^{2s(d-\eta)}+e^{2s\Phi}\Theta\right\}
\end{aligned}$$

for all $s > 0$. Therefore, by (4.18), (4.20) and (4.21), we arrive at

$$\begin{aligned}
&\int_Q(s|\widetilde{\mathbf{w}}|^2+s|\nabla_{x,t}\widetilde{\mathbf{w}}|^2+|\triangle\widetilde{\mathbf{w}}|^2+s|w_3|^2+|\nabla_{x,t}w_3|^2e^{2s\varphi}dxdt\\
\le& C_7\int_Q(s|\widetilde{\mathbf{v}}|^2+s|\nabla_{x,t}\widetilde{\mathbf{v}}|^2+|\triangle\widetilde{\mathbf{v}}|^2+s|v_3|^2+|\nabla_{x,t}v_3|^2)e^{2s\varphi}dxdt\\
&+C_8se^{2s\Phi}(\|(1-\chi_2)\widetilde{\mathbf{w}}\|^2_{\{H^2(Q_\omega)\}^2}+\|(1-\chi_2)w_3\|^2_{H^1(Q_\omega)})\\
\le& C_9\left\{\int_Q\left(\sum_{k=1}^2\left(|f_k|^2+|\nabla f_k|^2\right)+|f_3|^2\right)e^{2s\varphi}dxdt+e^{2s(d-\eta)}+se^{2s\Phi}\Theta\right\} \quad (4.23)
\end{aligned}$$

for all large $s > 0$.

On the other hand, by integral by parts, we have

$$\frac{1}{2}\int_\Omega |\nabla\widetilde{\mathbf{w}}(x,t_0)|^2 e^{2s\varphi(x,t_0)}dx = \frac{1}{2}\int_0^{t_0} \partial_t\left(\int_\Omega \sum_{j=1}^2 |\nabla w_j(x,t)|^2 e^{2s\varphi(x,t)}dx\right)dt$$

$$= \int_0^{t_0}\int_\Omega \left(\sum_{j,k=1}^2 (\partial_k w_j)(\partial_k\partial_t w_j) + s(\partial_t\varphi)|\nabla\widetilde{\mathbf{w}}|^2\right)e^{2s\varphi}dxdt$$

$$= \int_0^{t_0}\int_\Omega \left(s(\partial_t\varphi)|\nabla\widetilde{\mathbf{w}}|^2 - \sum_{j=1}^2 \partial_t w_j\left(\triangle w_j + 2s\sum_{k=1}^2(\partial_k w_j)(\partial_k\varphi)\right)\right)e^{2s\varphi}dxdt$$

$$\le C_{10}\int_Q (|\triangle\widetilde{\mathbf{w}}|^2 + s|\nabla_{x,t}\widetilde{\mathbf{w}}|^2)e^{2s\varphi}dxdt$$

for all $s > 0$. Here we have used (4.15) and $\widetilde{\mathbf{w}}(\cdot,0) = 0$ by (4.12). Furthermore we have

$$\frac{1}{2}\int_\Omega |\widetilde{\mathbf{w}}(x,t_0)|^2 e^{2s\varphi(x,t_0)}dx = \frac{1}{2}\int_0^{t_0}\partial_t\left(\int_\Omega\sum_{j=1}^2 |w_j(x,t)|^2 e^{2s\varphi(x,t)}dx\right)dt$$

$$= \int_0^{t_0}\int_\Omega\left(\sum_{j=1}^2 w_j(\partial_t w_j) + s(\partial_t\varphi)|\widetilde{\mathbf{w}}|^2\right)e^{2s\varphi}dxdt$$

$$\le C_{11}\int_Q (s|\widetilde{\mathbf{w}}|^2 + |\partial_t\widetilde{\mathbf{w}}|^2)e^{2s\varphi}dxdt$$

and

$$\frac{1}{2}\int_\Omega |w_3(x,t_0)|^2 e^{2s\varphi(x,t_0)}dx \le C_{12}\int_Q (s|w_3|^2 + |\partial_t w_3|^2)e^{2s\varphi}dxdt$$

for all $s > 0$. Therefore, we have

$$\int_\Omega (|\nabla\widetilde{\mathbf{w}}(x,t_0)|^2 + |\widetilde{\mathbf{w}}(x,t_0)|^2 + |w_3(x,t_0)|^2)e^{2s\varphi(x,t_0)}dx$$
$$\le C_{13}\int_Q \left(s\,|\widetilde{\mathbf{w}}|^2 + s\,|\nabla_{x,t}\widetilde{\mathbf{w}}|^2 + |\triangle\widetilde{\mathbf{w}}|^2 + s\,|w_3|^2 + |\partial_t w_3|^2\right)e^{2s\varphi}dxdt \tag{4.24}$$

for all $s > 0$. Moreover, by (4.1), (4.2), (4.3) and (4.12), we have

$$\widetilde{\mathbf{F}}(x,t_0) = \rho(x)\partial_t^2\widetilde{\mathbf{u}}(x,t_0) = \rho(x)\widetilde{\mathbf{w}}(x,t_0),$$

$$F_3(x,t_0) = \rho(x)\partial_t^2 u_3(x,t_0) = \rho(x) w_3(x,t_0), \quad x \in \Omega.$$

Therefore, noting (4.5) and (4.7), we have

$$f_j(x) = \frac{\rho(x)}{R_j(x,t_0)} w_j(x,t_0), \qquad j = 1,2,3,$$

and

$$\nabla f_k(x) = w_k(x,t_0)\nabla\left\{\frac{\rho(x)}{R_k(x,t_0)}\right\} + \frac{\rho(x)}{R_k(x,t_0)}\nabla w_k(x,t_0), \quad k=1,2,\ x \in \Omega.$$

Therefore, by (4.7), we have

$$\begin{aligned}&\int_\Omega \left(\sum_{k=1}^2 (|f_k|^2 + |\nabla f_k|^2) + |f_3|^2\right) e^{2s\varphi(x,t_0)} dx \\ \le & C_{14}\int_\Omega \left(|\nabla \widetilde{\mathbf{w}}(x,t_0)|^2 + |\widetilde{\mathbf{w}}(x,t_0)|^2 + |w_3(x,t_0)|^2\right) e^{2s\varphi(x,t_0)} dx\end{aligned} \tag{4.25}$$

for all $s > 0$. By (4.23), (4.24) and (4.25), we obtain

$$\begin{aligned}&\int_\Omega \left(\sum_{k=1}^2 \left(|f_k|^2 + |\nabla f_k|^2\right) + |f_3|^2\right) e^{2s\varphi(x,t_0)} dx \\ \le & C_{15}\left\{\int_Q \left(\sum_{k=1}^2 (|f_k|^2 + |\nabla f_k|^2) + |f_3|^2\right) e^{2s\varphi} dxdt + e^{2s(d-\eta)} + se^{2s\Phi}\Theta\right\}\end{aligned} \tag{4.26}$$

for all sufficiently large $s > 0$.

We will estimate the first term of the right hand side of (4.26) as follows. By (3.1), we have

$$\begin{aligned}&\varphi(x,t) - \varphi(x,t_0) = e^{\gamma(|x-x_0|^2 - \nu(t-t_0)^2)} - e^{\gamma|x-x_0|^2} \\ =& e^{\gamma|x-x_0|^2}\left(e^{-\gamma\nu(t-t_0)^2} - 1\right) \le e^{-\gamma\nu(t-t_0)^2} - 1 \le 0,\ t \in [0,T].\end{aligned}$$

$$\begin{aligned}&\int_Q \left(\sum_{k=1}^2 (|f_k|^2 + |\nabla f_k|^2) + |f_3|^2\right) e^{2s\varphi} dxdt \\ =& \int_\Omega \left(\int_0^T e^{2s(\varphi(x,t)-\varphi(x,t_0))} dt\right)\left(\sum_{k=1}^2 \left(|f_k|^2 + |\nabla f_k|^2\right) + |f_3|^2\right) e^{2s\varphi(x,t_0)} dx \\ \le & \int_\Omega \left(\int_0^T e^{2s\left(e^{-\gamma\nu(t-t_0)^2}-1\right)} dt\right)\left(\sum_{k=1}^2 \left(|f_k|^2 + |\nabla f_k|^2\right) + |f_3|^2\right) e^{2s\varphi(x,t_0)} dx\end{aligned} \tag{4.27}$$

for all large $s > 0$. Moreover, we have $\lim_{s\to\infty} \exp\{2s(\exp(-\gamma\nu(t-t_0)^2)-1)\} = 0$ for $t \neq t_0$ and $|\exp\{2s(\exp(-\gamma\nu(t-t_0)^2)-1)\}| \le 1$. Therefore, by the Lebesgue theorem, we have that there exists $s_1 > 0$ such that

$$C_{15}\int_0^T e^{2s(e^{-\gamma\nu(t-t_0)^2}-1)}dt \le \frac{1}{2} \text{ for all } s > s_1. \tag{4.28}$$

From (4.26), (4.27) and (4.28) it follows that there exists $s_2 > 0$ such that

$$\int_\Omega \left(\sum_{k=1}^2 \left(|f_k|^2 + |\nabla f_k|^2\right) + |f_3|^2\right) e^{2s\varphi(x,t_0)}dx \le 2C_{15}\left\{e^{2s(d-\eta)} + se^{2s\Phi}\Theta\right\}$$

for all $s > s_2$. Therefore, noting (4.10), we otain that there exists $s_3 > 0$ such that

$$\begin{aligned}&\int_\Omega \left(\sum_{k=1}^2 \left(|f_k|^2 + |\nabla f_k|^2\right) + |f_3|^2\right) dx\\ \le& e^{-2sd}\int_\Omega \left(\sum_{k=1}^2 \left(|f_k|^2 + |\nabla f_k|^2\right) + |f_3|^2\right) e^{2s\varphi(x,t_0)}dx\\ \le& 2C_{15}\left\{e^{-2s\eta} + se^{2s(\Phi-d)}\Theta\right\} \le 2C_{15}\left\{e^{-2s\eta} + e^{2s\Phi}\Theta\right\}\end{aligned}$$

for all $s > s_3$. Hence we have

$$\int_\Omega \left(\sum_{k=1}^2 \left(|f_k|^2 + |\nabla f_k|^2\right) + |f_3|^2\right) dx \le C_{16}\left(e^{-2s\eta} + e^{2s\Phi}\Theta\right) \text{ for all } s > 0,$$

where $C_{16} = 2C_{15}e^{2s_3\Phi}$.

In order to prove (4.9), we can assume that $\Theta < 1$, so that $-(\log\Theta)(2\eta+2\Phi) > 0$. We take

$$s = -\frac{\log\Theta}{2\eta + 2\Phi}.$$

Then it follows that

$$e^{-2s\eta} = e^{2s\Phi}\Theta = \Theta^{\frac{\eta}{\eta+\Phi}}.$$

Therefore, by (4.29), we have

$$\int_\Omega \left(\sum_{k=1}^2 \left(|f_k|^2 + |\nabla f_k|^2\right) + |f_3|^2\right) dx \le 2C_{16}\Theta^{\frac{\eta}{\eta+\Phi}}.$$

Noting (4.22), we obatin (4.9). The proof of Theorem 2 is completed. □

§5. Extensions.

Relying on this approach of Carleman estimates which have been proved successful, we are now considering the case of more general geometries to solve the inverse problems for elastic bodies whose equilibrium equations are of Koiter shells type (Li, Miara and Yamamoto [21]) and relax the condition of the shallowness introduced in [23].

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References

1. M. Bellassoued and M. Yamamoto, *Logarithmic stability in determination of a coefficient in an acoustic equation by arbitrary boundary observation*, J. Math. Pures Appl. **85** (2006), 193–224.
2. A.L. Bukhgeim and M.V. Klibanov, *Global uniqueness of a class of multidimensional inverse problems*, Soviet Math. Dokl. **24** (1981), 244–247.
3. J. Cagnol, I. Lasiecka, C. Lebiedzik and J.-P. Zolésio, *Uniform stability in structural acoustic models with flexible curved walls.*, to appear J. Diff. Eqns.
4. J. Cheng, V. Isakov, M. Yamamoto and Q. Zhou, *Lipschitz stability in the lateral Cauchy problem for elasticity system*, J. Math. Kyoto Univ. **43** (2003), 475-501.
5. P.G.Ciarlet and B.Miara, *Justification of the two-dimensional equations of a linearly elastic shallow shell*, Comm. Pure Appl. Math. **45** (1992), 327–360.
6. G. Geymonat, P. Loreti and V. Valente, *Exact controllability of thin elastic hemispherical shell via harmonic analysis*, in "Boundary Value Problems for Partial Differential Equations and Applications", Masson, Paris (1993).
7. G. Geymonat, P. Loreti and V. Valente, *Spectral problem for thin shells and exact controllability*, in Proceeding of the colloquium " Spectral Analysis of Complex Structures", Collection Travaux en cours, **49**, Herman, Paris. (1995), 327-360.
8. L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.
9. O.Yu. Imanuvilov, V. Isakov and M. Yamamoto, *An inverse problem for the dynamical Lame system with two sets of boundary data*, Comm. Pure Appl. Math. **56** (2003), 1366-1382.
10. O.Yu. Imanuvilov and M. Yamamoto, *Carleman estimates for the non-stationary Lamé system and the application to an inverse problem*, ESAIM Control Optim. Calc. Var. **11** (2005), 1–56 (electronic).
11. O.Yu. Imanuvilov and M. Yamamoto, *Carleman estimates for the three-dimensional nonstationary Lamé system and application to an inverse problem*, in "Control

theory of partial differential equations", Lect. Notes Pure Appl. Math., vol. 242, Chapman & Hall/CRC, Boca Raton, FL (2005), 337–374.

12. O.Yu. Imanuvilov and M. Yamamoto, *Carleman estimates for the Lamé system with stress boundary condtion*, to appear in Publ. RIMS Kyoto Univ. **44** (2008).
13. V. Isakov, *A nonhyperbolic Cauchy problem for* $\Box_b\Box_c$ *and its applications to elasticity theory*, Comm. Pure Appl. Math. **39** (1986), 747–767.
14. A. Kazemi and M.V. Klibanov, *Stability estimates for ill-posed Cauchy problems involving hyperbolic equations and inequalities*, Appl. Analysis **50** (1993), 93–102.
15. M.V. Klibanov, *Inverse problems and Carleman estimates*, Inverse Problems **8** (1992), 575–596.
16. V. Komornik, *On the zeros of Bessel type functions and applications to exact controllabiliy problems*, Asymptotic Analysis **5** (1991), 115-128.
17. J.E. Lagnese and J.-L.Lions, *Modelling analysis and control of thin plates*, Masson, Paris, 1990.
18. I. Lasiecka, *Mathematical Control Theory of Coupled PDE Systems*, SIAM, Philadelphia, 2000.
19. I. Lasiecka and R. Marchand, *Control and stabilization in nonlinear structural acoustic problems*, in Proceedings of SPIE – Volume 3039 Smart Structures and Materials 1997: Mathematics and Control in Smart Structures, Vasundara V. Varadan, Jagdish Chandra, Editors, June 1997, pp. 192-203.
20. I. Lasiecka, R. Triggiani and V.Valente, *Uniform stabilization of spherical shells by boundary dissipation*, Advances in Differential Equations **1-4** (1996), 635-674.
21. S. Li, B. Miara and M. Yamamoto, *Carleman estimates for Koiter shell equation and inverse problems*, work in progress.
22. J.-L. Lions, *Côntrolabilité exacte, Perturbations et stabilisation de systèmes distribués*, Masson, Paris, 1988.
23. B. Miara and V. Valente, *Exact controllability of a Koiter shell by a boundary action*, J. Elasticity **52** (1999), 267-287.
24. B. Miara and V. Valente, *Stabilization of a Koiter Shell by a Boundary Action*, in Proceedings of Second European Conference on Structural Control (2000).
25. D.L. Russell, *Exact boundary value controllability theorems for waves and heat processes in star-complemented regions*, in "Differential Games and Control Theory", Marcel-Dekker, New York (1974).
26. J. J. Telega and W.R. Bielski, *Controllability and stabilization in elasticity, heat conduction and thermoelasticity: Review of recent developments*, J. Global Optim. **17** (2000), 353-386.
27. V. Valente, *Relaxed exact spectral controllability of membrane shells*, J. Math. Pures Appl. **76** (1997), 551–562.
28. G. Yuan and M. Yamamoto, *Lipschitz stability in inverse problems for a Kirchhoff plate equation*, Asymptotic Analysis **53** (2007), 29-60.

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