UTMS 2008–28 September 8, 2008

Foundations of Algebraic Logic

by

Kensaku Gomi

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

Foundations of Algebraic Logic

Kensaku GOMI *

September 9, 2008

Abstract

Newly formulated foundations of algebraic logic will be presented. Although it aims at applications to mathematical psychology, results within this paper is universal and so is expected to be applied to other branches as well.

Key words: universal algebraic logic, sorted algebra.

2000 Mathematics Subject Classification: 03G99

1 Introduction

The purpose of this paper is to present newly formulated foundations of algebraic logic. Although it aims at applications to mathematical psychology as in [1], results within this paper is universal and so is expected to be applied to other branches as well.

We will define a **formal language** to be a certain **universal sorted algebra**, which is generated by **constants** and **variables** and in general has unary **variable operations** indexed by a formal product of a symbol and a variable. We will also define **denotable worlds** (cf. Remark 4.1) for the formal language to be certain sorted algebras similar to the **operational subalgebra** of the formal language obtained by removing the variable operations. Then the **denotations** of constants into each denotable world are defined to be **sort-consistent** mappings which associate each constant with an element of the denotable world of the same **type**. The denotations of variables are similarly defined, and we can furthermore construct the **power algebra** of the denotable world whose **exponent** is the set of the variable denotations into the denotable world. The power algebras are by definition similar to the denotable worlds. However, once the variable operations are **interpreted** as operations on the power algebra, then by the **universality** of the formal language, each constant denotation into the denotable world yields a sort-consistent homomorphism of the formal language into the power algebra, which we call a **meta-denotation**. Then two fundamental theorems named the **denotation theorem** and the

*Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914 Japan. email: mathpsy@mb.infoweb.ne.jp.

substitution-redenotation theorem will be proved concerning the interrelations among meta-denotation, free occurrences of variables, and substitution for variables. Finally, a **logical system** is defined to be a triple of a formal language, a domain of its denotable worlds, and a family of interpretations of the variable operations for the denotable worlds in the domain. Although not required as far as [1] is concerned, we will proceed to a study of the **functional expressions** of the elements of formal languages, and conclude this paper by the determination of the **denotable functions** on denotable worlds.

The foundations of algebraic logic are not completed by this paper. First of all, the above results are based on the theory of sorted algebras and related **based algebras** [3], which is also considered part of algebraic logic, although its organization is purely algebraic. Also, if a logical system has a **truth**, then it yields a **logical space**, which is defined to be a pair of a set and a set of its subsets, and a theory of completeness for the logical spaces is developed in [4], which is algebra-flavored. Thus [3], [4], the present paper, and potential successors constitute what I presently call the foundations of algebraic logic, and [1] is their first outcome in mathematical psychology .

In fact, the above-mentioned papers and the present one are abridged translations of an impermanent aspect of the author's personal electronic publication *Mathematical Psychology* [2], where work in progress has been shown for more than a decade by frequent revisions, and in particular, algebraic logic has been developed more elaborately than here and applied.

It should be noted that Horikawa [5] generalized the whole theory in the present paper by allowing variable operations to be indexed by sequences of arbitrary length of symbols and variables.

We expect that our set-theoretical notation and terminology will be standard except that we denote the set of all mappings of a set Y into a set Z by $Y \to Z$. Thus $f \in Y \to Z$ means $f : Y \to Z$.

2 Preliminaries on sorted algebras

Here is given an account of algebras to the extent necessary for the subsequent sections. For omitted proofs, we refer the reader to [2][3].

2.1 Basic definitions and remarks

For each set A and each natural number n, an n-ary **operation** on A is a mapping α of a subset D of A^n into A. The set D is called the **domain** of α and denoted by $\mathrm{Dom}\,\alpha$, while the image αD is denoted by $\mathrm{Im}\,\alpha$. The number n is called an **arity** of α, and so if $D = \emptyset$, every natural number is an arity of α. We say that α is **global** if $D = A^n$. A subset B of A is said to be **closed** under the operation α if $\alpha(a_1, \ldots, a_n) \in B$ for each $(a_1, \ldots, a_n) \in B^n \cap D$. If B is closed under α, the **restriction** $\alpha|_{B^n \cap D}$ of α to B becomes an operation on B.

An **algebra** is a set A equipped with a family $(\alpha_\lambda)_{\lambda\in\Lambda}$ of operations on A, which we call the **operation system** or **OS** of the algebra A. We often identify the operation α_λ with its index λ. The set A is called the **support** of the algebra $(A,(\alpha_\lambda)_{\lambda\in\Lambda})$. The algebra $(A,(\alpha_\lambda)_{\lambda\in\Lambda})$ is said to be **global** if α_λ is global for every $\lambda\in\Lambda$.

The algebra $(A,(\alpha_\lambda)_{\lambda\in\Lambda})$ has two kinds of **subalgebras**. The first is an algebra $(A,(\alpha_\mu)_{\mu\in M})$ obtained by reducing the OS of A from $(\alpha_\lambda)_{\lambda\in\Lambda}$ to $(\alpha_\mu)_{\mu\in M}$ for a subset M of Λ. Such an algebra will be called an **operational subalgebra** and denoted by A_M. Also, if a subset B of A is closed under α_λ for each $\lambda\in\Lambda$, then B becomes an algebra equipped with the operation system $(\beta_\lambda)_{\lambda\in\Lambda}$ consisting of restrictions β_λ of α_λ to B. Such an algebra $(B,(\beta_\lambda)_{\lambda\in\Lambda})$ is called a **support subalgebra**.

Let $(A,(\alpha_\lambda)_{\lambda\in\Lambda})$ be an algebra. Then the intersection of support subalgebras of A is also a support subalgebra of A, and A itself is a support subalgebra of A. Therefore, for each subset S of A, the intersection of all support subalgebras of A which contain S is the smallest of the support subalgebras of A which contain S. We denote it by $[S]$ and call it the **closure** of S. Define the subsets S_n $(n=0,1,\dots)$ of A inductively as follows. First $S_0=S$. Next for each $n\geq 1$, S_n is the set of all elements $\alpha_\lambda(a_1,\dots,a_m)$ with $\lambda\in\Lambda$, $(a_1,\dots,a_m)\in\mathrm{Dom}\,\alpha_\lambda$, and $a_i\in S_{l_i}$ $(i=1,\dots,m)$ for some non-negative integers $l_1,\dots,l_m$ such that $n=1+\sum_{i=1}^m l_i$. Then $[S]=\bigcup_{n\geq 0} S_n$. We call S_n $(n=0,1,\dots)$ the **descendants** of S. It also holds that an element $a\in A$ belongs to $[S]$ iff there exists an **S-generating sequence** of a, which is defined to be a sequence $a_1,\dots,a_n$ of elements of A satisfying $a_n=a$ and

$$a_i\in S\cup\bigcup_{\lambda\in\Lambda}\alpha_\lambda\big(\{a_1,\dots,a_{i-1}\}^{n_\lambda}\cap\mathrm{Dom}\,\alpha_\lambda\big)$$

for each $i\in\{1,\dots,n\}$, where n_λ is an arity of α_λ.

Two algebras A and B are said to be **similar**, if they have operation systems $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\beta_\lambda)_{\lambda\in\Lambda}$ indexed by the same set Λ, and α_λ and β_λ have a common arity for each $\lambda\in\Lambda$.

Let $(A,(\alpha_\lambda)_{\lambda\in\Lambda})$ and $(B,(\beta_\lambda)_{\lambda\in\Lambda})$ be similar algebras. Then a mapping f of A into B is called a **homomorphism** or a **Λ-homomorphism** if it satisfies the following two conditions for all $\lambda\in\Lambda$, where n_λ denotes an arity common to α_λ and β_λ.

- If $(a_1,\dots,a_{n_\lambda})\in\mathrm{Dom}\,\alpha_\lambda$, then $(fa_1,\dots,fa_{n_\lambda})\in\mathrm{Dom}\,\beta_\lambda$ and $f(\alpha_\lambda(a_1,\dots,a_{n_\lambda}))=\beta_\lambda(fa_1,\dots,fa_{n_\lambda})$.
- If $(a_1,\dots,a_{n_\lambda})\in A^{n_\lambda}$ and $(fa_1,\dots,fa_{n_\lambda})\in\mathrm{Dom}\,\beta_\lambda$, then $(a_1,\dots,a_{n_\lambda})\in\mathrm{Dom}\,\alpha_\lambda$.

A bijective homomorphism is called an **isomorphism**. If both A and B are global algebras, a mapping f of A into B is a homomorphism iff it satisfies the following condition for all $\lambda\in\Lambda$ and all $(a_1,\dots,a_{n_\lambda})\in A^{n_\lambda}$:

$$f(\alpha_\lambda(a_1,\dots,a_{n_\lambda}))=\beta_\lambda(fa_1,\dots,fa_{n_\lambda}).$$

A **sorted algebra** is an algebra A equipped with an algebra T similar to A and a homomorphism σ of A into T. We call T and σ the **type algebra** and the **sorting** of the sorted algebra A. For each $a \in A$, we call σa the **type of** a. For each subset S of A and each $t \in T$, we define the **t-part** S_t of S to be the inverse image $\{a \in S \mid \sigma a = t\}$ of t in S by σ.

Every global algebra A may be regarded as a sorted algebra (A, T, σ), where T is an arbitrary singleton made into a global algebra similar to A in the obvious unique manner and σ is the unique mapping of A into T. Conversely if (A, T, σ) is a sorted algebra with T global, then A is a global algebra.

Let (A, T, σ) and (B, T, τ) be sorted algebras with the same type algebra T. Then a mapping f of A into B is said to be **sort-consistent**, if it satisfies $\tau f = \sigma$, or equivalently $f(A_t) \subseteq B_t$ for all $t \in T$.

2.2 Universal sorted algebras

A sorted algebra (A, T, σ) is said to be **universal** or called a **USA** if A has a subset S which satisfies the following two conditions, the latter being called the **universality**.

- $A = [S]$.
- If (A', T, σ') is a sorted algebra and φ is a mapping of S into A' satisfying $\sigma'\varphi = \sigma|_S$, then there exists a sort-consistent homomorphism f of A into A' which extends φ.

We call S as above the set of the **primes** of A. It is known that every sorted algebra has at most one prime set and that f in the above condition is uniquely determined by φ.

If (A, T, σ, S) is a USA with T a global singleton, then A is a global algebra and satisfies the following conditions.

- $A = [S]$.
- If A' is a global algebra similar to A and φ is a mapping of S into A', then there exists a homomorphism of A into A' extending φ.

Conversely, assume that a global algebra A and its subset S satisfy these conditions. Then the sorted algebra (A, T, σ) made as before with a singleton T and a unique mapping $\sigma \in A \to T$ together with S constitutes a USA (A, T, σ, S). Therefore, we say that A is a **universal global algebra** or a UGA over S.

The following theorem is known to hold.

Theorem 2.1 (Unique Existence of USA) Let S be a set, T be an algebra, and τ be a mapping of S into T. Then there exists a USA (A, T, σ, S) with $\sigma|_S = \tau$. If (A', T, σ', S) is also a USA with $\sigma'|_S = \tau$, then there exists a sort-consistent isomorphism of A onto A' extending id_S.

Thus, in order to define a USA, we only need to define a set S, an algebra T, and a mapping τ of S into T. We call τ the **pre-sorting**.

In the course of the proof of Theorem 2.1 in [2][3], it is shown that if (A, T, σ, S) is a USA then A is the direct union of the descendants S_n ($n = 0, 1, \dots$) of S. Therefore, for each element a of A, there exists a unique non-negative integer n such that $a \in S_n$. We call it the **rank** of a. It is also shown that if $a \in S$ then a has no expression $a = \alpha(a_1, \dots, a_k)$ by an operation α in the OS of A, while if $a \in A - S$ then a has a unique such expression and $\operatorname{rank} a = 1 + \sum_{i=1}^{k} \operatorname{rank} a_i$.

2.3 Power algebras

Let (A, T, σ) be a sorted algebra and V be a non-empty set. Define $A^V = \bigcup_{t \in T}(V \to A_t)$. Then we can construct a sorted algebra (A^V, T, ρ) as follows. First define the sorting ρ of A^V into T by $\rho b = t$ for each $b \in V \to A_t$ and each $t \in T$. Then

$$\rho b = \sigma(bv)$$

for each $b \in A^V$ and each $v \in V$. Let $(\alpha_\lambda)_{\lambda \in \Lambda}$ and $(\tau_\lambda)_{\lambda \in \Lambda}$ be the OS's of A and T respectively, and let n_λ be an arity of α_λ and τ_λ. For each $\lambda \in \Lambda$, define the operation β_λ on A^V as follows. First define the domain of β_λ to be

$$D_\lambda = \big\{(b_1, \dots, b_{n_\lambda}) \in (A^V)^{n_\lambda} \mid (\rho b_1, \dots, \rho b_{n_\lambda}) \in \operatorname{Dom} \tau_\lambda\big\}.$$

If $(b_1, \dots, b_{n_\lambda}) \in D_\lambda$, then $\big(\sigma(b_1 v), \dots, \sigma(b_{n_\lambda} v)\big) = (\rho b_1, \dots, \rho b_{n_\lambda}) \in \operatorname{Dom} \tau_\lambda$ so $(b_1 v, \dots, b_{n_\lambda} v) \in \operatorname{Dom} \alpha_\lambda$ for each $v \in V$, and we can define the mapping $\beta_\lambda(b_1, \dots, b_{n_\lambda})$ of V into A by

$$\big(\beta_\lambda(b_1, \dots, b_{n_\lambda})\big)v = \alpha_\lambda(b_1 v, \dots, b_{n_\lambda} v)$$

for each $v \in V$. Furthermore

$$\sigma\big(\alpha_\lambda(b_1 v, \dots, b_{n_\lambda} v)\big) = \tau_\lambda\big(\sigma(b_1 v), \dots, \sigma(b_{n_\lambda} v)\big) = \tau_\lambda(\rho b_1, \dots, \rho b_{n_\lambda}),$$

and $t = \tau_\lambda(\rho b_1, \dots, \rho b_{n_\lambda})$ is not varied by $v \in V$, hence $\beta_\lambda(b_1, \dots, b_{n_\lambda}) \in V \to A_t \subseteq A^V$. Thus β_λ certainly is an operation on A^V for each $\lambda \in \Lambda$, and so $(A^V, (\beta_\lambda)_{\lambda \in \Lambda})$ becomes an algebra. Furthermore

$$\begin{aligned}\rho(\beta_\lambda(b_1, \dots, b_{n_\lambda})) &= \sigma\big(\big(\beta_\lambda(b_1, \dots, b_{n_\lambda})\big)v\big) \\ &= \sigma\big(\alpha_\lambda(b_1 v, \dots, b_{n_\lambda} v)\big) = \tau_\lambda(\rho b_1, \dots, \rho b_{n_\lambda})\end{aligned}$$

with any element $v \in V$, and so ρ is a homomorphism of A^V into T. Thus we have constructed the sorted algebra (A^V, T, ρ), which we call the **power algebra** of A with **exponent** V. Furthermore, it follows from the above definition that for each $v \in V$ the mapping $b \mapsto bv$ of A^V into A is a sort-consistent homomorphism, which we call the **projection** by v.

2.4 Occurrences and substitutions

2.4.1 Occurrences

Let $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ be an algebra. If, for two elements a and b of A, there exists an element $\lambda \in \Lambda$ such that $a = \alpha_\lambda(\dots, b, \dots)$, then we write $b \prec a$. If $b \prec a$ or $b = a$, we write $b \preceq a$. If there exists a sequence $b_0, \dots, b_n$ of elements of A such that $b_0 = a$, $b_n = b$ and $b_i \preceq b_{i-1}$ for $i = 1, \dots, n$, then we say that b **occurs** in a and call the sequence an **occurrence** of b in a. For each subset B of A and each element $a \in A$, we denote by B^a the set of the elements of B which occur in a. Furthermore, we define $\Lambda^a = \{\lambda \in \Lambda \mid (\operatorname{Im} \alpha_\lambda)^a \neq \emptyset\}$. If $\lambda \in \Lambda^a$, then we say that λ **occurs** in a.

Lemma 2.1 Let (A, T, σ, S) be a USA and $(\alpha_\lambda)_{\lambda\in\Lambda}$ be the OS of A. Then the following holds.

- For each element $a \in A$, S^a and Λ^a are finite sets.
- $\Lambda^a = \begin{cases} \emptyset & \text{when } a \in S, \\ \{\lambda\} \cup \bigcup_{k=1}^{n_\lambda} \Lambda^{a_k} & \text{when } a = \alpha_\lambda(a_1, \dots, a_{n_\lambda}). \end{cases}$

2.4.2 Notations and assumptions

In the rest of this subsection, let (A, T, σ, S) be a USA, and $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\tau_\lambda)_{\lambda\in\Lambda}$ be the OS's of A and T respectively. Furthermore, assume that Λ is contained in the set of the formal products of the elements of $\Gamma \amalg S$ for some set Γ. More precisely, Λ is a subset of the free semigroup over $\Gamma \amalg S$. For each element λ of Λ, S^λ is the set of the elements of S which occur in λ as defined in §2.4.1. Furthermore, we define $S^\Lambda = \bigcup_{\lambda\in\Lambda} S^\lambda$.

2.4.3 Free occurrences

Let $a \in A$ and $s \in S$. Then an occurrence $s_0, \dots, s_n$ of s in a is said to be **free**, if $\{s_0, \dots, s_n\} \cap \operatorname{Im} \alpha_\lambda = \emptyset$ for each $\lambda \in \Lambda$ such that $s \in S^\lambda$. If there exists a free occurrence of s in a, we say that s **occurs free** in a or write $s \ll a$. For each subset X of S, we define $X^a_{\text{free}} = \{x \in X \mid x \ll a\}$. Let $b \in A$. Then the occurrence $s_0, \dots, s_n$ of s in a is said to be **free from** b, if $\{s_0, \dots, s_n\} \cap \operatorname{Im} \alpha_\lambda = \emptyset$ for each $\lambda \in \Lambda$ such that $(S^\lambda)^b_{\text{free}} \neq \emptyset$. We say that s is **free from** b **in** a, if every free occurrence of s in a is free from b.

Lemma 2.2 The following holds.

- If $a = \alpha_\lambda(a_1, \dots, a_{n_\lambda}) \in A$, then $S^a_{\text{free}} = \bigcup_{k=1}^{n_\lambda} S^{a_k}_{\text{free}} - S^\lambda$.
- Let $a = \alpha_\lambda(a_1, \dots, a_{n_\lambda}) \in A$, $s \in S$, and $b \in A$. Then s is free from b in a iff either s is free from b in a_k for each $k \in \{1, \dots, n_\lambda\}$ and $(S^\lambda)^b_{\text{free}} = \emptyset$ or $s \not\ll a$.

- If $a = \alpha_\lambda(a_1, \dots, a_{n_\lambda}) \in A,\ s \in S - S^\lambda,\ b \in A$, and s is free from b in a, then s is free from b in a_k for each $k \in \{1, \dots, n_\lambda\}$.
- If $a, b \in A$ and $(S^\lambda)^b_{\text{free}} = \emptyset$ for each $\lambda \in \Lambda^a$, then every element of S is free from b in a.

2.4.4 Substitutions and occurrences

Let $s_1, \dots, s_n$ ($n \geq 0$) be distinct elements of S and $c_1, \dots, c_n$ be elements of A with $\sigma s_i = \sigma c_i$ ($i = 1, \dots, n$). Then, for each element a of A, we can define the element $a\left(\dfrac{s_1, \dots, s_n}{c_1, \dots, c_n}\right)$ of A with $\sigma\left(a\left(\dfrac{s_1, \dots, s_n}{c_1, \dots, c_n}\right)\right) = \sigma a$ by induction on the pairs (n, r) of n and $r = \operatorname{rank} a$ arranged in lexicographical order as follows. First of all, if $n = 0$, then we define $a\left(\dfrac{s_1, \dots, s_n}{c_1, \dots, c_n}\right) = a$. Therefore assume $n \geq 1$. If $r = 0$, then $a \in S$, and so we define

$$a\left(\frac{s_1, \dots, s_n}{c_1, \dots, c_n}\right) = \begin{cases} c_i & \text{if } a = s_i \text{ for some } i \in \{1, \dots, n\}, \\ a & \text{if } a \notin \{s_1, \dots, s_n\}, \end{cases} \tag{2.1}$$

hence $\sigma\left(a\left(\dfrac{s_1, \dots, s_n}{c_1, \dots, c_n}\right)\right) = \sigma a$ as desired. Therefore assume $r \geq 1$. Then a has a unique expression $a = \alpha_\lambda(a_1, \dots, a_{n_\lambda})$, and r is greater than the ranks of $a_1, \dots, a_{n_\lambda}$. Let $i_1, \dots, i_m$ ($0 \leq m \leq n$) be the numbers such that

$$\{s_1, \dots, s_n\} - S^\lambda = \{s_{i_1}, \dots, s_{i_m}\}, \qquad\qquad i_1 < \cdots < i_m.$$

Then we define

$$a\left(\frac{s_1, \dots, s_n}{c_1, \dots, c_n}\right) = \alpha_\lambda\left(a_1\left(\frac{s_{i_1}, \dots, s_{i_m}}{c_{i_1}, \dots, c_{i_m}}\right), \dots, a_{n_\lambda}\left(\frac{s_{i_1}, \dots, s_{i_m}}{c_{i_1}, \dots, c_{i_m}}\right)\right). \tag{2.2}$$

This is possible because, by induction, $a'_k = a_k\left(\dfrac{s_{i_1}, \dots, s_{i_m}}{c_{i_1}, \dots, c_{i_m}}\right)$ has already been defined and satisfies $\sigma a'_k = \sigma a_k$ for $k = 1, \dots, n_\lambda$, and so since $(\sigma a_1, \dots, \sigma a_{n_\lambda})$ belongs to $\operatorname{Dom} \tau_\lambda$, so does $(\sigma a'_1, \dots, \sigma a'_{n_\lambda})$, hence $(a'_1, \dots, a'_{n_\lambda}) \in \operatorname{Dom} \alpha_\lambda$. Moreover, even when $a\left(\dfrac{s_1, \dots, s_n}{c_1, \dots, c_n}\right) \neq a$, we have

$$\begin{aligned} \sigma\left(a\left(\frac{s_1, \dots, s_n}{c_1, \dots, c_n}\right)\right) &= \sigma\big(\alpha_\lambda(a'_1, \dots, a'_{n_\lambda})\big) \\ &= \tau_\lambda(\sigma a'_1, \dots, \sigma a'_{n_\lambda}) = \tau_\lambda(\sigma a_1, \dots, \sigma a_{n_\lambda}) = \sigma a \end{aligned}$$

as desired. The definition of $a\left(\dfrac{s_1, \dots, s_n}{c_1, \dots, c_n}\right)$ by induction is complete. We call the transformation $a \mapsto a\left(\dfrac{s_1, \dots, s_n}{c_1, \dots, c_n}\right)$ on A the (simultaneous) **substitution**

of $c_1, \ldots, c_n$ **for** $s_1, \ldots, s_n$. Since $\sigma\left(a\left(\dfrac{s_1, \ldots, s_n}{c_1, \ldots, c_n}\right)\right) = \sigma a$, the substitution is sort-consistent. Notice that the following does not always holds:

$$a\left(\frac{s_1, \ldots, s_n}{c_1, \ldots, c_n}\right) = a\left(\frac{s_1}{c_1}\right) \cdots \left(\frac{s_n}{c_n}\right).$$

Lemma 2.3 As for the substitution $\left(\dfrac{s_1, \ldots, s_n}{c_1, \ldots, c_n}\right)$, if s_i does not occur free in an element $a \in A$ for some $i \in \{1, \ldots, n\}$, then the following holds:

$$a\left(\frac{s_1, \ldots, s_n}{c_1, \ldots, c_n}\right) = a\left(\frac{s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n}{c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n}\right).$$

Lemma 2.4 Let $a \in A$ and $b = a\left(\dfrac{s_1, \ldots, s_n}{c_1, \ldots, c_n}\right)$. Then $S^b_{\text{free}} \subseteq \bigcup_{i=1}^n S^{c_i}_{\text{free}} \cup \left(S^a_{\text{free}} - \{s_1, \ldots, s_n\}\right)$.

Lemma 2.5 Let $a_1, \ldots, a_m \in A$, $B \subseteq A$, and $s_1, \ldots, s_n$ be distinct elements of S which satisfy $(S^\lambda)^b_{\text{free}} \subseteq \{s_1, \ldots, s_n\}$ for each $\lambda \in \Lambda$ and each $b \in B$. Assume that, for each $t \in T$ with $\{s_1, \ldots, s_n\}_t \cap S^\Lambda \neq \emptyset$, $S_t \cap S^\Lambda$ is enumerable. Then there exist distinct elements $r_1, \ldots, r_n \in S^\Lambda \cup (\{s_1, \ldots, s_n\} - S^\Lambda)$ which satisfy the following conditions.

- $\sigma r_i = \sigma s_i$ for each $i \in \{1, \ldots, n\}$.
- Each element of S is free from $b\left(\dfrac{s_1, \ldots, s_n}{r_1, \ldots, r_n}\right)$ in $a_1, \ldots, a_m$ for each element $b \in B$.

3 Formal languages

By definition of moderate generality, a **formal language** is a universal sorted algebra (A, T, σ, S) equipped with subsets C and X of S and a set Γ which satisfy the following three conditions.

- The set S is the direct sum $C \amalg X$ of C and $X \neq \emptyset$.
- Let $(\tau_\lambda)_{\lambda \in \Lambda}$ be the OS of the algebra T. Then its index set Λ is contained in the subset $\Gamma \cup \Gamma X$ of the free semigroup over $\Gamma \amalg S$.
- The arity of each operation τ_λ with $\lambda \in \Lambda \cap \Gamma X$ is equal to 1.

According to Theorem 2.1, any quintuple T, S, C, X, Γ satisfying the above conditions together with any pre-sorting $\tau \in S \to T$ determines a formal language $(A, T, \sigma, S, C, X, \Gamma)$ with $\sigma|_S = \tau$.

We call C and X the sets of the **constants** and **variables** respectively, and call Γ the **index basis**. Henceforth, we identify each index $\lambda \in \Lambda \cap \Gamma X$ with the operation τ_λ, call it a **variable operation**, and denote its domain by T_λ, because τ_λ is unary and so $T_\lambda \subseteq T$. Furthermore, we define

$$X' = \{x \in X \mid \Lambda \cap \Gamma x \neq \emptyset\}$$

and call X' the set of **qualifying variables**. We also define

$$\Lambda' = \Lambda \cap \Gamma,$$

call Λ' the set of **invariable indices**, and denote by T' the operational subalgebra $T_{\Lambda'}$ of T obtained by reducing the OS of T from $(\tau_\lambda)_{\lambda\in\Lambda}$ to $(\tau_\lambda)_{\lambda\in\Lambda'}$. While T' as an algebra is equal to T iff $\Lambda' = \Lambda$, T' as a set is equal to T.

Since Λ is a subset of the free semigroup over $\Gamma \amalg S$, we may discuss free occurrences and substitutions on (A, T, σ, S). Furthermore, since $\Lambda \subseteq \Gamma \cup \Gamma X$, the following holds:

$$S^\lambda = \begin{cases} \emptyset & \text{when } \lambda \in \Lambda', \\ \{x\} & \text{when } \lambda \in \Lambda \cap \Gamma x \ (x \in X), \end{cases} \tag{3.1}$$

$$S^\Lambda = X'. \tag{3.2}$$

Example 3.1 The propositional language may be defined to be the formal language $(A, T, \sigma, S, C, X, \Gamma)$ as follows. First, let the prime set S satisfy $S = X$, or $C = \emptyset$. Next, let the type algebra T be the singleton $\{\phi\}$ equipped with the OS consisting of the three binary global operations $\wedge, \vee, \Rightarrow$ and one unary global operation $\neg$. Necessarily, we have

$$\begin{aligned} &\mathrm{Dom} \wedge = \mathrm{Dom} \vee = \mathrm{Dom} \Rightarrow = \{\phi\}^2, && \mathrm{Dom} \neg = \{\phi\}, \\ &\phi \wedge \phi = \phi \vee \phi = \phi \Rightarrow \phi = \phi, && \neg\phi = \phi, \end{aligned} \tag{3.3}$$

and the pre-sorting τ maps every element of X to ϕ. Finally, let the index basis Γ be equal to the index set $\Lambda = \{\wedge, \vee, \Rightarrow, \neg\}$ of T.

Thus, the set $\Lambda' = \Lambda \cap \Gamma$ of invariable indices is equal to Λ, $T' = T_{\Lambda'}$ is equal to T not only as a set but also as an algebra, and there are no variable operations or qualifying variables.

Since the type algebra is a global singleton, the propositional language A thus defined is nothing but a UGA over a non-empty set X whose OS consists of binary operations $\wedge, \vee, \Rightarrow$ and an unary operation $\neg$. Consequently, A is the direct union of the descendants X_n $(n = 0, 1, \dots)$ of X, and X_n is inductively described as follows. First, $X_0 = X$. Next for $n \geq 1$, X_n is the set of the elements

$$a_1 \wedge a_2, \qquad a_1 \vee a_2, \qquad a_1 \Rightarrow a_2, \qquad \neg a,$$

where a_1, a_2, a satisfy the conditions

$$a_i \in X_{n_i} \ (i = 1, 2), \qquad n = 1 + n_1 + n_2, \qquad a \in X_{n-1}.$$

Therefore, the above definition of the propositional language is equivalent to the usual one.

Example 3.2 The first-order predicate language may be defined to be the formal language $(A, T, \sigma, S, C, X, \Gamma)$ as follows. First, we impose no additional conditions on the prime set S. Next, we let T be a set $\{\epsilon, \phi\}$ of two distinct elements ϵ, ϕ equipped with the OS consisting of the four operations $\wedge, \vee, \Rightarrow, \neg$ satisfying (3.3), and the two kinds of unary operations $\forall x, \exists x$ $(x \in X)$ satisfying

$$\mathrm{Dom}\,\forall x = \mathrm{Dom}\,\exists x = \{\phi\}, \qquad \forall x\phi = \exists x\phi = \phi, \tag{3.4}$$

and the two sets F and $P \neq \emptyset$ of operations of arbitrary arities such that

$$\mathrm{Dom}\,f = \{\epsilon\} \times \cdots \times \{\epsilon\}, \qquad f(\epsilon, \dots, \epsilon) = \epsilon, \tag{3.5}$$

$$\mathrm{Dom}\,p = \{\epsilon\} \times \cdots \times \{\epsilon\}, \qquad p(\epsilon, \dots, \epsilon) = \phi \tag{3.6}$$

for each $f \in F$ and each $p \in P$. Therefore, the index set Λ of T is equal to $\{\wedge, \vee, \Rightarrow, \neg, \forall x, \exists x \mid x \in X\} \cup F \cup P$. Next, we define $\Gamma = \{\wedge, \vee, \Rightarrow, \neg, \forall, \exists\} \cup F \cup P$. Then certainly $\Lambda \subseteq \Gamma \cup \Gamma X$ and the set $\Lambda \cap \Gamma X$ of the variable operations consists of the unary operations $\forall x, \exists x$ $(x \in X)$. Finally, we assume that the pre-sorting $\tau \in S \to T$ satisfies $\tau S = \{\epsilon\}$.

Thus, the set $\Lambda' = \Lambda \cap \Gamma$ of invariable indices is equal to $\{\wedge, \vee, \Rightarrow, \neg\} \cup F \cup P$, and the set X' of qualifying variables is equal to X.

In order to clarify the structure of the formal language A thus defined, first let B be the closure of S in the operational subalgebra A_F obtained by reducing the OS of A to F. Then as in Example 3.1, B is a union $\bigcup_{n=0}^{\infty} S_n$ of the descendants S_n $(n = 0, 1, \dots)$ of S, and since S is contained in the ϵ-part A_ϵ of A, it follows from (3.5) that $B \subseteq A_\epsilon$ and S_n is inductively described as follows. First, $S_0 = S$. Next for $n \geq 1$, S_n is the set of the elements $f(b_1, \dots, b_k)$, where $f \in F$ and $b_i \in S_{n_i}$ $(i = 1, \dots, k)$ for some non-negative integers $n_1, \dots, n_k$ satisfying $n = 1 + \sum_{i=1}^{k} n_i$.

Next, let C be the closure of B in the operational subalgebra A_P obtained by reducing the OS of A to P. Then as before, we have $C = \bigcup_{n=0}^{\infty} B_n$, and B_n $(n = 0, 1, \dots)$ are inductively described as follows. First, $B_0 = B$. Next for $n \geq 1$, B_n is the set of the elements $p(c_1, \dots, c_k)$, where $p \in P$, $(c_1, \dots, c_k) \in \mathrm{Dom}\,p$, and $c_i \in B_{n_i}$ $(i = 1, \dots, k)$ for some non-negative integers $n_1, \dots, n_k$ satisfying $n = 1 + \sum_{i=1}^{k} n_i$. In particular, B_1 consists of the elements $p(b_1, \dots, b_k)$ with $b_1, \dots, b_k \in B$. Since $P \neq \emptyset$, we have $B_1 \neq \emptyset$. Also, it follows from (3.6) that B_1 is contained in the ϕ-part A_ϕ of A. Hence it furthermore follows that $B_n = \emptyset$ for $n \geq 2$. For instance if $n = 2$, then since $2 = 1 + \sum_{i=1}^{k} n_i$, we have $n_j = 1$ for some $j \in \{1, \dots, k\}$, and since $c_i \in B_{n_i}$ $(i = 1, \dots, k)$, we have $c_j \in B_1 \subseteq A_\phi$ and so $(c_1, \dots, c_k) \notin \mathrm{Dom}\,p$. Therefore $B_2 = \emptyset$, and we can similarly prove $B_n = \emptyset$ for $n \geq 2$ by induction on n. Thus $C = B \cup B_1$.

Next, let D be the closure of B_1 in the operational subalgebra of A obtained by reducing the OS of A to $\{\wedge, \vee, \Rightarrow, \neg, \forall x, \exists x \mid x \in X\}$. Then as before, it follows from (3.3) and (3.4) that $D = \bigcup_{n=0}^{\infty} D_n \subseteq A_\phi$ and D_n $(n = 0, 1, \dots)$ are inductively described as follows. First, $D_0 = B_1$. Next for $n \geq 1$, D_n is the set of the elements

$$d_1 \wedge d_2, \qquad d_1 \vee d_2, \qquad d_1 \Rightarrow d_2, \qquad \neg d, \qquad \forall x d, \qquad \exists x d,$$

where $d_i \in D_{n_i}$ $(i = 1, 2)$, $n = 1 + n_1 + n_2$, $d \in D_{n-1}$, and $x \in X$.

In fact $A = B \cup D$. In order to prove this, since $A = [S]$ and $S \subseteq B \subseteq B \cup D$, we only need to show that $B \cup D$ is closed under every operation $\lambda \in \Lambda$. Therefore, suppose $(a_1, \ldots, a_k) \in (B \cup D)^k \cap \mathrm{Dom}\,\lambda$, and let $a = \lambda(a_1, \ldots, a_k)$. Recall $B \subseteq A_\epsilon$ and $D \subseteq A_\phi$. Hence if $\lambda \in F$, then $a_1, \ldots, a_k \in B$, and since B is closed under every operation in F, we have $a \in B$. Next if $\lambda \in P$, then also $a_1, \ldots, a_k \in B$, hence $a \in B_1 \subseteq D$. Finally if $\lambda \in \{\wedge, \vee, \Rightarrow, \neg, \forall x, \exists x \mid x \in X\}$, then $a_1, \ldots, a_k \in D$, and since D is closed under $\wedge, \vee, \Rightarrow, \neg, \forall x, \exists x$ $(x \in X)$, we have $a \in D$.

We have shown $A = B \cup D$. Since $B \subseteq A_\epsilon$, $D \subseteq A_\phi$, and $A_\epsilon \cap A_\phi = \emptyset$, we conclude that $B = A_\epsilon$ and $D = A_\phi$ hold. Furthermore, using a certain theorem in [3], we can prove that A_ϵ equipped with the operation system F is a UGA over S and that A_ϕ equipped with the operation system $\{\wedge, \vee, \Rightarrow, \neg, \forall x, \exists x \mid x \in X\}$ is a UGA over the set $\{p(a_1, \ldots, a_k) \mid p \in P,\ a_1, \ldots, a_k \in A_\epsilon\}$. Consequently as in Example 3.1, we have $A_\epsilon = \coprod_{n=0}^{\infty} S_n$ and $A_\phi = \coprod_{n=0}^{\infty} D_n$.

Thus, the predicate language defined above is the set of the terms and formulas in the usually defined predicate logic.

Example 3.3 The set of λ-terms in the typed λ-calculus may be defined to be the formal language $(A, T, \sigma, S, C, X, \Gamma)$ as follows. However for certain reasons, we write Ω for λ.

First, we impose no additional conditions on the prime set S and the presorting $\tau \in S \to T$. Next, we define the support of the type algebra T to be the support of a UGA over a set T_0 whose OS consists of one binary operation $\to$. Then $T = \coprod_{n=0}^{\infty} T_n$ and T_n $(n = 1, 2, \ldots)$ are the set of the elements $t_1 \to t_2$ with $t_i \in T_{n_i}$ $(i = 1, 2)$ and $n = 1 + n_1 + n_2$. Next, we define the OS of T to consist of the binary operation $\bullet$ and the unary operations Ωx $(x \in X)$ defined by

$$\mathrm{Dom}\,\bullet = \{(t \to u, t) \mid t, u \in T\}, \qquad (t \to u) \bullet t = u,$$
$$\mathrm{Dom}\,\Omega x = T, \qquad \Omega x\, t = \tau x \to t.$$

Therefore, the index set Λ of T is equal to $\{\bullet, \Omega x \mid x \in X\}$. Finally, we define $\Gamma = \{\bullet, \Omega\}$. Then certainly $\Lambda \subseteq \Gamma \cup \Gamma X$ and the set $\Lambda \cap \Gamma X$ of variable operations consists of the unary operations Ωx $(x \in X)$. Furthermore, the set $\Lambda' = \Lambda \cap \Gamma$ of invariable indices is equal to $\{\bullet\}$, and the set X' of qualifying variables is equal to X.

Example 3.4 The MPC language and C language investigated in [2][1] are formal languages.

4 Denotable worlds

Let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language, Λ be the index set of T, $\Lambda' = \Lambda \cap \Gamma$ be the set of the invariable indices, and T' be the operational subalgebra $T_{\Lambda'}$ of

T. Then, a sorted algebra W is called a **denotable world** for A, if it satisfies the following two conditions.

- The type algebra of W is equal to T'.
- $W_t \neq \emptyset$ for each $t \in \sigma S$, that is, for each $t \in T$ with $S_t \neq \emptyset$.

Remark 4.1 In [1][2], denotable worlds are alternatively called **cognizable worlds** from the viewpoint of mathematical psychology, and an arbitrarily chosen non-empty collection of cognizable worlds for A is called the domain of the **actual worlds** for A.

Example 4.1 Suppose $(A, T, \sigma, S, C, X, \Gamma)$ is the propositional language defined in Example 3.1. Then A is a UGA over X whose OS consists of the binary operations $\wedge, \vee, \Rightarrow$ and the unary operation $\neg$, and since $T' = T$ as algebras and $T = \{\phi\}$, the denotable worlds for A are the non-empty global algebras similar to A.

If $(A, T, \sigma, S, C, X, \Gamma)$ is the predicate language defined in Example 3.2, then $T = \{\epsilon, \phi\}$, and the denotable worlds W for A are the direct unions $W_\epsilon \amalg W_\phi$ of a non-empty algebra W_ϵ similar to A_ϵ and an algebra W_ϕ similar to a denotable world for the propositional language.

Example 4.2 Let $(A, T, \sigma, S, C, X, \Gamma)$ be the set of the λ-terms defined in Example 3.3. Here we construct a specific denotable world W for A.

Since W is the direct union of its t-parts W_t $(t \in T)$, we have to first define a family of sets W_t $(t \in T)$. Since T is a UGA over a set T_0 with respect to the binary operation $\to$, we define W_t by induction on the rank n of t with respect to $\to$. If $n = 0$, then $t \in T_0$, so we define W_t to be an arbitrary non-empty set. Suppose $n \geq 1$. Then $t = t_1 \to t_2$ with $t_i \in T_{n_i}$ $(i = 1, 2)$ and $n = 1 + n_1 + n_2$, and so since W_{t_i} $(i = 1, 2)$ have already been defined, we may define $W_t = W_{t_1} \to W_{t_2}$. Hence

$$W_{t \to u} = W_t \to W_u$$

for all $(t, u) \in T^2$. Since we have defined a family of sets W_t $(t \in T)$, we define

$$W = \coprod_{t \in T} W_t.$$

It now remains to define a binary operation $\bullet$ on W. First, we define

$$\mathrm{Dom} \bullet = \coprod_{t,u \in T} (W_{t \to u} \times W_t).$$

Next for $a \in W_{t \to u}$ and $b \in W_t$, since $W_{t \to u} = W_t \to W_u$, we may define

$$a \bullet b = ab,$$

where ab is the image in W_u of $b \in W_t$ by the mapping $a \in W_t \to W_u$. This completes the construction of a specific denotable world W for A.

Example 4.3 The MPC worlds and C worlds investigated in [2][1] are cognizable worlds respectively of the MPC language and C language mentioned in Example 3.4.

5 Denotations and meta-denotations

Let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language. Then for each denotable world W for A, a C-**denotation** or **constant denotation** into W is a mapping Φ of C into W which satisfies $\Phi C_t \subseteq W_t$ for each $t \in T$. There is at least one C-denotation. If $C = \emptyset$, then since $\emptyset \to W = \{\emptyset\}$ by the set-theoretical definition of $Y \to Z$, $\emptyset$ is the unique C-denotation. Similarly, an X-**denotation** or a **variable denotation** into W is a mapping ν of X into W which satisfies $\nu X_t \subseteq W_t$ for each $t \in T$. We denote the set of all X-denotations into W by $V_{X,W}$, because denotations are alternatively called **valuations**. Then $V_{X,W} \neq \emptyset$ because $W_t \neq \emptyset$ whenever $S_t \neq \emptyset$, and so we can construct the power algebra $(W^{V_{X,W}}, T', \rho)$ of W with exponent $V_{X,W}$ as described in §2, where $T' = T_{\Lambda'}$, $\Lambda' = \Lambda \cap \Gamma$, and Λ is the index set of T. Let $(\beta_\lambda)_{\lambda \in \Lambda'}$ be the OS of $W^{V_{X,W}}$. Recall that we identify each index $\lambda \in \Lambda \cap \Gamma X$ with the operation τ_λ, call it a variable operation, and denote its domain by T_λ because it is a subset of T.

Suppose that, for a denotable world W for A and for each variable operation $\lambda \in \Lambda \cap \Gamma X$ and the variable x such that $\lambda \in \Gamma x$, we are given a mapping

$$\lambda_W \in \left(\bigcup_{t \in T_\lambda} (W_{\sigma x} \to W_t) \right) \to W \tag{5.1}$$

which satisfies

$$\lambda_W (W_{\sigma x} \to W_t) \subseteq W_{\lambda t} \tag{5.2}$$

for each $t \in T_\lambda$. Then we can define the unary operation β_λ on $W^{V_{X,W}}$ for each $\lambda \in \Lambda \cap \Gamma X$ as follows, and extending the OS of $W^{V_{X,W}}$ from $(\beta_\lambda)_{\lambda \in \Lambda'}$ to $(\beta_\lambda)_{\lambda \in \Lambda}$, we can construct the sorted algebra $(W^{V_{X,W}}, T, \rho)$. First we define, for each pair x, w of $x \in X$ and $w \in W_{\sigma x}$, the transformation $\nu \mapsto (x/w)\nu$ on $V_{X,W}$ by

$$\big((x/w)\nu\big) y = \begin{cases} \nu y & \text{when } X \ni y \neq x, \\ w & \text{when } y = x. \end{cases} \tag{5.3}$$

We call the transformation (x/w) the **redenotation** for x by w. Next we define, for each quadruple t, φ, x, ν consisting of $t \in T$, $\varphi \in V_{X,W} \to W_t$, $x \in X$ and $\nu \in V_{X,W}$, the mapping $\varphi\big((x/\square)\nu\big)$ of $W_{\sigma x}$ into W_t by

$$\Big(\varphi\big((x/\square)\nu\big)\Big)\, w = \varphi\big((x/w)\nu\big) \tag{5.4}$$

for each $w \in W_{\sigma x}$. We finally define for each $\lambda \in \Lambda \cap \Gamma X$ the unary operation β_λ on $W^{V_{X,W}}$ as follows. Suppose $\lambda \in \Gamma x$ with $x \in X$. First we define

$$\operatorname{Dom} \beta_\lambda = \bigcup_{t \in T_\lambda} (V_{X,W} \to W_t). \tag{5.5}$$

Next for each $t \in T_\lambda$ and each $\varphi \in V_{X,W} \to W_t$ we define $\beta_\lambda\varphi$ to be the element of $V_{X,W} \to W_{\lambda t}$ such that

$$(\beta_\lambda\varphi)\nu = \lambda_W\big(\varphi\big((x/\square)\nu\big)\big) \tag{5.6}$$

for each $\nu \in V_{X,W}$. Since $\varphi\big((x/\square)\nu\big) \in W_{\sigma x} \to W_t$ and $\lambda_W(W_{\sigma x} \to W_t) \subseteq W_{\lambda t}$, certainly $(\beta_\lambda\varphi)\nu \in W_{\lambda t}$. Thus

$$\beta_\lambda(V_{X,W} \to W_t) \subseteq V_{X,W} \to W_{\lambda t} \tag{5.7}$$

for each $t \in T_\lambda$. Since $V_{X,W} \to W_t$ is the t-part of $W^{V_{X,W}}$ for each $t \in T$, we have thus constructed the sorted algebra $(W^{V_{X,W}}, T, \rho)$. We call the mapping λ_W used above for $\lambda \in \Lambda \cap \Gamma X$ an **interpretation** of λ on W.

Now let Φ be a C-denotation into W. Then we can construct the sort-consistent homomorhism Φ^* of A into $W^{V_{X,W}}$ as follows. First we define the mapping φ of S into $V_{X,W} \to W$ so that

$$(\varphi a)\nu = \begin{cases} \Phi a & \text{when } a \in C, \\ \nu a & \text{when } a \in X \end{cases}$$

for each $\nu \in V_{X,W}$. Then $\varphi S_t \subseteq V_{X,W} \to W_t$ for each $t \in T$ because $\Phi C_t \subseteq W_t$ and $\nu X_t \subseteq W_t$, and so φ maps S into $W^{V_{X,W}}$ and satisfies $\rho\varphi = \sigma|_S$. Therefore by the universality of A, there exists a unique sort-consistent homomorphism of A into $W^{V_{X,W}}$ which extends φ. We call it the **meta-denotation** determined by Φ and denote it by Φ^*. Since Φ^* is an extension of φ,

$$(\Phi^* a)\nu = \begin{cases} \Phi a & \text{when } a \in C, \\ \nu a & \text{when } a \in X \end{cases} \tag{5.8}$$

for each $\nu \in V_{X,W}$.

Remark 5.1 By definition, a **logical system** is a triple $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ of a formal language $(A, T, \sigma, S, C, X, \Gamma)$, a domain $\mathcal{W}$ of denotable worlds for A, and a family $(\lambda_W)_{\lambda,W}$ of interpretations λ_W of the variable operations $\lambda \in \Lambda \cap \Gamma X$ on $W \in \mathcal{W}$.

Suppose the logical system $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ satisfies the following condition.

- For an element $\phi \in T$, the ϕ-part A_ϕ of A is non-empty, and the ϕ-part W_ϕ of each $W \in \mathcal{W}$ is equal to $\mathbb{T} = \{0, 1\}$.

Then we call ϕ a **truth** and call the elements of A_ϕ the **sentences**.

Each logical system with a truth yields a "sentence logical space." A general theory of completeness for logical spaces is developed in [4], and a specific logical space is investigated in [1] from the viewpoint of mathematical psychology [2].

Example 5.1 Here is shown interpretations of the variable operations which are usually denoted by $\forall x$ and $\exists x$.

Assume that a variable operation $\lambda \in \Lambda \cap \Gamma x$ $(x \in X)$ of a formal language $(A, T, \sigma, S, C, X, \Gamma)$ satisfies $T_\lambda = \{\phi\}$ and $\lambda\phi = \phi$ and that a denotable world W for A satisfies $W_\phi = \mathbb{T}$ (cf. (3.4) in Example 3.2 and Remark 5.1). Then (5.1) and (5.2) show that the interpretation λ_W of λ on W is a mapping

$$\lambda_W \in (W_{\sigma x} \to W_\phi) \to W_\phi,$$

and (5.5) and (5.7) show that the unary operation β_λ on $W^{V_{X,W}}$ satisfies

$$\operatorname{Dom} \beta_\lambda = V_{X,W} \to W_\phi, \qquad \operatorname{Im} \beta_\lambda \subseteq V_{X,W} \to W_\phi.$$

For instance, define λ_W by

$$\lambda_W f = \inf\{fw \mid w \in W_{\sigma x}\} \tag{5.9}$$

for each $f \in W_{\sigma x} \to W_\phi$, where the infimum is taken with respect to the usual order on $W_\phi = \mathbb{T}$. Then (5.6) and (5.4) show that

$$(\beta_\lambda \varphi)v = \inf\big\{\varphi\big((x/w)v\big) \mid w \in W_{\sigma x}\big\} \tag{5.10}$$

for each $\varphi \in V_{X,W} \to W_\phi$ and $v \in V_{X,W}$, which implies that $(\beta_\lambda\varphi)v = 1$ iff $\varphi\big((x/w)v\big) = 1$ for all $w \in W_{\sigma x}$. Thus, it is reasonable to denote the variable operation $\lambda \in \Lambda \cap \Gamma x$ thus interpreted by $\forall x$, considering that it is the product of $\forall \in \Gamma$ and $x \in X$.

Similarly, if we define λ_W by

$$\lambda_W f = \sup\{fw \mid w \in W_{\sigma x}\}$$

for each $f \in W_{\sigma x} \to W_\phi$, then we have

$$(\beta_\lambda \varphi)v = \sup\big\{\varphi\big((x/w)v\big) \mid w \in W_{\sigma x}\big\}$$

for each $\varphi \in V_{X,W} \to W_\phi$ and $v \in V_{X,W}$, which implies that $(\beta_\lambda\varphi)v = 1$ iff there exists an element $w \in W_{\sigma x}$ such that $\varphi\big((x/w)v\big) = 1$. Thus, it is reasonable to denote the variable operation $\lambda \in \Lambda \cap \Gamma x$ thus interpreted by $\exists x$, considering that it is the product of $\exists \in \Gamma$ and $x \in X$.

Example 5.2 Here is shown an interpretation of the variable operations Ωx on the formal language defined in Example 3.3 on its denotable world defined in Example 4.2.

Assume more generally that a variable operation $\lambda \in \Lambda \cap \Gamma x$ $(x \in X)$ of a formal language $(A, T, \sigma, S, C, X, \Gamma)$ and its denotable world W satisfy $W_{\lambda t} = W_{\sigma x} \to W_t$ for each $t \in T_\lambda$. Then (5.1) and (5.2) show that the interpretation λ_W of λ on W is a transformation on $\bigcup_{t \in T_\lambda}(W_{\sigma x} \to W_t)$ satisfying

$$\lambda_W(W_{\sigma x} \to W_t) \subseteq W_{\sigma x} \to W_t$$

for each $t \in T_\lambda$, and (5.5) and (5.7) show that the unary operation β_λ on $W^{V_{X,W}}$ satisfies

$$\operatorname{Dom} \beta_\lambda = \bigcup_{t \in T_\lambda} (V_{X,W} \to W_t),$$
$$\beta_\lambda(V_{X,W} \to W_t) \subseteq V_{X,W} \to (W_{\sigma x} \to W_t).$$

for each $t \in T_\lambda$. For instance, define λ_W by

$$\lambda_W f = f \tag{5.11}$$

for all $f \in W_{\sigma x} \to W_t$. Then (5.6) and (5.4) show that

$$\big((\beta_\lambda \varphi)\nu\big)w = \varphi\big((x/w)\nu\big) \tag{5.12}$$

for each $\varphi \in \bigcup_{t \in T_\lambda}(V_{X,W} \to W_t)$, $\nu \in V_{X,W}$, and $w \in W_{\sigma x}$. We will denote the variable operation $\lambda \in \Lambda \cap \Gamma x$ thus interpreted by Ωx, considering that it is the product of $\Omega \in \Gamma$ and $x \in X$.

6 Fundamental theorems on denotations

Throughout this section, we let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language, W be its denotable world, and Φ be a C-denotation into W. We also assume that $W^{V_{X,W}}$ has been made into a sorted algebra with type algebra T by some interpretation λ_W on W of each variable operation λ on T. Then the meta-denotation $\Phi^* \in A \to W^{V_{X,W}}$ is defined. We denote the OS's of A, $W^{V_{X,W}}$, and W by $(\alpha_\lambda)_{\lambda \in \Lambda}$, $(\beta_\lambda)_{\lambda \in \Lambda}$, and $(\omega_\lambda)_{\lambda \in \Lambda'}$ respectively, where $\Lambda' = \Lambda \cap \Gamma$ because the type algebra of W is $T' = T_{\Lambda'}$.

Theorem 6.1 (Denotation theorem) Let $a \in A$ and $\nu, \nu' \in V_{X,W}$. Assume $\nu x = \nu' x$ for every variable x which occurs free in a. Then $(\Phi^* a)\nu = (\Phi^* a)\nu'$.

Proof Since A is a USA, we may argue by induction on the rank r of a.

Assume $r = 0$. Then $a \in S = C \cup X$. If $a \in C$, then

$$(\Phi^* a)\nu = \Phi a = (\Phi^* a)\nu'$$

by (5.8). If $a \in X$, then since $a \ll a$,

$$(\Phi^* a)\nu = \nu a = \nu' a = (\Phi^* a)\nu'$$

by (5.8) and our assumption. Therefore, $(\Phi^* a)\nu = (\Phi^* a)\nu'$ holds in this case.

Therefore assume $r \geq 1$. Then a has an expression $a = \alpha_\lambda(a_1, \ldots, a_{n_\lambda})$ and r is greater than the ranks of $a_1, \ldots, a_{n_\lambda}$. Since $\Phi^* \in A \to W^{V_{X,W}}$ is a Λ-homomorphism, we have

$$(\Phi^* a)\nu = \big(\beta_\lambda(\Phi^* a_1, \ldots, \Phi^* a_{n_\lambda})\big)\nu.$$

Assume $\lambda \in \Lambda'$. Then, since the projection by ν is a Λ'-homomorphism of $W^{V_{X,W}}$ into W, the above equation may be rewritten

$$(\Phi^* a)\nu = \omega_\lambda\big((\Phi^* a_1)\nu, \ldots, (\Phi^* a_{n_\lambda})\nu\big),$$

and a similar equation holds with ν replaced by ν'. Also since $\lambda \in \Lambda'$, $S^\lambda = \emptyset$ by (3.1), and so Lemma 2.2 shows that $X^{a_k}_{\text{free}} \subseteq X^{a}_{\text{free}}$, hence $(\Phi^* a_k)\nu = (\Phi^* a_k)\nu'$ for all $k \in \{1, \ldots, n_\lambda\}$ by the induction hypothesis. Therefore, $(\Phi^* a)\nu = (\Phi^* a)\nu'$ holds in this case.

Therefore assume $\lambda \notin \Lambda'$. Then $\lambda \in \Gamma x$ for some $x \in X$ and $n_\lambda = 1$, hence $a = \alpha_\lambda a_1$. Therefore

$$(\Phi^* a)\nu = \big(\beta_\lambda(\Phi^* a_1)\big)\nu = \lambda_W\Big((\Phi^* a_1)\big((x/\square)\nu\big)\Big)$$

by (5.6), and a similar equation holds with ν replaced by ν'. Therefore, we only need to show $(\Phi^* a_1)\big((x/\square)\nu\big) = (\Phi^* a_1)\big((x/\square)\nu'\big)$. In order to do so, in view of (5.4), we have to show

$$(\Phi^* a_1)\big((x/w)\nu\big) = (\Phi^* a_1)\big((x/w)\nu'\big)$$

for each $w \in W_{\sigma x}$. This will follow from the induction hypothesis, if we show that $\big((x/w)\nu\big)y = \big((x/w)\nu'\big)y$ for each $y \in X^{a_1}_{\text{free}}$. This certainly holds, because if $y = x$, then $\big((x/w)\nu\big)y = w = \big((x/w)\nu'\big)y$ by (5.3), while if $y \neq x$, then $y \notin S^\lambda$ by (3.1), and so $y \in X^a_{\text{free}}$ by Lemma 2.2, hence $\big((x/w)\nu\big)y = \nu y = \nu' y = \big((x/w)\nu'\big)y$ by (5.3) and our assumption. The proof is complete.

In order to state the next theorem, we generalize the redenotation $\nu \mapsto (x/w)\nu$ on $V_{X,W}$ defined by (5.3). Let $x_1, \ldots, x_n$ ($n \geq 0$) be distinct variables and let $w_i \in W_{\sigma x_i}$ ($i = 1, \ldots, n$). Then, for each X-denotation ν, there exists an X-denotation ν' which satisfies

$$\nu' x = \begin{cases} w_i & \text{if } x = x_i \text{ for some } i \in \{1, \ldots, n\}, \\ \nu x & \text{if } x \in X - \{x_1, \ldots, x_n\}. \end{cases}$$

We denote it by $\left(\dfrac{x_1, \ldots, x_n}{w_1, \ldots, w_n}\right)\nu$. Then the symbol $\left(\dfrac{x_1, \ldots, x_n}{w_1, \ldots, w_n}\right)$ may be regarded as denoting a transformation on $V_{X,W}$, which we call a **redenotation** for $(x_1, \ldots, x_n)$ by $(w_1, \ldots, w_n)$. When $n = 0$, it is the identity transformation on $V_{X,W}$. As an immediate consequence of this definition, we have

$$\begin{aligned} \left(\frac{x_1, \ldots, x_n}{w_1, \ldots, w_n}\right) &= \left(\frac{x_1, \ldots, x_{i-1}}{w_1, \ldots, w_{i-1}}\right)\left(\frac{x_i, \ldots, x_n}{w_i, \ldots, w_n}\right) \\ &= \left(\frac{x_i, \ldots, x_n}{w_i, \ldots, w_n}\right)\left(\frac{x_1, \ldots, x_{i-1}}{w_1, \ldots, w_{i-1}}\right) \end{aligned} \tag{6.1}$$

for each number $i \in \{2, \ldots, n\}$.

Theorem 6.2 (Substitution-redenotation theorem) Let $x_1, \ldots, x_n$ be distinct variables of A and $a, c_1, \ldots, c_n \in A$. Assume, for each $i \in \{1, \ldots, n\}$, that $\sigma x_i = \sigma c_i$ and x_i is free from c_i in a. Then the following holds for each $\nu \in V_{X,W}$.

$$\left(\Phi^*\left(a\left(\frac{x_1, \ldots, x_n}{c_1, \ldots, c_n}\right)\right)\right)\nu = (\Phi^* a)\left(\left(\frac{x_1, \ldots\ldots\ldots, x_n}{(\Phi^* c_1)\nu, \ldots, (\Phi^* c_n)\nu}\right)\nu\right) \tag{6.2}$$

Proof We argue by induction on the pairs (n, r) of n and $r = \operatorname{rank} a$ arranged in lexicographical order. If $n = 0$, then both the substitution $\left(\dfrac{x_1, \ldots, x_n}{c_1, \ldots, c_n}\right)$ and the redenotation $\left(\dfrac{x_1, \ldots\ldots\ldots, x_n}{(\Phi^* c_1)\nu, \ldots, (\Phi^* c_n)\nu}\right)$ are identity transformations, and so both sides of (6.2) are equal to $(\Phi^* a)\nu$. Therefore assume $n \geq 1$.

Assume $r = 0$. Then $a \in S = C \cup X$. If $a \in C$, then $a \notin \{x_1, \ldots, x_n\}$, and so the left-hand side of (6.2) is equal to $(\Phi^* a)\nu$ by (2.1), and therefore, both sides are equal to Φa by (5.8). If $a = x_i$ for some $i \in \{1, \ldots, n\}$, then the left-hand side is equal to $(\Phi^* c_i)\nu$ by (2.1), while the right-hand side is also equal to $(\Phi^* c_i)\nu$ by (5.8) and the definition of the redenotations. If $a \in X - \{x_1, \ldots, x_n\}$, then the left-hand side is equal to νa as in the case $a \in C$, while the right-hand side is also equal to νa by (5.8) and the definition of the redenotations.

Therefore assume $r \geq 1$. Then a has an expression $a = \alpha_\lambda(a_1, \ldots, a_{n_\lambda})$ and r is greater than the ranks of $a_1, \ldots, a_{n_\lambda}$. Define

$$
\begin{aligned}
u &= \left(\frac{x_1, \ldots\ldots\ldots, x_n}{(\Phi^* c_1)\nu, \ldots, (\Phi^* c_n)\nu}\right)\nu, \\
b &= a\left(\frac{x_1, \ldots, x_n}{c_1, \ldots, c_n}\right), \\
b_k &= a_k\left(\frac{x_1, \ldots, x_n}{c_1, \ldots, c_n}\right) \qquad\qquad (k = 1, \ldots, n_\lambda).
\end{aligned}
$$

We have to show $(\Phi^* b)\nu = (\Phi^* a)u$.

Suppose $x_i \not\ll a$ for some $i \in \{1, \ldots, n\}$. Then Lemma 2.3 shows that the left-hand side of (6.2) is unchanged by the deletion of $\dfrac{x_i}{c_i}$, while Theorem 6.1 shows that the right-hand side is unchanged by the deletion of $\dfrac{x_i}{(\Phi^* c_i)\nu}$. Therefore, (6.2) holds by the induction hypothesis.

Therefore assume $x_i \ll a$ for all $i \in \{1, \ldots, n\}$. Then $\{x_1, \ldots, x_n\} \cap S^\lambda = \emptyset$ by Lemma 2.2. Hence $b = \alpha_\lambda(b_1, \ldots, b_{n_\lambda})$ by (2.2), and x_i is free from c_i in a_k by Lemma 2.2 for each $i \in \{1, \ldots, n\}$ and each $k \in \{1, \ldots, n_\lambda\}$. Therefore $(\Phi^* b_k)\nu = (\Phi^* a_k)u$ by the induction hypothesis for each $k \in \{1, \ldots, n_\lambda\}$. Also, since $\Phi^* \in A \to W^{V_{X,W}}$ is a Λ-homomorphism, we have

$$
\begin{aligned}
(\Phi^* b)\nu &= \big(\beta_\lambda(\Phi^* b_1, \ldots, \Phi^* b_{n_\lambda})\big)\nu, \\
(\Phi^* a)u &= \big(\beta_\lambda(\Phi^* a_1, \ldots, \Phi^* a_{n_\lambda})\big)u.
\end{aligned}
$$

If $\lambda \in \Lambda'$, then since the projections by ν and u are Λ'-homomorphisms of $W^{V_{X,W}}$ into W, the above equations may be rewritten

$$
\begin{aligned}
(\Phi^* b)\nu &= \omega_\lambda\big((\Phi^* b_1)\nu, \ldots, (\Phi^* b_{n_\lambda})\nu\big), \\
(\Phi^* a)u &= \omega_\lambda\big((\Phi^* a_1)u, \ldots, (\Phi^* a_{n_\lambda})u\big).
\end{aligned}
$$

Since $(\Phi^* b_k)\nu = (\Phi^* a_k)u$ for each $k \in \{1, \ldots, n_\lambda\}$, we conclude that $(\Phi^* b)\nu = (\Phi^* a)u$ as desired.

Therefore assume $\lambda \notin \Lambda'$. Then $\lambda \in \Gamma x$ for some $x \in X$ and $n_\lambda = 1$, so $a = \alpha_\lambda a_1$ and $b = \alpha_\lambda b_1$. Therefore

$$(\Phi^* b)v = \big(\beta_\lambda(\Phi^* b_1)\big)v = \lambda_W\Big((\Phi^* b_1)\big((x/\Box)v\big)\Big),$$
$$(\Phi^* a)u = \big(\beta_\lambda(\Phi^* a_1)\big)u = \lambda_W\Big((\Phi^* a_1)\big((x/\Box)u\big)\Big)$$

by (5.6). Therefore, we only need to show $(\Phi^* b_1)\big((x/\Box)v\big) = (\Phi^* a_1)\big((x/\Box)u\big)$. In order to do so, in view of (5.4), we have to show

$$(\Phi^* b_1)\big((x/w)v\big) = (\Phi^* a_1)\big((x/w)u\big)$$

for any $w \in W_{\sigma x}$. Recall that x_i is free from c_i in a_1 for each $i \in \{1, \dots, n\}$. Therefore

$$(\Phi^* b_1)\big((x/w)v\big) = (\Phi^* a_1)\left(\left(\frac{x_1, \dots\dots\dots, x_n}{(\Phi^* c_1)v', \dots, (\Phi^* c_n)v'}\right)\big((x/w)v\big)\right)$$

by the induction hypothesis, where $v' = (x/w)v$. We are assuming that x_i is free from c_i in $a = \alpha_\lambda a_1$ and $x_i \ll a$ for each $i \in \{1, \dots, n\}$ and $x \in S^\lambda$. Therefore, $x \not\ll c_i$ by Lemma 2.2, and so $(\Phi^* c_i)v' = (\Phi^* c_i)v$ by Theorem 6.1 for each $i \in \{1, \dots, n\}$. Recall also that $\{x_1, \dots, x_n\} \cap S^\lambda = \emptyset$, while $x \in S^\lambda$. Hence $x_1, \dots, x_n, x$ are distinct. Therefore

$$\left(\frac{x_1, \dots\dots\dots, x_n}{(\Phi^* c_1)v', \dots, (\Phi^* c_n)v'}\right)\big((x/w)v\big) = \left(\frac{x_1, \dots\dots\dots, x_n}{(\Phi^* c_1)v, \dots, (\Phi^* c_n)v}\right)\big((x/w)v\big)$$
$$= (x/w)\left(\left(\frac{x_1, \dots\dots\dots, x_n}{(\Phi^* c_1)v, \dots, (\Phi^* c_n)v}\right)v\right) = (x/w)u.$$

by (6.1), hence $(\Phi^* b_1)\big((x/w)v\big) = (\Phi^* a_1)\big((x/w)u\big)$ as desired.

7 Functional expressions

Throughout this section as in §6, we let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language, W be its denotable world, and Φ be a C-denotation into W. We also assume that $W^{V_{X,W}}$ has been made into a sorted algebra with type algebra T by some interpretation λ_W on W of each variable operation λ on T. We denote the OS's of A, $W^{V_{X,W}}, T$, and W by $(\alpha_\lambda)_{\lambda \in \Lambda}$, $(\beta_\lambda)_{\lambda \in \Lambda}$, $(\tau_\lambda)_{\lambda \in \Lambda}$, and $(\omega_\lambda)_{\lambda \in \Lambda'}$ respectively, where $\Lambda' = \Lambda \cap \Gamma$.

7.1 The definition

Here is shown that the C-denotation Φ associates each element a of A with a function on W.

First of all, (A, T, σ, S) is a USA, and Λ is a subset of the free semigroup over $\Gamma \amalg S$. Therefore, the assumption in §2.4.2 is satisfied, and so we may apply all the definitions and lemmas in §2.4 to (A, T, σ, S). In particular, X^a_{free} is a finite set by Lemma 2.1, and so there exists a sequence $(x_1, \ldots, x_n)$ of distinct elements $x_1, \ldots, x_n$ of X of finite length $n \geq 0$ which satisfies

$$X^{a}_{\text{free}} \subseteq \{x_1, \ldots, x_n\}. \tag{7.1}$$

We call it a **free base** of a. Notice that it is not uniquely determined by a and that not all variables in it actually occur free in a. Notice also that (7.1) with $n = 0$ means $X^{a}_{\text{free}} = \emptyset$, in which case we say that a is **closed**.

Now we define a function $F \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_n} \to W_{\sigma a}$. If $n = 0$, then $W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$ is equal to $\{\emptyset\}$ by its set-theoretical definition, and so we regard $W_{\sigma x_1} \times \cdots \times W_{\sigma x_n} \to W_{\sigma a}$ as $W_{\sigma a}$. Therefore if $n = 0$, F is an element of $W_{\sigma a}$, which we regard as a 0-ary function.

The definition of F is as follows. If $(w_1, \ldots, w_n) \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$, then since $x_1, \ldots, x_n$ are distinct and x_i and w_i are of the same type, there exists an element $v \in V_{X,W}$ which satisfies $vx_i = w_i$ for each $i \in \{1, \ldots, n\}$. Using any one of such $v \in V_{X,W}$, we define $F(w_1, \ldots, w_n) = (\Phi^* a)v$. Since Φ^* is sort-consistent, we have $\Phi^* a \in V_{X,W} \to W_{\sigma a}$, and so the right-hand side of this equation certainly belongs to $W_{\sigma a}$. Moreover, Theorem 6.1 shows that the right-hand side does not depend on the choice of $v \in V_{X,W}$ which satisfies $vx_i = w_i$ for each $i \in \{1, \ldots, n\}$. Thus, the function F is determined by Φ, a, and $(x_1, \ldots, x_n)$. We call it the **functional expression** of a under Φ with respect to $(x_1, \ldots, x_n)$, and denote it by

$$a^{\Phi}(x_1, \ldots, x_n),$$

while we abbreviate its image at $(w_1, \ldots, w_n) \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$ to

$$a^{\Phi}(w_1, \ldots, w_n).$$

Therefore, $a^{\Phi}(w_1, \ldots, w_n)$ is defined formally by

$$a^{\Phi}(w_1, \ldots, w_n) = \bigl(a^{\Phi}(x_1, \ldots, x_n)\bigr)(w_1, \ldots, w_n)$$

and substantially by

$$a^{\Phi}(w_1, \ldots, w_n) = (\Phi^* a)v \qquad \bigl(v \in V_{X,W},\ vx_i = w_i\ (i = 1, \ldots, n)\bigr).$$

Two alternative definitions are as follows:

$$a^{\Phi}(vx_1, \ldots, vx_n) = (\Phi^* a)v \qquad \bigl(v \in V_{X,W}\bigr),$$
$$a^{\Phi}(w_1, \ldots, w_n) = (\Phi^* a)\left(\left(\frac{x_1, \ldots, x_n}{w_1, \ldots, w_n}\right)v\right) \qquad (v \in V_{X,W}).$$

As for the former one, notice $\{(vx_1, \ldots, vx_n) \mid v \in V_{X,W}\} = W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$.

Example 7.1 Suppose $a \in C$. Then, since a is closed, an arbitrary sequence $(x_1, \ldots, x_n)$ of distinct variables is a free base of a, and so the functional expression $a^{\Phi}(x_1, \ldots, x_n)$ of a is defined on $W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$. Since $(\Phi^* a)v = \Phi a$

for all $v \in V_{X,W}$ by (5.8), it follows that $a^{\Phi}(x_1, \ldots, x_n)$ is the constant function of value Φa.

Suppose $x \in X$. Then, since $X^x_{\text{free}} = \{x\}$, an arbitrary sequence $(x_1, \ldots, x_n)$ of distinct variables satisfying $x \in \{x_1, \ldots, x_n\}$ is a free base of x, and so the functional expression $x^{\Phi}(x_1, \ldots, x_n)$ of x is defined on $W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$. Let $i \in \{1, \ldots, n\}$ be the number for which $x = x_i$. Then since $(\Phi^* x)v = vx_i$ for all $v \in V_{X,W}$ by (5.8), it follows that $x^{\Phi}(x_1, \ldots, x_n)$ is the projection of $W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$ onto $W_{\sigma x_i}$.

7.2 Functional expressions and operations

Recall that, for each $\lambda \in \Lambda \cap \Gamma X$, we identify τ_λ with λ and denote its domain by T_λ, because τ_λ is unary and so $T_\lambda \subseteq T$.

Theorem 7.1 Let $a \in A$ and $(x_1, \ldots, x_n)$ be a free base of a. Then the following holds on the functional expression $a^{\Phi}(x_1, \ldots, x_n)$.

(1) If $a = \alpha_\lambda(a_1, \ldots, a_{n_\lambda})$ with $\lambda \in \Lambda'$, then $(x_1, \ldots, x_n)$ is a free base of a_k for each $k \in \{1, \ldots, n_\lambda\}$ and the functional expressions $a_k^{\Phi}(x_1, \ldots, x_n)$ $(k = 1, \ldots, n_\lambda)$ satisfy

$$a^{\Phi}(w_1, \ldots, w_n) = \omega_\lambda\big(a_1^{\Phi}(w_1, \ldots, w_n), \ldots, a_{n_\lambda}^{\Phi}(w_1, \ldots, w_n)\big)$$

for each $(w_1, \ldots, w_n) \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$.

(2) If $a = \alpha_\lambda b$ with $\lambda \in \Lambda \cap \Gamma x$ and $x \in X - \{x_1, \ldots, x_n\}$, then $(x, x_1, \ldots, x_n)$ is a free base of b and the functional expression $b^{\Phi}(x, x_1, \ldots, x_n)$ satisfies

$$a^{\Phi}(w_1, \ldots, w_n) = \lambda_W\big(b^{\Phi}(\Box, w_1, \ldots, w_n)\big)$$

for each $(w_1, \ldots, w_n) \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$, where $b^{\Phi}(\Box, w_1, \ldots, w_n)$ is the element of $W_{\sigma x} \to W_{\sigma b}$ which maps each $w \in W_{\sigma x}$ to $b^{\Phi}(w, w_1, \ldots, w_n)$.

(3) If $a = \alpha_\lambda b$ with $\lambda \in \Lambda \cap \Gamma x_i$ for some $i \in \{1, \ldots, n\}$, then $(x_1, \ldots, x_n)$ is a free base of b and the functional expression $b^{\Phi}(x_1, \ldots, x_n)$ satisfies

$$a^{\Phi}(w_1, \ldots, w_n) = \lambda_W\big(b^{\Phi}(w_1, \ldots, w_{i-1}, \Box, w_{i+1}, \ldots, w_n)\big)$$

for each $(w_1, \ldots, w_n) \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$, where $b^{\Phi}(w_1, \ldots, w_{i-1}, \Box, w_{i+1}, \ldots, w_n)$ is the element of $W_{\sigma x_i} \to W_{\sigma b}$ which maps each $w \in W_{\sigma x_i}$ to $b^{\Phi}(w_1, \ldots, w_{i-1}, w, w_{i+1}, \ldots, w_n)$.

Lemma 7.1 Let $x_1, \ldots, x_m, x'_1, \ldots, x'_n$ be distinct variables and assume that $(x_1, \ldots, x_m)$ is a free base of an element $a \in A$. Then, $(x'_1, \ldots, x'_n, x_1, \ldots, x_m)$ is also a free base of a, and so $a^{\Phi}(x_1, \ldots, x_m)$ and $a^{\Phi}(x'_1, \ldots, x'_n, x_1, \ldots, x_m)$ are defined. Denote them by G and H. Then,

$$G(w_1, \ldots, w_m) = H(w'_1, \ldots, w'_n, w_1, \ldots, w_m)$$

for each $(w'_1, \ldots, w'_n, w_1, \ldots, w_m) \in W_{\sigma x'_1} \times \cdots \times W_{\sigma x'_n} \times W_{\sigma x_1} \times \cdots \times W_{\sigma x_m}$.

Proof Let ν be an element of $V_{X,W}$ which satisfies $\nu x_i = w_i$ $(i = 1, \ldots, m)$ and $\nu x'_j = w'_j$ $(j = 1, \ldots, n)$. Then

$$G(w_1, \ldots, w_m) = (\Phi^* a)\nu = H(w'_1, \ldots, w'_n, w_1, \ldots, w_m).$$

Proof of Theorem 7.1 (1) Since $\lambda \in \Lambda'$, $S^\lambda = \emptyset$ by (3.1), and so $X^{a_k}_{\text{free}} \subseteq X^a_{\text{free}} \subseteq \{x_1, \ldots, x_n\}$ by Lemma 2.2 for each $k \in \{1, \ldots, n_\lambda\}$. Let ν be an X-denotation such that $\nu x_i = w_i$ for $i = 1, \ldots, n$. Then since Φ^* is a Λ-homomorphism and the projection by ν is a Λ'-homomorphism,

$$\begin{aligned} a^\Phi(w_1, \ldots, w_n) &= (\Phi^* a)\nu = \big(\Phi^*\big(\alpha_\lambda(a_1, \ldots, a_{n_\lambda})\big)\big)\nu \\ &= \omega_\lambda\big((\Phi^* a_1)\nu, \ldots, (\Phi^* a_{n_\lambda})\nu\big) \\ &= \omega_\lambda\big(a^\Phi_1(w_1, \ldots, w_n), \ldots, a^\Phi_{n_\lambda}(w_1, \ldots, w_n)\big). \end{aligned}$$

(2) Since $(x, x_1, \ldots, x_n)$ is also a free base of a, (2) is a consequence of (3) together with Lemma 7.1.

(3) Since $\lambda \in \Lambda \cap \Gamma x_i$, $S^\lambda = \{x_i\}$ by (3.1), and so $X^b_{\text{free}} \subseteq \{x_i\} \cup X^a_{\text{free}} \subseteq \{x_1, \ldots, x_n\}$ by Lemma 2.2. Since $a = \alpha_\lambda b$ with $\lambda \in \Lambda \cap \Gamma X$, we have $\sigma a = \tau_\lambda(\sigma b) = \lambda(\sigma b)$ by our notational convention, so $t = \sigma b$ belongs to T_λ and $\sigma a = \lambda t$. Therefore $\lambda_W(W_{\sigma x_i} \to W_{\sigma b}) \subseteq W_{\sigma a}$ by (5.2). Let ν be an X-denotation such that $\nu x_i = w_i$ for $i = 1, \ldots, n$. Then

$$(\Phi^* b)\big((x_i/w)\nu\big) = b^\Phi(w_1, \ldots, w_{i-1}, w, w_{i+1}, \ldots, w_n)$$

for each $w \in W_{\sigma x_i}$, and so

$$(\Phi^* b)\big((x_i/\square)\nu\big) = b^\Phi(w_1, \ldots, w_{i-1}, \square, w_{i+1}, \ldots, w_n)$$

by (5.4) and the definition of the right-hand side. Therefore

$$\begin{aligned} a^\Phi(w_1, \ldots, w_n) &= (\Phi^* a)\nu = \big(\Phi^*(\alpha_\lambda b)\big)\nu \\ &= \big(\beta_\lambda(\Phi^* b)\big)\nu \\ &= \lambda_W\Big((\Phi^* b)\big((x_i/\square)\nu\big)\Big) && \text{(by (5.6))} \\ &= \lambda_W\big(b^\Phi(w_1, \ldots, w_{i-1}, \square, w_{i+1}, \ldots, w_n)\big). \end{aligned}$$

Remark 7.1 Let $x_1, \ldots, x_n$ be distinct variables of A and define

$$\begin{aligned} A(x_1, \ldots, x_n) &= \big\{a \in A \mid X^a_{\text{free}} \subseteq \{x_1, \ldots, x_n\}\big\}, \\ W(x_1, \ldots, x_n) &= W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}. \end{aligned}$$

Then, $A(x_1, \ldots, x_n)$ is a support subalgebra of A by Lemma 2.2, and Theorem 7.1 (1) implies that the mapping $a \mapsto a^\Phi(x_1, \ldots, x_n)$ of $A(x_1, \ldots, x_n)$ into the power algebra $W^{W(x_1, \ldots, x_n)}$ is a Λ'-homomorphism.

Corollary 7.1.1 If $a = \alpha_\lambda(x_1, \ldots, x_{n_\lambda})$ with $\lambda \in \Lambda'$ and $x_1, \ldots, x_{n_\lambda}$ are distinct variables, then $(x_1, \ldots, x_{n_\lambda})$ is a free base of a and the functional expression $a^\Phi(x_1, \ldots, x_{n_\lambda})$ satisfies

$$a^\Phi(w_1, \ldots, w_{n_\lambda}) = \omega_\lambda(w_1, \ldots, w_{n_\lambda}).$$

for each $(w_1, \ldots, w_{n_\lambda}) \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_{n_\lambda}}$.

Proof This is a consequence of Theorem 7.1 (1) and Example 7.1.

Example 7.2 Assume as in Example 5.1 that a variable operation $\lambda \in \Lambda \cap \Gamma x$ $(x \in X)$ satisfies $T_\lambda = \{\phi\}$ and $\lambda\phi = \phi$ and that W satisfies $W_\phi = \mathbb{T}$. Interpret λ by (5.9), denote it by $\forall x$, and identify it with α_λ and β_λ. Then, for each $a \in A_\phi$, we have $\forall x a \in A_\phi$, hence $\Phi^* a,\ \Phi^*(\forall x a) \in V_{X,W} \to \mathbb{T}$, and

$$\big(\Phi^*(\forall x a)\big)v = \big(\forall x(\Phi^* a)\big)v = \inf\big\{(\Phi^* a)\big((x/w)v\big) \mid w \in W_{\sigma x}\big\}$$

by (5.10) for each $v \in V_{X,W}$. Assume that $(x, x_2, \ldots, x_n)$ is a free base of a. Then $(x_2, \ldots, x_n)$ is a free base of $\forall x a$ by Lemma 2.2, and the above equation shows that the functional expressions $a^\Phi(x, x_2, \ldots, x_n)$ and $(\forall x a)^\Phi(x_2, \ldots, x_n)$ are both $\mathbb{T}$-valued and satisfy

$$(\forall x a)^\Phi(w_2, \ldots, w_n) = \inf\big\{a^\Phi(w, w_2, \ldots, w_n) \mid w \in W_{\sigma x}\big\}$$

for each $(w_2, \ldots, w_n) \in W_{\sigma x_2} \times \cdots \times W_{\sigma x_n}$, that is, $(\forall x a)^\Phi(w_2, \ldots, w_n) = 1$ iff $a^\Phi(w, w_2, \ldots, w_n) = 1$ for all $w \in W_{\sigma x}$.

Example 7.3 Assume as in Example 5.2 that a variable operation $\lambda \in \Lambda \cap \Gamma x$ $(x \in X)$ and W satisfy $W_{\lambda t} = W_{\sigma x} \to W_t$ for each $t \in T_\lambda$. Interpret λ by (5.11), denote it by Ωx, and identify it with α_λ and β_λ. Then, for each $a \in A_t$, we have $\Phi^* a \in V_{X,W} \to W_t,\ \Phi^*(\Omega x\, a) \in V_{X,W} \to (W_{\sigma x} \to W_t)$, and

$$\big(\big(\Phi^*(\Omega x\, a)\big)v\big)w = \big(\big(\Omega x\,(\Phi^* a)\big)v\big)w = (\Phi^* a)\big((x/w)v\big)$$

by (5.12) for each $v \in V_{X,W}$ and $w \in W_{\sigma x}$. Assume that $(x, x_2, \ldots, x_n)$ is a free base of a. Then $(x_2, \ldots, x_n)$ is a free base of $\Omega x\, a$ by Lemma 2.2, and the above equation shows that the functional expressions $a^\Phi(x, x_2, \ldots, x_n)$ and $(\Omega x\, a)^\Phi(x_2, \ldots, x_n)$ are W_t-valued and $(W_{\sigma x} \to W_t)$-valued respectively, and satisfy

$$\big((\Omega x\, a)(w_2, \ldots, w_n)\big)w = a^\Phi(w, w_2, \ldots, w_n)$$

for each $(w, w_2, \ldots, w_n) \in W_{\sigma x} \times W_{\sigma x_2} \times \cdots \times W_{\sigma x_n}$.

7.3 Substitution-composition theorem

Theorem 7.2 Let $a, c_1, \dots, c_m \in A$ and let $(x_1, \dots, x_m)$ be a free base of a. Assume $\sigma c_i = \sigma x_i$ for each $i \in \{1, \dots, m\}$ and define

$$b = a\left(\frac{x_1, \dots, x_m}{c_1, \dots, c_m}\right).$$

Assume that $(y_1, \dots, y_n)$ is a free base of each of $c_1, \dots, c_m$. Then $(y_1, \dots, y_n)$ is a free base of b. If furthermore x_i is free from c_i in a for each $i \in \{1, \dots, m\}$, then $b^\Phi(y_1, \dots, y_n)$ is equal to the composite function of $a^\Phi(x_1, \dots, x_m)$ and ${c_1}^\Phi(y_1, \dots, y_n), \dots, {c_m}^\Phi(y_1, \dots, y_n)$, that is,

$$b^\Phi(w_1, \dots, w_n) = a^\Phi\left({c_1}^\Phi(w_1, \dots, w_n), \dots, {c_m}^\Phi(w_1, \dots, w_n)\right)$$

for each $(w_1, \dots, w_n) \in W_{\sigma y_1} \times \cdots \times W_{\sigma y_n}$.

Proof That $(y_1, \dots, y_n)$ is a free base of b is an immediate consequence of Lemma 2.4. If x_i is free from c_i in a for each $i \in \{1, \dots, m\}$, then we can argue as follows for each $v \in V_{X,W}$ by using Theorem 6.2:

$$\begin{aligned} b^\Phi(vy_1, \dots, vy_n) = (\Phi^* b)v &= \left(\Phi^*\left(a\left(\frac{x_1, \dots, x_m}{c_1, \dots, c_m}\right)\right)\right)v \\ &= (\Phi^* a)\left(\left(\frac{x_1, \dots\dots\dots, x_m}{(\Phi^* c_1)v, \dots, (\Phi^* c_m)v}\right)v\right) \\ &= a^\Phi\left((\Phi^* c_1)v, \dots, (\Phi^* c_m)v\right) \\ &= a^\Phi\left({c_1}^\Phi(vy_1, \dots, vy_n), \dots, {c_m}^\Phi(vy_1, \dots, vy_n)\right) \end{aligned}$$

The proof is complete.

Corollary 7.2.1 Let $a \in A$ and let $(x_1, \dots, x_n)$ be a free base of a. Assume that $y_1, \dots, y_n$ are distinct variables such that $\sigma y_i = \sigma x_i$ for each $i \in \{1, \dots, n\}$, and define

$$b = a\left(\frac{x_1, \dots, x_n}{y_1, \dots, y_n}\right).$$

Then $(y_1, \dots, y_n)$ is a free base of b. If furthermore x_i is free from y_i in a for each $i \in \{1, \dots, n\}$, then $b^\Phi(y_1, \dots, y_n) = a^\Phi(x_1, \dots, x_n)$.

Proof This is a consequence of Theorem 7.2 with $m = n$ and $c_i = y_i$ for each $i \in \{1, \dots, n\}$ together with Example 7.1.

8 Denotable functions

Throughout this section as in §6 and §7, we let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language, W be its denotable world, and Φ be a C-denotation into W. We also assume that $W^{V_X,w}$ has been made into a sorted algebra with type algebra T by some interpretation λ_W on W of each variable operation λ on T. We denote the OS's of $A, W^{V_X,w}, T$, and W by $(\alpha_\lambda)_{\lambda\in\Lambda}$, $(\beta_\lambda)_{\lambda\in\Lambda}$, $(\tau_\lambda)_{\lambda\in\Lambda}$, and $(\omega_\lambda)_{\lambda\in\Lambda'}$ respectively, where $\Lambda' = \Lambda \cap \Gamma$. Recall that, for each $\lambda \in \Lambda \cap \Gamma X$, we identify τ_λ with λ and denote its domain by T_λ. Also, X' is the set of the qualifying variables.

In §7, we have seen that if $(x_1, \ldots, x_n)$ is a free base of an element $a \in A$, then $a^\Phi(x_1, \ldots, x_n)$ is a function whose domain is equal to $W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$ and whose image is contained in $W_{\sigma a}$. In view of this fact, we make the following definition.

Definition 8.1 A **type function** on W is a function whose domain is equal to $W_{t_1} \times \cdots \times W_{t_n}$ for some elements $t_1, \ldots, t_n \in T$ $(n \geq 0)$ and whose image is contained in W_t for some element $t \in T$. The 0-ary type functions on W are the elements of W. We denote by $\mathcal{F}$ the set of the type functions on W:

$$\mathcal{F} = \coprod_{n=0}^{\infty} \left(\coprod_{t_1,\ldots,t_n,t\in T} (W_{t_1} \times \cdots \times W_{t_n} \to W_t) \right).$$

Example 8.1 The constant function of domain $W_{t_1} \times \cdots \times W_{t_n}$ and value $w \in W$ belongs to $\mathcal{F}$, which we denote by $(t_1, \ldots, t_n \to w)$.

The projection of $W_{t_1} \times \cdots \times W_{t_n}$ $(n \geq 1)$ onto W_{t_i} belongs to $\mathcal{F}$, which we denote by $(t_1, \ldots, t_n \downarrow i)$ for each $i \in \{1, \ldots, n\}$.

If $W_{t_1} \times \cdots \times W_{t_n} \subseteq \mathrm{Dom}\, \omega_\lambda$, then the restriction of ω_λ to $W_{t_1} \times \cdots \times W_{t_n}$ is a type function whose image is contained in $W_{\tau_\lambda(t_1,\ldots,t_n)}$.

Definition 8.2 An element $F \in \mathcal{F}$ is said to be **Φ-denoted** by an element $a \in A$ if $F = a^\Phi(x_1, \ldots, x_n)$ for some free base $(x_1, \ldots, x_n)$ of a. Also, the F is said to be **Φ-denotable** by A if F is Φ-denoted by an element of A. We denote by $\mathcal{F}^\Phi$ the set of the Φ-denotable functions in $\mathcal{F}$.

The purpose of this section is to pin down the structure of $\mathcal{F}^\Phi$. To that end, we first list primitive Φ-denotable functions.

Example 8.2 The functional expression of each element $a \in A$ under Φ is Φ-denoted by a. As shown in Example 7.1, if $x_1, \ldots, x_n$ are distinct variables, then for each element $w \in \Phi C$, the constant function $(\sigma x_1, \ldots, \sigma x_n \to w)$ is Φ-denoted by an element $a \in C$ such that $w = \Phi a$. Also, if $n \geq 1$, the projection $(\sigma x_1, \ldots, \sigma x_n \downarrow i)$ is Φ-denoted by x_i for each $i \in \{1, \ldots, n\}$.

In view of Example 8.2, we make the following definition.

Definition 8.3 The **Φ-primitive functions** are the following two kinds of functions made of the sequences $(x_1, \ldots, x_n)$ of distinct variables.

- The constant functions $(\sigma x_1, \ldots, \sigma x_n \to w)$ with $w \in \Phi C$ and $n \geq 0$.

- The projections $(\sigma x_1, \dots, \sigma x_n \downarrow i)$ with $n \geq 1$ and $i \in \{1, \dots, n\}$.

We denote by $\mathcal{P}^\Phi$ the set of the Φ-primitive functions.

Example 8.2 shows that the Φ-primitive functions are Φ-denotable, that is, $\mathcal{P}^\Phi \subseteq \mathcal{F}^\Phi$. It will be shown in Theorems 8.1 and 8.2 that every Φ-denotable function is generated from Φ-primitive functions by certain operations. In order to state the results, we make $\mathcal{F}$ into the algebra whose OS consists of the following five families of operations.

(0) The family of permutations p $(p \in \coprod_{n=1}^{\infty} \mathfrak{S}_n)$ Here $\mathfrak{S}_n$ is the symmetric group on the letters $1, \dots, n$. The permutation $p \in \mathfrak{S}_n$ transforms each n-ary type function $G \in W_{t_1} \times \cdots \times W_{t_n} \to W_t$ into the type function $pG \in W_{t_{p1}} \times \cdots \times W_{t_{pn}} \to W_t$ defined by

$$(pG)(w_{p1}, \dots, w_{pn}) = G(w_1, \dots, w_n).$$

(1) The family of compositions $\circ_m$ $(m = 1, 2, \dots)$ The composition $\circ_m$ transforms each $(m+1)$-tuple $(G, H_1, \dots, H_m)$ of type functions $G \in W_{u_1} \times \cdots \times W_{u_m} \to W_t$ and $H_i \in W_{t_1} \times \cdots \times W_{t_n} \to W_{u_i}$ $(i = 1, \dots, m)$ into the type function $G \circ (H_1, \dots, H_m) \in W_{t_1} \times \cdots \times W_{t_n} \to W_t$ defined by

$$\begin{aligned}&\bigl(G \circ (H_1, \dots, H_m)\bigr)(w_1, \dots, w_n)\\ &= G\bigl(H_1(w_1, \dots, w_n), \dots, H_m(w_1, \dots, w_n)\bigr).\end{aligned}$$

(2) The family of operations λ $(\lambda \in \Lambda', \ \mathrm{Dom}\,\omega_\lambda \neq \emptyset)$ Let n_λ be the arity of ω_λ. Then the operation λ transforms each n_λ-tuple $(G_1, \dots, G_{n_\lambda})$ of type functions $G_k \in W_{t_1} \times \cdots \times W_{t_n} \to W_{u_k}$ $(k = 1, \dots, n_\lambda)$ with $W_{u_1} \times \cdots \times W_{u_{n_\lambda}} \subseteq \mathrm{Dom}\,\omega_\lambda$ into the type function $\lambda(G_1, \dots, G_{n_\lambda}) \in W_{t_1} \times \cdots \times W_{t_n} \to W_{\tau_\lambda(u_1, \dots, u_{n_\lambda})}$ defined by

$$\begin{aligned}&\bigl(\lambda(G_1, \dots, G_{n_\lambda})\bigr)(w_1, \dots, w_n)\\ &= \omega_\lambda\bigl(G_1(w_1, \dots, w_n), \dots, G_{n_\lambda}(w_1, \dots, w_n)\bigr).\end{aligned}$$

(3) The family of operations $\flat_{\lambda,i}$ $(\lambda \in \Lambda \cap \Gamma X, \ i = 1, 2, \dots)$ The operation $\flat_{\lambda,i}$ with $\lambda \in \Lambda \cap \Gamma x$ $(x \in X)$ transforms each type function $G \in W_{t_1} \times \cdots \times W_{t_n} \to W_t$ with $t \in T_\lambda$, $i \leq n$, and $\sigma x = t_i$ into the type function $\flat_{\lambda,i} G \in W_{t_1} \times \cdots \times W_{t_{i-1}} \times W_{t_{i+1}} \times \cdots \times W_{t_n} \to W_{\lambda t}$ defined by

$$\begin{aligned}&(\flat_{\lambda,i} G)(w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n)\\ &= \lambda_W\bigl(G(w_1, \dots, w_{i-1}, \Box, w_{i+1}, \dots, w_n)\bigr),\end{aligned}$$

where $G(w_1, \dots, w_{i-1}, \Box, w_{i+1}, \dots, w_n)$ is the element of $W_{\sigma x} \to W_t$ which maps each element $w \in W_{\sigma x} = W_{t_i}$ to $G(w_1, \dots, w_{i-1}, w, w_{i+1}, \dots, w_n) \in W_t$. This definition makes sense because $\lambda_W(W_{\sigma x} \to W_t) \subseteq W_{\lambda t}$ by (5.2).

(4) The family of operations $\sharp_{t,i}$ $(t \in T,\ X'_t \neq \emptyset,\ i = 1, 2, \ldots)$ The operation $\sharp_{t,i}$ transforms each type function $G \in W_{t_1} \times \cdots \times W_{t_n} \to W_u$ such that $i - 1 \leq n$ and $u \in \mathrm{Im}\,\lambda$ for some $\lambda \in \Lambda \cap \Gamma X'_t$ into the type function $\sharp_{t,i} G \in W_{t_1} \times \cdots \times W_{t_{i-1}} \times W_t \times W_{t_i} \times \cdots \times W_{t_n} \to W_u$ defined by

$$(\sharp_{t,i} G)(w_1, \ldots, w_{i-1}, w, w_i, \ldots, w_n) = G(w_1, \ldots, w_n).$$

Definition 8.4 $[\mathcal{P}^\Phi]$ denotes, as usual, the closure of $\mathcal{P}^\Phi$ in the algebra $\mathcal{F}$, while $\langle \mathcal{P}^\Phi \rangle$ denotes the closure of $\mathcal{P}^\Phi$ in the operational subalgebra of $\mathcal{F}$ obtained by deleting the permutations and the compositions from the OS of $\mathcal{F}$.

Theorem 8.1 $\mathcal{F}^\Phi \subseteq \langle \mathcal{P}^\Phi \rangle$.

Lemma 8.1 If $\lambda \in \Lambda'$ and $a_1, \ldots, a_{n_\lambda} \in A$, then the following conditions are equivalent, and $\mathrm{Dom}\,\omega_\lambda \neq \emptyset$ under these conditions.

(1) $(a_1, \ldots, a_{n_\lambda}) \in \mathrm{Dom}\,\alpha_\lambda$.

(2) $W_{\sigma a_1} \times \cdots \times W_{\sigma a_{n_\lambda}} \subseteq \mathrm{Dom}\,\omega_\lambda$.

Proof Let ρ be the sorting of W, and assume (1). Then $(\sigma a_1, \ldots, \sigma a_{n_\lambda}) \in \mathrm{Dom}\,\tau_\lambda$ because σ is a homomorphism. Let $(w_1, \ldots, w_\lambda) \in W_{\sigma a_1} \times \cdots \times W_{\sigma a_{n_\lambda}}$. Then $(\rho w_1, \ldots, \rho w_\lambda) = (\sigma a_1, \ldots, \sigma a_{n_\lambda}) \in \mathrm{Dom}\,\tau_\lambda$, and so $(w_1, \ldots, w_\lambda) \in \mathrm{Dom}\,\omega_\lambda$ because ρ is a homomorphism. Thus (2) holds.

Conversely assume (2). Since Φ^* and the projections by X-denotations are sort-consistent, it follows that $W_t \neq \emptyset$ for each $t \in \sigma A$. Therefore, there exists an element $(w_1, \ldots, w_\lambda) \in W_{\sigma a_1} \times \cdots \times W_{\sigma a_{n_\lambda}}$, and $(w_1, \ldots, w_\lambda) \in \mathrm{Dom}\,\omega_\lambda$ by (2). Hence $(\sigma a_1, \ldots, \sigma a_{n_\lambda}) = (\rho w_1, \ldots, \rho w_\lambda) \in \mathrm{Dom}\,\tau_\lambda$ because ρ is a homomorphism. Thus (1) holds because σ is a homomorphism.

Proof of Theorem 8.1 We only need to show that $F = a^\Phi(x_1, \ldots, x_n)$ belongs to $\langle \mathcal{P}^\Phi \rangle$ for each $a \in A$. We argue by induction on the rank r of a.

Assume $r = 0$. Then $a \in S = C \cup X$. If $a \in C$, then $F = (\sigma x_1, \ldots, \sigma x_n \to \Phi a)$ by Example 7.1. If $a \in X$, then $F = (\sigma x_1, \ldots, \sigma x_n \downarrow i)$ for some $i \in \{1, \ldots, n\}$ by Example 7.1. In either case, $F \in \mathcal{P}^\Phi \subseteq \langle \mathcal{P}^\Phi \rangle$.

Therefore we assume $r \geq 1$. Then there are three cases discussed in Theorem 7.1. Let $\sigma x_i = t_i$ $(i = 1, \ldots, n)$.

First assume $a = \alpha_\lambda(a_1, \ldots, a_{n_\lambda})$ with $\lambda \in \Lambda'$. Then by Theorem 7.1, $(x_1, \ldots, x_n)$ is a free base of a_k for each $k \in \{1, \ldots, n_\lambda\}$, and defining $G_k = a_k^\Phi(x_1, \ldots, x_n)$, we have $G_k \in W_{t_1} \times \cdots \times W_{t_n} \to W_{\sigma a_k}$ and

$$\begin{aligned} F(w_1, \ldots, w_n) &= a^\Phi(w_1, \ldots, w_n) \\ &= \omega_\lambda\big(a_1^\Phi(w_1, \ldots, w_n), \ldots, a_{n_\lambda}^\Phi(w_1, \ldots, w_n)\big) \\ &= \omega_\lambda\big(G_1(w_1, \ldots, w_n), \ldots, G_{n_\lambda}(w_1, \ldots, w_n)\big) \end{aligned}$$

for each $(w_1, \dots, w_n) \in W_{t_1} \times \cdots \times W_{t_n}$. Furthermore, $W_{\sigma a_1} \times \cdots \times W_{\sigma a_{n_\lambda}} \subseteq \mathrm{Dom}\,\omega_\lambda \neq \emptyset$ by Lemma 8.1. Thus $F = \lambda(G_1 \dots, G_{n_\lambda})$. Since $G_k \in \langle \mathcal{P}^\Phi \rangle$ for each $k \in \{1, \dots, n_\lambda\}$ by the induction hypothesis, $F \in \langle \mathcal{P}^\Phi \rangle$ as desired.

Next assume $a = \alpha_\lambda b$ with $\lambda \in \Lambda \cap \Gamma x$ and $x \in X - \{x_1, \dots, x_n\}$. Then by Theorem 7.1, $(x, x_1, \dots, x_n)$ is a free base of b, and defining $G = b^\Phi(x, x_1, \dots, x_n)$ and $t = \sigma b$, we have $G \in W_{\sigma x} \times W_{t_1} \times \cdots \times W_{t_n} \to W_t,\ t \in T_\lambda$, and

$$\begin{aligned} F(w_1, \dots, w_n) &= a^\Phi(w_1, \dots, w_n) \\ &= \lambda_W\big(b^\Phi(\Box, w_1, \dots, w_n)\big) \\ &= \lambda_W\big(G(\Box, w_1, \dots, w_n)\big) \\ &= (\flat_{\lambda,1} G)(w_1, \dots, w_n), \end{aligned}$$

for each $(w_1, \dots, w_n) \in W_{t_1} \times \cdots \times W_{t_n}$. Thus $F = \flat_{\lambda,1} G$. Since $G \in \langle \mathcal{P}^\Phi \rangle$ by the induction hypothesis, we have $F \in \langle \mathcal{P}^\Phi \rangle$ as desired.

Finally assume $a = \alpha_\lambda b$ with $\lambda \in \Lambda \cap \Gamma x_i$ for some $i \in \{1, \dots, n\}$. Then by Theorem 7.1, $(x_1, \dots, x_n)$ is a free base of b, and defining $G = b^\Phi(x_1, \dots, x_n)$ and $t = \sigma b$, we have $G \in W_{t_1} \times \cdots \times W_{t_n} \to W_t$, $t \in T_\lambda$, and

$$\begin{aligned} F(w_1, \dots, w_n) &= a^\Phi(w_1, \dots, w_n) \\ &= \lambda_W\big(b^\Phi(w_1, \dots, w_{i-1}, \Box, w_{i+1}, \dots, w_n)\big) \\ &= \lambda_W\big(G(w_1, \dots, w_{i-1}, \Box, w_{i+1}, \dots, w_n)\big) \\ &= (\flat_{\lambda,i} G)(w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n) \end{aligned}$$

for each $(w_1, \dots, w_n) \in W_{t_1} \times \cdots \times W_{t_n}$. Since $\flat_{\lambda,i} G \in W_{t_1} \times \cdots \times W_{t_{i-1}} \times W_{t_{i+1}} \times \cdots \times W_{t_n} \to W_{\lambda t}$ and $x_i \in X'_{t_i}$, we furthermore have

$$(\flat_{\lambda,i} G)(w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n) = \big(\sharp_{t_i,i}(\flat_{\lambda,i} G)\big)(w_1, \dots, w_n).$$

Thus $F = \big(\sharp_{t_i,i}(\flat_{\lambda,i} G)\big)$. Since $G \in \langle \mathcal{P}^\Phi \rangle$ by the induction hypothesis, we conclude that $F \in \langle \mathcal{P}^\Phi \rangle$ as desired.

Theorem 8.2 $\mathcal{F}^\Phi = \langle \mathcal{P}^\Phi \rangle = [\mathcal{P}^\Phi]$ holds under the following three conditions.

(1) For each $t \in T$, X'_t is either empty or enumerable.

(2) For each $t \in T$, there exists a subset Γ_t of Γ which satisfies $\Lambda \cap \Gamma X_t = \Gamma_t X'_t$.

(3) If variable operations λ and λ' are **similar** in the sense that there exist an element $\nabla \in \Gamma$ and variables $x, x' \in X'$ which satisfy $\lambda = \nabla x$, $\lambda' = \nabla x'$, and $\sigma x = \sigma x'$, then $T_\lambda = T_{\lambda'}$ and the interpretations λ_W and λ'_W of λ and λ' on W are equal.

The Lemmas 8.2, 8.3, and 8.4 hold without the conditions (1) (2) (3) of Theorem 8.2.

Lemma 8.2 Let $(x_1, \dots, x_n)$ be a free base of an element $a \in A$ and $p \in \mathfrak{S}_n$. Then, $(x_{p1}, \dots, x_{pn})$ is also a free base of a, and so $a^\Phi(x_1, \dots, x_n)$ and $a^\Phi(x_{p1}, \dots, x_{pn})$ are defined. Denote them by G and H. Then $H = pG$.

Proof Let $(w_1, \dots, w_n) \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$ and let ν be a X-denotation such that $\nu x_i = w_i$ for $i = 1, \dots, n$. Then $\nu x_{pi} = w_{pi}$ for $i = 1, \dots, n$, hence

$$H(w_{p1}, \dots, w_{pn}) = (\Phi^* a)\nu = G(w_1, \dots, w_n) = (pG)(w_{p1}, \dots, w_{pn}).$$

Therefore $H = pG$.

Lemma 8.3 Let G be a type function on W and assume that $\flat_{\lambda,i} G$ is defined for some $i > 1$. Then, $\flat_{\lambda,1}(pG)$ is also defined for some permutation p, and $\flat_{\lambda,i} G = \flat_{\lambda,1}(pG)$ holds.

Proof Let p be the cycle $(i, \dots, 2, 1)$. Then, since $G \in W_{t_1} \times \cdots \times W_{t_n} \to W_t$ for some $t_1, \dots, t_n, t \in T$ with $i \le n$, we have

$$pG \in W_{t_i} \times W_{t_1} \times \cdots \times W_{t_{i-1}} \times W_{t_{i+1}} \times \cdots \times W_{t_n} \to W_t.$$

Since furthermore $t \in T_\lambda$ and $\lambda \in \Lambda \cap \Gamma x$ for some $x \in X$ with $\sigma x = t_i$, $\flat_{\lambda,1}(pG)$ is also defined, and both $\flat_{\lambda,i} G$ and $\flat_{\lambda,1}(pG)$ belong to $W_{t_1} \times \cdots \times W_{t_{i-1}} \times W_{t_{i+1}} \times \cdots \times W_{t_n} \to W_{\lambda t}$. Since

$$(pG)(w_i, w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n) = G(w_1, \dots, w_n)$$

for each $(w_1, \dots, w_n) \in W_{t_1} \times \cdots \times W_{t_n}$,

$$(pG)(\Box, w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n) = G(w_1, \dots, w_{i-1}, \Box, w_{i+1}, \dots, w_n)$$

for each $(w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n) \in W_{t_1} \times \cdots \times W_{t_{i-1}} \times W_{t_{i+1}} \times \cdots \times W_{t_n}$. Hence the following, which proves $\flat_{\lambda,i} G = \flat_{\lambda,1}(pG)$:

$$\begin{aligned}
&\big(\flat_{\lambda,1}(pG)\big)(w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n) \\
&= \lambda_W\big((pG)(\Box, w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n)\big) \\
&= \lambda_W\big(G(w_1, \dots, w_{i-1}, \Box, w_{i+1}, \dots, w_n)\big) \\
&= (\flat_{\lambda,i} G)(w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n).
\end{aligned}$$

Lemma 8.4 Let G be a type function on W and assume that $\sharp_{t,i} G$ is defined for some $i > 1$. Then $\sharp_{t,1} G$ is also defined and there exists a permutation p which satisfies $\sharp_{t,i} G = p(\sharp_{t,1} G)$.

Proof Let us denote t also by t_0. Then $G \in W_{t_1} \times \cdots \times W_{t_n} \to W_u$, $i-1 \le n$, $u \in \operatorname{Im} \lambda$ for some $\lambda \in \Lambda \cap \Gamma X'_t$, and $\sharp_{t,i} G \in W_{t_1} \times \cdots \times W_{t_{i-1}} \times W_{t_0} \times W_{t_i} \times \cdots \times W_{t_n} \to W_u$. Therefore, $\sharp_{t,1} G$ is also defined and belongs to $W_{t_0} \times W_{t_1} \times \cdots \times W_{t_n} \to W_u$. Since $0 \le i-1 \le n$, we can define the permutation p on the letters $0, 1, \dots, n$ by $p = (0, 1, \dots, i-1)$, and we have

$$\begin{aligned}
&(\sharp_{t,i} G)(w_1, \dots, w_{i-1}, w_0, w_i, \dots, w_n) = G(w_1, \dots, w_n) \\
&\quad = (\sharp_{t,1} G)(w_0, w_1, \dots, w_n) = \big(p(\sharp_{t,1} G)\big)(w_1, \dots, w_{i-1}, w_0, w_i, \dots, w_n)
\end{aligned}$$

for each $(w_1, \dots, w_{i-1}, w_0, w_i, \dots, w_n) \in W_{t_1} \times \cdots \times W_{t_{i-1}} \times W_{t_0} \times W_{t_i} \times \cdots \times W_{t_n}$. Thus $\sharp_{t,i} G = p(\sharp_{t,1} G)$.

Lemma 8.5 Let $a_1, \dots, a_m \in A$, $B \subseteq A$, and assume that $(y_1, \dots, y_n)$ is a free base of every element of B. Then there exist distinct variables $x_1, \dots, x_n$ which satisfy the following conditions.

- $\sigma x_i = \sigma y_i$ for each $i \in \{1, \dots, n\}$.
- Each element of S is free from $b\left(\dfrac{y_1, \dots, y_n}{x_1, \dots, x_n}\right)$ in $a_1, \dots, a_m$ for each element $b \in B$.

Proof Let $\lambda \in \Lambda$ and $b \in B$. Then $(S^\lambda)^b_{\text{free}} \subseteq X^b_{\text{free}} \subseteq \{y_1, \dots, y_n\}$ by (3.1) and our assumption. For each $t \in T$ with $\{y_1, \dots, y_n\}_t \cap S^\Lambda \neq \emptyset$, we have $X'_t = S_t \cap S^\Lambda \neq \emptyset$ by (3.2), and so $S_t \cap S^\Lambda$ is enumerable by the condition (1) of Theorem 8.2. Now we can apply Lemma 2.5.

Lemma 8.6 Let $a_1, \dots, a_m \in A$ and $y_1, \dots, y_n$ be distinct variables. Then, there exist distinct variables $x_1, \dots, x_n$ which satisfy the following conditions.

- $\sigma x_i = \sigma y_i$ for each $i \in \{1, \dots, n\}$.
- Each element of S is free from $x_1, \dots, x_n$ in $a_1, \dots, a_m$.

Proof This is a consequence of Lemma 8.5 with $B = \{y_1, \dots, y_n\}$.

Proof of Theorem 8.2 Since $\mathcal{P}^\Phi \subseteq \mathcal{F}^\Phi \subseteq \langle \mathcal{P}^\Phi \rangle \subseteq [\mathcal{P}^\Phi]$ by Example 8.2 and Theorem 8.1, we only need to show that $\mathcal{F}^\Phi$ is closed under the five kinds of operations in the OS of $\mathcal{F}$. Lemma 8.2 shows that $\mathcal{F}^\Phi$ is closed under the permutations. In view of this together with Lemma 8.3 and Lemma 8.4, we only need to show $F \in \mathcal{F}^\Phi$ in each of the following four cases.

(1) $F = G \circ (H_1, \dots, H_m)$ and $G, H_1, \dots, H_m \in \mathcal{F}^\Phi$.

(2) $F = \lambda(G_1, \dots, G_{n_\lambda})$ and $G_1, \dots, G_{n_\lambda} \in \mathcal{F}^\Phi$.

(3) $F = \flat_{\lambda,1} G$ and $G \in \mathcal{F}^\Phi$.

(4) $F = \sharp_{t,1} G$ and $G \in \mathcal{F}^\Phi$.

(1) Here $G \in W_{u_1} \times \cdots \times W_{u_m} \to W_t$, $H_i \in W_{t_1} \times \cdots \times W_{t_n} \to W_{u_i}$ ($i = 1, \dots, m$), $F \in W_{t_1} \times \cdots \times W_{t_n} \to W_t$, and

$$F(w_1, \dots, w_n) = G\big(H_1(w_1, \dots, w_n), \dots, H_m(w_1, \dots, w_n)\big)$$

for each $(w_1, \dots, w_n) \in W_{t_1} \times \cdots \times W_{t_n}$. Also, there exist an element $a \in A$ and its free base $(x_1, \dots, x_m)$ such that $G = a^\Phi(x_1, \dots, x_m)$, hence $\sigma x_i = u_i$ ($i = 1, \dots, m$). Also, for each $i \in \{1, \dots, m\}$, there exist an element $a_i \in A$ and its free base $(x^i_1, \dots, x^i_n)$ such that $H_i = a_i^\Phi(x^i_1, \dots, x^i_n)$, hence $\sigma x^i_j = t_j$ for each $j \in \{1, \dots, n\}$.

Lemma 8.6 applied to $a_1, \dots, a_m$ and $x^1_1, \dots, x^1_n$ shows that there exist distinct variables $y_1, \dots, y_n$ which satisfy the following conditions.

- $\sigma y_j = t_j$ for each $j \in \{1, \dots, n\}$.
- x^i_j is free from y_j in a_i for each $i \in \{1, \dots, m\}$ and each $j \in \{1, \dots, n\}$.

Therefore, for each $i \in \{1, \dots, m\}$, we may define $b_i = a_i \left(\dfrac{x^i_1, \dots, x^i_n}{y_1, \dots, y_n} \right)$, and Corollary 7.2.1 shows that $(y_1, \dots, y_n)$ is a free base of b_i and $H_i = a_i^\Phi(x^i_1, \dots, x^i_n) = b_i^\Phi(y_1, \dots, y_n)$, hence $\sigma y_j = t_j$ for each $j \in \{1, \dots, n\}$.

By using Lemma 8.5 for $a, b_1, \dots, b_m$ and $B = \{b_1, \dots, b_m, y_1, \dots, y_n\}$, we have that there exist distinct variables $z_1, \dots, z_n$ which satisfy the following conditions.

- $\sigma z_j = t_j$ for each $j \in \{1, \dots, n\}$.
- Each element of S is free from $b_i \left(\dfrac{y_1, \dots, y_n}{z_1, \dots, z_n} \right)$ and z_j in $a, b_1, \dots, b_m$ for each $i \in \{1, \dots, m\}$ and each $j \in \{1, \dots, n\}$.

Define $c_i = b_i \left(\dfrac{y_1, \dots, y_n}{z_1, \dots, z_n} \right)$ for each $i \in \{1, \dots, m\}$. Then Corollary 7.2.1 shows that $(z_1, \dots, z_n)$ is a free base of c_i and $H_i = b_i^\Phi(y_1, \dots, y_n) = c_i^\Phi(z_1, \dots, z_n)$, hence $\sigma c_i = u_i = \sigma x_i$ for each $i \in \{1, \dots, m\}$. Define $b = a \left(\dfrac{x_1, \dots, x_m}{c_1, \dots, c_m} \right)$. Then Theorem 7.2 shows that $(z_1, \dots, z_n)$ is a free base of b and $b^\Phi(z_1, \dots, z_n)$ satisfies

$$\begin{aligned} b^\Phi(w_1, \dots, w_n) &= a^\Phi\big(c_1{}^\Phi(w_1, \dots, w_n), \dots, c_m{}^\Phi(w_1, \dots, w_n)\big) \\ &= G\big(H_1(w_1, \dots, w_n), \dots, H_m(w_1, \dots, w_n)\big) \end{aligned}$$

for each $(w_1, \dots, w_n) \in W_{t_1} \times \cdots \times W_{t_n}$. Thus $F = b^\Phi(z_1, \dots, z_n) \in \mathcal{F}^\Phi$ as desired.

(2) Here $\lambda \in \Lambda'$, $G_k \in W_{t_1} \times \cdots \times W_{t_n} \to W_{u_k}$ $(k = 1, \dots, n_\lambda)$, $W_{u_1} \times \cdots \times W_{u_{n_\lambda}} \subseteq \operatorname{Dom} \omega_\lambda$, $F \in W_{t_1} \times \cdots \times W_{t_n} \to W_{\tau_\lambda(u_1, \dots, u_{n_\lambda})}$, and

$$F(w_1, \dots, w_n) = \omega_\lambda\big(G_1(w_1, \dots, w_n), \dots, G_{n_\lambda}(w_1, \dots, w_n)\big)$$

for each $(w_1, \dots, w_n) \in W_{t_1} \times \cdots \times W_{t_n}$. Also, for each $k \in \{1, \dots, n_\lambda\}$, there exist an element $a_k \in A$ and its free base $(x^k_1, \dots, x^k_n)$ such that $G_k = a_k^\Phi(x^k_1, \dots, x^k_n)$, hence $\sigma x^k_i = t_i$ for each $i \in \{1, \dots, n\}$.

Lemma 8.6 applied to $a_1, \dots, a_{n_\lambda}$ and $x^1_1, \dots, x^1_n$ shows that there exist distinct variables $x_1, \dots, x_n$ which satisfy the following conditions.

- $\sigma x_i = t_i$ for each $i \in \{1, \dots, n\}$.
- x^k_i is free from x_i in a_k for each $k \in \{1, \dots, n_\lambda\}$ and each $i \in \{1, \dots, n\}$.

Therefore, for each $k \in \{1, \dots, n_\lambda\}$, we may define $b_k = a_k \left(\dfrac{x^k_1, \dots, x^k_n}{x_1, \dots, x_n} \right)$, and Corollary 7.2.1 shows that $(x_1, \dots, x_n)$ is a free base of b_k and $G_k =$

$a_k^\Phi(x_1^k,\dots,x_n^k) = b_k^\Phi(x_1,\dots,x_n)$, hence $\sigma b_k = u_k$. Consequently $W_{\sigma b_1} \times \cdots \times W_{\sigma b_{n_\lambda}} \subseteq \operatorname{Dom}\omega_\lambda$, and so $(b_1,\dots,b_{n_\lambda}) \in \operatorname{Dom}\alpha_\lambda$ by Lemma 8.1. Define $a = \alpha_\lambda(b_1,\dots,b_{n_\lambda})$. Then, $(x_1,\dots,x_n)$ is a free base of a by Lemma 2.2, and Theorem 7.1 shows that the functional expression $a^\Phi(x_1,\dots,x_n)$ satisfies

$$\begin{aligned} a^\Phi(w_1,\dots,w_n) &= \omega_\lambda\big(b_1^\Phi(w_1,\dots,w_n),\dots,b_{n_\lambda}^\Phi(w_1,\dots,w_n)\big) \\ &= \omega_\lambda\big(G_1(w_1,\dots,w_n),\dots,G_{n_\lambda}(w_1,\dots,w_n)\big) \\ &= \lambda(G_1,\dots,G_{n_\lambda})(w_1,\dots,w_n) \end{aligned}$$

for each $(w_1,\dots,w_n) \in W_{t_1} \times \cdots \times W_{t_n}$. Thus $F = a^\Phi(x_1,\dots,x_n) \in \mathcal{F}^\Phi$ as desired.

(3) Here $\lambda = \nabla x$ ($\nabla \in \Gamma$, $x \in X'$), $G \in W_{t'} \times W_{t_1} \times \cdots \times W_{t_n} \to W_t$, $t \in T_\lambda$, $\sigma x = t'$, $F \in W_{t_1} \times \cdots \times W_{t_n} \to W_{\lambda t}$, and

$$F(w_1,\dots,w_n) = \lambda_W\big(G(\Box,w_1,\dots,w_n)\big)$$

for each $(w_1,\dots,w_n) \in W_{t_1} \times \cdots \times W_{t_n}$. Also, there exist an element $b \in A$ and its free base $(y,x_1,\dots,x_n)$ such that $G = b^\Phi(y,x_1,\dots,x_n)$, hence $\sigma b = t$, $\sigma y = t'$, and $\sigma x_i = t_i$ for each $i \in \{1,\dots,n\}$.

Since $x \in X'_{t'}$, $X'_{t'}$ is enumerable by the condition (1) of Theorem 8.2. Therefore, it follows from Lemma 2.1 and (3.1) that there exists an element $x' \in X'_{t'} - \big(\bigcup_{\mu\in\Lambda^b} S^\mu \cup \{x_1,\dots,x_n\}\big)$. Since $\sigma y = t' = \sigma x'$, we may define $c = b\left(\dfrac{y,x_1,\dots,x_n}{x',x_1,\dots,x_n}\right)$. Since $S^{x'}_{\text{free}} = \{x'\}$, it follows that $(S^\mu)^{x'}_{\text{free}} = \emptyset$ for each $\mu \in \Lambda^b$. Therefore y is free from x' in b by Lemma 2.2. Also, for each $i \in \{1,\dots,n\}$, x_i is free from x_i in b. Furthermore, $x',x_1,\dots,x_n$ are distinct. Therefore, $(x',x_1,\dots,x_n)$ is a free base of c and $G = b^\Phi(y,x_1,\dots,x_n) = c^\Phi(x',x_1,\dots,x_n)$ by Corollary 7.2.1.

Since $\lambda = \nabla x \in \Lambda \cap \Gamma X_{t'}$ and $\Lambda \cap \Gamma X_{t'} = \Gamma_{t'} X'_{t'}$ by the condition (2) of Theorem 8.2, $\lambda' = \nabla x'$ is also a variable operation, which is similar to λ because $\sigma x = t' = \sigma x'$. Therefore, by the condition (3) of Theorem 8.2, $T_\lambda = T_{\lambda'}$ and the interpretations λ_W and λ'_W of λ and λ' on W are equal. Since $\sigma c = t \in T_\lambda = T_{\lambda'}$, we have $c \in \operatorname{Dom}\alpha_{\lambda'}$. Define $a = \alpha_{\lambda'} c$. Then Lemma 2.2 shows that $(x_1,\dots,x_n)$ is a free base of a. Therefore by Theorem 7.1,

$$\begin{aligned} a^\Phi(w_1,\dots,w_n) &= \lambda'_W\big(c^\Phi(\Box,w_1,\dots,w_n)\big) \\ &= \lambda_W\big(G(\Box,w_1,\dots,w_n)\big) = F(w_1,\dots,w_n) \end{aligned}$$

for each $(w_1,\dots,w_n) \in W_{t_1} \times \cdots \times W_{t_n}$. Thus $F \in \mathcal{F}^\Phi$ as desired.

(4) Here $t \in T$, $X'_t \neq \emptyset$, $G \in W_{t_1} \times \cdots \times W_{t_n} \to W_u$, and

$$F(w,w_1,\dots,w_n) = G(w_1,\dots,w_n)$$

for each $(w,w_1,\dots,w_n) \in W_t \times W_{t_1} \times \cdots \times W_{t_n}$. Also, there exist an element $a \in A$ and its free base $(x_1,\dots,x_n)$ such that $G = a^\Phi(x_1,\dots,x_n)$.

Furthermore, by the condition (1) of Theorem 8.2, there exists an element $y \in X'_t - \{x_1, \ldots, x_n\}$. Define $H = a^\Phi(y, x_1, \ldots, x_n)$. Then by Lemma 7.1,

$$G(w_1, \ldots, w_n) = H(w, w_1, \ldots, w_n)$$

for each $(w, w_1, \ldots, w_n) \in W_t \times W_{t_1} \times \cdots \times W_{t_n}$. Thus $F = H \in \mathcal{F}^\Phi$ as desired.

References

[1] Gomi, K., and Y. Mizumura, "A completeness theorem for the logical system MPCL designed for mathematical psychology," preprint, http://homepage3.nifty.com/gomiken/english/mathpsy.htm.

[2] Gomi, K., *Mathematical Psychology* (in Japanese), personal electronic publication, http://homepage3.nifty.com/gomiken/index.htm, 1997.

[3] Gomi, K., "Sorted algebras and based algebras," preprint, http://homepage3.nifty.com/gomiken/english/mathpsy.htm.

[4] Gomi, K., "Theory of completeness for logical spaces," preprint, http://homepage3.nifty.com/gomiken/english/mathpsy.htm.

[5] Horikawa, S., *Basic Theory of Formal Languages Whose Variable Operations Have an Arbitrary Number of Qualifying Variables* (in Japanese), master's thesis, Graduate School of Mathematical Sciences, Univ. Tokyo, 2006, uploaded to http://homepage3.nifty.com/gomiken/mathpsy/horikawa.dvi.

UTMS

2008–17 Takashi Tsuboi: *On the group of real analytic diffeomorphisms.*

2008–18 Takefumi Igarashi and Noriaki Umeda: *Nonexistence of global solutions in time for reaction-diffusion systems with inhomogeneous terms in cones.*

2008–19 Oleg Yu. Imanouilov, Gunther Uhlmann, and Masahiro Yamamoto: *Partial data for the Calderón problem in two dimensions.*

2008–20 Xuefeng Liu and Fumio Kikuchi: *Analysis and estimation of error constants for P_0 and P_1 interpolations over triangular finite elements.*

2008–21 Fumio Kikuchi, Keizo Ishii and Issei Oikawa: *Discontinuous Galerkin FEM of Hybrid displacement type – Development of polygonal elements –.*

2008–22 Toshiyuki Kobayashi and Gen Mano: *The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p, q)$.*

2008–23 Kensaku Gomi and Yasuaki Mizumura: *A Completeness Theorem for the Logical System MPCL Designed for Mathematical Psychology.*

2008–24 Kensaku Gomi: *Theory of Completeness for Logical Spaces.*

2008–25 Oleg Yu. Imanuvilov, Gunther Uhlmann, and Masahiro Yamamoto: *Uniqueness by Dirichlet-to-Neumann map on an arbitrary part of boundary in two dimensions.*

2008–26 Takashi Tsuboi: *On the uniform simplicity of diffeomorphism groups.*

2008–27 A. Benabdallah, M. Cristofol, P. Gaitan and M. Yamamoto: *Inverse problem for a parabolic system with two components by measurements of one component.*

2008–28 Kensaku Gomi: *Foundations of Algebraic Logic.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012