UTMS 2009–13 June 19, 2009

Carleman estimates for parabolic equations and applications

by

Masahiro Yamamoto

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

CARLEMAN ESTIMATES FOR PARABOLIC EQUATIONS AND APPLICATIONS

Masahiro Yamamoto

Department of Mathematical Sciences, The University of Tokyo
Komaba Meguro Tokyo 153-8914 Japan
e-mail: myama@ms.u-tokyo.ac.jp

Abstract. In this article, concerning parabolic equations, we give self-contained descriptions on:

(1) derivations of Carleman estimates
(2) methods for applications of the Carleman estimates to estimates of solutions and to inverse problems

Moreover limiting to parabolic equations, we survey the previous and recent results in view of applicability of the Carleman estimate.

We do not intend to pursue any general treatments of the Carleman estimate itself but by showing it in a direct manner, we mainly aim at demonstrating the applicability of the Carleman estimate to estimation of solutions and inverse problems.

Contents.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

§1. Introduction.

For an inverse problem, in spite of the ill-posedness, one can prove conditional stability estimates which assure that one can restore the stability if one can restrict a class of solutions within an a priori bounded set. In practise, such an a priori bounded set can be interpreted as a physically acceptable constraint set. The conditional stability is not only theoretically interesting but also is important for stable numerics. There are several methods for proving the conditional stability, and a method by Carleman estimates is one of them.

Recently in the fields of the inverse problem and the control theory, Carleman estimates are applied in various ways to produce remarkable results. The purposes of this article are

(1) self-contained descriptions for deriving Carleman estimates
(2) discussions of typical methodologies for the application of a Carleman estimate in establishing the uniqueness and the stability for estimation problems of solutions and inverse problems of determining coefficients and source terms.
(3) an overview of classical and recent results for Carleman estimates and the applications to the estimation problem and the inverse problem

There have been already rich amounts of works for the theory of Carleman estimates, and a general theory is completed but here we will expose a direct method for proving a Carleman estimate. Such a direct derivation may give hints for Carleman estimates for other types of partial differential equations.

Moreover there are many works under process as well as established works in various fields even for equations of parabolic type, and so we will not intend a perfect list in overviewing and we will make an overview in terms of the inverse problem.

It is a common and important feature that we should discuss the theory of Carleman estimate and the application to the inverse problem uniformly for a wide class of partial differential equations including equations of hyperbolic type. However in the current article, we will restrict ourselves to equations of parabolic type.

Notations.

$\Omega \subset \mathbb{R}^n$: a bounded spatial domain with smooth boundary $\partial\Omega$, $Q = \Omega \times (0,T)$. We understand $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $t \geq 0$ respectively as the spatial and the time variables. $x' = (x_2, ..., x_n) \in \mathbb{R}^{n-1}$, $\xi' = (\xi_2, ..., \xi_n) \in \mathbb{R}^{n-1}$,

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_j = \frac{\partial}{\partial x_j},$$

$$\nabla = (\partial_1, \cdots, \partial_n), \quad \Delta = \partial_1^2 + \cdots + \partial_n^2.$$

ω: an arbitrarily fixed subdomain of Ω. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index with $\alpha_j \in \mathbb{N} \cup \{0\}$. We set $\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and $\nu = \nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ is the outward unit normal vector to $\partial\Omega$ at x. Let $\frac{\partial}{\partial\nu} = \nu \cdot \nabla$. Let $D \subset \Omega \times (0,T)$ be a domain with smooth boundary ∂D and let $\widetilde{\nu}$ be the outward unit normal vector to ∂D. On ∂D, we set $\frac{\partial u}{\partial \widetilde{\nu}} = \widetilde{\nu} \cdot \nabla_{x,t} u$,

$$|\widetilde{\nabla}_{x,t} u| = \left(\left| \frac{\partial u}{\partial \widetilde{\nu}} \right|^2 + \left| \frac{\partial u}{\partial \widetilde{\tau}} \right|^2 \right)^{\frac{1}{2}}$$

where $\frac{\partial u}{\partial \widetilde{\tau}}$ is the orthogonal component of $(\nabla u, \partial_t u)$ to $\frac{\partial u}{\partial \widetilde{\nu}}$. For example, $|\widetilde{\nabla}_{x,t} u| = \left(|\partial_t u|^2 + \left|\frac{\partial u}{\partial \nu}\right|^2 + \left|\frac{\partial u}{\partial \tau}\right|^2 \right)^{\frac{1}{2}}$ on $\partial\Omega \times (0,T) \subset \partial D$ and $|\widetilde{\nabla}_{x,t} u| = (|\partial_t u|^2 + |\nabla u|^2)^{\frac{1}{2}}$ on $\Omega \times \{t\} \subset \partial D$, where $\frac{\partial u}{\partial \tau}$ is the orthogonal component of ∇u to $\frac{\partial u}{\partial \nu}$. We use usual function spaces $C^1(\overline{Q})$, $H^2(\Omega)$ (e.g., Adams [1]),

$$H^{1,0}(Q) = \{u \in L^2(Q);\ \nabla u \in L^2(Q)\}$$

and for $m \in \mathbb{N}$,

$$H^{2m,m}(Q) = \{u \in L^2(Q);\, \partial_x^\alpha \partial_t^{\alpha_{n+1}} u \in L^2(Q), \quad |\alpha| + 2\alpha_{n+1} \le 2m\}$$

and $\|\cdot\|_{H^{2m,m}(Q)}$ is the corresponding norms. For $\mathbf{a} \in \mathbb{R}^n$, by $\mathbf{a}^T$ we denote the transpose vector, and by $[\mathbf{a}]_k$ we denote the k-th component of $\mathbf{a}$. By $C(\lambda)$ we denote generic constants which depend on other parameter λ.

If we will not specially state, then we always that

$$a_{ij} \in C^1(\overline{Q}), \quad a_{ij} = a_{ji}, \quad 1 \le i,j \le n, \tag{1.1}$$

and that the coefficients $\{a_{ij}\} \equiv \{a_{ij}\}_{1\le i,j\le n}$ satisfy the uniform ellipticity: there exists a constant $\sigma_1 > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \ge \sigma_1|\zeta|^2, \qquad \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n, \quad (x,t) \in \overline{Q}. \tag{1.2}$$

Let $b_k, c \in L^\infty(Q)$, $1 \le k \le n$. A typical parabolic operator which we will discuss, is

$$\begin{aligned}(Lu)(x,t) = \partial_t u(x,t) - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j u(x,t)\\ - \sum_{k=1}^n b_k(x,t)\partial_k u - c(x,t)u, \quad (x,t) \in Q.\end{aligned} \tag{1.3}$$

§2. What is a Carleman estimate?

A Carleman estimate is an L^2-weighted estimate with large parameter for a solution to a partial diifferential equation. Here in place of a statement of a general theorem for the Carleman estimate, we start with a very direct derivation for a simplest heat equation:

$$\partial_t u(x,t) = \Delta u(x,t) + f(x,t), \quad x \in \Omega,\ t > 0. \tag{2.1}$$

Our aim is to find an L^2-weighted estimate with large parameter s in some domain $D \subset Q$, which is called a Carleman estimate: We choose a suitable function $\varphi(x,t)$ satisfying: there exist constants $C >$ and $s_0 > 0$ such that

$$\int_D s(|\nabla u(x,t)|^2 + |u(x,t)|^2) e^{2s\varphi(x,t)} dxdt \le C \int_D |f(x,t)|^2 e^{2s\varphi(x,t)} dxdt \tag{2.2}$$

for all $s > s_0$ and all $u \in C_0^\infty(D)$. We note that in (2.2), the estimate is valid uniformly for all large $s > 0$, i.e., $s \ge s_0$: a fixed constant. In other words, the constant $C > 0$ is independent of $s > s_0$ and $u \in C_0^\infty(D)$. For applications, the parameter s plays an essential role and it is also important how to choose a weight function $\varphi(x,t)$.

The Carleman estimate was first established by Carleman [30] for proving the unique continuation for a two-dimensional elliptic equation. Since then there have been great concerns for the Carleman estimate and the applications, and there are remarkable general treatments by Egorov [43], Hörmander [65], [66], Isakov [81], [82], [86], Tataru [128], Taylor [129], Tréves [130]. However, here for easier understanding not only by speacialist, we will give a heuristic derivation, which may be useful for the insight for the characters of a Carleman estimate.

We first consider a simple heat equation (2.1). Let us assume that we alreay find a weight function $\varphi(x,t)$. For treating the weighted L^2-norms, we introduce

$$w(x,t) = e^{s\varphi(x,t)} u(x,t), \quad Pw(x,t) = e^{s\varphi}(\partial_t - \Delta)(e^{-s\varphi} w).$$

Then we rewrite the right-hand side of (2.2) by

$$\int_D f^2 e^{2s\varphi} dxdt = \int_D |Pw(x,t)|^2 dxdt.$$

Therefore our task is a lower estimate of $\|Pw\|^2_{L^2(D)}$. Direct calculations yield

$$Pw = \partial_t w - \Delta w + 2s\nabla\varphi \cdot \nabla w + (-s\partial_t\varphi - s^2|\nabla\varphi|^2 + s\Delta\varphi) w.$$

Let us consider formally. That is, assuming that $u \in C_0^\infty(D)$, we will take integration by parts as we like. One traditional way for obtaining a lower estimate for $\|Pw\|^2_{L^2(D)}$, is the decomposion of the operator P into the symmetric part P_+ and the antisymmetric part P_-: $Pw = P_+w + P_-w$. See e.g., Bukhgeim [19] for a one-dimensional Schrödinger equation.

We consider the formal adjoint operator P^* to P:

$$(Pw, v)_{L^2(D)} = (w, P^*v)_{L^2(D)}, \quad v, w \in C_0^\infty(D).$$

For example, we have

$$(\partial_t w, v)_{L^2(D)} = -(w, \partial_t v)_{L^2(D)}$$

and

$$(-\Delta w, v)_{L^2(D)} = (w, \Delta v)_{L^2(D)}$$

by integration by parts, the Green theorem and $v, w \in C_0^\infty(D)$. Hence we see that

$$P^*w = -\partial_t w - \Delta w - 2s\nabla\varphi \cdot \nabla w - (s\Delta\varphi + s^2|\nabla\varphi|^2 + s(\partial_t\varphi))w.$$

We define the symmetric part P_+ and the antisymmetric part P_- of P by

$$P_+ = \frac{1}{2}(P + P^*), \quad P_- = \frac{1}{2}(P - P^*).$$

Then we have $Pw = P_+w + P_-w$, and

$$P_+w = -\Delta w - (s^2|\nabla\varphi|^2 + s\partial_t\varphi)w$$

and

$$P_-w = \partial_t w + 2s\nabla\varphi \cdot \nabla w + s(\Delta\varphi)w.$$

Hence

$$
\begin{aligned}
&\int_D f^2 e^{2s\varphi} dxdt = \|P_+ w + P_- w\|^2_{L^2(D)} \\
(2.3)\qquad &= \|P_+ w\|^2_{L^2(D)} + \|P_- w\|_{L^2(D)} + 2(P_+ w, P_- w)_{L^2(D)} \ge 2(P_+ w, P_- w)_{L^2(D)}.
\end{aligned}
$$

That is, we will estimate the right-hand side of (2.2) from the below by means of $2(P_+ w, P_- w)_{L^2(D)}$. We note here that we discarded other terms $\|P_+ w\|^2_{L^2(D)}$ and $\|P_- w\|^2_{L^2(D)}$ although there may be better possibilities for decomposing of Pw (see section 3 for a general parabolic equation).

We have

$$
\begin{aligned}
&2(P_+ w, P_- w)_{L^2(D)} = 2(-\Delta w - (s^2|\nabla\varphi|^2 + s\partial_t\varphi)w, \quad \partial_t w + 2s\nabla\varphi\cdot\nabla w + s(\Delta\varphi)w)_{L^2(D)} \\
=&2(-\Delta w, \partial_t w)_{L^2(D)} + 2(-\Delta w, 2s\nabla\varphi\cdot\nabla w)_{L^2(D)} + 2(-\Delta w, s(\Delta\varphi)w)_{L^2(D)} \\
&-2((s^2|\nabla\varphi|^2 + s\partial_t\varphi)w, \partial_t w)_{L^2(D)} - 2((s^2|\nabla\varphi|^2 + s\partial_t\varphi)w, 2s\nabla\varphi\cdot\nabla w)_{L^2(D)} \\
&-2((s^2|\nabla\varphi|^2 + s\partial_t\varphi)w, s(\Delta\varphi)w)_{L^2(D)}.
\end{aligned}
$$

By the integration by parts and $w \in C_0^\infty(D)$, we will reduce the orders of derivatives of w. Henceforth $C > 0$ denotes generic constants which are independent of s and may change line by line. For example, the calculations are as follows:

$$
\begin{aligned}
&-2((s^2|\nabla\varphi|^2 + s\partial_t\varphi)w, 2s\nabla\varphi\cdot\nabla w)_{L^2(D)} = -4s\sum_{i=1}^n \int_D \{(s^2|\nabla\varphi|^2 + s\partial_t\varphi)w\}(\partial_i\varphi)(\partial_i w)dxdt \\
=&-2s\sum_{i=1}^n \int_D (s^2|\nabla\varphi|^2 + s\partial_t\varphi)(\partial_i\varphi)\partial_i(w^2)dxdt \\
=&2s\sum_{i=1}^n \int_D \partial_i((s^2|\nabla\varphi|^2 + s\partial_t\varphi)\partial_i\varphi)w^2 dxdt \\
=&2s\int_D \{\nabla(s^2|\nabla\varphi|^2 + s\partial_t\varphi)\cdot\nabla\varphi + (s^2|\nabla\varphi|^2 + s\partial_t\varphi)\Delta\varphi\}w^2 dxdt.
\end{aligned}
$$

Next the Green formula yields

$$
\begin{aligned}
&2(-\Delta w, s(\Delta\varphi)w)_{L^2(D)}\\
=&2s\int_D \nabla w\cdot\nabla((\Delta\varphi)w)dxdt = 2s\int_D(\Delta\varphi)|\nabla w|^2dxdt + 2s\int_D\nabla(\Delta\varphi)\cdot w\nabla wdxdt,
\end{aligned}
$$

and

$$
\left|s\int_D \nabla(\Delta\varphi)\cdot w\nabla wdxdt\right| \le Cs\int_D |w||\nabla w|dxdt.
$$

Hence

$$
2(-\Delta w, s(\Delta\varphi)w)_{L^2(D)} \ge 2s\int_D(\Delta\varphi)|\nabla w|^2dxdt - Cs\int_D|w||\nabla w|dxdt.
$$

Next, noting $2(\partial_k w)(\partial_k\partial_j w) = \partial_j(|\partial_k w|^2)$ and integration by parts, we have

$$
\begin{aligned}
&2(-\Delta w, 2s\nabla\varphi\cdot\nabla w)_{L^2(D)} = 2\sum_{j,k=1}^n(-\partial_k^2 w, 2s(\partial_j w)\partial_j\varphi)_{L^2(D)}\\
=&2\sum_{j,k=1}^n(\partial_k w, 2s(\partial_k\partial_j w)(\partial_j\varphi))_{L^2(D)} + (\partial_k w, 2s(\partial_j w)(\partial_k\partial_j\varphi))_{L^2(D)}\\
=&-2s\sum_{j,k=1}^n\int_D(\partial_j^2\varphi)|\partial_k w|^2dxdt + 4s\sum_{j,k=1}^n\int_D(\partial_j w)(\partial_k w)(\partial_j\partial_k\varphi)dxdt.
\end{aligned}
$$

Therefore, noting that s^3 is the maximal order of the term w^2 and s is the maximal order of the term $|\nabla w|^2$, we have

$$
\begin{aligned}
&\frac{1}{2}(P_+w, P_-w)_{L^2(D)}\\
\ge& s^3\int_D\{\nabla(|\nabla\varphi|^2)\cdot\nabla\varphi\}w^2dxdt + 2s\sum_{j,k=1}^n\int_D(\partial_j w)(\partial_k w)(\partial_j\partial_k\varphi)dxdt\\
&-C\int_D s^2w^2dxdt - Cs\int|w||\nabla w|dxdt\\
\ge& s^3\int_D\{\nabla(|\nabla\varphi|^2)\cdot\nabla\varphi\}w^2dxdt + 2s\sum_{j,k=1}^n\int_D(\partial_j w)(\partial_k w)(\partial_j\partial_k\varphi)dxdt\\
&-C\int_D(|\nabla w|^2 + s^2w^2)dxdt.
\end{aligned}
$$

At the last inequality we used also

$$
s|\nabla w||w| \le \frac{1}{2}s^2|w|^2 + \frac{1}{2}|\nabla w|^2.
$$

Hence, since we can consider only sufficiently large $s > 0$, noting the maximum powers in s for the terms of $|w|^2$ and $|\nabla w|^2$, we can absorb terms of lower powers, so that if φ satisfies

$$\{\partial_i \partial_j \varphi\}_{1\le i,j\le n} \text{ is positive definite} \tag{2.4}$$

and

$$\text{there exists a constant } r_1 > 0 \text{ such that} \quad \nabla(|\nabla\varphi|^2)\cdot\nabla\varphi \ge r_1 \quad \text{on } \overline{D}, \tag{2.5}$$

then there exist constants $C > 0$ and $s_0 > 0$ such that

$$\int_D (s|\nabla w|^2 + s^3|w|^2)dxdt \le C\int_D f^2 e^{2s\varphi}dxdt$$

for all $s \ge s_0$ and all $w \in C_0^\infty(D)$. Noting $w = e^{s\varphi}u$, we rewrite in terms of u, and

$$\int_D (s|\nabla u|^2 + s^3|u|^2)e^{2s\varphi}dxdt \le C\int_D f^2 e^{2s\varphi}dxdt \tag{2.6}$$

for all $s \ge s_0$ and all $u \in C_0^\infty(D)$.

The next important step is the choice of the weight function φ. The weight function has to satisfy not only (2.4) and (2.5) but also some geometric condition for meaningful applications to the unique continuation and the inverse problem. More precisely, in applying a Carleman estimate, as D we usually consider a level set defined by $\{(x,t);\, \varphi(x,t) > \delta\}$ with some constant $\delta > 0$, and such a level set should be a bounded domain at least (see sections 5 and 6).

For Carleman estimate (2.6), our possible choice is not flexible. That is, we are restricted to the function:

$$\varphi(x,t) = |x - x_0|^2 - \beta(t - t_0)^2. \tag{2.7}$$

Here we assume that $|x-x_0| \ne 0$ for any $(x,t) \in \overline{D}$, and $\beta > 0$ and $t_0 \in (0,T)$ are arbitrarily fixed. Then we have $\nabla(|\nabla\varphi|^2)\cdot\nabla\varphi = 16|x-x_0|^2 > 0$ on $\overline{D}$ and $\{\partial_i\partial_j\varphi\}_{1\le i,j\le n} = 2E_n$, where $E_n \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$. That is, the conditions (2.4) and (2.5) are satisfied.

As is seen from sections 5 and 6, the Carleman estimate produces the uniqueness and the stability in a level set $\{(x,t);\ \varphi(x,t) > \delta\}$. Let $\Gamma \subset \partial\Omega$ be a subboundary. By the choice (2.7) in applying a Carleman estimate, we can see that near Γ, the domain Ω has to be convex for proving that the solution to $\partial_t u - \Delta u = 0$ satisfying $|u| = |\nabla u| = 0$ on $\Gamma \times (0,T)$, vanishes in a subdomain including Γ. However by the parabolicity, we can expect that the uniqueness should hold without such a convexity assumption.

§3. A direct derivation of a Carleman estimate for a parabolic equation.

Let $D \subset Q$ be a bounded domain whose boundary ∂D is composed of a finite number of smooth surfaces. As is discussed in detail in section 7, there are many papers deriving Carleman estimates even though we are restricted to parabolic equations, and as for derivations, I refer to:

(1) general way: Eller and Isakov [45], Isakov [80, 82, 84, 85], Tataru [128].

(2) direct way: Chae, Imanuvilov and Kim [32], Fursikov and Imanuvilov [58], Imanuvilov [68], Lavrent'ev, Romanov and Shishat·skǐ[107], Yuan and Yamamoto [136].

As for the direct derivations for hyperbolic equations, see also [100] and [107], but we will not discuss the hyperbolic case.

In this survey, we will mainly explain a direct method, because of its flexibility. The key tool of the direct method is the integration by parts and suitable grouping the terms according to the orders of the parameter s in a Carleman estimate. Here we will make motivating explanations for the grouping in order that the explanations may be more friendly to the readers and they may be able to apply the direct method to other types of equations to obtain possible Carleman estimates.

Before starting the derivations, we note that it is sufficient to prove a Carleman estimate for one of two types of parabolic equations:

$$\rho(x,t)\partial_t u(x,t) - \sum_{i,j=1}^n \partial_i(\widetilde{a}_{ij}(x,t)\partial_j u(x,t)) - \sum_{k=1}^n \widetilde{b}_k(x,t)\partial_k u(x,t) - \widetilde{c}(x,t)u(x,t) = \widetilde{f}(x,t)$$

and

$$\partial_t u(x,t) - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j u(x,t) - \sum_{k=1}^n b_k(x,t)\partial_k u(x,t) - c(x,t)u(x,t) = f(x,t).$$

Here $\rho \in C^1(\overline{D})$ with $\rho > 0$ on $\overline{D}$ and $b_k, \widetilde{b}_k, c, \widetilde{c} \in L^\infty(D)$, we assume that

$$\begin{cases} \widetilde{a}_{ij} \in C^1(\overline{D}), \quad \widetilde{a}_{ij} = \widetilde{a}_{ji},\ 1 \le i,j \le n, \\ \displaystyle\sum_{i,j=1}^n \widetilde{a}_{ij}(x,t)\xi_i\xi_j \ge \sigma_1 \sum_{i=1}^n \xi_i^2, \quad (x,t) \in \overline{D},\ \xi_1, ..., \xi_n \in \mathbb{R}. \end{cases}$$

This is seen because:

$$\rho(x,t)\partial_t u(x,t) - \sum_{i,j=1}^n \partial_i(\widetilde{a}_{ij}(x,t)\partial_j u(x,t)) - \sum_{k=1}^n \widetilde{b}_k(x,t)\partial_k u(x,t) - \widetilde{c}(x,t)u(x,t) = \widetilde{f}(x,t)$$

if and only if

$$\partial_t u(x,t) - \sum_{i,j=1}^n \frac{\widetilde{a}_{ij}}{\rho}\partial_i\partial_j u - \sum_{k=1}^n \frac{1}{\rho}\left(\widetilde{b}_k + \sum_{i=1}^n \partial_i \widetilde{a}_{ik}\right)\partial_k u - \frac{\widetilde{c}}{\rho}u = \frac{\widetilde{f}}{\rho}.$$

Let us set

$$Lu(x,t) = \partial_t u - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j u(x,t) - \sum_{k=1}^n b_k(x,t)\partial_k u(x,t) - c(x,t)u(x,t) \quad \text{in } Q$$

and

$$L_0 u = \partial_t u - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j u(x,t) \quad \text{in } Q.$$

Here we assume that a_{ij}, $1 \le i,j \le n$ satisfy (1.1) and (1.2), and $b_k, c \in L^\infty(Q)$, $1 \le k \le n$.

We consider a parabolic equation $Lu = f$. Our purpose is to establish a Carleman estimate

$$\begin{aligned}&\int_D \left\{ \frac{1}{s\varphi}\left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i\partial_j u|^2 \right) + s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3 u^2 \right\} e^{2s\varphi} dxdt\\ \le& C\int_D |Lu|^2 e^{2s\varphi} dxdt\end{aligned}$$

for all large $s > 0$ and $\lambda > 0$ and all $u \in H^{2,1}(Q)$ satisfying $\mathrm{supp}\, u \in D$ For it, it suffices to prove the estimate for L_0. Because $|L_0u|^2 \le 2|Lu|^2 + 2|\sum_{k=1}^n b_k\partial_k u + cu|^2$ in Q, that is,

$$\begin{aligned}&\int_D |L_0 u|^2 e^{2s\varphi} dxdt\\ \le& 2\int_D |Lu|^2 e^{2s\varphi} dxdt + 4\int_D \left(\sum_{k=1}^n \|b_k\|^2_{L^\infty(D)} |\nabla u|^2 + \|c\|^2_{L^\infty(D)} |u|^2 \right) e^{2s\varphi} dxdt.\end{aligned}$$

Hence the Carleman estiamte for L_0 yields

$$\begin{aligned}&\int_D \left\{ \frac{1}{s\varphi}\left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i\partial_j u|^2 \right) + s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3 u^2 \right\} e^{2s\varphi} dxdt\\ \le& C\int_D |Lu|^2 e^{2s\varphi} dxdt\\ +& C\sum_{k=1}^n \|b_k\|^2_{L^\infty(D)} \int_D |\nabla u|^2 e^{2s\varphi} dxdt + C\|c\|^2_{L^\infty(D)} \int_D |u|^2 e^{2s\varphi} dxdt.\end{aligned}$$

Therefore we choose $s > 0$ sufficiently large and we can absorb the second and the third terms on the right-hand side into the left-hand side. Thus for the Carleman estimate, only the terms with derivatives of highest orders in x and t are important if the coefficients of lower-order terms are in $L^\infty(Q)$.

The simple method in section 2, is based on the decomposition into the symmetric and the antisymmetric parts and may be transparent, but for the general parabolic equation with variable coefficients, it does not work. In this section, we explain a direct method for deriving a Carleman estimate for a general parabolic equation. Here, for a weight function φ, the important factor is the second large parameter $\lambda > 0$ and we search for the weight function φ in the form of $e^{\lambda\psi}$. This form has been recognized as useful and see e.g., Hörmander [65], section 8.6, and is essentially used in several references: Eller and Isakov [45], Imanuvilov [68], Isakov [80], Isakov and Kim [87], [88]. Also see section 7.4. Thanks to the form $e^{\lambda\psi}$, it is easier to guarantee the positivity of the coefficients of $|w|^2$ and $|\nabla w|^2$ in estimating $\|Pw\|^2_{L^2(D)}$. This kind of form is very useful also in section 9.

Let $d \in C^2(\overline{D})$ and $|\nabla d| \neq 0$ on $\overline{D}$ and let us set

$$\psi(x,t) = d(x) - \beta(t-t_0)^2 + c_0$$

with $t_0 \in (0,T)$, $c_0, \beta > 0$ such that $\inf_{(x,t)\in Q} \psi(x,t) > 0$ and

$$\varphi(x,t) = e^{\lambda\psi(x,t)}.$$

It is not essential that $\psi > 0$ in Q, but the positivity is convenient in the succeeding arguments.

Remark. For $Lu = f$, we consider a Carleman estimate, and our argument holds also for the parabolic inequality

$$\left| \partial_t u(x,t) - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j u(x,t) \right| \le C(|\nabla u(x,t)| + |u(x,t)| + f(x,t)|)$$

in Q.

First we assume

$$u \in C_0^\infty(D).$$

We further set

$$\sigma(x,t) = \sum_{i,j=1}^{n} a_{ij}(x,t)(\partial_i d)(x)(\partial_j d)(x), \quad (x,t) \in \overline{Q}$$

and

$$w(x,t) = e^{s\varphi(x,t)}u(x,t)$$

and

$$Pw(x,t) = e^{s\varphi}L_0(e^{-s\varphi}w) = e^{s\varphi}L_0u.$$

The derivation argument consists of

(1) the decomposition of P into the part P_1 and P_2, where P_1 is composed of second-order and zeroth-order terms in x, and P_2 is composed of first-order terms in t and first-order terms in x. Here the terms in Pw are classified by the highest order of s, λ and φ, and not by the symmetric and the antisymmetric parts. Compare P_1, P_2 (see (3.1) and (3.2)) with P_+, P_- in section 2.

(2) Estimation of $\int_D(|P_2w|^2 + 2(P_1w)(P_2w))dxdt$ from the below.

(3) Another estimate for

$$\int_D Pw \times [\text{the term } u \text{ with second highest order of } s, \lambda, \varphi \text{ among } Pw].$$

By our decomposition, we have to estimate the L^2-term of $\partial_t u$, and so we need to estimate also the term $\int_D |P_2w|^2 dxdt$ in step (2). Moreover the estimate in the second step produces the estimate of u with desirable order of s, λ, φ but not the term of ∇u. This is a natural consequence with different orders of the derivatives of terms of under consideration. Therefore another estimate in the third step is necessary. This kind of double estimates is also used in section 9 and in proving the observability inequality by the multiplier method. As for the multiplier method, see e.g., Komornik [102], pp.36-39 where the wave equation is considerd but a principle is similar: the two estimates are obtained from

$$\int_D (\partial_t^2 u - \Delta u - c(x)u)(h(x) \cdot \nabla u) dxdt$$

and

$$\int_D (\partial_t^2 u - \Delta u - c(x)u) u dxdt$$

with a suitable vector-valued function $h(x)$, and added them to obtain an L^2-estimate of u. The second estimate for the wave equation by the multipiier method has a purpose similar to step (3) in our case.

We have

$$Pw = \partial_t w - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j w + 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(x,t)(\partial_i d)\partial_j w$$
$$-s^2\lambda^2\varphi^2\sigma w + s\lambda^2\varphi\sigma w + s\lambda\varphi w \sum_{i,j=1}^n a_{ij}\partial_i\partial_j d - s\lambda\varphi w(\partial_t\psi) \text{ in } D.$$

Here we note that we have specified all the dependency of coefficients on s, λ and φ. We set

$$A_1 = s\lambda^2\varphi\sigma + s\lambda\varphi \sum_{i,j=1}^n a_{ij}\partial_i\partial_j d - s\lambda\varphi(\partial_t\psi)$$
$$\equiv s\lambda^2\varphi a_1(x,t;s,\lambda).$$

Then

$$Pw = \partial_t w - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j w + 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(x,t)(\partial_i d)\partial_j w$$
$$-s^2\lambda^2\varphi^2\sigma w + A_1 w \quad \text{in } D.$$

We note that a_1 depends on s and λ but

$$|a_1(x,t;s,\lambda)| \le C \quad \text{for } (x,t) \in \overline{D} \text{ and all sufficiently large } \lambda > 0 \text{ and } s > 0.$$

Here and henceforth by C, C_1, etc., we denote generic constants which are independent of s, λ and φ but may change line by line.

Then taking into consideration the orders of (s, λ, φ), we divide Pw as follows:

$$P_1 w = -\sum_{i,j=1}^{n} a_{ij}(x,t)\partial_i\partial_j w - s^2\lambda^2\varphi^2 w\sigma(x,t) + A_1 w \tag{3.1}$$

and

$$P_2 w = \partial_t w + 2s\lambda\varphi \sum_{i,j=1}^{n} a_{ij}(x,t)(\partial_i d)\partial_j w. \tag{3.2}$$

By $\|fe^{s\varphi}\|^2_{L^2(D)} = \|P_1 w + P_2 w\|^2_{L^2(D)}$, we have

$$2\int_D (P_1 w)(P_2 w)dxdt + \|P_2 w\|^2_{L^2(D)} \le \int_D f^2 e^{2s\varphi} dxdt. \tag{3.3}$$

We estimate:

$$\begin{aligned}
&\int_D (P_1 w)(P_2 w)dxdt \\
= &-\sum_{i,j=1}^{n}\int_D a_{ij}(\partial_i\partial_j w)(\partial_t w)dxdt - \sum_{i,j=1}^{n}\int_D a_{ij}(\partial_i\partial_j w)2s\lambda\varphi\sum_{k,\ell=1}^{n} a_{k\ell}(\partial_k d)(\partial_\ell w)dxdt \\
&-\int_D s^2\lambda^2\varphi^2\sigma w(\partial_t w)dxdt - \int_D 2s^3\lambda^3\varphi^3\sigma w\sum_{i,j=1}^{n} a_{ij}(\partial_i d)(\partial_j w)dxdt \\
&+\int_D (A_1 w)(\partial_t w)dxdt + \int_D (A_1 w)2s\lambda\varphi\sum_{i,j=1}^{n} a_{ij}(\partial_i d)(\partial_j w)dxdt \\
&\equiv \sum_{k=1}^{6} J_k.
\end{aligned} \tag{3.4}$$

Now, applying the intregration by parts, $a_{ij} = a_{ji}$ and $u \in C_0^\infty(D)$ and assuming that $\lambda > 1$ and $s > 1$ are sufficiently large, we reduce all the derivatives of w to

$w, \partial_i w, \partial_t w$. We continue the estimation of J_k, $k = 1, ..., 6$.

$$
\begin{aligned}
|J_1| = & \left| - \sum_{i,j=1}^{n} \int_D a_{ij} (\partial_i \partial_j w)(\partial_t w) dxdt \right| \\
= & \left| \sum_{i,j=1}^{n} \int_D (\partial_i a_{ij})(\partial_j w)(\partial_t w) dxdt + \sum_{i,j=1}^{n} \int_D a_{ij} (\partial_j w)(\partial_i \partial_t w) dxdt \right| \\
= & \Bigg| \sum_{i,j=1}^{n} \int_D (\partial_i a_{ij})(\partial_j w)(\partial_t w) dxdt \\
& + \left(\sum_{i>j} \int_D a_{ij} ((\partial_j w)(\partial_i \partial_t w) + (\partial_i w)(\partial_j \partial_t w)) dxdt \right. \\
& \left. + \int_D \sum_{i=1}^{n} a_{ii} (\partial_i w)(\partial_i \partial_t w) dxdt \right) \Bigg| \\
\le & C \int_D |\nabla w| |\partial_t w| dxdt + \frac{1}{2} \left| \int_D \sum_{i,j=1}^{n} (\partial_t a_{ij})(\partial_i w)(\partial_j w) dxdt \right| \\
\le & C \int_D |\nabla w| |\partial_t w| dxdt + C \int_D |\nabla w|^2 dxdt.
\end{aligned} \tag{3.5}
$$

Here we used

$$
\begin{aligned}
& \left(\sum_{i>j} \int_D a_{ij} ((\partial_j w)(\partial_i \partial_t w) + (\partial_i w)(\partial_j \partial_t w)) dxdt \right. \\
& \left. + \int_D \sum_{i=1}^{n} a_{ii} (\partial_i w)(\partial_i \partial_t w) dxdt \right) = \frac{1}{2} \sum_{i,j=1}^{n} \int_D a_{ij} \partial_t ((\partial_j w)(\partial_i w)) dxdt.
\end{aligned}
$$

Next

$$
\begin{aligned}
J_2 = & - \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{n} \int_D 2s\lambda \varphi a_{ij} a_{k\ell} (\partial_k d)(\partial_\ell w)(\partial_i \partial_j w) dxdt \\
= & 2s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{n} \lambda (\partial_i d) \varphi a_{ij} a_{k\ell} (\partial_k d)(\partial_\ell w)(\partial_j w) dxdt \\
& + 2s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{n} \varphi \partial_i (a_{ij} a_{k\ell} \partial_k d)(\partial_\ell w)(\partial_i w) dxdt \\
& + 2s\lambda \int_D \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{n} \varphi a_{ij} a_{k\ell} (\partial_k d)(\partial_i \partial_\ell w)(\partial_j w) dxdt.
\end{aligned}
$$

We have

$$
[\text{first term}] = 2s\lambda^2 \int_D \varphi \left| \sum_{i,j=1}^{n} a_{ij} (\partial_i d)(\partial_j w) \right|^2 dxdt \ge 0,
$$

and similarly to J_1, we can estimate

$$[\text{third term}] = s\lambda \sum_{i,j=1}^n \sum_{k,\ell=1}^n \int_D \varphi a_{ij}a_{k\ell}(\partial_k d)\partial_\ell((\partial_i w)(\partial_j w))$$

$$= - s\lambda^2 \int_D \varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w) dxdt - s\lambda \int_D \varphi \sum_{i,j=1}^n \sum_{k,\ell=1}^n \partial_\ell(a_{ij}a_{k\ell}\partial_k d)(\partial_i w)(\partial_j w) dxdt.$$

Hence

$$J_2 \ge - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w) dxdt$$

$$-C\int_D s\lambda\varphi|\nabla w|^2 dxdt + 2s\lambda^2 \int_D \varphi \left| \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)\right|^2 dxdt$$

$$\ge - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w) dxdt - C\int_D s\lambda\varphi|\nabla w|^2 dxdt. \tag{3.6}$$

$$|J_3| = \left| - \int_D \frac{1}{2} s^2\lambda^2\varphi^2\sigma\partial_t(w^2) dxdt \right|$$

$$= \left| \int_D s^2\lambda^2\varphi\{\lambda(\partial_t\psi)\varphi\}\sigma w^2 dxdt + \frac{1}{2}\int_D s^2\lambda^2\varphi^2(\partial_t\sigma)w^2 dxdt\right|$$

$$\le C\int_D s^2\lambda^3\varphi^2 w^2 dxdt. \tag{3.7}$$

$$J_4 = - \int_D 2s^3\lambda^3\varphi^3\sigma w \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w) dxdt$$

$$= - \int_D s^3\lambda^3\varphi^3 \sum_{i,j=1}^n \sigma a_{ij}(\partial_i d)\partial_j(w^2) dxdt$$

$$= \int_D s^3\lambda^3 \sum_{i,j=1}^n 3\varphi^2\{\lambda(\partial_j d)\varphi\}\sigma a_{ij}(\partial_i d) w^2 dxdt$$

$$+ \int_D s^3\lambda^3\varphi^3 \sum_{i,j=1}^n \partial_j(\sigma a_{ij}\partial_i d) w^2 dxdt$$

$$\ge \int_D 3s^3\lambda^4\varphi^3\sigma^2 w^2 dxdt - C\int_D s^3\lambda^3\varphi^3 w^2 dxdt. \tag{3.8}$$

$$|J_5| = \left|\int_D (A_1 w)(\partial_t w) dxdt\right| = \left|\int_D s\lambda^2\varphi a_1 w(\partial_t w) dxdt\right|$$

$$= \frac{1}{2}\left|\int_D s\lambda^2\varphi a_1\partial_t(w^2) dxdt\right|$$

$$= \frac{1}{2}\left|\int_D s\lambda^2\varphi(\partial_t a_1)w^2 dxdt + \int_D s\lambda^3\varphi(\partial_t\psi)a_1 w^2 dxdt\right|$$

$$\le C\int_D s\lambda^3\varphi w^2 dxdt. \tag{3.9}$$

$$
\begin{aligned}
|J_6| =& \left| \int_D s\lambda^2\varphi a_1 \times 2s\lambda\varphi w \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)dxdt \right| \\
=& \left| \int_D 2a_1 s^2\lambda^3\varphi^2 \sum_{i,j=1}^n a_{ij}(\partial_i d)w(\partial_j w)dxdt \right| \\
=& \left| \int_D a_1 s^2\lambda^3\varphi^2 \sum_{i,j=1}^n a_{ij}(\partial_i d)\partial_j(w^2)dxdt \right| \\
=& \left| -\int_D \partial_j(a_1 s^2\lambda^3\varphi^2 a_{ij}(\partial_i d))w^2 dxdt \right| \\
\le& C\int_D s^2\lambda^4\varphi^2 w^2 dxdt. \qquad (3.10)
\end{aligned}
$$

We remark that in estimating $|J_6|$, we need integration by parts. If we will apply a simpler way by the Cauchy-Schwarz inequality to estimate like $\int_D s^2\lambda^3\varphi^2|\nabla w||w|dxdt$, then we lose orders of s, λ and we can not continue the estimation.

Hence, by (3.4) - (3.10), we obtain

$$
\begin{aligned}
&\int_D (P_1 w)(P_2 w)dxdt \ge 3\int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dxdt - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dxdt \\
&-C\int_D s\lambda\varphi|\nabla w|^2 dxdt - C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2)w^2 dxdt - C\int_D |\nabla w||\partial_t w|dxdt.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&3\int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dxdt - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dxdt \\
\le& \int_D (P_1 w)(P_2 w)dxdt + C\int_D s\lambda\varphi|\nabla w|^2 dxdt \\
&+C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2)w^2 dxdt + C\int_D |\nabla w||\partial_t w|dxdt. \qquad (3.11)
\end{aligned}
$$

Moreover for all large $s > 0$, by the definition of P_2 and an inequality: $|\alpha+\beta|^2 \ge \frac{1}{2}|\alpha|^2 - |\beta|^2$, we obtain

$$
\begin{aligned}
&\int_D |P_2 w|^2 dxdt \ge \int_D \frac{1}{s\varphi}|P_2 w|^2 dxdt = \int_D \frac{1}{s\varphi}\left| \partial_t w + 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)\right|^2 dxdt \\
\ge& \frac{1}{2}\int_D \frac{1}{s\varphi}|\partial_t w|^2 dxdt - C\int_D s\lambda^2\varphi \left| \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)\right|^2 dxdt,
\end{aligned}
$$

that is,

$$\varepsilon \int_D \frac{1}{s\varphi}|\partial_t w|^2 dxdt \le C \int_D |P_2 w|^2 dxdt + C\varepsilon \int_D s\lambda^2\varphi|\nabla w|^2 dxdt$$

for any $\varepsilon > 0$. Hence by (3.11) and (3.3), we have

$$\begin{aligned}
&3\int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dxdt - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w) dxdt \\
&+\varepsilon \int_D \frac{1}{s\varphi}|\partial_t w|^2 dxdt \\
&\le C\int_D f^2 e^{2s\varphi} dxdt + C\int_D s\lambda\varphi|\nabla w|^2 dxdt + C\varepsilon \int_D s\lambda^2\varphi|\nabla w|^2 dxdt \\
&+C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2) w^2 dxdt + C\int_D |\nabla w||\partial_t w| dxdt.
\end{aligned}$$

Now we note that the factor with the maximal order in s, λ, φ of w^2 is $s^3\lambda^4\varphi^3\sigma^2$, the maximal factor of $|\nabla w|^2$ is $s\lambda^2\varphi\sigma$, and the maximal order of $|\partial_t w|^2$ is $\frac{1}{s\varphi}$. For example, since we can choose s, λ large, the term $(s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2)w^2$ is of lower order.

Here, since the Cauchy-Schwarz inequality implies

$$\begin{aligned}
&|\partial_t w||\nabla w| = s^{-\frac{1}{2}}\varphi^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}|\partial_t w| s^{\frac{1}{2}}\varphi^{\frac{1}{2}}\lambda^{\frac{1}{2}}|\nabla w| \\
&\le \frac{1}{2}\frac{1}{s\lambda\varphi}|\partial_t w|^2 + \frac{1}{2}s\lambda\varphi|\nabla w|^2,
\end{aligned}$$

we have

$$\begin{aligned}
&3\int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dxdt - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w) dxdt \\
&+\left(\varepsilon - \frac{C}{\lambda}\right)\int_D \frac{1}{s\varphi}|\partial_t w|^2 dxdt \\
&\le C\int_D f^2 e^{2s\varphi} dxdt + C\int_D s\lambda\varphi|\nabla w|^2 dxdt + C\varepsilon \int_D s\lambda^2\varphi|\nabla w|^2 dxdt \\
&+C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2) w^2 dxdt.
\end{aligned} \tag{3.12}$$

The first and the second terms on the left-hand side have different signs and so we need another estimate. Thus we will execute another estimation for

$$\int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dxdt$$

by means of

$$\int_D (P_1 w + P_2 w) \times (s\lambda^2\varphi\sigma w)dxdt.$$

Here we have chosen the factor $s\lambda^2\varphi\sigma w$ for obtaining the term of $|\nabla w|^2$ with desirable (s, λ, φ)-factor $s\lambda^2\varphi$. That is, multiplying

$$\partial_t w + 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w) - \sum_{i,j=1}^n a_{ij}\partial_i\partial_j w - s^2\lambda^2\varphi^2\sigma w + A_1 w = fe^{s\varphi}$$

with $s\lambda^2\varphi\sigma w$, we have

$$\begin{aligned}
&\int_D (\partial_t w)(s\lambda^2\varphi\sigma w)dxdt + \int_D 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)s\lambda^2\varphi\sigma w dxdt \\
&- \int_D \left(\sum_{i,j=1}^n a_{ij}\partial_i\partial_j w\right) s\lambda^2\varphi\sigma w dxdt - \int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dxdt \\
&+ \int_D (A_1 w)(s\lambda^2\varphi\sigma w)dxdt \\
&\equiv \sum_{k=1}^5 I_k = \int_D fe^{s\varphi}s\lambda^2\varphi\sigma w dxdt. \qquad (3.13)
\end{aligned}$$

Now, in terms of the integration by parts and $w \in C_0^2(D)$, noting that $|\partial_t\varphi| = |\lambda(\partial_t\psi)\varphi| \le C\lambda\varphi$ and $\partial_i\varphi = \lambda(\partial_i d)\varphi$, etc., we estimate the terms.

$$|I_1| = \left|\int_D \frac{1}{2}s\lambda^2\varphi\sigma\partial_t(w^2)dxdt\right| \le C\int_D s\lambda^3\varphi w^2 dxdt. \qquad (3.14)$$

$$\begin{aligned}
|I_2| &= \left|\int_D s^2\lambda^3\varphi^2\sigma \sum_{i,j=1}^n a_{ij}(\partial_i d)\partial_j(w^2)dxdt\right| \\
&= \left| - \int_D \sum_{i,j=1}^n s^2\lambda^3\{2\lambda(\partial_j d)\varphi^2\}\sigma a_{ij}(\partial_i d)w^2 dxdt \right. \\
&\left. - \sum_{i,j=1}^n s^2\lambda^3\varphi^2\partial_j(\sigma a_{ij}(\partial_i d))w^2 dxdt \right| \\
&\le C\int_D s^2\lambda^4\varphi^2 w^2 dxdt. \qquad (3.15)
\end{aligned}$$

$$
\begin{aligned}
I_3 &= -\int_D s\lambda^2 \sum_{i,j=1}^n \varphi\sigma a_{ij} w(\partial_i\partial_j w)dxdt \\
&= \int_D s\lambda^2 \sum_{i,j=1}^n \varphi\sigma a_{ij}(\partial_i w)(\partial_j w)dxdt + \int_D s\lambda^2 \sum_{i,j=1}^n \partial_i(\varphi\sigma a_{ij}) w(\partial_j w)dxdt \\
&\ge \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dxdt - C\int_D s\lambda^3\varphi|\nabla w||w|dxdt.
\end{aligned} \tag{3.16}
$$

$$
I_4 = -\int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dxdt. \tag{3.17}
$$

$$
|I_5| \le C\left|\int_D s\lambda^2\varphi \times s\lambda^2\varphi\sigma w^2 dxdt\right| \le C\int_D s^2\lambda^4\varphi^2 w^2 dxdt. \tag{3.18}
$$

Hence, by (3.13) - (3.18), we obtain

$$
\begin{aligned}
&\int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dxdt - \int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dxdt \\
\le& C\int_D |fe^{s\varphi} s\lambda^2\varphi\sigma w|dxdt + C\int_D s^2\lambda^4\varphi^2 w^2 dxdt + C\int_D s\lambda^3\varphi|\nabla w||w|dxdt \\
\le& C\int_D f^2 e^{2s\varphi}dxdt + C\int_D s^2\lambda^4\varphi^2 w^2 dxdt + C\int_D \lambda^2|\nabla w|^2 dxdt.
\end{aligned} \tag{3.19}
$$

At the last inequality, we argue as follows: By

$$
s\lambda^3\varphi|\nabla w||w| = (s\lambda^2\varphi|w|)(\lambda|\nabla w|) \le \frac{1}{2}s^2\lambda^4\varphi^2 w^2 + \frac{1}{2}\lambda^2|\nabla w|^2,
$$

we have

$$
\int_D s\lambda^3\varphi|\nabla w||w|dxdt \le \frac{1}{2}\int_D (s^2\lambda^4\varphi^2 w^2 + \lambda^2|\nabla w|^2)dxdt.
$$

Furthermore

$$
\begin{aligned}
&|fe^{s\varphi} s\lambda^2\varphi\sigma w| \\
\le& \frac{1}{2}f^2 e^{2s\varphi} + \frac{1}{2}s^2\lambda^4\varphi^2\sigma^2 w^2 \le \frac{1}{2}f^2 e^{2s\varphi} + Cs^2\lambda^4\varphi^2 w^2.
\end{aligned}
$$

Finally we consider $2\times(3.19)+(3.12)$. Using (1.2) and $\sigma_0 \equiv \inf_{(x,t)\in Q} \sigma(x,t) > 0$, we obtain

$$\begin{aligned}
&\int_D s^3\lambda^4\varphi^3\sigma_0^2 w^2 dxdt + (\sigma_0\sigma_1 - C\varepsilon)\int_D s\lambda^2\varphi|\nabla w|^2 dxdt \\
&+\left(\varepsilon - \frac{C}{\lambda}\right)\int_D \frac{1}{s\varphi}|\partial_t w|^2 dxdt \\
\le& C\int_D f^2 e^{2s\varphi} dxdt \\
&+C\int_D (s\lambda\varphi + \lambda^2)|\nabla w|^2 dxdt + C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2) w^2 dxdt.
\end{aligned} \tag{3.20}$$

Therefore, first choosing $\varepsilon > 0$ sufficiently small such that $\sigma_0\sigma_1 - C\varepsilon > 0$ and then taking $\lambda > 0$ sufficiently large such that $\varepsilon - \frac{C}{\lambda} > 0$, we can absorb the second and the third terms on the right-hand side of (3.20) into the left-hand side and we obtain

$$\begin{aligned}
&\int_D s^3\lambda^4\varphi^3 w^2 dxdt + \int_D s\lambda^2\varphi|\nabla w|^2 dxdt + \int_D \frac{1}{s\varphi}|\partial_t w|^2 dxdt \\
\le& C\int_D f^2 e^{2s\varphi} dxdt.
\end{aligned} \tag{3.21}$$

Noting $w = ue^{s\varphi}$, we have

$$\begin{aligned}
&\int_D \left(\frac{1}{s\varphi}|\partial_t u|^2 + s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3 u^2\right) e^{2s\varphi} dxdt \\
\le& C\int_D f^2 e^{2s\varphi} dxdt.
\end{aligned} \tag{3.22}$$

Moreover we assume that for $t \in [0,T]$, the boundary of the domain $D\cap\{t\} \subset \mathbb{R}^n$ is composed of a finite number of smooth surfaces. Then we can include the terms of $\partial_i\partial_j u$ in the Carleman estimate by means of the a priori estimate for an elliptic equation as follows. By the representation of P and $|A_1(x,t)| \le Cs\lambda^2\varphi$, we have

$$\begin{aligned}
&\left|\sum_{i,j=1}^n a_{ij}\partial_i\partial_j w\right|^2 \\
\le& C(|\partial_t w|^2 + s^2\lambda^2\varphi^2|\nabla w|^2 + s^4\lambda^4\varphi^4 w^2) \quad \text{in } Q.
\end{aligned}$$

Hence, by (3.21),

$$\int_D \frac{1}{s\varphi}\left|\sum_{i,j=1}^n a_{ij}\partial_i\partial_j w\right|^2 dxdt$$

(3.23)

$$\le C\int_D \left(\frac{1}{s\varphi}|\partial_t w|^2 + s\lambda^2\varphi|\nabla w|^2 + s^3\lambda^4\varphi^3 w^2\right) dxdt \le C\int_D f^2 e^{2s\varphi} dxdt$$

for all large $s > 0$ and $\lambda > 0$.

Moreover we have

$$\partial_i\partial_j\left(\frac{w}{\sqrt{\varphi}}\right) = \frac{\partial_i\partial_j w}{\sqrt{\varphi}} - \frac{\partial_i\partial_j\varphi}{2\varphi^{\frac{3}{2}}}w$$

(3.24)

$$-\frac{1}{2\varphi^{\frac{3}{2}}}\{(\partial_j w)(\partial_i\varphi) + (\partial_i w)(\partial_j\varphi)\} + \frac{3}{4\varphi^{\frac{5}{2}}}(\partial_i\varphi)(\partial_j\varphi)w, \quad 1 \le i, j \le n,$$

and

$$\sum_{i,j=1}^n a_{ij}\partial_i\partial_j\left(\frac{w}{\sqrt{\varphi}}\right)$$

$$=\frac{g}{\sqrt{\varphi}} - \frac{\sum_{i,j=1}^n a_{ij}\partial_i\partial_j\varphi}{2\varphi^{\frac{3}{2}}}w + \frac{3}{4\varphi^{\frac{5}{2}}}w\sum_{i,j=1}^n a_{ij}(\partial_i\varphi)(\partial_j\varphi) - \frac{1}{\varphi^{\frac{3}{2}}}\sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j\varphi)$$

where we set $g = \sum_{i,j=1}^n a_{ij}\partial_i\partial_j w$. Since $w(\cdot, t) \in H_0^1(D\cap\{t\})$ for all $t \in [0, T]$, we apply a usual a priori estimate for the Dirichlet problem for the elliptic equation (e.g., Gilbarg and Trudinger [60]), so that

$$\int_{D\cap\{t\}}\sum_{i,j=1}^n \left|\partial_i\partial_j\left(\frac{w}{\sqrt{\varphi}}\right)\right|^2 (x,t)dx$$

$$\le C\int_{D\cap\{t\}}\frac{g(x,t)^2}{\varphi}dx + C\int_{D\cap\{t\}}\frac{\left|\sum_{i,j=1}^n a_{ij}\partial_i\partial_j\varphi\right|^2}{\varphi^3}|w(x,t)|^2 dx$$

$$+C\int_{D\cap\{t\}}\frac{w(x,t)^2}{\varphi^5}\left|\sum_{i,j=1}^n a_{ij}(\partial_i\varphi)(\partial_j\varphi)\right|^2 dx$$

(3.25)

$$+C\int_{D\cap\{t\}}\frac{1}{\varphi^3}\left|\sum_{i,j=1}^n a_{ij}(\partial_i w)\partial_j\varphi\right|^2 dx.$$

On the other hand, (3.24) yields

$$
\begin{aligned}
&\int_{D\cap\{t\}} \frac{1}{\varphi}|\partial_i\partial_j w(x,t)|^2 dx\\
\le &C\int_{D\cap\{t\}} \Biggl\{\left|\partial_i\partial_j\left(\frac{w}{\sqrt{\varphi}}\right)\right|^2 + \frac{|\partial_i\partial_j\varphi|^2}{\varphi^3}w^2 \\
&+\frac{1}{\varphi^3}(|\partial_j w|^2|\partial_i\varphi|^2 + |\partial_i w|^2|\partial_j\varphi|^2) + \frac{1}{\varphi^5}|\partial_i\varphi|^2|\partial_j\varphi|^2 w^2\Biggr\}(x,t)dx. \qquad (3.26)
\end{aligned}
$$

Since $\partial_i\varphi = \lambda(\partial_i d)\varphi$ and $\partial_i\partial_j\varphi = \lambda(\partial_i\partial_j d)\varphi + \lambda^2(\partial_i d)(\partial_j d)\varphi$, we see by $\lambda > 1$ that

$$
\begin{aligned}
&|\partial_i\varphi(x,t)| \le C\lambda\varphi(x,t),\\
&|\partial_i\partial_j\varphi(x,t)| \le C\lambda^2\varphi(x,t), \quad 1\le i,j\le n,\ (x,t)\in\overline{D}. \qquad (3.27)
\end{aligned}
$$

Hence, by $\varphi \ge 1$, estimates (3.25) and (3.26) yield

$$
\begin{aligned}
&\sum_{i,j=1}^n \int_{D\cap\{t\}} \frac{1}{\varphi(x,t)}|\partial_i\partial_j w(x,t)|^2 dx \le C\int_{D\cap\{t\}} \frac{g^2(x,t)}{\varphi(x,t)}dx\\
&+C\int_{D\cap\{t\}} (\lambda^4 w^2 + \lambda^2|\nabla w|^2)(x,t)dx.
\end{aligned}
$$

Integrating in t, we have

$$
\begin{aligned}
&\sum_{i,j=1}^n \int_D \frac{1}{s\varphi}|\partial_i\partial_j w(x,t)|^2 dxdt\\
\le &C\int_D \frac{1}{s\varphi}\left|\sum_{i,j=1}^n a_{ij}\partial_i\partial_j w\right|^2 dxdt + C\int_D (\lambda^4 w^2 + \lambda^2|\nabla w|^2)dxdt.
\end{aligned}
$$

With (3.21) and (3.23), we obtain

$$
\int_D \frac{1}{s\varphi}\sum_{i,j=1}^n |\partial_i\partial_j w|^2 dxdt \le C\int_D f^2 e^{2s\varphi}dxdt
$$

for all large $s > 0$ and $\lambda > 0$. Thus we can complete the derivation of a Carleman estimate for $u \in C_0^\infty(D)$. Noting that

$$
\begin{aligned}
&e^{s\varphi}\partial_i\partial_j u = \partial_i\partial_j w - s\lambda\varphi((\partial_i d)(\partial_j w) + (\partial_j d)(\partial_i w))\\
&+\{s^2\lambda^2\varphi^2(\partial_i d)(\partial_j d) - s\lambda^2\varphi(\partial_i d)(\partial_j d) - s\lambda\varphi(\partial_i\partial_j d)\}w,
\end{aligned}
$$

and writing the estimate in terms of u, we state the Carleman estimate here as a theorem.

Theorem 3.1. *Let $d \in C^2(\overline{\Omega})$ satisfy $|\nabla d| \neq 0$ on $\overline{\Omega}$ and let*

$$\psi(x,t) = d(x) - \beta(t-t_0)^2$$

with $\beta > 0$ and $0 < t_0 < T$. We assume that ∂D is smooth and for $t \in [0,T]$, the boundary of the domain $D \cap \{t\} \subset \mathbb{R}^n$ is composed of a finite number of smooth surfaces. There exists a constant $\lambda_0 > 0$ such that for arbitrary $\lambda \geq \lambda_0$, we can choose a constant $s_0(\lambda) > 0$ satisfying: there exists a constant $C = C(s_0, \lambda_0) > 0$ such that

$$\begin{aligned} &\int_D \left\{ \frac{1}{s\varphi}\left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right) + s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3 u^2 \right\} e^{2s\varphi} dxdt \\ &\leq C \int_D |Lu|^2 e^{2s\varphi} dxdt \end{aligned} \tag{3.28}$$

for all $s > s_0$ and all u satisfying

$$u \in H^{2,1}(Q), \quad supp\, u \in D. \tag{3.29}$$

The constant $C > 0$ in (3.28) depends continuously on $\max_{1\leq i,j\leq n} \|a_{ij}\|_{C^1(\overline{Q})}$, $\|b_i\|_{L^\infty(Q)}$, $\|c\|_{L^\infty(Q)}$. This dependency holds also in Theorems 3.2 and 3.3.

Here we note:

(1) By a usual density argument (i.e., the approximation of any u satisfying (3.29) by a sequence $u_n \in C_0^\infty(D)$), we can transfer (3.28) for $u_n \in C_0^\infty(D)$ to the Carleman estimate for all u satisfying (3.29).

(2) In the case of $\widetilde{\psi}(x,t) = d(x) - \beta(t-t_0)^2 + c_0$ where $\inf_{(x,t)\in Q} \widetilde{\psi}(x,t) > 0$, the proof is already completed. In the case of $c_0 = 0$, we have $e^{2s\varphi} = e^{2se^{\lambda\psi}} =$

$\exp(2(se^{-\lambda c_0})e^{\lambda\widetilde{\psi}})$, so that by replacing $s_0(\lambda)$ by $s_0(\lambda)e^{\lambda c_0}$, we can reduce the case of $c_0 = 0$ to the previous case.

Thus Theorem 3.1 follows.

Even if supp $u \subset D$ does not hold, we can follow the previous argument without omitting boundary integral terms which are produced by each integration by parts, and we can prove the following Carleman estimate:

Theorem 3.2. *Let $d \in C^2(\overline{\Omega})$ satisfy $|\nabla d| \ne 0$ on $\overline{\Omega}$ and let $\psi(x,t) = d(x) - \beta(t-t_0)^2$. We assume that ∂D is smooth and for $t \in [0,T]$, the boundary of the domain $D \cap \{t\} \subset \mathbb{R}^n$ is composed of a finite number of smooth surfaces. There exists a constant $\lambda_0 > 0$ such that for arbitrary $\lambda \ge \lambda_0$, we can choose a constant $s_0(\lambda) > 0$ satisfying: there exists a constant $C = C(s_0, \lambda_0) > 0$ such that*

$$\begin{aligned}&\int_D \left\{ \frac{1}{s\varphi}\left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i\partial_j u|^2 \right) + s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3 u^2 \right\} e^{2s\varphi} dxdt \\ \le & C\int_D |Lu|^2 e^{2s\varphi} dxdt \\ &+ Ce^{C(\lambda)s}\int_{\partial D} (|\widetilde{\nabla}_{x,t}u|^2 + |u|^2) dSdt \end{aligned} \tag{3.30}$$

for all $s > s_0$ and all $u \in H^{2,1}(D)$.

See Notations in section 1 as for the definition of $|\widetilde{\nabla}_{x,t}u|^2$.

Thanks to the large parameters $\lambda > 0$ and $s > 0$, we can derive a Carleman estimate for a weakly coupled parabolic system whose principal parts are not coupled: Let $\mathbf{u} = (u_1, ..., u_N)^T$ and $a^k_{ij} \in C^1(\overline{Q})$, $1 \le i,j \le n$, $1 \le k \le N$, satisfy (1.1) and (1.2), and $b^{k\ell}_i, c^{k\ell} \in L^\infty(Q)$, $1 \le i \le n$, $1 \le k,\ell \le N$. We set

$$\begin{aligned}[A\mathbf{u}]_\ell = & \sum_{i,j=1}^n a^\ell_{ij}\partial_i\partial_j u_\ell \\ & - \sum_{k=1}^N\sum_{i=1}^n b^{k\ell}_i \partial_i u_k - \sum_{k=1}^N c^{k\ell}u_k, \quad 1 \le \ell \le N. \end{aligned} \tag{3.31}$$

Then

Theorem 3.3. *Let* $d \in C^2(\overline{\Omega})$ *satisfy* $|\nabla d| \neq 0$ *on* $\overline{\Omega}$ *and let* $\psi(x,t) = d(x) - \beta(t - t_0)^2$*. We assume that* ∂D *is smooth and for* $t \in [0,T]$*, the boundary of the domain* $D \cap \{t\} \subset \mathbb{R}^n$ *is composed of a finite number of smooth surfaces. There exists a constant* $\lambda_0 > 0$ *such that for arbitrary* $\lambda \geq \lambda_0$*, we can choose a constant* $s_0(\lambda) > 0$ *satisfying: there exists a constant* $C = C(s_0, \lambda_0) > 0$ *such that*

$$\begin{aligned}&\int_D \left\{ \frac{1}{s\varphi}\left(|\partial_t \mathbf{u}|^2 + \sum_{i,j=1}^n |\partial_i\partial_j \mathbf{u}|^2 \right) + s\lambda^2\varphi|\nabla \mathbf{u}|^2 + s^3\lambda^4\varphi^3|\mathbf{u}^2| \right\} e^{2s\varphi} dxdt \\ &\le C \int_D |\partial_t \mathbf{u} - A\mathbf{u}|^2 e^{2s\varphi} dxdt \end{aligned} \tag{3.32}$$

for all $s > s_0$ *and all* u *satisfying*

$$\mathbf{u} \in H^{2,1}(Q)^N, \quad \mathit{supp}\, \mathbf{u} \in D.$$

§4. Global Carleman estimate.

In sections 2 and 3, we prove Carleman estimates in a domain D which is not necessarily same as $\Omega \times (0,T)$. Moreover for applications to inverse problems (see sections 5.1 and 6.1), D is given by φ. In other words, when we apply the Carleman estimate in section 3, first we have to choose φ and then D is determined where the results concerning the inverse problems are valid, and not vice versa. Imanuvilov [68] proved a global Carleman estimate which holds over $\Omega \times (0,T)$ for functions without compact supports. See also Chae, Imanuvilov and Kim [32], Fursikov and Imanuvilov [58], Imanuvilov [67]. The global Carleman estimate is very useful for proving an observability inequality which yields the null exact controllability and a stability estimate in determinining coefficients over Ω. In this section, we present the global Carleman estimate by Imanuvilov.

We recall that

$$H^{2,1}(Q) = \{u \in L^2(Q);\, \partial_t u, \partial_i u, \partial_i\partial_j u \in L^2(Q),\, 1 \le i,j \le n\}.$$

We consider a boundary value problem for a parabolic operator:

$$Lu(x,t) \equiv \partial_t u - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j u$$

$$- \sum_{i=1}^n b_i(x,t)\partial_i u - c(x,t)u = f \quad \text{in } Q \tag{4.1}$$

$$l_1(x)\frac{\partial u}{\partial \nu_A} + l_2(x)u = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{4.2}$$

Here we define the conormal derivative with respect to a_{ij} by $\frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^n a_{ij}(\partial_i u)\nu_j$. We assume (1.2) and

$$a_{ij} \in C^1(\overline{Q}), \quad a_{ij} = a_{ji}, \quad b_i, c \in L^\infty(Q), \quad 1 \le i,j \le n.$$

Suppose that $l_1, l_2 \in C^2(\partial\Omega)$ and

$$\text{either } l_1 > 0 \quad \text{or} \quad l_1 = 0 \text{ and } l_2 = 1 \quad \text{on } \partial\Omega. \tag{4.3}$$

For the global Carleman estimate, we need a special weight function. The existence of such a function is proved in [58], [68], [73]. Let $\omega \subset \Omega$ be an arbitrary subdomain.

Lemma 4.1. *Let ω_0 be an arbitrarily fixed subdomain of Ω such that $\overline{\omega_0} \subset \omega$. Then there exists a fucntion $d \in C^2(\overline{\Omega})$ such that*

$$d(x) > 0 \quad x \in \Omega, \quad d|_{\partial\Omega} = 0, \quad |\nabla d(x)| > 0, \quad x \in \overline{\Omega \setminus \omega_0}. \tag{4.4}$$

Example. Let $\Omega = \{x;\, |x| < 1\}$ and let $0 \in \omega_0$. Then $d(x) = 1 - |x|^2$ satisfies (4.4).

We set

$$\varphi(x,t) = \frac{e^{\lambda d(x)}}{t(T-t)}, \quad \alpha(x,t) = \frac{e^{\lambda d(x)} - e^{2\lambda\|d\|_{C(\overline{\Omega})}}}{t(T-t)}, \tag{4.5}$$

where $\lambda > 0$. Moreover we set

$$\sigma_2 = \sum_{i,j=1}^{n} \|a_{ij}\|_{C^1(\overline{Q})} + \sum_{i=1}^{n} \|b_i\|_{L^\infty(Q)} + \|c\|_{L^\infty(Q)}, \quad \sigma_3 = \sum_{i,j=1}^{n} \|a_{ij}\|_{C^1(\overline{Q})}. \tag{4.6}$$

The following is the global Carleman estimate:

Theorem 4.1. *There exists a number $\lambda_0 > 0$ such that for an arbitrary $\lambda \ge \lambda_0$, we can choose a constant $s_0(\lambda) \ge 0$ satisfying: there exists a constant $C = C(s_0, \lambda_0) > 0$ such that*

$$\begin{aligned}&\int_Q \left\{ \frac{1}{s\varphi}\left(|\partial_t u|^2 + \sum_{i,j=1}^{n} |\partial_i \partial_j u|^2 \right) + s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3|u|^2 \right\} e^{2s\alpha} dxdt \\ \le & C\int_Q |Lu|^2 e^{2s\alpha} dxdt + C\int_{\omega\times(0,T)} s^3\lambda^4\varphi^3 u^2 e^{2s\alpha} dxdt\end{aligned}$$

for all $s > s_0$ and all $u \in H^{2,1}(Q)$ satisfying (4.2). Here the constant $C > 0$ depends continuously on σ_2, λ_0, but independent of s and the constant λ_0 depends continuously on σ_3.

By (4.5), we note that

$$\lim_{t\downarrow 0} \varphi(x,t)^k e^{2s\alpha(x,t)} = \lim_{t\uparrow T} \varphi(x,t)^k e^{2s\alpha(x,t)} = 0$$

for any $k \ge 0$ and near $t = 0, T$, the weight function $e^{2s\alpha(x,t)}$ behaves with the same order of $\exp\left(-\frac{C}{t(T-t)}\right)$ with $C > 0$. Thanks to the this decay of the weight function, we need not assume that u vanishes at $t = 0, T$. Moreover for the Carleman estimate, it is sufficient that u satisfies one boundary condition (4.2) on the lateral boundary $\partial\Omega \times (0,T)$. Therefore this Carleman estimate holds over $\Omega \times (0,T)$ for u without compact supports.

For the case where we are given overdetermining data on an arbitrary part $\Gamma \subset \partial\Omega$, we need another weight function and see [68], [74].

Lemma 4.2. *Let $\Gamma \neq \emptyset \subset \partial\Omega$ be an arbitrary relatively open subset. Then there exists a function $d_0 \in C^2(\overline{\Omega})$ such that*

$$\begin{aligned} & d_0(x) > 0, \quad x \in \Omega, \qquad |\nabla d_0(x)| > 0, \quad x \in \overline{\Omega}, \\ & \sum_{i,j=1}^{n} a_{ij}(x,t)(\partial_i d_0)(x)\nu_j(x) \le 0, \quad x \in \partial\Omega \setminus \Gamma,\ 0 < t < T \end{aligned} \tag{4.7}$$

if $a_{ij} \in C^1(\overline{Q})$, $1 \le i,j \le n$ satisfy (1.1) and (1,2).

We note that d_0 satisfies (4.7) for all a_{ij} satisfying (1.1) and (1.2). In other words, d_0 does not depend on choices of a_{ij}.

We set

$$\varphi_0(x,t) = \frac{e^{\lambda d_0(x)}}{t(T-t)}, \quad \alpha_0(x,t) = \frac{e^{\lambda d_0(x)} - e^{2\lambda \|d_0\|_{C(\overline{\Omega})}}}{t(T-t)}. \tag{4.8}$$

Example. Let us consider a special case where $a_{ij} = 0$ if $i \neq j$ and $a_{ii} = 1$ and

$$\Omega = \{x \in \mathbb{R}^n;\ |x| < R\}, \qquad \Gamma = \{x \in \partial\Omega;\ (x - x_0, \nu(x)) \ge 0\}$$

with an arbitrarily fixed $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$. Here $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$. Then we can take $d_0(x) = |x - x_0|^2$.

By α, we have

Theorem 4.2. *There exists a number $\lambda_0 > 0$ such that for an arbitrary $\lambda \ge \lambda_0$, we can choose a constant $s_0(\lambda) \ge 0$ satisfying: there exists a constant $C = C(s_0, \lambda_0) > 0$ such that*

$$\begin{aligned} & \int_Q \left\{ \frac{1}{s\varphi_0} \left(|\partial_t u|^2 + \sum_{i,j=1}^{n} |\partial_i \partial_j u|^2 \right) + s\lambda^2 \varphi_0 |\nabla u|^2 + s^3 \lambda^4 \varphi_0^3 |u|^2 \right\} e^{2s\alpha_0} dxdt \\ \le & C \int_Q |Lu|^2 e^{2s\alpha_0} dxdt + Ce^{C(\lambda)s} \int_{\Gamma \times (0,T)} (|\partial_t u|^2 + |\nabla u|^2 + |u|^2) dSdt \end{aligned}$$

for all $s > s_0$ and all $u \in H^{2,1}(Q)$ satisfying (4.2). Here the constant $C > 0$ depends continuously on σ_2, λ_0, but independent of s and the constant λ_0 depends continuously on σ_3.

Here on $\Gamma \times (0,T)$ we note that $|\nabla u|^2 = \left|\frac{\partial u}{\partial \nu}\right|^2 + \left|\frac{\partial u}{\partial \tau}\right|^2$ where $\frac{\partial u}{\partial \tau}$ is the orthogonal component of ∇u to $\frac{\partial u}{\partial \nu}$.

The proof is done on the basis of the decomposition of the operator $Pw = e^{s\alpha}L(e^{-s\alpha}w)$ into P_1 and P_2 defined by (3.1) and (3.2). In fact, the proof in section 3 is an imitation of the original proofs of Theorems 4.1 and 4.2.

We conclude this section with the corresponding Carleman estimates to Theorems 4.1 and 4.2 for a weakly coupled parabolic system:

$$\partial_t \mathbf{u} = A\mathbf{u} \quad \text{in } Q$$

with

$$l_1(x) \sum_{i,j=1}^{n} a_{ij}^{\ell}(\partial_i u_\ell)\nu_j + l_2(x)u_\ell = 0 \quad \text{on } \partial\Omega \times (0,T),\ 1 \le \ell \le N. \tag{4.9}$$

Here A is defined by (3.31) and let $\varphi_0, \alpha_0, \varphi$ and α be defined by (4.5) and (4.8). Then

Theorem 4.3. *There exists a number $\lambda_0 > 0$ such that for an arbitrary $\lambda \ge \lambda_0$, we can choose a constant $s_0(\lambda) \ge 0$ satisfying: there exists a constant $C = C(s_0, \lambda_0) > 0$ such that*

$$\begin{aligned}
&\int_Q \left\{ \frac{1}{s\varphi}\left(|\partial_t \mathbf{u}|^2 + \sum_{i,j=1}^{n} |\partial_i\partial_j \mathbf{u}|^2 \right) + s\lambda^2\varphi|\nabla \mathbf{u}|^2 + s^3\lambda^4\varphi^3|\mathbf{u}|^2 \right\} e^{2s\alpha} dxdt \\
\le & C\int_Q |\partial_t \mathbf{u} - A\mathbf{u}|^2 e^{2s\alpha} dxdt + C\int_{\omega\times(0,T)} s^3\lambda^4\varphi^3|\mathbf{u}|^2 e^{2s\alpha} dxdt
\end{aligned}$$

for all $s > s_0$ and all $\mathbf{u} \in H^{2,1}(Q)^N$ satisfying (4.9). Here the constant $C > 0$ depends continuously on σ_2, λ_0, but independent of s and the constant λ_0 depends continuously on σ_3.

Theorem 4.4. *There exists a number $\lambda_0 > 0$ such that for an arbitrary $\lambda \ge \lambda_0$, we can choose a constant $s_0(\lambda) \ge 0$ satisfying: there exists a constant $C = C(s_0, \lambda_0) > 0$ such that*

$$\int_Q \left\{ \frac{1}{s\varphi_0} \left(|\partial_t \mathbf{u}|^2 + \sum_{i,j=1}^n |\partial_i \partial_j \mathbf{u}|^2 \right) + s\lambda^2 \varphi_0 |\nabla \mathbf{u}|^2 + s^3\lambda^4\varphi_0^3 |\mathbf{u}|^2 \right\} e^{2s\alpha_0} dxdt$$
$$\le C \int_Q |\partial_t \mathbf{u} - A\mathbf{u}|^2 e^{2s\alpha_0} dxdt + Ce^{C(\lambda)s} \int_{\Gamma\times(0,T)} \left(|\partial_t \mathbf{u}|^2 + |\nabla \mathbf{u}|^2 + |\mathbf{u}|^2 \right) dSdt$$

for all $s > s_0$ and all $\mathbf{u} \in H^{2,1}(Q)^N$ satisfying (4.9). Here the constant $C > 0$ depends continuously on σ_2, λ_0, but independent of s and the constant λ_0 depends continuously on σ_3.

As for the derivation of a Carleman estimate, see also Choulli [33], section 3.2.

§5. Applications to the unique continuation and the observability inequality.

Originally the Carleman estimate has been invented for proving the uniqueness in a Cauchy problem for an ellitpic equation by Carleman [30] and as first application of the Carleman estimate in this secion, we will discuss the methodology for the uniqueness and the conditional stability for a parabolic equation by a local version of Carleman estimate: Theorem 3.2.

§5.1. Conditional stability for the Cauchy problem.

Let $\Gamma \subset \partial\Omega$ be an arbitrary subboundary of $\partial\Omega$. That is, let Γ contain a non-empty set $\{x \in \mathbb{R}^n;\ |x - x_0| < \rho\} \cap \partial\Omega$ with some $x_0 \in \mathbb{R}^n$ and $\rho > 0$. We consider a Cauchy problem for a parabolic equation:

$$(Lu)(x,t) \equiv \partial_t u - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j u$$
$$- \sum_{i=1}^n b_i(x,t)\partial_i u - c(x,t)u = f, \quad (x,t) \in Q, \tag{5.1}$$

$$u|_{\Gamma\times(0,T)} = g, \quad \frac{\partial u}{\partial \nu_A}|_{\Gamma\times(0,T)} = h, \tag{5.2}$$

where a_{ij}, $1 \le i,j \le n$ satisfy (1.1) and (1.2), and

$$b_i, c \in L^\infty(Q), \quad 1 \le i,j \le n. \tag{5.3}$$

Remark. As is seen in Theorem 7.4, we have an H^{-1} Carleman estimate if a_{ij} are Lipschitz continuous on $\overline{Q}$, and so for all the discussions in section 5, we can relax the regularity (1.1) to the Lipschitz continuous a_{ij} on $\overline{Q}$.

Cauchy problem. Determine u in some domain $D \subset Q$ by $u|_{\Gamma\times(0,T)}$ and $\frac{\partial u}{\partial \nu_A}|_{\Gamma\times(0,T)}$.

As for the uniqueness results, there are very many works. Therefore we will not list up comprehensively even though we restrict to parabolic equations. Mizohata [116] is one of early papers for the parabolic case, and we refer to John [89], Landis [105], Saut and Scheurer [124], Sogge [125]. See also the monographs Egorov [43], Hörmander [65], [66], Isakov [81], [86], Klibanov and Timonov [100], Zuilly [138], and a survey paper [133] by Vessella, and the references therein. In this subsection, we give accounts for methods for applying Carleman estimates for proving stability results in the Cauchy problem. One introduces a suitable cut-off function and extend Cauchy data in a suitable Sobolev space to reduce the problem to functions with compact supports and then one can apply a local Carleman estimate (e.g., Theorem 3.1). This argument is quite traditional and is valid for other types of partial differential equations, and see e.g., sections 3.2 and 3.3 in Isakov [86]. However the extension argument for gaining compact supports, breaks the best possibility of the regularity of Cauchy data. Therefore on the basis of Theorem 3.2 which is a Carleman estimate for functions without compact supports, we show other version. Moreover our choice of subdomain for the stability is more flexible.

Theorem 5.1 (conditional stability). *Let $M > 0$ be arbitrarily given. For any $\varepsilon > 0$ and an arbitrary bounded domain Ω_0 such that $\overline{\Omega}_0 \subset \Omega \cup \Gamma$, $\partial\Omega_0 \cap \partial\Omega$ is a non-empty open subset of $\partial\Omega$ and $\partial\Omega_0 \cap \partial\Omega \subsetneqq \Gamma$, there exist constants $C > 0$ and $\theta \in (0,1)$ such that*

$$\|u\|_{H^{2,1}(\Omega_0\times(\varepsilon,T-\varepsilon))} \le C\|u\|^{1-\theta}_{H^{1,0}(Q)}(\|f\|_{L^2(Q)} + \|g\|_{H^1(\Gamma\times(0,T))} + \|h\|_{L^2(\Gamma\times(0,T))})^\theta$$

$$+C(\|f\|_{L^2(Q)} + \|g\|_{H^1(\Gamma\times(0,T))} + \|h\|_{L^2(\Gamma\times(0,T))}).$$

In the theorem, we are given Cauchy data $u, \nabla u$ on $\Gamma \times (0,T)$, and we estimate u in an interior domain Ω_0 such that $\partial\Omega_0 \cap \partial\Omega \subsetneqq \Gamma$ over a time interval $(\varepsilon, T-\varepsilon)$. This is a kind of interior estimate and usually the interior estimate is of Hölder type.

Proof. We choose a bouned domain Ω_1 with smooth boundary such that

$$\overline{\Omega}_0 \subsetneqq \Omega_1, \quad \Gamma = \partial\Omega \cap \Omega_1.$$

We note that Ω_1 is not a subset of Ω and $\Omega_1 \setminus \overline{\Omega}$ contains some non-empty open set. Choosing $\overline{\omega}_0 \subset \Omega_1 \setminus \overline{\Omega}_0$, we apply Lemma 4.1 to obtain $d \in C^2(\overline{\Omega}_1)$ satisfying

$$d(x) > 0, \quad x \in \Omega_1, \quad d(x) = 0, \quad x \in \partial\Omega_1 \cap \overline{\Omega}, \quad |\nabla d(x)| > 0, \quad x \in \overline{\Omega}_1 \cap \overline{\Omega}.$$

Then we can choose a sufficiently small $\delta > 0$ such that

$$\{x \in \mathbb{R}^n;\, d(x) > 4\delta\} \cap \overline{\Omega} \supset \Omega_0. \tag{5.5}$$

Moreover we choose $\beta > 0$ sufficiently large, so that

$$\|d\|_{C(\overline{\Omega}_1)} - 4\delta + \beta\left(\frac{T}{2} - \varepsilon\right)^2 < \beta\frac{T^2}{4}. \tag{5.6}$$

We set $\varphi(x,t) = e^{\lambda\psi(x,t)}$ with parameter $\lambda > 0$, $\mu_k = \exp\left(\lambda\left(k\delta - \beta\left(\frac{T}{2} - \varepsilon\right)^2\right)\right)$, $k = 1, 2, 3, 4$, and

$$\psi(x,t) = d(x) - \beta\left(t - \frac{T}{2}\right)^2, \quad D = \{(x,t);\, x \in \overline{\Omega}, \quad \varphi(x,t) > \mu_4\}.$$

Then, by (5.6) we can verify that

$$\overline{\Omega} \times (0,T) \supset D \supset \Omega_0 \times (\varepsilon, T-\varepsilon). \tag{5.7}$$

In fact, by (5.5) and the definition of μ_4, it directly follows that $D \supset \Omega_0 \times (\varepsilon, T-\varepsilon)$. Next if $(x,t) \in D$, then

$$\|d\|_{C(\overline{\Omega}_1)} - \beta\left(t - \frac{T}{2}\right)^2 \ge d(x) - \beta\left(t - \frac{T}{2}\right)^2 > 4\delta - \beta\left(\frac{T}{2} - \varepsilon\right)^2,$$

which implies $0 < t < T$ by (5.6).

By (5.7) we have

$$\partial D = \Sigma_1 \cup \Sigma_2,$$

$$\Sigma_1 \subset \Gamma \times (0,T), \quad \Sigma_2 = \{(x,t);\, x \in \Omega,\, \varphi(x,t) = \mu_4\}. \tag{5.8}$$

We apply Theorem 3.2 in D with suitably fixed $\lambda > 0$ and $C > 0$ denotes generic constants depending on λ, but independent of s and a respective choice of g, h, u. For it, we need a cut-off function because we have no data on $\partial D \setminus (\Gamma \times (0,T))$. Let $\chi \in C^\infty(\mathbb{R}^{n+1})$ such that $0 \le \chi \le 1$ and

$$\chi(x,t) = \begin{cases} 1, & \varphi(x,t) > \mu_3, \\ 0, & \varphi(x,t) < \mu_2. \end{cases} \tag{5.9}$$

We set $v = \chi u$, and have

$$\begin{aligned} Lv = \chi f + \chi' u - 2\sum_{i,j=1}^n a_{ij}(\partial_i\chi)\partial_j u \\ - \left(\sum_{i,j=1}^n a_{ij}\partial_i\partial_j\chi\right) u - \left(\sum_{i=1}^n b_i\partial_i\chi\right) u \quad \text{in } D. \end{aligned}$$

By (5.8) and (5.9), we see that

$$v = |\nabla v| = 0 \qquad \text{on } \Sigma_2.$$

Hence Theorem 3.2 yields

$$
\begin{aligned}
&\int_D \left\{ \frac{1}{s}\left(|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v|^2 \right) + s|\nabla v|^2 + s^3|v|^2 \right\} e^{2s\varphi} dxdt \\
\le & C\int_D f^2 e^{2s\varphi} dxdt \\
&+ C\int_D \left| \chi' u - 2\sum_{i,j=1}^n a_{ij}(\partial_i\chi)\partial_j u - \left(\sum_{i,j=1}^n a_{ij}\partial_i\partial_j\chi \right) u \right. \\
&\left. - \left(\sum_{i=1}^n b_i\partial_i\chi \right) u \right|^2 e^{2s\varphi} dxdt \\
&+ Ce^{Cs}\int_{\Sigma_1} (|\partial_t v|^2 + |\nabla v|^2 + |v|^2) dSdt
\end{aligned} \tag{5.10}
$$

for all $s \ge s_0$. By (5.9), the second integral on the right-hand side does not vanish only if $\mu_2 \le \varphi(x,t) \le \mu_3$ and so

$$
\begin{aligned}
&\int_D \left| \chi' u - 2\sum_{i,j=1}^n a_{ij}(\partial_i\chi)\partial_j u - \left(\sum_{i,j=1}^n a_{ij}\partial_i\partial_j\chi \right) u - \left(\sum_{i=1}^n b_i\partial_i\chi \right) u \right| e^{2s\varphi} dxdt \\
\le & Ce^{2s\mu_3}\|u\|^2_{H^{1,0}(Q)}.
\end{aligned}
$$

By (5.5) and (5.6), we can directly verify that if $(x,t) \in \Omega_0 \times (\varepsilon, T-\varepsilon)$, then $\varphi(x,t) > \mu_4$. Therefore, noting (5.7), we see that

$$
\begin{aligned}
&[\text{the left-hand side of (5.10)}] \\
\ge & \int_\varepsilon^{T-\varepsilon}\int_{\Omega_0} \left\{ \frac{1}{s}\left(|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v|^2 \right) + s|\nabla v|^2 + s^3|v|^2 \right\} e^{2s\varphi} dxdt \\
\ge & e^{2s\mu_4}\int_\varepsilon^{T-\varepsilon}\int_{\Omega_0} \left\{ \frac{1}{s}\left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i\partial_j u|^2 \right) + s|\nabla u|^2 + s^3|u|^2 \right\} dxdt.
\end{aligned}
$$

Hence (5.10) yields

$$e^{2s\mu_4}\int_\varepsilon^{T-\varepsilon}\int_{\Omega_0}\left\{\frac{1}{s}\left(|\partial_t u|^2+\sum_{i,j=1}^n|\partial_i\partial_j u|^2\right)+s|\nabla u|^2+s^3|u|^2\right\}dxdt$$
$$\le C\int_D f^2e^{2s\varphi}dxdt+Ce^{2s\mu_3}\|u\|^2_{H^{1,0}(Q)}$$
$$+Ce^{Cs}\int_{\Gamma\times(0,T)}(|\partial_t u|^2+|\nabla u|^2+|u|^2)dSdt.$$

Setting

$$F=\|f\|_{L^2(Q)}+\|g\|_{H^1(\Gamma\times(0,T))}+\|h\|_{L^2(\Gamma\times(0,T))}, \tag{5.11}$$

we have

$$\int_\varepsilon^{T-\varepsilon}\int_{\Omega_0}\left\{|\partial_t u|^2+\sum_{i,j=1}^n|\partial_i\partial_j u|^2+|\nabla u|^2+|u|^2\right\}dxdt$$
$$\le Cse^{-2s(\mu_4-\mu_3)}\|u\|^2_{H^{1,0}(Q)}+Ce^{Cs}F^2$$

for all $s\ge s_0$. By $\sup_{s>0}se^{-s(\mu_4-\mu_3)}<\infty$, we estimate $se^{-2s(\mu_4-\mu_3)}$ by $e^{-s(\mu_4-\mu_3)}$ on the right-hand side. Moreover, replacing C by Ce^{Cs_0}, we can have

$$\|u\|^2_{H^{2,1}(\Omega_0\times(\varepsilon,T-\varepsilon))}\le Ce^{-s(\mu_4-\mu_3)}\|u\|^2_{H^{1,0}(Q)}+Ce^{Cs}F^2 \tag{5.12}$$

for all $s\ge 0$. First let $F=0$. Then letting $s\to\infty$ in (5.12), we see that $u=0$ in $\Omega_0\times(0,T)$, so that the conclusion of Theorem 5.1 holds true. Next let $F\ne 0$. First let $F\ge\|u\|_{H^{2,1}(Q)}$. Then (5.12) implies $\|u\|_{H^{2,1}(\Omega_0\times(0,T))}\le Ce^{Cs}F$ for $s\ge 0$, which already proves the theorem. Second let $F<\|u\|_{H^{2,1}(Q)}$. We choose $s>0$ minimizing the right-hand side of (5.12), that is,

$$e^{-s(\mu_4-\mu_3)}\|u\|^2_{H^{1,0}(Q)}=e^{Cs}F^2.$$

By $F\ne 0$, we can choose

$$s=\frac{2}{C+\mu_4-\mu_3}\log\frac{\|u\|_{H^{1,0}(Q)}}{F}>0.$$

Then (5.12) gives

$$\|u\|_{H^{2,1}(\Omega_0\times(0,T))} \le 2C\|u\|_{H^{1,0}(Q)}^{\frac{C}{C+\mu_4-\mu_3}} F^{\frac{\mu_4-\mu_3}{C+\mu_4-\mu_3}}.$$

The the proof of Theorem 5.1 is completed.

For better estimation, we used Theorem 3.2 for functions without compact supports. If we use a Carleman estimate for functions with compact supports, then we have to extend g and h to functions H in Q such that $H|_{\Gamma\times(0,T)} = g$ and $\frac{\partial H}{\partial \nu_A}|_{\Gamma\times(0,T)} = h$. Setting $\widetilde{u} = u - H$ and taking the cut-off function χ defined by (5.9), we can obtain a stability estimate for the Cauchy problem with stronger norms of data: $\|g\|_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))\cap H^1(0,T;L^2(\Gamma))}$ and $\|h\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))\cap H^1(0,T;L^2(\Gamma))}$ (see, e.g., Theorems 3.2.2, 3.2.3 and 3.3.10 in Isakov [86]). One can apply Theorem 3.2 locally with generous choice of φ. More precisely, one can apply Theorem 3.2 by dividing Ω_0 into a family of subdomains Ω^j, but as an a priori bound, $\|u\|_{H^{1,0}(Q)}$ is not sufficient. In that case, the choice of the weight function can be generous and Ω^j is defined by a level set $\{\varphi(x,t) > \delta\}$. However, for each step in a domain Q^j, we need to estimate $\|u\|_{H^{1,1}(\partial Q_j)}$ for estimating $\|u\|_{H^{2,1}(Q^j)}$ and so the repeat of estimation can not be continued without a priori bound by a stronger norm than $\|u\|_{H^{1,0}(Q)}$. For one-time estimation, we need a special weight function in Lemma 4.1.

Next we consider an estimate of $u(x_0,t_0)$ for $x_0 \in \partial\Omega \setminus \Gamma$ and $t_0 > 0$. An argument for obtaining an estimate on the boundary seems not popular. Here we consider a simple case for convenience. Let

$$\Omega = (0,2) \times \{x';\ |x'| < R\}$$

and let us be given data of u in $(1,2) \times \{x';\ |x'| < R\} \times (0,T)$. Then we would like to estimate $u\left(0, 0, \frac{T}{2}\right)$. We set

$$
\begin{cases}
D_T = \{(x', t);\ |x'| < R,\ 0 < t < T\}, \\
F = \sup_{1 \le x_1 \le 2} (\|u(x_1, \cdot, \cdot)\|_{H^1(D_T)} + \|\partial_1 u(x_1, \cdot, \cdot)\|_{L^2(D_T)}).
\end{cases}
\tag{5.13}
$$

Let $m \in \mathbb{N}$ satisfy

$$
m > \max\left\{\frac{n+1}{2}, 3\right\}.
$$

Then our estimate is

Theorem 5.2.

$$
\left|u\left(0, 0, \frac{T}{2}\right)\right| \le \frac{CM}{\left(\log \frac{1}{F}\right)^{\frac{m-3}{2(m+1)^2}}}.
$$

provided that $F < 1$ *and*

$$
\|u\|_{H^{m+2}(Q)} \le M.
$$

Since we are interested in the rate of the stability estimate as $F \to 0$, it is sufficient to consider the case of $F < 1$.

The estimate at a boundary point is of logarithmic rate and much weaker than the interior estimate given by Theorem 5.1, and we need an a priori bound of solutions in Sobolev space H^{m+2} of higher order. We note that F is not a norm of boundary data but by the method of Theorem 5.1, we can estimate F in terms of boundary data on $x_1 = 2$ after we enlarge the domain Q in x and t and assume the parabolic equation with Cauchy data for the extended domain. Moreover we can repeatedly apply the argument here to obtain an estimate for $u(0, x', t)$ for $|x'| < R - \varepsilon$ and $\varepsilon < t < T - \varepsilon$ with arbitrarily fixed $\varepsilon > 0$. For this kind of estimates on a subboundary for elliptic equations, see Eller and Yamamoto [46], Takeuchi and Yamamoto [126].

Proof. The proof consists of two steps:

(1) Hölder stability estimate in a subdomain with distance ρ from the subboundary $x_1 = 0$ whose Hölder exponent depends on ρ. See (5.23).

(2) integration of the Hölder estimates over the distance variable ρ.

The step (i) is a direct consequence of the Carleman estimate Theorem 3.2 in view of a suitable cut-off function.

We choose $\beta > 0$ and $\gamma > 0$ such that

$$\gamma R^2 > 1, \quad \frac{\beta T^2}{4} > 1. \tag{5.14}$$

We set

$$\psi(x,t) = x_1 - \gamma |x'|^2 - \beta \left(t - \frac{T}{2} \right)^2$$

and

$$Q(\xi) = \{(x,t);\, x_1 < 1 + \xi,\, \psi(x,t) > \xi\}.$$

Then, by (5.14), we see that $Q(\xi) \subset \Omega \times (0,T)$ for $\xi \in (0,1)$. For applying Theorem 3.2, for $\eta \in \left(0, \frac{1}{4}\right)$, we need a cut-off function $\chi_\eta \in C^\infty(\mathbb{R}^{n+1})$ satisfying $0 \le \chi_\eta \le 1$ and

$$\chi_\eta(x,t) = \begin{cases} 1, & (x,t) \in Q(3\eta), \\ 0, & (x,t) \in \mathbb{R}^{n+1} \setminus Q(2\eta). \end{cases} \tag{5.15}$$

Such χ_η can be constructed as follows. Let $\widetilde{\chi} \in C^\infty(\mathbb{R})$ satisfy $0 \le \widetilde{\chi} \le 1$ and $\widetilde{\chi}(\xi) = \begin{cases} 0, & \xi \le 0, \\ 1, & \xi \ge 1 \end{cases}$. By setting $\chi_\eta(x,t) = \widetilde{\chi}\left(\frac{\psi(x,t) - 2\eta}{\eta}\right)$, the condition (5.15) is satisfied.

Moreover

$$\begin{aligned} &|\nabla_{x,t}\chi_\eta(x,t)| \le \frac{C}{\eta}, \\ &|\partial_i\partial_j\chi_\eta(x,t)| \le \frac{C}{\eta^2}, \quad 1 \le i,j \le n,\ (x,t) \in \mathbb{R}^{n+1}. \end{aligned} \tag{5.16}$$

We set

$$v_\eta = \chi_\eta u.$$

Then

$$v_\eta = |\nabla v_\eta| = 0 \quad \text{on } \partial Q(\eta) \setminus \{(1+\eta, x', t); (x', t) \in D_T\}. \tag{5.17}$$

Direct calculations yield

$$\begin{aligned} Lv_\eta = &\chi_\eta' u - 2\sum_{i,j=1}^n a_{ij}(\partial_i \chi_\eta)\partial_j u \\ &- \left(\sum_{i,j=1}^n a_{ij}\partial_i\partial_j\chi_\eta\right) u - \left(\sum_{i=1}^n b_i \partial_i \chi_\eta\right) u \quad \text{in } Q(\eta). \end{aligned}$$

In terms of (5.16), fixing $\lambda > 0$ we can apply Theorem 3.2 to have

$$\begin{aligned} &\int_{Q(\eta)} \left(\frac{1}{s}|\partial_t v_\eta|^2 + s|\nabla v_\eta|^2 + s^3|v_\eta|^2\right) e^{2s\varphi} dxdt \\ \le &C\int_{Q(\eta)} \left| \chi_\eta' u - 2\sum_{i,j=1}^n a_{ij}(\partial_i\chi_\eta)\partial_j u - \left(\sum_{i,j=1}^n a_{ij}\partial_i\partial_j\chi_\eta\right) u \right. \\ &\left. - \left(\sum_{i=1}^n b_i\partial_i\chi_\eta\right) u \right|^2 e^{2s\varphi} dxdt \\ &+ Ce^{Cs}F^2 \end{aligned} \tag{5.18}$$

for all $s \ge s_0$. Here F is defined by (5.13). Here and henceforth $C > 0$ denotes generic constants which are dependent on λ but independent of s and a respective choice of u.

By (5.15) and (5.16), the first term on the right-hand side of (5.18) is bounded by

$$\frac{C}{\eta^4} e^{2s\mu_1} \|u\|^2_{H^{1,0}(Q)},$$

where we set $\mu_1 = e^{2\lambda\eta}$ and $\mu_2 = e^{3\lambda\eta}$. Since

$$\begin{aligned} &e^{2s\mu_2} \int_{Q(3\eta)\cap Q(\eta)} |u|^2 dxdt \le \int_{Q(\eta)} |v|^2 e^{2s\varphi} dxdt \\ \le &\frac{C}{\eta^4} e^{2s\mu_1} \|u\|^2_{H^{1,0}(Q)} + Ce^{Cs}F^2 \end{aligned}$$

for all $s \ge s_0$. Hence

$$\|u\|^2_{L^2(Q(3\eta)\cap Q(\eta))} \le \frac{C}{\eta^4} e^{-2s(\mu_2-\mu_1)} M^2 + Ce^{Cs}F^2 \tag{5.19}$$

for all $s \ge s_0$. Since $Q(3\eta) = (Q(3\eta) \cap Q(\eta)) \cup (Q(3\eta) \cap \{1+3\eta > x_1 \ge 1+\eta\})$ and $\|u\|_{L^2(Q(3\eta)\cap\{1+3\eta>x_1>1+\eta\})} \le F$ by (5.13) and $0 < \eta < \frac{1}{3}$, we obtain

$$\|u\|^2_{L^2(Q(3\eta))} \le \frac{C}{\eta^4} e^{-2s(\mu_2-\mu_1)} M^2 + Ce^{Cs}F^2 + CF^2$$
$$\le \frac{C_1}{\eta^4} e^{-2s(\mu_2-\mu_1)} M^2 + C_1 e^{C_1 s}F^2$$

for all $s \ge s_0$. Replacing C_2 by $C_1 e^{s_0 C_1}$, we have

$$\|u\|^2_{L^2(Q(3\eta))} \le \frac{C_2}{\eta^4} e^{-2s(\mu_2-\mu_1)} M^2 + C_2 e^{C_2 s}F^2 \tag{5.20}$$

for all $s \ge 0$.

Let first $F = 0$. Then Theorem 5.1 implies that $u\left(0, 0, \frac{T}{2}\right) = 0$, and so already the conclusion of the theorem is verified. Next we assume that $F \ne 0$. Without loss of generality, we can assume that $M > 1$. Since $\mu_2 - \mu_1 = e^{2\lambda\eta}(e^{\lambda\eta} - 1) \ge e^{\lambda\eta} - 1 \ge \lambda\eta$, in terms of (5.20), we have

$$\|u\|_{L^2(Q(3\eta))} \le \frac{C_2}{\eta^2} e^{-s\lambda\eta} M + C_2 e^{C_2 s}F \tag{5.21}$$

for all $s \ge 0$. We choose $s > 0$ minimizing the right-hand side of (5.21): $e^{-\lambda\eta s}M = e^{C_2 s}F$, that is,

$$s = \frac{1}{C_2 + \lambda\eta} \log \frac{M}{F} > 0.$$

Then

$$\|u\|_{L^2(Q(3\eta))} \le \frac{C_3}{\eta^2} M^{\frac{C_2}{C_2+\lambda\eta}} F^{\frac{\lambda\eta}{C_2+\lambda\eta}}, \tag{5.22}$$

where $C_3 > 0$ is independent of $\eta \in \left(0, \frac{1}{3}\right)$.

We apply the interpolation inequality and the Sobolev embedding in terms of $2m > n+1$ (e.g., [1]), we see that

$$\begin{aligned}&\|u\|_{L^\infty(Q(3\eta))} \le C_4\|u\|_{H^m(Q(3\eta))}\\ \le&C_5\|u\|^{\frac{m}{m+1}}_{H^{m+1}(Q(3\eta))}\|u\|^{\frac{1}{m+1}}_{L^2(Q(3\eta))}.\end{aligned}$$

Here, since $Q(3\eta)$ is congruent each other in $\eta \in \left(0, \frac{1}{3}\right)$ by translation in x_1, the constants C_4 and C_5 are independent of $\eta \in \left(0, \frac{1}{3}\right)$. Consequently noting $M > 1$, we have

$$\begin{aligned}&\sup_{(x,t)\in\overline{Q(3\eta)}} |u(x,t)| \le C_6 M^{\frac{m}{m+1}} \left(\eta^{-2} M^{\frac{C_2}{C_2+\lambda\eta}} F^{\frac{\lambda\eta}{C_2+\lambda\eta}}\right)^{\frac{1}{m+1}}\\ \le&C_7 M\eta^{-\frac{2}{m+1}} F^{\frac{\lambda\eta}{(m+1)(C_2+\lambda\eta)}}.\end{aligned}$$

Since $F < 1$, we see that

$$F^{\frac{\lambda\eta}{(m+1)(C_2+\lambda\eta)}} \le F^{C_8\eta}$$

with some constant $C_8 > 0$ which is independent of η. Hence

$$\sup_{(x,t)\in\overline{Q(3\eta)}} |u(x,t)| \le C_9 M\eta^{-\frac{2}{m+1}} F^{C_8\eta}.$$

Let $\xi' = (\xi_2, ..., \xi_n) \in \mathbb{R}^{n-1}$. In particular,

$$\left|u\left(3\eta + \gamma|\xi'|^2, \xi', \frac{T}{2}\right)\right| \le C_9 M\eta^{-\frac{2}{m+1}} F^{C_8\eta}, \quad |\xi'| < R,\ 0 < \eta < \frac{1}{3}. \tag{5.23}$$

In terms of the change of independent variables: $(x_1, x') \longrightarrow (\eta, \xi')$ defined by

$$x_1 = 3\eta + \gamma|\xi'|^2, \quad x' = \xi',$$

we have

$$\begin{aligned}&\int_{\gamma|x'|^2<x_1<1,|x'|<R} \left|u\left(x, \frac{T}{2}\right)\right|^2 dx\\ =&3\int_{|\xi'|<R}\left(\int_0^{\frac{1}{3}-\frac{\gamma}{3}|x'|^2} \left|u\left(3\eta + \gamma|\xi'|^2, \xi', \frac{T}{2}\right)\right|^2 d\eta\right) d\xi'.\end{aligned}$$

Therefore by (5.23) we obtain

$$\int_{\gamma|x'|^2<x_1<1,|x'|<R}\left|u\left(x,\frac{T}{2}\right)\right|^2 dx \le \frac{1}{3}C_9^2\int_0^{\frac{1}{3}}\eta^{-\frac{4}{m+1}}M^2F^{2C_8\eta}d\eta. \tag{5.24}$$

Here by noting that $F<1$ and $\frac{4}{m+1}<1$, we see that

$$\int_0^{\frac{1}{3}}\eta^{-\frac{4}{m+1}}F^{2C_8\eta}d\eta=\int_0^{\frac{1}{3}}\eta^{-\frac{4}{m+1}}e^{-2C_8\left(\log\frac{1}{F}\right)\eta}d\eta$$
$$\le\int_0^{\infty}\eta^{-\frac{4}{m+1}}e^{-C_{10}\left(\log\frac{1}{F}\right)\eta}d\eta=\Gamma\left(\frac{m-3}{m+1}\right)\left(C_{10}\log\frac{1}{F}\right)^{-\frac{m-3}{m+1}}.$$

Here $\Gamma(\cdot)$ is the gamma function. Consequently

$$\left\|u\left(\cdot,\frac{T}{2}\right)\right\|_{L^2\left(Q(0)\cap\left\{\frac{T}{2}\right\}\right)}\le C_{11}M\left(\log\frac{1}{F}\right)^{-\frac{m-3}{2(m+1)}}.$$

Applying the interpolation inequality in $Q(0)\cap\left\{\frac{T}{2}\right\}$, we obtain

$$\sup_{x\in Q(0)\cap\left\{\frac{T}{2}\right\}}\left|u\left(x,\frac{T}{2}\right)\right|^2\le C_{12}\left\|u\left(\cdot,\frac{T}{2}\right)\right\|_{H^m\left(Q(0)\cap\left\{\frac{T}{2}\right\}\right)}$$
$$\le C_{13}\left\|u\left(\cdot,\frac{T}{2}\right)\right\|^{\frac{m}{m+1}}_{H^{m+1}\left(Q(0)\cap\left\{\frac{T}{2}\right\}\right)}\left\|u\left(\cdot,\frac{T}{2}\right)\right\|^{\frac{1}{m+1}}_{L^2\left(Q(0)\cap\left\{\frac{T}{2}\right\}\right)}.$$

Since $(0,0)\in\overline{Q(0)\cap\left\{\frac{T}{2}\right\}}$ and

$$\left\|u\left(\cdot,\frac{T}{2}\right)\right\|_{H^{m+1}(\Omega)}\le C'_{13}\|u\|_{C([0,T];H^{m+1}(\Omega))}\le C_{13}\|u\|_{H^1(0,T;H^{m+1}(\Omega))}$$

by the Sobolev embedding, proof of the theorem is completed.

§5.2. Observability inequality.

In section 5.1, we consider Cauchy problems where we are not given boundary values on some part of the boundary $\partial\Omega$. In this subsection, assuming that we know the boundary condition on the whole lateral boundary $\partial\Omega\times(0,T)$, but not initial value, we discuss the estimation of solution by extra boundary data or interior data of solution.

We consider a boundary value problem for a parabolic operator:

$$Lu(x,t) \equiv= \partial_t u - \sum_{i,j=1}^{n} a_{ij}(x,t)\partial_i\partial_j u$$

$$- \sum_{i=1}^{n} b_i(x,t)\partial_i u - c(x,t)u = 0 \quad \text{in } Q \tag{5.25}$$

$$l_1(x)\frac{\partial u}{\partial \nu_A} + l_2(x)u = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{5.26}$$

We assume (1.2) and

$$a_{ij} \in C^1(\overline{Q}), \quad a_{ij} = a_{ji}, \quad b_i, c \in L^\infty(Q), \quad 1 \le i,j \le n.$$

Suppose that $l_1, l_2 \in C^2(\partial\Omega)$ and

$$\text{either } l_1 > 0 \quad \text{or} \quad l_1 = 0 \text{ and } l_2 = 1 \quad \text{on } \partial\Omega.$$

Estimation of solution.

Let $0 < t_0 < T$ be given, and $\Gamma \subset \partial\Omega$, $\omega \subset \Omega$ be a subdomain. Let u satisfy (5.25) and (5.26). Estimate $u(\cdot, t_0)$ in Ω by means of data of u on $\Gamma \times (0,T)$ or in $\omega \times (0,T)$.

The global Carleman estimate in section 4 provides answers. That is, by (4.5) and (4.8), we see in Theorems 4.1 and 4.2 that for any $\varepsilon > 0$,

$$\alpha_0(x,t), \alpha(x,t) \ge \frac{-c_0}{\varepsilon(T-\varepsilon)}, \quad \varepsilon < t < T - \varepsilon,$$

where $c_0 = \max_{x\in\overline{\Omega}}(e^{2\lambda\|d_0\|_{C(\overline{\Omega})}} - e^{\lambda d_0(x)})$ or $c_0 = \max_{x\in\overline{\Omega}}(e^{2\lambda\|d\|_{C(\overline{\Omega})}} - e^{\lambda d(x)})$. Therefore Theorem 4.1 and Theorem 4.2 imply

$$\|u\|_{H^1(\varepsilon,T-\varepsilon;L^2(\Omega))} \le C\|u\|_{L^2(\omega\times(0,T)} \tag{5.27}$$

and

$$\|u\|_{H^1(\varepsilon,T-\varepsilon;L^2(\Omega))} \le C\left(\|u\|_{H^1(\Gamma\times(0,T)} + \left\|\frac{\partial u}{\partial \nu_A}\right\|_{L^2(\Gamma\times(0,T))}\right) \tag{5.28}$$

respectively. These establish the Lipschitz stability in determining $u(\cdot, t_0)$, $t_0 > 0$. As for other types of results on the estimation of solutions, see section 8. We note that a Carleman estimate yields an energy estimate called an observability inequality for conservative systems such as hyperbolic equations and see Kazemi and Klibanov [91], Klibanov and Malinsky [98], Klibanov and Timonov [100], Tataru [127].

Finally we should mention the null exact controllability as very important application of the global Carleman estimate. More precisely, we consider

$$Ly(x,t) = q(x,t), \qquad (x,t) \in Q \tag{5.29}$$

with the same boundary condition (5.26) and

$$y(x,0) = y_0(x), \qquad x \in \Omega. \tag{5.30}$$

Null exact controllability.

Find $q \in L^2(Q)$ such that supp $q \subset \overline{\omega} \times (0,T)$ and $y(x,T) = 0$ for $x \in \Omega$.

This problem has been considered comprehensively for many years and Fattorini and Russell [52], Russell [122], [123] are early papers. We can discuss the null exact controllability also for semilinear parabolic equations, and the Navier-Stokes equations, the Burgers equation, the Boussinesq equation. Imanuvilov [67], [68] applied the global Carleman estimate to prove the null exact controllability: there exists such a control q and the norm is estimated by y_0. See also Lebeau and Robbiano [108] as an early paper.

Also for hyperbolic equations which we do not treat in this survey, there are many results on the exact controllability. See Ho [64], and Komornik [102], Lions [114] and the references therein. Since Imanuvilov [67], there have been substantial amounts of works for the null exact controllability for the parabolic equations on the basis of the Carleman estimates. As very limited references on the basis of the Carleman estimate, we refer to the following papers:

(1) Parabolic equations: Coron and Guerrero [36], Fernández-Cara and de Teresa [53], Fernández-Cara and Guerrero [54], Glass and Guerrero [61], González-Burgos and de Teresa [62], Rosier and Zhang [121].

(2) parabolic equations whose principal coefficients are discontinuous or of bounded variations: Benabdallah, Dermenjian and Le Rousseau [13] - [15], Doubova, Osses and Puel [41], Le Rousseau [110].

(3) the Navier-Stokes equations:Coron and Guerrero [37], Fabre and Lebeau [51], Fernández-Cara, Guerrero, Imanuvilov and Puel [55], Fursikov and Imanuvilov [58], Guerrero [63], Imanuvilov [69], [70],

§6. Applications to inverse problems for parabolic equations.

§6.1. Local Hölder stability for an inverse coefficient problem.

We consider

$$\partial_t u = \mathrm{div}\,(p(x)\nabla u(x,t)), \quad (x,t) \in Q. \tag{6.1}$$

Let $0 < t_0 < T$ be fixed arbitrarily and let $\Omega_0 \subset \Omega$ be a subdomain such that $\Gamma_0 \equiv \partial\Omega_0 \cap \partial\Omega \ne \emptyset$ is a subboundary (i.e., a relatively open subset of $\partial\Omega$).

Now we discuss

Inverse Coefficient Problem. Determine $p(x)$, $x \in \Omega$ from $u(\cdot, t_0)$ in Ω and $u|_{\Gamma_0\times(0,T)}$ and $\nabla u|_{\Gamma_0\times(0,T)}$.

Our subjects are the uniqueness and the stability for the inverse coefficient problem. This is an inverse problem with a finite number of measurements which requires a couple of a single input of initial value and boundary value (i.e., $u|_{\Gamma_0\times(0,T)}$)

and the measurements of the corresponding data of $u(\cdot, t_0)$ and $\nabla u|_{\Gamma_0\times(0,T)}$. On the other hand, we can consider other formulation with many boundary measurements which is based on the Dirichlet-to-Neumann map, and such inverse problems have been studied comprehensively. However the inverse problems by the Dirichlet-to-Neumann map are outside of the scope of this article and see e.g., a monograph by Isakov [86] as a reference book.

The inverse problems with a finite number of measurements have been solved firstly by Bukhgeim and Klibanov [20], whose methodology is composed of a Carleman estimate and an integral inequality by the maximality of the weight function at $t = t_0$ when we are given spatial data $u(\cdot, t_0)$. For a wide class of partial differential equations, their method is valid to yield the uniqueness in the inverse problem, provided that a suitable Carleman estimate is prepared. Since [20], there have been many works by their methodology: [74], [77], [78], [80], [81], [83], [92-95], [101] and see more references in section 8. As for the parabolic case, Isakov [81] is one of earlier works on the uniqueness, and Imanuvilov and Yamamoto [74] proved the Lipschitz stability which holds over the whole domain Ω. For the determination of $p(x)$ in a hyperbolic equation $\partial_t^2 u = \mathrm{div}\,(p(x)\nabla u)$, see [101].

In this section, we present a method on the basis of Carleman estimates for inverse coefficient problems with a finite number of measurements. Imanuvilov and Yamamoto [74], [77], [78] modified the method in [20] and proved the Lipschitz stability for an inverse coefficient problem for a hyperbolic equation. We modify the method in [77] to discuss an inverse parabolic problem. We will apply two versions of Carleman estimates shown in sections 3 and 4, and the local Carleman estimate in section 3 yields the local estimate of Hölder type in determining a coefficient, while the global Carleman estimate in section 4 produces a global Lipschitz estimate over Ω.

Let u satisfy (6.1) and v satisfy

$$\partial_t v = \mathrm{div}\,(q(x)\nabla u(x,t)), \quad (x,t) \in Q. \tag{6.2}$$

We define an admissible set of unknown coefficients p, q by

$$\mathcal{A} = \{p \in C^2(\overline{\Omega});\, p > 0 \,\mathrm{on}\, \overline{\Omega}, \quad \|p\|_{C^2(\overline{\Omega})} \le M\}, \tag{6.3}$$

where $M > 0$ is an arbitrarily fixed constant. This means that unknown coefficients are uniformly bounded by the $C^2(\overline{\Omega})-$ norms. Moreover we assume that the both solutions u, v are sufficiently smooth, so that we can take t-derivatives necessary times.

For the application of the Carleman estimate Theorem 3.2, we introduce a weight function and a subdomain defined by the weight function. We set $x' = (x_2, ..., x_n)$, $x = (x_1, x') \in \mathbb{R}^n$, and

$$\Omega(\delta) = \left\{(x, x');\, 0 < x_1 < -\frac{1}{\gamma}|x'|^2 + \frac{\delta}{\gamma}\right\} \tag{6.4}$$

with some $\gamma > 0$ and $\delta > 0$. We assume that $\Omega(4\delta) \subset \Omega$ and $\Gamma_0 \subset \{x_1 = 0\}$. For a given Ω, we can apply the argument in §5.1 with the special choice d for the weight function, but for simplicity, we consider the domain in the form $\Omega(\delta)$.

Our weight function is:

$$\varphi(x,t) = e^{\lambda\psi(x,t)}, \quad \psi(x,t) = -\gamma x_1 - |x'|^2 - \beta(t - t_0)^2. \tag{6.5}$$

We set

$$\begin{aligned} Q(\delta) &= \{(x,t);\, x_1 > 0,\, \varphi(x,t) > e^{-\lambda\delta}\} \\ &= \{(x,t);\, x_1 > 0,\, \psi(x,t) > -\delta\} \end{aligned} \tag{6.6}$$

for $\delta > 0$. We note that $Q(\delta) \cap \{t = t_0\} = \Omega(\delta)$. We choose $\beta > 0$ and $\delta > 0$ such that

$$\max\{t_0^2, (T - t_0)^2\} = \frac{4\delta}{\beta}. \tag{6.7}$$

Then, noting that $\Omega(4\delta) \subset \Omega$, we see that $Q(4\delta) \subset Q$.

We set

$$y = u - v, \quad f = p - q, \quad R = v \quad \text{in } Q.$$

Then

$$\partial_t y = \operatorname{div}(p\nabla y) + \operatorname{div}(f\nabla R) \quad \text{in } Q. \tag{6.8}$$

Since y has not a compact support, we need a cut-off function. By $Q(4\delta) \supset Q(3\delta) \supset Q(2\delta) \supset Q(\delta)$, there exists $\chi \in C^\infty(\mathbb{R}^{n+1})$ such that $0 \le \chi \le 1$ and

$$\chi(x,t) = \begin{cases} 1, & (x,t) \in Q(2\delta), \\ 0, & (x,t) \in Q(4\delta) \setminus \overline{Q(3\delta)}. \end{cases} \tag{6.9}$$

Thanks to the sufficient smoothness of u and v, we can take

$$z_1 = \chi\partial_t y, \qquad z_2 = \chi\partial_t^2 y.$$

Direct calculations yield

$$\begin{aligned} &\partial_t z_k - \operatorname{div}(p\nabla z_k) = \chi\operatorname{div}(f\nabla\partial_t^k R) \\ +&(\partial_t\chi)(\partial_t^k y) - p(\Delta\chi)\partial_t^k y - 2p(\nabla\chi)\cdot\nabla(\partial_t^k y) - (\nabla p\cdot\nabla\chi)\partial_t^k y, \ k = 1, 2. \end{aligned} \tag{6.10}$$

Setting

$$\begin{aligned} &G_k(\nabla_{x,t}\chi, \Delta\chi)(x,t) = G_k(x,t) \\ =&(\partial_t\chi)(\partial_t^k y) - p(\Delta\chi)\partial_t^k y - 2p(\nabla\chi)\cdot\nabla(\partial_t^k y) - (\nabla p\cdot\nabla\chi)\partial_t^k y, \ k = 1, 2, \end{aligned} \tag{6.11}$$

we see that

$$G_k(x,t) = 0 \quad \text{if } \nabla_{x,t}\chi(x,t) = \Delta\chi(x,t) = 0,$$

$$|G_1(x,t)| + |G_2(x,t)| \le C\left(\sum_{k=1}^{2} |\partial_t^k y(x,t)| + |\nabla \partial_t^k y(x,t)|\right). \tag{6.12}$$

Moreover (6.4) and (6.9) imply

$$z_k = |\nabla z_k| = 0 \quad \text{on } \partial Q(4\delta),\ k = 1, 2. \tag{6.13}$$

We set

$$F^2 = \sum_{k=1}^{3} \|\partial_t^k(u-v)\|^2_{L^2(\Gamma_0\times(0,T))} + \sum_{k=1}^{2}\left(\left\|\frac{\partial}{\partial\nu}\partial_t^k(u-v)\right\|^2_{L^2(\Gamma_0\times(0,T))} + \left\|\frac{\partial}{\partial\tau}\partial_t^k(u-v)\right\|^2_{L^2(\Gamma_0\times(0,T))}\right).$$

Hence, by fixing $\lambda > 0$ large, Theorem 3.2 implies

$$\int_{Q(4\delta)} (s|\nabla z_k|^2 + s^3 z_k^2)e^{2s\varphi}dxdt \le C\int_{Q(4\delta)} \chi^2|\mathrm{div}\,(f\nabla\partial_t^k R)|^2 e^{2s\varphi}dxdt + C\int_{Q(4\delta)} G_k^2 e^{2s\varphi}dxdt + Ce^{Cs}F^2, \quad k = 1, 2$$

for all large $s > 0$. By $\nabla z_k = (\nabla\chi)\partial_t^k y + \chi\nabla\partial_t^k y$, we have

$$\int_{Q(4\delta)} \chi^2(s|\nabla\partial_t^k y|^2 + s^3|\partial_t^k y|^2)e^{2s\varphi}dxdt \le C\int_{Q(4\delta)} \chi^2(|\nabla f|^2 + f^2)e^{2s\varphi}dxdt + C\int_{Q(4\delta)} (G_k^2 + s|\nabla\chi|^2|\partial_t^k y|^2)e^{2s\varphi}dxdt + Ce^{Cs}F^2, \quad k = 1, 2$$

for all large $s > 0$. By (6.9) and (6.12), we see that

$$[\text{the left-hand side}] \le C\int_{Q(4\delta)} (|\nabla(\chi f)|^2 + |\chi f|^2)e^{2s\varphi}dxdt + CM_1^2 e^{2s\mu_1} + Ce^{Cs}F^2,$$

where

$$\begin{cases} M_1 = \sum_{k=0}^{2} (\|\partial_t^k y\|^2_{L^2(Q)} + \|\nabla\partial_t^k y\|^2_{L^2(Q)})^{\frac{1}{2}} + M, \\ \mu_1 = e^{-2\lambda\delta} > 0. \end{cases} \tag{6.14}$$

Therefore

$$
\begin{aligned}
&\int_{Q(4\delta)} \chi^2(s|\nabla\partial_t^2 y|^2 + s|\nabla\partial_t y|^2 + s^3|\partial_t^2 y|^2 + s^3|\partial_t y|^2)e^{2s\varphi}dxdt \\
\le& C\int_{Q(4\delta)} (|\nabla(\chi f)|^2 + |\chi f|^2)e^{2s\varphi}dxdt + CM_1^2 e^{2s\mu_1} + Ce^{Cs}F^2
\end{aligned}
\tag{6.15}
$$

for all large $s > 0$. We set

$$
a(x) = (u-v)(x,t_0), \quad b(x) = v(x,t_0), \quad x \in \Omega. \tag{6.16}
$$

Equation (6.8) yields

$$
\begin{aligned}
&\mathrm{div}\,(f\nabla b) = \partial_t y(\cdot,t_0) - \mathrm{div}\,(p\nabla a), \\
&\nabla \mathrm{div}\,(f\nabla b) = \nabla\partial_t y(\cdot,t_0) - \nabla\mathrm{div}\,(p\nabla a) \quad \text{in } \Omega.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\mathrm{div}\,(\chi f\nabla b) = \chi\partial_t y(\cdot,t_0) - \chi\mathrm{div}\,(p\nabla a) - \nabla\chi\cdot f\nabla b, \\
&\partial_i\mathrm{div}\,(\chi f\nabla b) = \chi\partial_i\partial_t y(\cdot,t_0) + (\partial_i\chi)\partial_t y(\cdot,t_0) - \partial_i(\chi\mathrm{div}\,p\nabla a) - \partial_i(\nabla\chi\cdot f\nabla b).
\end{aligned}
\tag{6.17}
$$

Moreover by (6.9) and (6.12), equations (6.17) yield

$$
\begin{aligned}
&\int_{\Omega(4\delta)} (|\mathrm{div}\,(\chi f\nabla b)|^2 + |\nabla\mathrm{div}\,(\chi f\nabla b)|^2)e^{2s\varphi(x,t_0)}dx \\
\le& C\int_{\Omega(4\delta)} \chi^2(|\partial_t y(x,t_0)|^2 + |\nabla\partial_t y(x,t_0)|^2)e^{2s\varphi(x,t_0)}dx + CM_1^2 e^{2s\mu_1} \\
&+Ce^{Cs}\|a\|^2_{H^3(\Omega(4\delta))}.
\end{aligned}
\tag{6.18}
$$

On the other hand, without loss of generality, we may assume that $t_0 > T - t_0$. Then (6.7) implies $t_0^2 = \frac{4\delta}{\beta}$. Therefore the set $\{(x_1,x');\, x_1 > 0,\, \gamma x_1 + |x'|^2 < 4\delta - \beta(t-t_0)^2\}$ is empty for $t = 0$, and we have

$$
\begin{aligned}
&\int_{\Omega(4\delta)} |\chi(\nabla^j\partial_t y)(x,t_0)|^2 e^{2s\varphi(x,t_0)}dx \\
=&\int_0^{t_0} \frac{\partial}{\partial t}\left(\int_{x_1>0,\gamma x_1+|x'|^2<4\delta-\beta(t-t_0)^2} |\chi(\nabla^j\partial_t y)(x,t)|^2 e^{2s\varphi(x,t)}dx\right)dt, \quad j = 0,1.
\end{aligned}
$$

Hence

$$\begin{aligned}
&\left|\int_{\Omega(4\delta)} |\chi(\nabla^j\partial_t y)(x,t_0)|^2 e^{2s\varphi(x,t_0)}dx\right|\\
=&\left|\int_0^{t_0}\int_{x_1>0,\gamma x_1+|x'|^2<4\delta-\beta(t-t_0)^2} \{2\chi^2(\nabla^j\partial_t y\cdot\nabla^j\partial_t^2 y)+\chi^2|\nabla^j\partial_t y|^2 2s(\partial_t\varphi)\}e^{2s\varphi}dxdt\right.\\
&+\left.\int_0^{t_0}\int_{x_1>0,\gamma x_1+|x'|^2<4\delta-\beta(t-t_0)^2} 2\chi(\partial_t\chi)|\nabla^j\partial_t y|^2 e^{2s\varphi}dxdt\right|\\
\le& C\int_{Q(4\delta)}\chi^2(s|\nabla^j\partial_t y|^2+|\nabla^j\partial_t^2 y|^2)e^{2s\varphi}dxdt+CM_1^2e^{2s\mu_1}.
\end{aligned}$$

Here we used the Cauchy-Schwarz inequality: $|(\nabla^j\partial_t y\cdot\nabla^j\partial_t^2 y)|\le|\nabla^j\partial_t y||\nabla^j\partial_t^2 y|$. Consequently (6.15) implies

$$\begin{aligned}
&\int_{\Omega(4\delta)}|\chi(\nabla^j\partial_t y)(\cdot,t_0)|^2e^{2s\varphi(x,t_0)}dx\\
\le& C\int_{Q(4\delta)}(|\chi f|^2+|\nabla(\chi f)|^2)e^{2s\varphi}dxdt+CM_1^2e^{2s\mu_1}+Ce^{Cs}F^2,\quad j=0,1
\end{aligned}\tag{6.19}$$

for all large $s>0$. Equations (6.18) and (6.19) imply

$$\begin{aligned}
&\int_{\Omega(4\delta)}(|\mathrm{div}\,(\chi f\nabla b)|^2+|\nabla\mathrm{div}\,(\chi f\nabla b)|^2)e^{2s\varphi(x,t_0)}dx\\
\le& C\int_{Q(4\delta)}(|\chi f|^2+|\nabla(\chi f)|^2)e^{2s\varphi}dxdt\\
&+Ce^{Cs}(F^2+\|a\|^2_{H^3(\Omega(4\delta))})+CM_1^2e^{2s\mu_1}.
\end{aligned}\tag{6.20}$$

We have to estimate $\|f\|^2_{H^1(\Omega(\delta))}$ by the left-hand side. For it, we need an estimate for the first-order partial differential operator. This can be done by the method of characteristics, but we use another Carleman estimate because we have to estimate with the weight $e^{2s\varphi(x,t_0)}$.

Lemma 6.1. *We set*

$$(P_0g)(x)=div(g\nabla b),\quad x\in\Omega(4\delta).$$

Assume that

$$\begin{cases} \gamma\partial_1 b(x) + 2\sum_{i=2}^{n}(\partial_i b)x_i < 0, & x \in \overline{\Omega(4\delta)}, \\ \partial_1 b(0,x') > 0, & (0,x') \in \Gamma_0. \end{cases} \tag{6.21}$$

Then there exists a constant $C > 0$ such that

$$\begin{aligned} &\int_{\Omega(4\delta)} s^2(|\nabla g|^2 + g^2)e^{2s\varphi(x,t_0)}dx \\ \le C &\int_{\Omega(4\delta)} (|\nabla(P_0 g)(x)|^2 + |(P_0 g)(x)|^2)e^{2s\varphi(x,t_0)}dx \end{aligned} \tag{6.22}$$

for all large $s > 0$ and all $g \in H^2(\Omega(4\delta))$ such that $|g| = |\nabla g| = 0$ on $\partial\Omega(4\delta)\setminus\{x_1 = 0\}$.

For the proof, we will show the following lemma for a general case.

Lemma 6.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and let $\nu = (\nu_1, ..., \nu_n)$ be the unit outward normal vector to $\partial\Omega$ and let us consider a first-order partial differential operator*

$$(P_0 g)(x) = \sum_{j=1}^{n} p_j(x)\partial_j g(x),$$

where $p_j \in C^1(\overline{\Omega})$, $j = 1, ..., n$. We assume that $\varphi_0 \in C^1(\overline{\Omega})$ satisfies

$$\sum_{j=1}^{n} p_j(x)\partial_j\varphi_0(x) > 0, \quad x \in \overline{\Omega}. \tag{6.23}$$

Then there exists $s_1 > 0$ and $C_1 = C_1(s_0, \Omega) > 0$ such that

$$\int_\Omega s^2(|\nabla g|^2 + |g|^2)e^{2s\varphi_0(x)}dx \le C_1\int_\Omega (|\nabla P_0 g|^2 + |P_0 g|^2)e^{2s\varphi_0(x)}dx$$

for all $s > s_0$ and $g \in H^2(\Omega)$ satisfying

$$|\nabla g| = |g| = 0 \quad on \left\{x \in \partial\Omega;\ \sum_{j=1}^{n} p_j(x)\nu_j(x) \ge 0\right\}. \tag{6.24}$$

Proof of Lemma 6.2. We set $w = e^{s\varphi_0}g$, $Q_0 w = e^{s\varphi_0}P_0(e^{-s\varphi_0}w)$. Then

$$\int_\Omega |P_0 g|^2 e^{2s\varphi_0(x)}dx = \int_\Omega |Q_0 w|^2 dx.$$

We have

$$Q_0 w = P_0 w - sq_0 w,$$

where $q_0(x) = \sum_{j=1}^n p_j(x)\partial_j\varphi_0(x)$. Therefore, by (6.23), (6.24) and integration by parts, we obtain

$$\begin{aligned}
&\|Q_0 w\|^2_{L^2(\Omega)} = \|P_0 w\|^2_{L^2(\Omega)} + s^2\|q_0 w\|^2_{L^2(\Omega)} - 2s\int_\Omega \sum_{j=1}^n p_j(\partial_j w)q_0 w dx\\
\ge& s^2\int_\Omega q_0(x)^2 w(x)^2 dx - s\int_\Omega \sum_{j=1}^n p_j q_0 \partial_j(w^2)dx\\
\ge& C_2 s^2\int_\Omega w(x)^2 dx - s\int_{\partial\Omega}\sum_{j=1}^n p_j q_0 w^2 \nu_j dS + s\int_\Omega \sum_{j=1}^n \partial_j(p_j q_0)w^2 dx\\
\ge& (C_2 s^2 - C_3 s)\int_\Omega w^2 dx - s\int_{\partial\Omega\cap\{\sum_{j=1}^n p_j\nu_j \le 0\}}\left(\sum_{j=1}^n p_j\nu_j\right) q_0 w^2 dS\\
\ge& (C_2 s^2 - C_3 s)\int_\Omega w^2 dx.
\end{aligned}$$

By (6.23), we have $q_0 > 0$ on $\partial\Omega$, so that the right-hand side is greater than or equal to $(C_2 s^2 - C_3 s)\int_\Omega w^2 dx$. Thus by taking $s > 0$ sufficiently large, we have

$$\int_\Omega s^2|g|^2 e^{2s\varphi_0(x)}dx \le C_4\int_\Omega |P_0 g|^2 e^{2s\varphi_0(x)}dx.$$

Next, setting $h = P_0 g$, we observe

$$P_0(\partial_k g) = \partial_k h - \sum_{j=1}^n (\partial_k p_j)(x)(\partial_j g)(x), \quad k = 1, ..., n.$$

By (6.24), we can apply the previous argument, so that we have

$$\begin{aligned}
&s^2\int_\Omega |\nabla g|^2 e^{2s\varphi_0}dx\\
\le& C_4\int_\Omega |\nabla h|^2 e^{2s\varphi_0}dx + C_4\int_\Omega \left|\sum_{j,k=1}^n (\partial_k p_j)(x)(\partial_j g)(x)\right|^2 e^{2s\varphi_0}dx\\
\le& C_5\int_\Omega |\nabla(P_0 g)|^2 e^{2s\varphi_0}dx + C_5\int_\Omega |\nabla g|^2 e^{2s\varphi_0}dx.
\end{aligned}$$

Consequently, choosing $s > 0$ large, we can absorb the second term on the right-hand side into the left-hand side, so that we can obtain

$$s^2 \int_{\Omega} |\nabla g|^2 e^{2s\varphi_0} dx \le C_6 \int_{\Omega} |\nabla (P_0 g)|^2 e^{2s\varphi_0} dx.$$

Thus the proof of Lemma 6.2 is completed.

In Lemma 6.2, we set $\Omega = \Omega(4\delta)$, $p_j = \partial_j b$, $j = 1, ..., n$, $\varphi_0(x) = -\gamma x_1 - |x'|^2$, and Lemma 6.1 follows.

Applying Lemma 6.1 to χf and $P_0 g = \mathrm{div}\,(g\nabla b)$, in terms of (6.20), we have

$$\begin{aligned}&\int_{\Omega(4\delta)} s^2(|\nabla(\chi f)|^2 + |\chi f|^2) e^{2s\varphi(x,t_0)} dx\\ \le& C\int_{Q(4\delta)} (|\nabla(\chi f)|^2 + |\chi f|^2) e^{2s\varphi} dxdt + Ce^{Cs}(F^2 + \|a\|^2_{H^3(\Omega(4\delta))}) + CM_1^2 e^{2s\mu_1}\end{aligned}$$

for all large $s > 0$. Since

$$\varphi(x,t) \le \varphi(x,t_0), \quad x \in \overline{\Omega},\ 0 \le t \le T, \tag{6.25}$$

we see that

$$\begin{aligned}&s^2\int_{\Omega(4\delta)} (|\nabla(\chi f)|^2 + |\chi f|^2) e^{2s\varphi(x,t_0)} dx\\ \le& CT\int_{\Omega(4\delta)} (|\nabla(\chi f)|^2 + |\chi f|^2) e^{2s\varphi(x,t_0)} dxdt + Ce^{Cs}(F^2 + \|a\|^2_{H^3(\Omega(4\delta))}) + CM_1^2 e^{2s\mu_1},\end{aligned}$$

and taking $s > 0$ sufficiently large such that $s \ge s_0$, we can absorb the first term on the right-hand side into the left-hand side, and

$$\begin{aligned}&s^2\int_{\Omega(4\delta)} (|\nabla(\chi f)|^2 + |\chi f|^2) e^{2s\varphi(x,t_0)} dx\\ \le& Ce^{Cs}(F^2 + \|a\|^2_{H^3(\Omega(4\delta))}) + CM_1^2 e^{2s\mu_1}\end{aligned}$$

for all $s \ge s_0$. We set $F_1^2 = F^2 + \|a\|^2_{H^3(\Omega(4\delta))}$. Since $\Omega(\delta) \subset \Omega(4\delta)$ and $e^{2s\varphi(x,t_0)} \ge 1$ for $x \in \overline{\Omega}$, by (6.9) we obtain

$$\begin{aligned}&e^{2se^{-\lambda\delta}} \int_{\Omega(\delta)} (|\nabla f|^2 + |f|^2) dx \le \int_{\Omega(4\delta)} (|\nabla(\chi f)|^2 + |\chi f|^2) e^{2s\varphi(x,t_0)} dx\\ \le& Ce^{Cs} F_1^2 + CM_1^2 e^{2s\mu_1}\end{aligned}$$

for all $s \ge s_0$. We set $\mu_2 = e^{-\lambda\delta}$. Then

$$\int_{\Omega(\delta)} (|\nabla f|^2 + |f|^2)dx \le Ce^{Cs}F_1^2 + CM_1^2 e^{-2s(\mu_2-\mu_1)} \tag{6.26}$$

for all $s \ge s_0$. We note that $\mu_2 - \mu_1 > 0$.

Replace C by Ce^{Cs_0}, we obtain (6.26) for all $s \ge 0$. We can assume that $F_1 \ne 0$. We choose $s \ge 0$ minimizing the right-hand side of (6.26):

$$e^{Cs}F_1^2 = M_1^2 e^{-2s(\mu_2-\mu_1)},$$

that is,

$$s = \frac{2}{C + 2(\mu_2 - \mu_1)} \log \frac{M_1}{F_1}.$$

Then by (6.26), we obtain

$$\|f\|_{H^1(\Omega(\delta))} \le \sqrt{2}CM_1^{\frac{C}{C+2(\mu_2-\mu_1)}} F_1^{\frac{2(\mu_2-\mu_1)}{C+2(\mu_2-\mu_1)}}.$$

To sum up, we state

Theorem 6.1. *Let u and v satisfy (6.1) and (6.2) respectively. For $0 < \delta < \delta'$, let $\Omega(\delta') \subset \Omega$, $\Gamma_0 \subset \{x_1 = 0\}$. We assume*

$$\sum_{k=0}^{2} (\|\partial_t^k u\|_{L^\infty(Q)} + \|\partial_t^k v\|_{L^\infty(Q)} + \|\nabla\partial_t^k u\|_{L^\infty(Q)} + \|\nabla\partial_t^k v\|_{L^\infty(Q)}) \le M$$

and let $\mathcal{A}$ and $\Omega(\delta)$ be defined by (6.3) and (6.4) with $\delta > 0$. Moreover we assume that $b = u(\cdot, t_0)$ or $b = v(\cdot, t_0)$ satisfies (6.21). Then, for $\varepsilon > 0$, there exist constants $C > 0$ and $\theta \in (0,1)$ such that

$$\begin{aligned} \|p - q\|_{H^1(\Omega(\delta))} \le CM^{1-\theta}\Bigg\{ &\|(u-v)(\cdot,t_0)\|_{H^3(\Omega(\delta'))} \\ &+ \sum_{k=1}^{3} \|\partial_t^k (u-v)\|_{L^2(\Gamma_0\times(0,T))} \\ &+ \sum_{k=1}^{2} \left(\left\| \frac{\partial}{\partial\nu}\partial_t^k(u-v)\right\|_{L^2(\Gamma_0\times(0,T))} + \left\|\frac{\partial}{\partial\tau}\partial_t^k(u-v)\right\|_{L^2(\Gamma_0\times(0,T))}\right)\Bigg\}^{\theta}. \end{aligned} \tag{6.27}$$

for $p, q \in \mathcal{A}$.

In the estimation in $\Omega(\delta)$, we need data of solution at $t = t_0 > 0$ in a larger domain $\Omega(\delta')$. For the uniqueness, we need not take such a larger domain. That is, the above argument produces the uniqueness: if $u = v$ and $\nabla u = \nabla v$ on $\Gamma_0 \times (0, T)$ and $u(\cdot, t_0) = v(\cdot, t_0)$ in $\Omega(\delta)$, then $p(x) = q(x)$ for $x \in \Omega(\delta)$. Estimate (6.27) of coefficients holds if unknown coefficients and solutions are in a priori bounded subsets in suitable norms and (6.27) is called a conditional stability estimate. The exponent θ in (6.27) depends on the a priori bound M, and $\theta \to 0$ as $M \to \infty$, which means that (6.27) becomes worse if M is larger.

For applying a Carleman estimate, we can not choose $t_0 = 0$. Therefore this is not an inverse problem to a usual initial value/boundary value problem. The uniqueness in the case of $t_0 = 0$ is a longstanding open problem. If the coefficients are independent of t and the observation area is sufficiently large, then one can change the inverse parabolic problem to an inverse problem for a hyperbolic equation by an integral transform in t, and prove the uniqueness and the stability for the case $t_0 = 0$ (see Klibanov [95], Theorem 4.7, and Klibanov and Timonov [100]). Also see Isakov [86], section 9.2. Moreover for the inverse problems by Carleman estimates, we always need a positivity condition such as (6.21).

We can apply the global Carleman estimate Theorems 4.1 and 4.2 to prove the Lipschitz stability over Ω in determining $p(x)$. As for the Lipschitz stability in determining principal coefficients in hyperbolic equations, see Klibanov and Yamamoto [101], but for the parabolic case, the argument for the Lipschitz stability is simpler than those hyperbolic case, thanks to the global Carleman estimate with a singular weight function.

§6.2 Global Lipschitz stability for an inverse source problem.

In this section, as one example of the application of the global Carleman estimate in section 4, we discuss an inverse source problem for a weakly coupled system of parabolic equations. In particular, if a boundary condition is given on the whole $\partial\Omega \times (0,T)$, then the global Carleman estimate gives the Lipschitz stability rather directly.

We consider

$$\begin{aligned} \partial_t \mathbf{u}(x,t) = &\left(\sum_{i,j=1}^{n} a_{ij}^1(x)\partial_i\partial_j u_1, ... \sum_{i,j=1}^{n} a_{ij}^N \partial_i\partial_j u_N \right)^T \\ &+ \sum_{i=1}^{n} B_i(x,t)\cdot\partial_i \mathbf{u}(x,t) + C(x,t)\mathbf{u}(x,t) + R(x,t)\mathbf{f}(x), \quad (x,t)\in Q, \end{aligned} \tag{6.28}$$

$$\mathbf{u}|_{\partial\Omega\times(0,T)} = 0, \tag{6.29}$$

where $\mathbf{u} = (u_1, ..., u_N)^T$, $\mathbf{f} = (f_1, ..., f_N)^T$, and B_i, $1 \le i \le n$ and C are $N \times N$ matrix functions and $a_{ij}^k \in C^1(\overline{\Omega})$, $1 \le i, j \le n, 1 \le k \le N$, satisfy (1.1) and (1.2), $B_i \in L^\infty(Q)^{N\times N}$, $C \in L^\infty(Q)^{N\times N}$, R is a given $N \times N$ matrix function. A weakly coupled parabolic system (6.28) appears for example, as linearized reaction-diffusion equations.

For describing an application argument by a global Carleman estimate to an inverse problem, we discuss

Inverse source problem. Let $\Gamma \subset \partial\Omega$ be an arbitrarily fixed subboundary and let $t_0 \in (0,T)$ be any fixed time. Determine $\mathbf{f}(x)$, $x \in \Omega$ by $u(x,t_0)$, $x \in \Omega$ and $\frac{\partial u}{\partial \nu_A}(x,t)$, $x \in \Gamma, 0 < t < T$.

The main result is stated as follows:

Theorem 6.2. *We assume that* $\mathbf{u}, \partial_t \mathbf{u} \in H^{2,1}(Q)$, $R, \partial_t R \in L^\infty(Q)$ *and that*

$$det\, R(x, t_0) \neq 0, \quad x \in \overline{\Omega}. \tag{6.30}$$

Then there exists a constant $C > 0$ *depending on* $\Omega, R, a_{ij}^k, B_i, C$ *such that*

$$\|\mathbf{f}\|_{L^2(\Omega)^N} \le C \left(\|\mathbf{u}(\cdot, t_0)\|_{H^2(\Omega)^N} + \left\| \frac{\partial}{\partial \nu_A}(\partial_t \mathbf{u}) \right\|_{L^2(\Gamma \times (0,T))^N} \right) \tag{6.31}$$

for any $\mathbf{f} \in L^2(\Omega)^N$.

Proof. The proof is a simplification of Imanuvilov and Yamamoto [74]. Setting $\mathbf{z} = \partial_t \mathbf{u}$ and

$$A\mathbf{u} = \left(\sum_{i,j=1}^n a_{ij}^1 \partial_i \partial_j u_1, ..., \sum_{i,j=1}^n a_{ij}^N \partial_i \partial_j u_N \right)^T + \sum_{i=1}^n B_i \partial_i \mathbf{u} + C\mathbf{u},$$

we have

$$\partial_t \mathbf{z} = A\mathbf{z} + (\partial_t R)\mathbf{f} \quad \text{in } Q \tag{6.32}$$

$$\mathbf{z} = 0 \quad \text{on } \partial\Omega \times (0, T) \tag{6.33}$$

and

$$R(x, t_0)\mathbf{f}(x) = \partial_t \mathbf{u}(x, t_0) - (A\mathbf{u})(x, t_0), \quad x \in \Omega. \tag{6.34}$$

For the proof, it is sufficient to prove the estimate of $\mathbf{f}$ by the Neumann data over a shorter time interval centred at t_0:

$$\|\mathbf{f}\|_{L^2(\Omega)^N} \le C \left(\|\mathbf{u}(\cdot, t_0)\|_{H^2(\Omega)^N} + \left\| \frac{\partial}{\partial \nu_A}(\partial_t \mathbf{u}) \right\|_{L^2(\Gamma \times (t_0 - t_1, t_0 + t_1))^N} \right)$$

with $0 < t_0 - t_1 < t_0 + t_1 < T$. Hence by the change of t-variables, it sufficient to prove (6.31) in the case of $t_0 = \frac{T}{2}$.

Fixing $\lambda > 0$ in Theorem 4.4 and applying to (6.32) with (6.33),

$$\begin{aligned}
&\int_Q \left(\frac{1}{s\varphi_0}|\partial_t^2 \mathbf{u}|^2 + s\varphi_0|\nabla \partial_t \mathbf{u}|^2 + s^3\varphi_0^3|\partial_t \mathbf{u}|^2\right)e^{2s\alpha_0}dxdt \\
&\le C\int_Q |(\partial_t R)\mathbf{f}|^2 e^{2s\alpha_0}dxdt + Ce^{Cs}\int_{\Gamma\times(0,T)}\left|\frac{\partial}{\partial \nu_A}(\partial_t \mathbf{u})\right|^2 dSdt
\end{aligned} \tag{6.35}$$

for all large $s > 0$.

On the other hand, in terms of (6.34), for estimating of $\mathbf{f}$, we have to estimate $\partial_t \mathbf{u}(\cdot, t_0)$ with the weight function:

$$\int_\Omega \left|\partial_t \mathbf{u}\left(x, \frac{T}{2}\right)\right|^2 e^{2s\alpha_0\left(x, \frac{T}{2}\right)}dx.$$

By $\lim_{t\to 0} e^{2s\alpha_0(x,t)} = 0$ for $x \in \overline{\Omega}$, we have

$$\begin{aligned}
&\int_\Omega \left|\partial_t \mathbf{u}\left(x, \frac{T}{2}\right)\right|^2 e^{2s\alpha_0\left(x, \frac{T}{2}\right)}dx = \int_0^{\frac{T}{2}} \frac{\partial}{\partial t}\left(\int_\Omega |\partial_t \mathbf{u}(x,t)|^2 e^{2s\alpha_0(x,t)}dx\right)dt \\
&= \int_\Omega\int_0^{\frac{T}{2}} (2(\partial_t \mathbf{u}(x,t)\cdot \partial_t^2 \mathbf{u}(x,t)) + 2s(\partial_t \alpha_0)|\partial_t \mathbf{u}(x,t)|^2)e^{2s\alpha_0}dxdt \\
&\le \int_Q (2|\partial_t \mathbf{u}(x,t)||\partial_t^2 \mathbf{u}(x,t)| + Cs\varphi_0^2|\partial_t \mathbf{u}(x,t)|^2)e^{2s\alpha_0}dxdt.
\end{aligned}$$

Here we used $|\partial_t \alpha_0(x,t)| \le C\varphi_0(x,t)^2$, $(x,t) \in \overline{Q}$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&|\partial_t \mathbf{u}(x,t)||\partial_t^2 \mathbf{u}(x,t)| = \frac{1}{s\sqrt{\varphi_0}}|\partial_t \mathbf{u}(x,t)| \times s\sqrt{\varphi_0}|\partial_t^2 \mathbf{u}(x,t)| \\
&\le \frac{1}{s^2\varphi_0}|\partial_t^2 \mathbf{u}(x,t)|^2 + s^2\varphi_0|\partial_t \mathbf{u}(x,t)|^2,
\end{aligned}$$

and the application of (6.35) yields

$$\begin{aligned}
&\int_\Omega \left|\partial_t \mathbf{u}\left(x, \frac{T}{2}\right)\right|^2 e^{2s\alpha_0\left(x, \frac{T}{2}\right)}dx \le C\int_Q \left(\frac{1}{s^2\varphi_0}|\partial_t^2 \mathbf{u}|^2 + s^2\varphi_0^2|\partial_t \mathbf{u}|^2\right)e^{2s\alpha_0}dxdt \\
&\le \frac{C}{s}\int_Q |(\partial_t R)\mathbf{f}|^2 e^{2s\alpha_0}dxdt + Ce^{Cs}\int_{\Gamma\times(0,T)}\left|\frac{\partial}{\partial \nu_A}(\partial_t \mathbf{u})\right|^2 dSdt.
\end{aligned}$$

By (6.34), (6.30) and $\partial_t R \in L^\infty(Q)$, we see that

$$\int_\Omega |\mathbf{f}(x)|^2 e^{2s\alpha_0\left(x,\frac{T}{2}\right)} dx \le \frac{C}{s} \int_Q |\mathbf{f}(x)|^2 e^{2s\alpha_0} dxdt$$
$$+e^{Cs} \left\| u\left(\cdot, \frac{T}{2}\right)\right\|^2_{H^2(\Omega)^N} + Ce^{Cs} \int_{\Gamma\times(0,T)} \left| \frac{\partial}{\partial \nu_A}(\partial_t \mathbf{u})\right|^2 dSdt.$$

By the definition of α_0, we have

$$\alpha_0(x,t) \le \alpha_0\left(x, \frac{T}{2}\right), \quad (x,t) \in \overline{Q}. \tag{6.36}$$

Therefore

$$\int_Q |\mathbf{f}(x)|^2 e^{2s\alpha_0(x,t)} dxdt \le T \int_\Omega |\mathbf{f}(x)|^2 e^{2s\alpha_0\left(x,\frac{T}{2}\right)} dx,$$

that is,

$$\left(1 - \frac{C}{s}\right) \int_\Omega |\mathbf{f}(x)|^2 e^{2s\alpha_0\left(x,\frac{T}{2}\right)} dx$$
$$\le Ce^{Cs} \left(\left\| u\left(\cdot, \frac{T}{2}\right)\right\|^2_{H^2(\Omega)^N} + \int_{\Gamma\times(0,T)} \left| \frac{\partial}{\partial \nu_A}(\partial_t \mathbf{u})\right|^2 dSdt \right)$$

for all large $s > 0$. Taking $s > 0$ sufficiently large, we can complete the proof of Theorem 6.2.

Technical comments I.

Although the both are based on Carleman estimates, the original method since Bukhgeim and Klibanov [20] is different from ours and does not work in our inverse problems.

We explain their method for the determination of $\rho(x)$ in

$$\partial_t u = \frac{1}{\rho(x)} \Delta u \quad \text{in } Q.$$

Let $\widetilde{u}$ satisfy

$$\partial_t \widetilde{u} = \frac{1}{\widetilde{\rho}(x)} \Delta \widetilde{u} \quad \text{in } Q.$$

We assume that $u = \widetilde{u}$ and $\nabla u = \nabla\widetilde{u}$ on $\Gamma_0\times(0,T)$ and $u(\cdot,t_0) = \widetilde{u}(\cdot,t_0)$, $\Delta u(\cdot,t_0) \ne 0$ on $\overline{\Omega}$. Setting $y = u - \widetilde{u}$, $R = \Delta\widetilde{u}$ and $f = \widetilde{\rho} - \rho$, we obtain

$$\partial_t y = \frac{1}{\rho(x)}\Delta y + \frac{f}{\rho\widetilde{\rho}}R \quad \text{in } Q$$

and

$$y = |\nabla y| = 0 \quad \text{on } \Gamma_0 \times (0,T).$$

By $\Delta u(\cdot,t_0) \ne 0$ on $\overline{\Omega}$, we choose $t_1 > 0$ such that $R(x,t) \ne 0$ for $x \in \overline{\Omega}$ and $0 < t_0 - t_1 \le t \le t_0 + t_1 < T$. The key in Bukhgeim and Klibanov [20] is first division by R and second differentiation in t as follows: Setting $y = Rz$, we obtain

$$\partial_t z = \frac{1}{\rho(x)}\Delta z - \frac{\partial_t R}{R}z + \frac{2\nabla R}{\rho R}\cdot\nabla z + \frac{1}{\rho}\frac{\Delta R}{R}z + \frac{f}{\rho\widetilde{\rho}} \quad \text{in } \Omega\times(t_0-t_1,t_0+t_1).$$

Since $\frac{f}{\widetilde{\rho}\rho}$ is independent of t and $z(x,t) = \int_{t_0}^t \partial_t z(x,\eta)d\eta$ by $z(\cdot,t_0) = 0$, setting $w = \partial_t z$ and differentiating in t, we obtain

$$\begin{aligned}\partial_t w = {} & \frac{1}{\rho(x)}\Delta w - \frac{\partial_t R}{R}w + \frac{2\nabla R}{\rho R}\cdot\nabla w + \frac{1}{\rho}\frac{\Delta R}{R}w - \partial_t\left(\frac{\partial_t R}{R}\right)\int_{t_0}^t w d\eta \\ & + \frac{2}{\rho}\partial_t\left(\frac{\nabla R}{R}\right)\cdot\int_{t_0}^t \nabla w d\eta + \frac{1}{\rho}\partial_t\left(\frac{\Delta R}{R}\right)\int_{t_0}^t w d\eta \quad \text{in } \Omega\times(t_0-t_1,t_0+t_1).\end{aligned} \tag{6.37}$$

and

$$w = |\nabla w| = 0 \quad \text{on } \Gamma_0 \times (0,T).$$

Equation (6.37) is a parabolic equation with memory term. Thanks to an inequality

$$\int_{Q(\delta)}\left|\int_{t_0}^t w(x,\eta)d\eta\right|^2 e^{2s\varphi}dxdt \le \frac{C}{s}\int_{Q(\delta)}|w(x,t)|^2 e^{2s\varphi}dxdt. \tag{6.38}$$

where $\varphi \in C^2(\overline{Q})$ and

$$\varphi(x,t) < \varphi(x,t_0) \quad x \in \overline{\Omega},\ t \ne t_0,$$

we can apply the argument by a Carleman estimate in proving the unique continuation also to (6.37) and we can conclude that $w = 0$ in $\Omega(\delta)$ with $\delta > 0$, which implies that $\rho = \widetilde{\rho}$ in $\Omega(\delta)$. In fact, inequality (6.38) is crucial in the paper [20] by Bukhgeim and Klibanov and essential for the application of a Carleman estimate to inverse problems. As for the proof of (6.38), see Klibanov [95], Klibanov and Timonov [100].

Remark. Under a weaker condition $\varphi(x,t) \le \varphi(x,t_0)$ for $(x,t) \in Q(\delta)$, we can prove

$$\int_{Q(\delta)} \left| \int_{t_0}^t w(x,\eta) d\eta \right|^2 e^{2s\varphi} dxdt \le C \int_{Q(\delta)} |w(x,t)|^2 e^{2s\varphi} dxdt$$

([95]). For this inverse problem, the factor $\frac{1}{s}$ on the right-hand side of (6.38) is not necessary. However for some inverse problems, the factor is important (see, e.g., Cavaterra, Lorenzi and Yamamoto [31]).

The above is the original argument in Bukhgeim and Klibanov [20], but their method does not work for the inverse problems with a single measurement for (6.1) involving derivatives of an unknown coefficient and a parabolic system (6.28). It is easy to understand the situation for (6.1). As for (6.28) we will explain below. By (6.30), we can choose $t_1 > 0$ such that det $R(x,t) \ne 0$, $x \in \overline{\Omega}$, $0 < t_0 - t_1 \le t \le t_0 - t_1 < T$. For simplicity, we assume that $\mathbf{u}(x,t_0) = 0$ in Ω. Setting $\mathbf{u} = R\mathbf{v}$ and dividing by R, we obtain

$$\partial_t \mathbf{v} = R^{-1}AR\mathbf{v} - (R^{-1}\partial_t R)\mathbf{v} + \mathbf{f} \quad \text{in } Q. \tag{6.39}$$

Let $R = (r_{ij})_{1\le i,j\le n}$ and $R^{-1} = (r^{ij})_{1\le i,j\le n}$. Then

$$[R^{-1}AR\mathbf{v}]_p = \sum_{i,j=1}^n \left(\sum_{k,\ell=1}^N a_{ij}^k r^{pk} r_{k\ell} \right) \partial_i\partial_j v_\ell + [\text{lower-order terms}].$$

If a_{ij}^k depend on $k \in \{1,...,N\}$, then $R^{-1}AR\mathbf{v}$ can not be weakly coupled, so that we can not apply Theorem 4.4 to the resulting system (6.39).

Technical comments II.

For the inverse problem, it is essential that the weight function φ attains the maximum at $t = t_0$ where the spatial data of solutions are given. See (6.25) and (6.36). The maximality at $t = t_0$ is an alternative essential fact to inequality (6.38) and the corresponding part in the method by Bukgheim and Klibanov [20].

§7. Overview for Carleman estimates for parabolic equations.

§7.1. pointwise Carleman estimate

§7.2. general treatment

§7.3. global Carleman estimate

§7.4. Carleman estimate with second large parameter

§7.5. H^{-1}-Carleman estimates

§7.6. Carleman estimates for less regular principal terms

§7.7. Carleman estimate for degenerate parabolic equations

§7.8. Carleman estimates for parabolic systems

In sections 5 and 6, we show how to apply the Carleman estimates in sections 3 and 4 to estimation of solutions to parabolic equations and inverse problems. We know that relevant Carleman estimates produce stability results for these problems. In this section, also for possible novel applications, we summarize the existing results on Carleman estimates themselves for parabolic equations. Carleman estimates are used for various subjects and the number of related papers is very large, so that we do not intend any perfect list of the papers. Moreover it is reasonable to treat Carleman estimates not only for parabolic equations but also for a wider class of partial differential equations, but we concentrate on parabolic equations.

§7.1. Pointwise Carleman estimate.

In Lavrent'ev, Romanov and Shishat·skiĭ[107] (also Klibanov and Timonov [100]), a pointwise Carleman estimate was proved for a parabolic equation. According to [100], [107], we will present the pointwise Carleman estimate. First we choose a weight function similar to (6.5) whose level set is of a parabolic shape:

$$\psi(x,t) = x_1 + \frac{|x'|^2}{c_1} + \frac{(t-t_0)^2}{c_2} + \frac{1}{4} \tag{7.1}$$

$$D = \left\{(x,t);\, x_1 > 0,\, \psi(x,t) < \frac{3}{4}\right\}, \tag{7.2}$$

where $c_1 > 0$ and $c_2 > 0$ are suitable chosen constants. The boundary ∂D consists of two parts:

$$\begin{aligned}
&\partial D = \Gamma_1 \cup \Gamma_2,\\
&\Gamma_1 = \left\{(x,t);\, x_1 = 0,\, \frac{|x'|^2}{c_1} + \frac{(t-t_0)^2}{c_2} < \frac{1}{2}\right\},\\
&\Gamma_2 = \left\{(x,t);\, x_1 > 0,\, x_1 + \frac{|x'|^2}{c_1} + \frac{(t-t_0)^2}{c_2} = \frac{1}{2}\right\}.
\end{aligned}$$

We set

$$\varphi(x,t) = \psi(x,t)^{-\lambda} \tag{7.3}$$

where $\lambda > 0$ is a large second parameter in the Carleman estimate in a different form from the Carleman estimates in sections 2-4. We note that $\psi < 1$ on $\overline{D}$.

Theorem 7.1 (Klibanov and Timonov [100], Lavrent'ev, Romanov and Shishat·skiĭ[107]). *There exist sufficiently large positive constants λ_0 and s_0 which depend on on $\sigma_1, \max_{1\le i,j\le n} \|a_{ij}\|_{C^1(\overline{Q})}$, ψ, D, such that if $\lambda \ge \lambda_0$, then for all $u \in C^{2,1}(\overline{D})$ and all $s \ge s_0$, we have the following pointwise Carleman estimate:*

$$\begin{aligned}
&|Lu(x,t)|^2 e^{2s\psi^{-\lambda}}\\
\ge &C\left(\frac{1}{s}\left(|\partial_t u(x,t)|^2 + \sum_{i,j=1}^{n} |\partial_i\partial_j u(x,t)|^2\right) + s|\nabla u(x,t)|^2 + s^3|u(x,t)|^2\right) e^{2s\psi^{-\lambda}}\\
&+ divU + \partial_t V \quad in\ D,
\end{aligned} \tag{7.4}$$

where the vector-valued function (U, V) satisfies

$$|(U,V)(x,t)| \le Cs^3(|\partial_t u|^2 + |\nabla u|^2 + |u|^2)e^{2s\psi^{-\lambda}} \tag{7.5}$$
$$+\frac{C}{s}\sum_{i,j=1}^{n} |\partial_i\partial_j u(x,t)|^2 e^{2s\psi^{-\lambda}},$$

and the constant $C > 0$ depends on $\sigma_1, \max_{1\le i,j\le n} \|a_{ij}\|_{C^1(\overline{Q})}$, ψ, D.

The estimate without the terms $\partial_t u(x,t)$ and $\partial_i\partial_j u(x,t)$ was proved in [107], and in Klibanov and Timonov [100] the estimate with such terms is shown thanks to the a priori estiamte for a boundary value problems for the elliptic equation.

If we take the integrations of (7.4) over D, then in terms of the Gauss theorem for the last two terms on the right-hand side of (7.4), we can obtain a Carleman estimate similar to (2.6) and Theorems 3.1 - 3.2. The pointwise Carleman estimate is very useful because

(1) one can estimate the boundary terms and so we need not assume that functions have compact supports.

(2) it holds also for a parabolic inequality:

$$|Lu(x,t)| \le C(|u(x,t)| + |\nabla u(x,t)|), \quad (x,t) \in D.$$

For other types of equations, such pointwise Carleman estimate were proved in [107] and see also Amirov [5], Fu [57], Klibanov and Timonov [100], Zhang [137]. In [5], Carleman estimates are proved for ultrahyperbolic equations.

§7.2. General treatment.

According to Isakov [86], we overview the general treatments. Let $m = (m_1, ..., m_{n+1}) \in \mathbb{N}^{n+1}$ such that

$$m_1 = \cdots = m_q > m_{q+1} \ge \cdots \ge m_{n+1}, \quad \nabla_q = (\partial_1, ..., \partial_q, 0, ..., 0)$$

with some $q \in \{1, 2, ..., n+1\}$. Moreover let $t = x_{n+1}$, $\alpha = (\alpha_1, ..., \alpha_{n+1}) \in (\mathbb{N} \cap \{0\})^{n+1}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_{n+1}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \partial_{n+1}^{\alpha_{n+1}}$. We set $|\alpha : m| = \frac{\alpha_1}{m_1} + \cdots + \frac{\alpha_{n+1}}{m_{n+1}}$.

We consider an partial differential operator:

$$A = \sum_{|\alpha:m| \le 1} a_\alpha(x,t) \partial^\alpha. \tag{7.6}$$

If $m_1 = \cdots = m_{n+1}$, then the operator (7.6) is called isotropic and otherwise call (7.6) anisotropic. For hyperbolic and elliptic operators, we have $m_1 = m_2 = \cdots = m_{n+1} = 2$, while in the parabolic case, $m_1 = \cdots = m_n = 2$ and $m_{n+1} = 1$. For the isotropic case, a general theory for Carleman estimates has been completed for functions with compact supports (Hörmander [65]): a Carleman estimate holds if a quadratic form made by the weight function and the coefficients of (7.6) satisfies some positivity condition called the strong pseudo-convexity (e.g., Hörmander [65], section 8.5). For the anisotropic case, Isakov [82], [86] established a general theory for Carleman estimates for functions with compact supports. For the partial differential operator (7.6), we set

(7.7)
$$A_m(x,t,\zeta) = \sum_{|\alpha:m|=1} a_\alpha(x,t) \sqrt{-1}^{|\alpha|} \zeta_1^{\alpha_1} \cdots \zeta_{n+1}^{\alpha_{n+1}}, \quad \zeta = (\zeta_1, ..., \zeta_{n+1}) \in \mathbb{C}^{n+1}.$$

We assume that $a_\alpha \in C^1(\overline{D})$ if $|\alpha : m| = 1$ and $a_\alpha \in L^\infty(D)$ if $|\alpha : m| < 1$. Let $\varphi \in C^2(\overline{D})$ such that $\nabla_q \varphi \ne 0$ on $\overline{D}$. Then

Theorem 7.2 (Isakov [86]). *Suppose that either*

$$A_m(x,\xi) \ne 0 \quad \text{for all } \xi \in \mathbb{R}^{n+1} \setminus \{0\}$$

or

the coefficients of A_m are real-valued.

We assume

$$\sum_{i,j=1}^{q}\left((\partial_i\partial_j\varphi)\frac{\partial A_m}{\partial\zeta_i}\overline{\frac{\partial A_m}{\partial\zeta_j}}\right)+\frac{1}{s}Im\left(\sum_{k=1}^{q}(\partial_k A_m)\overline{\frac{\partial A_m}{\partial\zeta_k}}\right)>0 \tag{7.8}$$

for all $(x,t)\in\overline{D}$*, if*

$$\zeta=\xi+\sqrt{-1}s\nabla_q\varphi, \quad \zeta\ne 0, \quad A_m(x,\zeta)=0.$$

Then there exists a constant $C>0$ *such that*

$$\sum_{|\alpha:m|<1} s\int_D|\partial^\alpha u|^2e^{2s\varphi}dxdt\le C\int_D|Au|^2e^{2s\varphi}dxdt \tag{7.9}$$

for all large $s>0$ *and* $u\in C_0^\infty(D)$*.*

See also Isakov [84] and [85]. It is not always simple to find a weight function satisfying (7.8). For the realization for a parabolic equation, one can apply the Holmgren transform of the variables (e.g., Saut-Schereur [124]). We can easily show that our choice (6.5) satisfies (7.8) if $\gamma>0$ is sufficiently large. As is seen in section 6, the stability can be proved in a level set bounded by $\psi(x,t)$. The level set by ψ with large γ is a flat paraboloidal domain whose volume is proportional to $\frac{1}{\gamma}$, and for large γ, the volume is small. Therefore the subdomain where the uniqueness holds, is smaller if we choose large $\gamma>0$, and in terms of the parabolicity of the equation, it is not desirable to choose large γ. In order to prove the uniqueness in a larger domain, we need to divide such a domain into several congruent parabolic subdomains and repeat the estimates (see e.g., pp.63-64 in [86]). However, as is remarked after the proof of Theorem 5.1, such a patchwork argument may lose the best regularity. Thus, for Carleman estimates, we have to choose ψ not only for validating a Carleman estiamte but also for gaining a desirable shape of the level set where we can work for inverse problems. As for a general theory for Carleman estimates for functions without compact supports, see Tataru [128].

§7.3. Global Carleman estimate.

In section 4, we stated a global Carleman estimate by Imanuvilov [68] and here we refer to the succeeding papers. In the general theory, under requirements for a weight function, we can deduce a Carleman estimates. In other words, it is another difficult task to verify that a Carleman estimate holds true for given (x,t)-domain (e.g., $\Omega\times(0,T)$). Thus it is usually necessary to construct a special weight function. Among the achievements after [68], we present a Carleman estimate with a regular weight function, which is proved in Yuan and Yamamoto [136]. More precisely, a global Carleman estimate with the same type of Theorem 4.2 is proved similarly:

Theorem 7.3. *Assume that $\Gamma\subset\partial\Omega$ is an arbitrary subboundary, $a_{ij}\in C^1(\overline{\Omega})$, $a_{ij}=a_{ji}$, $1\le i,j\le n$ satisy (1.1) and (1.2). Let $d_0\in C^2(\overline{\Omega})$ be a function constructed in Lemma 4.2 and $0<t_0<T$. Let $\varphi(t,x)=e^{\lambda(d_0(x)-\beta|t-t_0|^2+M_1)}$, where $M_1>\sup_{0<t<T}\beta(t-t_0)^2$. Then there exist positive constants λ_0, s_0 and $C=C(\lambda_0,s_0)$ such that*

$$\begin{aligned}&\int_Q\left\{\frac{1}{s\varphi}\left(|\partial_t u|^2+\sum_{i,j=1}^n|\partial_i\partial_j u|^2\right)+s\lambda^2\varphi|\nabla u|^2+s^3\lambda^4\varphi^3|u|^2\right\}e^{2s\varphi}dxdt\\ \le& C\int_Q|Lu|^2e^{2s\varphi}dxdt+Cs\lambda\int_{\Gamma\times(0,T)}\varphi\left|\frac{\partial u}{\partial\nu_A}\right|^2e^{2s\varphi}d\Sigma\end{aligned}$$

for all $s>s_0$, $\lambda>\lambda_0$ and all $u\in H^{2,1}(Q)$ satisfying

$$u(\cdot,0)=u(\cdot,T)=0\quad\text{in }\Omega\quad u=0\quad\text{on }\partial\Omega\times(0,T).$$

The constants λ_0, s_0 and C continuously depend on $\sum_{i,j=1}^n\|a_{ij}\|_{C^1(\overline{\Omega})}$, T, Ω, σ_1.

We recall the definition of σ_1 in (1.2).

§7.4. Carleman estimate with second large parameter.

Carleman estimates are valid for parabolic systems such as a thermoelasticity system and a elasticity system with residual stress. For this system, by the coupling in terms of the highest-order derivatives, when we use a weight function with only a paramter $s > 0$, unlike a weakly coupled system (6.28), we can not obtain a Carleman estimate. As the weight function φ, we search for $\varphi = e^{\lambda\psi}$ with large parameter $\lambda > 0$ (see e.g., Theorem 3.1), and λ is the second large parameter. With the Carleman estimate with the second large parameter $\lambda > 0$, we can still manage the highest-order terms to obtain a Carleman estimate. As general treatment proving a Carleman estimate with the second large parameter, see Isakov and Kim [87], [88]. By our direct derivation, Carleman estimates with second large parameter are direct consequences (Theorems 3.1-3.3, 4.1-4.4 and 7.3).

§7.5. H^{-1}-Carleman estimates.

So far, we establish Carleman estimates for $Lu = f$ by $L^2(Q)$-norm of f and we call it an L^2-Carleman estimate. For the exact null controllability for a semilinear parabolic equation with nonlinear term $F(x, t, u, \nabla u)$ and an inverse source problem of determining $f \in H^{-\ell}(\Omega)$ with $\ell > 0$, an L^2-Carleman estimate does not work. Let the right-hand side of $Lu = f$ be given by

$$f = f_0 + \sum_{i=1}^{n} \partial_i f_i, \quad f_1, ..., f_n \in L^2(Q).$$

We are interested in a Carleman estimate where the right-hand side is estimated by $\|f_0 e^{s\varphi}\|_{L^2(0,T;H^{-1}(\Omega))} + \sum_{i=1}^{n} \|f_i\|_{L^2(Q)}$, which we call an H^{-1}-Carleman estimate. More precisely, we consider

$$Lu = \partial_t u - \sum_{i,j=1}^{n} \partial_i(a_{ij}(x,t)\partial_j u)$$

$$-\sum_{i=1}^{n} \partial_i(b_i(x,t)u) - c(x,t)u = f \qquad \text{in } Q, \tag{7.10}$$

$$u|_{\partial\Omega\times(0,T)} = 0, \quad u(\cdot,0) = u_0. \tag{7.11}$$

In addition to (1.1) and (1.2), assume

$$\begin{cases} a_{ij},\ 1 \le i,j \le n \text{ are Lipschitz continuous on } \overline{Q},\ a_{ij} = a_{ji}, \\ b_i \in L^\infty(0,T;L^r(\Omega)), \quad r > 2n,\ 1 \le i \le n, \\ c \in L^\infty(0,T;W^{-\mu}_{r_1}(\Omega)), \quad 0 \le \mu < \dfrac{1}{2},\ r_1 > \max\left\{\dfrac{2n}{3-2\mu}, 1\right\}. \end{cases} \tag{7.12}$$

We say that $u \in L^2(Q)$ is a weak solution to the problem (7.10) - (7.11) if for any $z \in L^2(0,T;H^2(\Omega))$ with $L^*z \in L^2(Q)$, $z|_{\partial\Omega\times(0,T)} = 0$ and $z(\cdot,T) = 0$, the following equality holds:

$$(u, L^*z)_{L^2(Q)} = (f,z)_{L^2(Q)} + (u_0, z(\cdot,0))_{L^2(\Omega)}.$$

Let d be given in Lemma 4.1 and let φ, α and σ_2 be defined by (4.5) and (4.6). In Imanuvilov and Yamamoto [76], [79], the following H^{-1}- Carleman estimate is proved.

Theorem 7.4.

(i) There exists a positive constant λ_0 such that for an arbitrary $\lambda \ge \lambda_0$, we can choose $s_0(\lambda) > 0$ satisfying: there exists a constant $C > 0$ such that

$$\begin{aligned} &\int_Q \{(s\varphi)^{1-2\ell}|\nabla u|^2 + (s\varphi)^{3-2\ell}|u|^2\}e^{2s\alpha}dxdt \\ \le& C\left(\|fe^{s\alpha}\|^2_{L^2(0,T;H^{-\ell}(\Omega))} + \int_{\omega\times(0,T)} (s\varphi)^{3-2\ell}|u|^2e^{2s\alpha}dxdt\right) \end{aligned} \tag{7.13}$$

for all $s > s_0$, $\ell \in [0,1]$ and each solution $u \in L^2(Q)$ to (7.10) - (7.11). Here the constants λ_0, s_0 and C are continuously dependent on σ_2 and independent of s.

(ii) If

$$f(x,t) = f_0(x,t) + \sum_{i=1}^n \partial_i f_i(x,t)$$

with $f_i \in L^2(Q)$, $1 \le i \le n$, then for $\ell \in \mathbb{R}$, there exists a positive constant λ_0 such that for an arbitrary $\lambda \ge \lambda_0$, we can choose $s_0(\lambda) > 0$ satisfying: there exists a constant $C > 0$ such that

$$\begin{aligned}&\int_Q \{(s\varphi)^{\ell-1}|\nabla u|^2 + (s\varphi)^{\ell+1}|u|^2)e^{2s\alpha}dxdt\\ \le& C\left(\|f_0(s\varphi)^{\frac{\ell}{2}}e^{s\alpha}\|^2_{L^2(0,T;H^{-1}(\Omega))} + \sum_{i=1}^n \|f_i(s\varphi)^{\frac{\ell}{2}}e^{s\alpha}\|^2_{L^2(Q)}\right.\\ &\left.+ \int_{\omega\times(0,T)} (s\varphi)^{1+\ell}|u|^2 e^{2s\alpha}dxdt\right) \end{aligned} \tag{7.14}$$

for all $s > s_0$, and each solution $u \in L^2(Q)$ to (7.10) - (7.11). Here the constants λ_0, s_0 and C are continuously dependent on σ_2 and independent of s.

Moreover if we further assume that $a_{ij} \in W^{1,\infty}(\Omega)$ and $b_i = 0$, $1 \le i \le n$ and $u|_{\omega\times(0,T)} = 0$, then

$$\begin{aligned}&\frac{1}{s}\|\sqrt{t(T-t)}\partial_t(ue^{s\alpha})\|^2_{L^2(0,T;H^{-1}(\Omega))}\\ &+ \int_Q \left(\frac{t(T-t)}{s}|\nabla u|^2 + \frac{s}{t(T-t)}|u|^2\right) e^{2s\alpha}dxdt\\ \le& C\|fe^{s\alpha}\|^2_{L^2(0,T;H^{-1}(\Omega))}\end{aligned} \tag{7.15}$$

for all $s \ge s_0$ and each solution $u \in L^2(Q)$ to (7.10) - (7.11).

Moreover we refer to Imanuvilov, Puel and Yamamoto [73] which proves an H^{-1} Carleman estimate for functions with non-zero Dirichlet boundary values on $\partial\Omega \times (0,T)$. In the Carleman estimate in [73], the term $\left(\frac{1}{s\varphi}|\nabla u|^2 + s\varphi|u|^2\right)e^{2s\alpha}$ is estimated with sharp norm of $u|_{\partial\Omega\times(0,T)}$. Furthermore we refer to Imanuvilov, Isakov and Yamamoto [72] as for a transferring argument from an L^2 Carleman estimate to an H^{-1} Carleman estimate for functions with compact supports. See Imanuvilov [71] for H^{-1}-Carleman estimated for hyperbolic equations.

§7.6. Carleman estimates for less regular principal terms.

For the direct derivation of a Carleman estimate in section 3, we assume that $a_{ij} \in C^1(\overline{Q})$. In H^{-1} Carleman estimate, we relax the regularity of a_{ij} to the Lipschitz continuity on $\overline{Q}$ (see (7.12)). In applications, it is also significant that the coefficients of the principal term are discontinuous. We consider

$$\partial_t u = \mathrm{div}\,(a(x)\nabla u(x,t)) \qquad \text{in } Q$$

where a is of piecewise C^1. Under some geometrical constraints on the interface of the break of the continuity of a, Carleman estimates are proved: Benabdallah, Dermenjian and Le Rousseau [13], [15], Benabdallah, Gaitan and Le Rousseau [16], Doubova, Osses and Puel [41], Le Rousseau and Robbiano [111] - [112], Poisson [120], Especially Le Rousseau and Robbiano [111] is a most updated work which establishes a Carleman estimate with generous geometrical constraints on the interface. See Benabdallah, Dermenjian and Le Rousseau [14], Le Rousseau [110] for a case of coefficients of bounded variations.

§7.7. Carleman estimate for degenerate parabolic equations.

So far, we assume that $\{a_{ij}\}_{1\le i,j\le n}$ is positive definite, that is, that (1.2) holds with $\sigma_1 > 0$. In some applications such as laminar flow on flat plates and population genetics, degenerate parabolic equations appear and see Cannarsa, Fragnelli and Rocchetti [21], [22], Cannarsa, Martinez and Vancostenoble [23], [24], where Carleman estimates are proved for degenerate parabolic equations and applied to the controllability. Here we state it according to Cannarsa and Yamamoto [25] in a simple case.

Let $1 < \alpha < 2$ and let us consider

$$\partial_t u(x,t) = \frac{\partial}{\partial x}\left(x^\alpha \frac{\partial u}{\partial x}\right) + f(x,t), \quad 0 < x < 1,\ t > 0, \tag{7.16}$$

$$u(1,t)=0, \quad \left(x^\alpha \frac{\partial u}{\partial x}\right)(0,t)=0, \quad t>0. \tag{7.17}$$

We set

$$a(x)=x^\alpha, \qquad 0\le x\le 1,$$

$$H^1_a(0,1)=\{u\in L^2(0,1);\ u \text{ is absolutely continuous in } (0,1],$$
$$\sqrt{a}\frac{\partial u}{\partial x}\in L^2(0,1),\ u(1)=0\},$$

and

$$\mathcal{D}(A)=\{u\in L^2(0,1);\ a\frac{\partial u}{\partial x}\in H^1(0,1)\},$$

$$Lu=\partial_t u-\frac{\partial}{\partial x}\left(x^\alpha\frac{\partial u}{\partial x}\right), \quad \varphi(x,t)=\frac{x^{2-\alpha}-2}{(2-\alpha)^2t^4(T-t)^4}, \quad 0\le x\le 1,\ 0<t<T.$$

In terms of the classical Hardy inequality, we can prove a Carleman estimate for a degenerate parabolic equation (7.16).

Theorem 7.5. *Let $1<\alpha<2$. There exist constants $s_0>0$ and $C>0$ such that*

$$\int_Q\left\{s^3\left(\frac{1}{t(T-t)}\right)^{12}x^{2-\alpha}u^2+s\left(\frac{1}{t(T-t)}\right)^4x^{\alpha-2}u^2+s\left(\frac{1}{t(T-t)}\right)^4x^\alpha|\partial_x u|^2\right.$$
$$\left.+s^{-2}(t(T-t))^8|\partial_t u|^2\right\}e^{2s\varphi}dxdt$$
$$\le C\int_Q|Lu|^2e^{2s\varphi}dxdt+C_1\int_0^T s\left(\frac{1}{t(T-t)}\right)^4|u_x(1,t)|^2e^{2s\varphi(1,t)}dt$$

for all $s\ge s_0$ and $u\in C([0,T];H^1_a(0,1))\cap L^2(0,T;\mathcal{D}(A))\cap H^1(0,T;L^2(0,1))$ satisfying (7.16) and (7.17).

§7.8. Carleman estimates for parabolic systems.

For weakly coupled parabolic systems (i.e., (6.28)), as is seen in sections 3 and 4, Carleman estimates are easily derived under the assumption that the coefficients of terms of lower orders are in $L^\infty(Q)$. However the Carleman estimate is very difficult for strongly coupled systems whose principal terms are different:

$$\partial_t u_\ell = \sum_{k=1}^{N} \sum_{i,j=1}^{n} a_{ij}^{k\ell}(x,t)\partial_i\partial_j u_k + f_\ell, \quad 1 \le \ell \le N. \tag{7.18}$$

Only a general theory by Calderón is available for proving Carleman estimates for systems of partial differential equations (e.g., Egorov [43], Zuilly [138]), but the Calderón theorem seems not applied to general strongly coupled parabolic systems. The difficulty for proving a Carleman estimate for a general parabolic system (7.18) consists in that the estimate in step (2) in section 3 does not work, so that we can not obtain an estimate similar to (3.12) unlike a single parabolic equation. However the estimate in step (3) in section 3 can be executed to obtain a estimate corresponding to (3.19). Here we will show it. We first assume conditions for $a_{ij}^{k\ell}$:

$$a_{ij}^{k\ell} \in C^1(\overline{Q}), \quad 1 \le i,j \le n,\ 1 \le k,\ell \le N, \tag{7.19}$$

there exists a constant $\sigma_4 > 0$ such that

$$\sum_{k,\ell=1}^{N} \sum_{i,j=1}^{n} a_{ij}^{k\ell}(x,t)\xi_i^k\xi_j^\ell \ge \sigma_4 \sum_{k=1}^{N}\sum_{i=1}^{n} |\xi_i^k|^2, \quad (x,t) \in \overline{Q},\ \xi_i^k \in \mathbb{R},\ 1 \le i \le n,\ 1 \le k \le N, \tag{7.20}$$

$$a_{ij}^{k\ell} = a_{ji}^{k\ell} = a_{ij}^{\ell k}, \quad 1 \le i,j \le n,\ 1 \le k,\ell \le N. \tag{7.21}$$

Condition (7.20) means that the part of second-order derivatives in x is uniformly elliptic and satisfies the Legendre condition (e.g., Giaquinta [59]). Condition (7.21) is the symmetry of $a_{ij}^{k\ell}$ also in k,ℓ, and is satisfied for example by a parabolic system describing a compressible fluid dynamics.

As in section 3, let $D \subset Q$ be a domain with smooth boundary ∂D, $d \in C^2(\overline{D})$ with $|\nabla d| \ne 0$ on $\overline{D}$, and let

$$\psi(x,t) = d(x) - \beta(t-t_0)^2 + c_0$$

with $t_0 \in (0,T)$, $\beta > 0$, $c_0 > 0$ satisfying $\inf_{(x,t)\in Q} \psi(x,t) > 0$. Then we have

Proposition 7.1. *There exist constants* $\lambda_0 > 0$ *and* $s_0 > 0$ *such that we can choose* $C = C(s, \lambda) > 0$ *such that*

$$\begin{aligned}&\int_D s\lambda^2\varphi|\nabla \mathbf{u}|^2 e^{2s\varphi}dxdt\\ \le& C\int_D \sum_{\ell=1}^N \left| \partial_t u_\ell - \sum_{k=1}^N\sum_{i,j=1}^n a_{ij}^{k\ell}\partial_i\partial_j u_k\right|^2 e^{2s\varphi}dxdt\\ +&C\int_D s^3\lambda^4\varphi^3|\mathbf{u}|^2e^{2s\varphi}dxdt\end{aligned}$$

for all $s > s_0$, $\lambda > \lambda_0$ *and* $\mathbf{u} \in H^{2,1}(Q)^N$ *such that supp* $\mathbf{u} \in D$.

This is not a Carleman estimate because the right-hand side contains the zeroth order term $\int_D s^3\lambda^4\varphi^3|\mathbf{u}|^2e^{2s\varphi}dxdt$. This is similar to (3.19) and unlike a single parabolic equation, we can not estimate in step (2) in section 3 (i.e., (3.12)) by means of more coupling with $a_{ij}^{k\ell}$, so that we can not absorb the term $\int_D s^3\lambda^4\varphi^3|\mathbf{u}|^2e^{2s\varphi}dxdt$ into the left-hand side. The proposition corresponds to the pseudoconvexity in Hörmander [65] (p.203). Similarly to Theorem 8.7.1 and Corollary 8.7.1 in [65], by our proposition we can prove: Let $a_{ij}^{k\ell} \in C^\infty(\overline{Q})$. There exists a finite dimensioncal space $\mathcal{M}$ in $H^{2,1}(D)^N$ such that if $\mathbf{u} \in H^{2,1}(D)^N$ satisfies

$$\partial_t u_\ell = \sum_{k=1}^N\sum_{i,j=1}^n a_{ij}^{k\ell}\partial_i\partial_j u_k \quad \text{in } D,\ 1 \le \ell \le N$$

and

$$|\mathbf{u}| = |\nabla \mathbf{u}| = 0 \quad \text{on } \partial D,$$

then $\mathbf{u} \in \mathcal{M}$. If we have a Carleman estimate, then we can conclude that $\mathcal{M} = \{0\}$, that is, the unique continuation from ∂D holds. Moreover if $\mathcal{M} = \{0\}$, then we can prove a conditional stability estimate by an argument in section 5.

Proof. We set

$$\mathbf{w} = (w_1, ..., w_N)^T = e^{s\varphi}\mathbf{u}, \quad \varphi = e^{\lambda\psi},$$

$$P\mathbf{w} = e^{s\varphi}\left(\partial_t(e^{-s\varphi}\mathbf{w}) - \left(\sum_{k=1}^N\sum_{i,j=1}^n a_{ij}^{k1}\partial_i\partial_j(e^{-s\varphi}w_k), ..., \sum_{k=1}^N\sum_{i,j=1}^n a_{ij}^{kN}\partial_i\partial_j(e^{-s\varphi}w_k)\right)^T\right).$$

Moreover we set

$$\sigma_{\ell k}(x,t) = \sum_{i,j=1}^n a_{ij}^{k\ell}(x,t)(\partial_i d)(x)\partial_j d(x), \quad \mu_{\ell k}(x,t) = \sum_{i,j=1}^n a_{ij}^{k\ell}(x,t)\partial_i\partial_j d(x)$$

and

$$A_\ell = s\lambda^2\varphi\sum_{k=1}^N \sigma_{\ell k}w_k + s\lambda\varphi\sum_{k=1}^N \mu_{\ell k}w_k - s\lambda\varphi(\partial_t\psi)w_\ell, \quad 1 \le \ell \le N.$$

Direct calculations yield

$$\begin{aligned}[P\mathbf{w}]_\ell = {} & \partial_t w_\ell - \sum_{k=1}^N\sum_{i,j=1}^n a_{ij}^{k\ell}\partial_i\partial_j w_k + 2s\lambda\varphi\sum_{k=1}^N\sum_{i,j=1}^n a_{ij}^{k\ell}(\partial_i d)(\partial_j w_k) \\ & -s^2\lambda^2\varphi^2\sum_{k=1}^N \sigma_{\ell k}w_k + A_\ell, \quad 1 \le \ell \le N.\end{aligned} \tag{7.22}$$

Here we recall that $[P\mathbf{w}]_\ell$ denotes the ℓ-th component of $P\mathbf{w}$. Now as in step (3) in section 3, for $1 \le \ell \le N$, we multiply $[P\mathbf{w}]_\ell = f_\ell e^{s\varphi}$ with $s\lambda^2\varphi w_\ell$, sum up over $\ell = 1, ..., N$ and integrate by parts over $(x,t) \in D$. The estimation is done by using (7.19) - (7.21), similarly to estimates (3.14) - (3.18). Henceforth let $\lambda > 1$ and $s > 1$. For example, we have

$$\begin{aligned} & -\sum_{k,\ell=1}^N\sum_{i,j=1}^n\int_D a_{ij}^{k\ell}(\partial_i\partial_j w_k)s\lambda^2\varphi w_\ell dxdt \\ = & \sum_{k,\ell=1}^N\sum_{i,j=1}^n\int_D s\lambda^2(\partial_i\varphi)a_{ij}^{k\ell}(\partial_j w_k)w_\ell dxdt + \sum_{k,\ell=1}^N\sum_{i,j=1}^n\int_D s\lambda^2\varphi a_{ij}^{k\ell}(\partial_j w_k)(\partial_i w_\ell)dxdt \\ \ge & \sigma_4\int_D s\lambda^2\varphi|\nabla\mathbf{w}|^2 dxdt - C\int_D s\lambda^3\varphi|\nabla\mathbf{w}||\mathbf{w}|dxdt \end{aligned}$$

and

$$\begin{aligned}
&\sum_{k,\ell=1}^{N}\sum_{i,j=1}^{n} a_{ij}^{k\ell}(\partial_i d)w_\ell(\partial_j w_k)\\
=&\sum_{k>\ell}^{N}\sum_{i,j=1}^{n} a_{ij}^{k\ell}(\partial_i d)(w_\ell(\partial_j w_k)+w_k(\partial_j w_\ell)) + \sum_{k=1}^{N}\sum_{i,j=1}^{n} a_{ij}^{k\ell}(\partial_i d)w_k(\partial_j w_k)\\
=&\frac{1}{2}\sum_{k\neq\ell}^{N}\sum_{i,j=1}^{n} a_{ij}^{k\ell}(\partial_i d)\partial_j(w_k w_\ell) + \frac{1}{2}\sum_{k=\ell}^{N}\sum_{i,j=1}^{n} a_{ij}^{k\ell}(\partial_i d)\partial_j(w_k w_\ell)\\
=&\frac{1}{2}\sum_{k,\ell=1}^{N}\sum_{i,j=1}^{n} a_{ij}^{k\ell}(\partial_i d)\partial_j(w_k w_\ell),
\end{aligned}$$

so that

$$\begin{aligned}
&\left|\sum_{k,\ell=1}^{N}\sum_{i,j=1}^{n}\int_D s^2\lambda^3\varphi^2 a_{ij}^{k\ell}(\partial_i d)w_\ell(\partial_j w_k)dxdt\right|\\
=&\frac{1}{2}\left|\sum_{k,\ell=1}^{N}\sum_{i,j=1}^{n} s^2\lambda^3\varphi^2 a_{ij}^{k\ell}(\partial_i d)\partial_j(w_k w_\ell)dxdt\right|\\
=&\frac{1}{2}\left|\sum_{k,\ell=1}^{N}\sum_{i,j=1}^{n} s^2\lambda^3\partial_j(\varphi^2 a_{ij}^{k\ell}(\partial_i d))w_k w_\ell dxdt\right|\\
\le& C\int_D s^2\lambda^4\varphi^2|\mathbf{w}|^2 dxdt.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\sigma_4\int_D s\lambda^2\varphi|\nabla\mathbf{w}|^2 dxdt - C\int_D s\lambda^3\varphi|\nabla\mathbf{w}||\mathbf{w}|dxdt - C\int_D (s^3\lambda^4\varphi^3 + s^2\lambda^4\varphi^2)|\mathbf{w}|^2 dxdt\\
\le&\int_D s\lambda^2\varphi|(\mathbf{f}e^{s\varphi}\cdot\mathbf{w})|dxdt.
\end{aligned}$$

Moreover

$$s\lambda^3\varphi|\nabla\mathbf{w}||\mathbf{w}| = s\lambda^2\varphi|\mathbf{w}|\lambda|\nabla\mathbf{w}| \le \frac{1}{2}s^2\lambda^4\varphi^2|\mathbf{w}|^2 + \frac{1}{2}\lambda^2|\nabla\mathbf{w}|^2$$

and

$$|s\lambda^2\varphi(\mathbf{f}e^{s\varphi}\cdot\mathbf{w})| \le \frac{1}{2}|\mathbf{f}e^{s\varphi}|^2 + \frac{1}{2}s^2\lambda^4\varphi^2|\mathbf{w}|,$$

noting that $s > 1$ and $\lambda > 1$ we can choose only terms with the maximal orders in s, λ to obtain

$$\left(\sigma_4 - \frac{C}{s\varphi}\right) \int_D s\lambda^2\varphi|\nabla \mathbf{w}|^2 dxdt$$
$$\le C \int_D |\mathbf{f}|^2 e^{2s\varphi} dxdt + C \int_D s^3\lambda^4\varphi^3|\mathbf{w}|^2 dxdt.$$

Choosing $s > 0$ large, we can

$$\int_D s\lambda^2\varphi|\nabla \mathbf{w}|^2 dxdt \le C \int_D |\mathbf{f}|^2 e^{2s\varphi} dxdt + C \int_D s^3\lambda^4\varphi^3|\mathbf{w}|^2 dxdt.$$

Rewriting the obtained estimate in terms of $\mathbf{u}$, we complete the proof of the proposition.

Here we look over other types of parabolic systems which are not strongly coupled but important in the mathematical physics.

I. the Navier-Stokes equations.

We consider the linearized Navier-Stokes equations describing the velocity field $\mathbf{v} = (v_1, v_2, v_3)^T$ in the incompressible viscous fluid:

$$\partial_t \mathbf{v} - \gamma\Delta\mathbf{v} + (\mathbf{b}(x,t)\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}(x,t), \quad (x,t) \in Q \tag{7.23}$$

$$\operatorname{div} \mathbf{v}(x,t) = 0, \qquad (x,t) \in Q \tag{7.24}$$

$$\mathbf{v}(x,t) = 0, \qquad x \in \partial\Omega,\ 0 < t < T. \tag{7.25}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $\gamma > 0$ is a constant describing the viscosity, and for simplicity we assume that the density is one, and we set $\mathbf{b} = (b_1, ..., b_n)^T$, $[(\mathbf{b}\cdot\nabla)\mathbf{v}]_\ell = \sum_{j=1}^n b_j\partial_j v_\ell$.

Setting $\mathbf{b} = \mathbf{v}$, equation (7.23) is the Navier-Stokes equations. We assume that

$$\mathbf{b} \in L^\infty(0,T; W^{1,\infty}(\Omega))^3, \quad \|\mathbf{b}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))^3} \le M$$

with arbitrarily fixed constant $M > 0$. Let $\omega \subset \Omega$ be a subdomain such that $\partial\omega \supset \partial\Omega$. Let $d \in C^2(\overline{\Omega})$ be constructed in Lemma 4.1 and let α be defined by (4.5).

Theorem 7.6. *There exists a constant $\lambda_0 = \lambda_0(\Omega, \omega, T) > 0$ such that for $\lambda \ge \lambda_0$, we can choose constants $C = C(\lambda, M) > 0$ and $s_0 = s_0(\lambda_0, M) > 0$ such that*

$$\int_Q \left(\frac{s^5}{t^5(T-t)^5}|\mathbf{v}|^2 + \frac{s^3}{t^3(T-t)^3}|\nabla \mathbf{v}|^2 + \frac{s^4}{t^4(T-t)^4}|rot\,\mathbf{v}|^2 \right.$$
$$\left. + \frac{s^2}{t^2(T-t)^2}|\nabla rot\,\mathbf{v}|^2 \right) e^{2s\alpha} dxdt$$
$$\le C \int_Q \frac{s}{t(T-t)}|rot\,\mathbf{f}|^2 e^{2s\alpha} dxdt + Ce^{Cs}(\|\partial_t(rot\,\mathbf{v})\|^2_{L^2(\omega\times(0,T))} + \|\mathbf{v}\|^2_{L^2(0,T;H^3(\omega))})$$

for all $s \ge s_0$ and all $\mathbf{v} \in H^{2,1}(Q)$ satisfying (7.23) - (7.25) and $rot\,\mathbf{v} \in H^{2,1}(Q)$.

In Theorem 7.6 we have to assume that $\partial\omega \supset \partial\Omega$ and it is quite strong. For any subdomain $\omega \subset \Omega$, we can prove the following Carleman estimate by a Carleman estimate for a single parabolic equation giving also a sharp estimate of the boundary value which is proved in Imanuvilov, Puel and Yamamoto [73]. See also Choulli, Imanuvilov and Yamamoto [34]. We set

$$H = \{\mathbf{v} \in L^2(Q)^n;\ \mathrm{div}\,\mathbf{v} = 0,\ (\mathbf{v}\cdot\nu) = 0 \,\text{on}\, \partial\Omega\}$$

and

$$V = \{\mathbf{v} \in H^1_0(Q)^n;\ \mathbf{v} \in H\}.$$

Theorem 7.7. *Let $\ell \in C^\infty[0,T]$, $\ell(t) > 0$ for $0 < t < T$, and* $\ell(t) = \begin{cases} t, & 0 \le t \le \frac{T}{4}, \\ T-t, & \frac{3}{4}T \le t \le T, \end{cases}$ *and*

$$\alpha_1(x,t) = \frac{e^{\lambda d(x)} - e^{2\lambda\|d\|_{C(\overline{\Omega})}}}{\ell^8(t)}.$$

We assume that $\mathbf{f} \in L^2(0,T;H)$. *There exists a constant* $\lambda_0 > 0$ *such that for* $\lambda > \lambda_0$, *there exist constants* $C > 0$ *and* $s_0 > 0$ *satisfying*

$$\begin{aligned}
&\int_Q s\varphi|rot\,\mathbf{v}|^2 e^{2s\alpha_1}dxdt + \int_Q s^2\varphi^2|\mathbf{v}|^2 e^{2s\alpha_1}dxdt \\
\le& C\int_Q |\mathbf{f}|^2 e^{2s\alpha_1}dxdt \\
&+ \int_0^T\int_\omega (s\varphi|rot\,\mathbf{v}|^2 + s^2\varphi^2|\mathbf{v}|^2)e^{2s\alpha_1}dxdt
\end{aligned}$$

for all $s \ge s_0$ *and* $\mathbf{v} \in L^2(0,T;V) \cap H^{2,1}(Q)^n$ *with* $\mathbf{v}(\cdot,0) \in V$.

These Carleman estimate can be applied to inverse problems by a method in section 6 and see [34].

The Carleman estimates for the Navier-Stokes equations have been studied related with the controllability and see: Coron and Guerrero [37], Fabre and Lebeau [51], Fernández-Cara, Guerrero, Imanuvilov and Puel [55], Fursikov and Imanuvilov [58], Guerrero [63], Imanuvilov [69], [70].

II. Carleman estimates for parabolic systems by data of one component.

Let $(c_{ij})_{1\le i,j\le 2} \in L^\infty(Q)$ and $(A_{ij})_{1\le i,j\le 2} \in L^\infty(Q)^n$. Consider the following reaction-diffusion system with convection terms:

$$(7.26)\qquad \begin{cases} \partial_t u = \Delta u + c_{11}\; u + c_{12}\; v + A_{11}\cdot\nabla u + A_{12}\cdot\nabla v + f \quad \text{in } Q, \\ \partial_t v = \Delta v + c_{21}\; u + c_{22}\; v + A_{21}\cdot\nabla u + A_{22}\cdot\nabla v + g \quad \text{in } Q, \\ u = v = 0 \qquad \text{on } \partial\Omega\times(0,T). \end{cases}$$

Our main interest is to derive a Carleman estimate of solution (u,v) of (7.26) by solely observing u in $\omega\times(0,T)$. We assume

$$(7.27)\qquad \begin{cases} \text{Let } \omega\subset\Omega \text{ with } \partial\omega\cap\partial\Omega = \gamma \text{ and } |\gamma| \ne 0, \\ |A_{12}(x,t)\cdot\nu(x)| \ne 0, \qquad (x,t)\in\gamma\times(0,T), \\ \|A_{12}\|_{C^2(\overline{\omega\times(0,T)})^n}, \quad \|c_{12}\|_{C^2(\overline{\omega\times(0,T)})}, \quad \|A_{11}\|_{C^1(\overline{\omega\times(0,T)})^n} \le M, \end{cases}$$

where $M > 0$ is an arbitrarily fixed constant. Then in Benabdallah, Cristofol, Gaitan and Yamamoto [12], the following Carleman estimate is proved:

Theorem 7.8. *Let $\omega \subset \Omega$ be a subdomain such that $\overline{\omega} \subset \Omega$. Under (7.27), there exist $\psi_\omega \in C^2(\overline{\Omega})$ with $\psi_\omega < 0$ on $\overline{\Omega}$ and two positive constants s_0 and C which depend on T, M, Ω, ω and the $L^\infty(Q)$-norms of c_{ij}, A_{ij}, such that there exist positive constants $C_1(s)$ and C such that the following Carleman estimate holds:*

$$\begin{aligned}&\int_Q (s\rho)^{-1}e^{2s\alpha_\omega}(|\partial_t u|^2 + |\partial_t v|^2 + |\Delta u|^2 + |\Delta v|^2\\ &+(s\rho)^2|\nabla u|^2 + (s\rho)^2|\nabla v|^2 + (s\rho)^4|u|^2 + (s\rho)^4|v|^2)dxdt\\ &\le C_1(s)(\|u\|^2_{H^{2,1}(\omega\times(0,T))} + \|f\|^2_{L^2(\omega\times(0,T))}) + C\int_Q e^{2s\alpha_\omega}(|f|^2+|g|^2)dxdt\end{aligned}$$

for all $s \ge s_0$ and any solution (u, v) to (7.26). Here we set

$$\alpha_\omega(x,t) = \frac{\psi_\omega(x)}{t(T-t)}, \quad \rho(t) = \frac{1}{t(T-t)}. \tag{7.28}$$

This is a Carleman estimate for a 2-component system with extra data in $\omega \times (0, T)$ of only one component. In [6] and [38], it is assumed that $A_{11} = A_{12} = 0$, and the proof can be completed by directly substituting v by means of u in $\omega \times (0, T)$. By the first-order coupling in the parabolic system (7.26), we need the first and the second conditions in (7.27). In [12], also the following Carleman estimate for a reaction-diffusion system with 3 components by one component observation. That is, we consider

$$\begin{cases}\partial_t u(x,t) = \Delta u + c_{11}(x,t)u + c_{12}(x,t)v + c_{13}(x,t)w + f(x,t) & \text{in } Q,\\ \partial_t v(x,t) = \Delta v + c_{21}(x,t)u + c_{22}(x,t)v + c_{23}(x,t)w + g(x,t) & \text{in } Q,\\ \partial_t w(x,t) = \Delta w + c_{31}(x,t)u + c_{32}(x,t)v + c_{33}(x,t)w + h(x,t) & \text{in } Q,\\ u = v = w = 0 \qquad \text{in } \partial\Omega\times(0,T). \end{cases} \tag{7.29}$$

We assume

$$
(7.30) \qquad \begin{cases} (c_{ij})_{1\le i,j\le 3} \in W^{2,\infty}(Q), \quad \|c_{ij}\|_{W^{2,\infty}(Q)} \le M, \\ \omega \text{ is of class } C^2,\ \partial\omega \cap \partial\Omega = \gamma \text{ and } |\gamma| \ne 0, \\ \left|(\nabla c_{12} - \dfrac{c_{12}}{c_{13}}\nabla c_{13})\cdot \nu\right| \ne 0 \text{ on } \gamma \times [0,T], \\ \|c_{12}\|_{W^{3,\infty}(\omega\times(0,T))}, \|c_{13}\|_{W^{3,\infty}(\omega\times(0,T))} \le M, \quad c_{13} \ne 0 \quad \text{on } \overline{Q}. \end{cases}
$$

Then we show a Carleman estimate with extra data of one component.

Theorem 7.9. *Under (7.30), there exist $\psi_\omega \in C^2(\overline{\Omega})$ with $\psi_\omega < 0$ on $\overline{\Omega}$ and a constant $s_0 > 0$ which depends on T, M, Ω, ω and the $L^\infty(\Omega)$-norms of c_{ij}, $1 \le i, j \le 3$ such that we can choose constants $C_1(s) > 0$ and $C > 0$ satisfying:*

$$
\begin{aligned}
&\int_Q (s\rho)^{-1} e^{2s\alpha_\omega}(|\partial_t u|^2 + |\partial_t v|^2 + |\partial_t w|^2 + |\Delta u|^2 + |\Delta v|^2 + |\Delta w|^2 \\
&+(s\rho)^2|\nabla u|^2 + (s\rho)^2|\nabla v|^2 + (s\rho)^2|\nabla w|^2 + (s\rho)^4 u^2 + (s\rho)^4 v^2 + (s\rho)^4 w^2)dxdt \\
&\le C_1(s)(\|u\|^2_{H^{4,2}(\omega\times(0,T))} + \|f\|^2_{H^{2,1}(\omega\times(0,T))} + \|g\|^2_{L^2(\omega\times(0,T))} + \|h\|^2_{L^2(\omega\times(0,T))}) \\
&+C\int_Q (|f^2| + |g|^2 + |h|^2) e^{2s\alpha_\omega} dxdt
\end{aligned}
$$

for all $s \ge s_0$ and all (u, v, w) satisfying (7.29). Here α_ω and ρ are defined by (7.28).

We refer also to Ammar-Khodja, Benabdallah, Dupaix and González- Burgos [7] as for a Carleman estimate with observation data of a limited number of components.

§8. Overview of results by parabolic Carleman estimates.

§8.1. Estimation of solutions to parabolic equations.

Let L be a parabolic operator defined by (1.3) and let us consider a parabolic equation $Lu = f$ in Q. In addition to a classical initial value/boundary value problems, there are several possibilities of formulations, which are meaningful also from the practial viewpoints.

1. Continuation problem. Let $\Gamma \subset \partial\Omega$ be a subboundary and let D be a given subdomain in Q. Determine $u|_D$ by extra data such as $u|_{\Gamma\times(0,T)}$ or $\frac{\partial u}{\partial \nu_A}|_{\Gamma\times(0,T)}$.

2. State estimation. Let boundary values be given over the whole $\partial\Omega \times (0,T)$. Let $0 \le t_0, t_1 \le T$ be given. Determine $u(x,t_0)$, $x \in \Omega$ by extra data such as $u|_{\omega\times(t_1,T)}$.

3. Backward problem. Determine $u(x,t_0)$, $x \in \Omega$ by $u(x,T)$, $x \in \Omega$.

As other papers on related problems, see Cannon [26], Chapters 10 and 11, John [89]. The continuation problem is important for example for the inverse heat conduction problem (e.g., Alifanov [3]) and the state estimation is related with the controllability.

For the continuation problem, as is stated in section 5, the Carleman estimate is a main method. In the case of $\overline{D} \subset Q$, a Carleman estimate and the cut-off technique yield a Hölder stability estimate. As other cases, we have

(1) $u(x,t_0)$, $x \in \partial\Omega$, $t_0 > 0$.

(2) $u(x,0)$, $x \in \Omega$.

(3) $u(x,0)$, $x \in \partial\Omega$.

Case (i) is discussed in Theorem 5.2, and we can argue case (ii) similarly, and in the both cases, the stability is of single logrithmic rate. Case (iii) needs more delicate application of the method of Theorem 5.2 and we can expect the stability of double logarithmic rate.

As for related results with the continuation problem, see Canuto, Rosset and Vessella [27], Di Cristo, Rondi and Vessella [40], Escauriaza and Vessella [49], Vessella [131-133]. In [40], assuming that $u|_{\Gamma\times(0,T)} = 0$, the authors prove a Hölder stability estimate for $\|u\|_{L^2(D\times(0,T))}$ where ∂D touches Γ and $T_1 < T$, and apply the estimate for the determination of shapes of unknown subboundary. The situation in [40] is different from Theorem 5.2 because $u|_{\Gamma\times(0,T)}$ is given, and so the stability is much better than Theorem 5.2.

For the state estimation, we refer to Klibanov [97], Li, Yamamoto and Zou [113], Xu and Yamamoto [134]. In Li, Yamamoto and Zou [113], assuming the zero Dirichlet boundary condition on $\partial\Omega \times (0,T)$, a logarithmic conditional stability estimate is proved in determining $u(x,0)$, $x \in \Omega$ by $u|_{\omega\times(t_1,T)}$ or $\frac{\partial u}{\partial \nu_A}|_{\Gamma\times(t_1,T)}$, where $t_1 > 0$ is given, $\omega \subset \Omega$ is an arbitrary subdomain and $\Gamma \subset \partial\Omega$ is an arbitrary subboundary. In Xu and Yamamoto [134], the case of $t_1 = 0$ is considered. For a parabolic inequality $|Lu| \le |f|$ in Q, Klibanov [97] proves a logarithmic conditional stability estimate in determining $u(x,0)$, $x \in \Omega$ by $u|_{\partial\Omega\times(0,T)}$ and $\frac{\partial u}{\partial \nu_A}|_{\partial\Omega\times(0,T)}$. See also Klibanov and Tikhonravov [99].

§8.2. Inverse problems for parabolic equations.

Since Bukhgeim and Klibanov [20] applying Carleman estimates to inverse problems of determining coefficients or source terms, there are many papers not only for parabolic equations but also for other types of equations such as hyperbolic equations. However we here restrict the scope to inverse problems for parabolic equations by Carleman estimates. A typical formulation for the inverse problem is as follows. Let

$$\partial_t u = \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + c(x)u + R(x,t)f(x) \quad \text{in } Q.$$

Let ω be an arbitrary subdomain, $\gamma, \Gamma \subset \partial\Omega$ be subboundaries. Let $0 < t_0 < T$ and R be given. Determine $f(x)$ or $c(x)$ or $a_{ij}(x)$ by

$$\{u(\cdot,t_0)|_{\Omega},\, u|_{\omega\times(0,T)},\, u|_{\Gamma\times(0,T)}\}$$

or

$$\left\{u(\cdot,t_0)|_\Omega,\ \frac{\partial u}{\partial \nu_A}|_{\gamma\times(0,T)},\ u|_{\Gamma\times(0,T)}\right\}.$$

After Bukhgeim and Klibanov [20], we should refer to Bukhgeim [19], Isakov [81], [83], [86], Klibanov [93], [95], [100]. In particular, Theorem 6.4.1 (p.152) in [81] proves the uniqueness in inverse problems for a parabolic inverse problem by the method in [20]. Khaĭdarov [92] proves a stability estimate locally in Ω and also see Isakov [83]. As monographs, see Choulli [33], section 3, Isakov [86], Chapter 9 as for accounts for wider classes of inverse problems for parabolic equations. Imanuvilov and Yamamoto [74] establishes the Lipschitz stability in determining $f(x)$ over Ω in the case of $\Gamma = \partial\Omega$. The key is the global Carleman estimate (Theorems 4.1 and 4.2) and see the proof of Theorem 6.2 as a simplified argument in [74]. Yamamoto and Zou [135] proves the Lipschitz stability in determining $c(x)$, $x \in \Omega$ and for it, the assumption $u(x,t_0) > 0$ for $x \in \overline{\Omega}$ is essential. This condition is very restrictive and there are two disadvantages in the results by Carleman estimates:

(1) Some positivity at t_0 is indispensable.

(2) We can not choose $t_0 = 0$.

We can not choose $t_0 = 0$, because as is pointed in section 6, for the application of a Carleman estimate, the weight function must gain the maximum at t_0 when spatial data $u(\cdot,t_0)$ are given and so if we take $t_0 = 0$, then the solution u must be continued over $t = 0$ but the parabolic equation is not reversible in time and the continuation is impossible in general.

The condition that $t_0 > 0$ means that we have to guarantee the positivity condition at an intermediate time t_0 of the heat process. In the case of determination of $c(x)$, by positive Dirichlet boundary values, the maximum principle guarantees that $u(\cdot, t_0) > 0$ on $\overline{\Omega}$.

In Yuan and Yamamoto [136], the determination of multiple coefficients $a_{ij}(x)$, $1 \le i, j \le n$, $x \in \overline{\Omega}$ is discussed. Since we have to determine $\frac{n(n+1)}{2}$ functions a_{ij}, we need data by repeating observations suitably and keep some independency or non-degeneracy of repeated $u(\cdot, t_0)$ at $t_0 > 0$. The values of u at t_0 can not be chosen directly, so that in [136], we add ℓ-times control function $h_i(x,t)$ supported in $\omega \times (0,T)$, $i = 1, ..., \ell$:

$$Lu = h_i, \quad i = 1, ..., \ell,$$

in order that the corresponding solutions u_i satisfy the desired non-degeneracy condition. The non-degeneracy conditon is achieved in view of the approximate controllability. The Lipschitz stability requires $\frac{(n+1)^2 n}{2}$-times suitable controls and the corresponding observation data, and with special choice we can reduce the number of the observations to $\frac{n(n+3)}{2}$.

As for the inverse problems of determining coefficients in parabolic systems with data of limited numbers of components, we refer to Benabdallah, Cristofol, Gaitan and Yamamoto [12], Cristofol, Gaitan and Ramoul [38], Cristofol, Gaitan, Ramoul and Yamamoto [39]. The key Carleman estimates are Theorems 7.8 and 7.9.

Choulli, Imanuvilov and Yamamoto [34], Imanuvilov and Yamamoto [75] discuss inverse problems of determining source terms in the linearized Navier-Stokes equations on the basis of Theorem 7.6.

As for works concerning the determination of nonlinearity in a parabolic equation by Carleman estimates, see Boulakia, Grandmont and Osses [18], Egger, Engl and Klibanov [42], Kaltenbacher and Klibanov [90], Klibanov [94], [96], Klibanov and Timonov [100], Chapter 4.

As for the inverse problems for parabolic equations with discontinuous coefficients of the principal term, see Bellassoued and Yamamoto [11], Benabdallah, Dermenjian and Le Rousseau [15], Benabdallah, Gaitan and Le Rousseau [16], Poisson [120].

Finally we mention a method for the numerical reconstruction of a coefficient by changing the coefficient inverse problem to the unique continuation for partial differential equation (6.37) with integral terms. We refer to Beilina and Klibanov [9], [10].

§9. Carleman estimates with x-independent weight functions.

In addition to a "space-like" Carleman estimate considered so far, one can consider a "time-like" Carleman estimate. Such a Carleman estimate is discussed in section 2 of Chapter IV of [107] and Lees and Protter [109] with different form of the weight function $(\alpha - t)^{-2s}$ with some $\alpha \in \mathbb{R}$ and applied to a backward parabolic problem. In this section, we will prove Carelman estimates with x-independent weight functions, and apply them to prove the conditional stability for a backward parabolic problem in time and an inverse problem of determining a source term.

We consider the parabolic operator whose ellitpic part is the divergence form:

$$Lu(x,t) = \partial_t u(x,t) - \sum_{i,j=1}^{n} \partial_i (a_{ij}(x,t)\partial_j u) = f(x,t).$$

In this section, in place of (1.1) we assume

$$a_{ij} \in C^1([0,T]; L^\infty(\Omega)), \quad a_{ij} = a_{ji}, \quad 1 \le i,j \le n \tag{9.1}$$

and we keep the assumption (1.2).

Remark. By the density argument (e.g., Corollary 2.1 in Imanuvilov and Yamamoto [79], all the following arguments work if a_{ij} are Lipschitz continuous in $t \in [0,T]$ to $L^\infty(\Omega)$.

§9.1. Carleman estimate and an application to a parabolic equation backward in time.

First we use a simple weight function: we set

$$\varphi(t) = e^{\lambda t}$$

where $\lambda > 0$ is fixed suitably. We note that $\partial_t \varphi = \lambda \varphi$.

Remark. As a method with a similar spirit, we can refer to the weight energy method. On the weight energy method, there are many papers and see monographs Ames and Straughan [4], Bloom [17], Lees and Protter [109], Payne [118], and the references therein. Except for Murray and Protter [117] for equations of hyperbolic types, all the papers use just t as weight function, and do not use the second large parameter. In Murray and Protter [117], the weight function e^{st^λ} is used to prove properties for the asymptotic behaviour.

The serious diffference is only with the introduction of the second large parameter $\lambda > 0$. Such a second large parameter is very flexible and gains a lot of possibility for better estimates. Also in this section, we see that the weight function $e^{\lambda t}$ gives estimate of Carleman type but the simple weight t or another choice without the second large parameter can not produce such estimates. As is shown in this section, our Carleman estimate yields conditional stability estimates by a usual cut-off argument (see e.g., Hörmander [65]) for parabolic equations backward in time whose elliptic part is not symmetric and the coefficients are also dependent on t.

Set

$$v = e^{s\varphi} u, \quad Pv = e^{s\varphi} L(e^{-s\varphi} v) = e^{s\varphi} f.$$

Assume that

$$u|_{\partial\Omega\times(0,T)} = 0. \tag{9.2}$$

Then $e^{s\varphi}\partial_t(e^{-s\varphi}v) = \partial_t v - s\lambda\varphi v$, $e^{s\varphi}\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j(ve^{-s\varphi})) = \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j v)$, and

$$Pv = e^{s\varphi}L(e^{-s\varphi}v) = \partial_t v - \left(s\lambda\varphi v + \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j v)\right) = e^{s\varphi}f. \tag{9.3}$$

We have

$$\begin{aligned}
&\|e^{s\varphi}f\|^2_{L^2(Q)} \\
&= \int_Q |\partial_t v|^2 dxdt + 2\int_Q (\partial_t v)\left(-s\lambda\varphi v - \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j v)\right) dxdt \\
&+ \int_Q \left| s\lambda\varphi v + \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j v)\right|^2 dxdt \\
&\ge \int_Q |\partial_t v|^2 dxdt + 2\int_Q \partial_t v\left(-\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j v)\right) dxdt \\
&+2\int_Q (\partial_t v)(-s\lambda\varphi)v dxdt \\
&\equiv \int_Q |\partial_t v|^2 dxdt + I_1 + I_2.
\end{aligned} \tag{9.4}$$

Thus

$$\int_Q f^2 e^{2s\varphi} dxdt \ge I_1 + I_2 \tag{9.5}$$

and

$$\int_Q |\partial_t v|^2 dxdt \le \int_Q f^2 e^{2s\varphi} dxdt + |I_1 + I_2|. \tag{9.6}$$

Henceforth $C_j > 0$ denotes generic constants which are independent of s, λ. We assume that $s > 1$ and $\lambda > 1$. By noting $a_{ij} = a_{ji}$, the assumption (9.2) and integration by parts yield

$$|I_1| = \left|-2\int_Q (\partial_t v)\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j v)dxdt\right| = \left|2\int_Q \sum_{i,j=1}^n (\partial_i\partial_t v)a_{ij}\partial_j v dxdt\right|$$

$$= \left|2\int_Q \sum_{i>j} a_{ij}((\partial_j v)\partial_i\partial_t v + (\partial_i v)\partial_j\partial_t v)dxdt + 2\sum_{i=1}^n \int_Q a_{ii}(\partial_i v)\partial_i\partial_t v dxdt\right|$$

$$= \left|2\int_Q \sum_{i>j} a_{ij}\partial_t((\partial_i v)(\partial_j v))dxdt + \int_Q \sum_{i=1}^n a_{ii}\partial_t((\partial_i v)^2)dxdt\right|$$

$$= \left|\int_Q \sum_{i,j=1}^n a_{ij}\partial_t((\partial_i v)\partial_j v)dxdt\right|$$

$$= \left|-\int_Q \sum_{i,j=1}^n (\partial_t a_{ij})(\partial_i v)\partial_j v dxdt + \sum_{i,j=1}^n \left[a_{ij}(\partial_i v)(\partial_j v)\right]_{t=0}^{t=T} dx\right|$$

(9.7)

$$\le C_1\int_Q |\nabla v|^2 dxdt + C_1\int_\Omega (|\nabla v(x,T)|^2 + |\nabla v(x,0)|^2)dx.$$

On the other hand,

$$I_2 = -s\lambda\int_Q 2(\partial_t v)v\varphi dxdt = -s\lambda\int_Q \partial_t(v^2)\varphi dxdt$$

$$= s\lambda\int_Q (\partial_t\varphi)v^2 dxdt - s\lambda\int_\Omega \left[\varphi v^2\right]_{t=0}^{t=T} dx$$

$$\ge s\lambda^2\int_Q \varphi v^2 dxdt - s\lambda\int_\Omega (e^{\lambda T}|v(x,T)|^2 + |v(x,0)|^2)dx. \tag{9.8}$$

Hence

$$\|e^{s\varphi}f\|^2_{L^2(Q)} \ge s\lambda^2\int_Q \varphi v^2 dxdt - C_1\int_Q |\nabla v|^2 dxdt$$

(9.9)

$$-s\lambda\int_\Omega (e^{\lambda T}|v(x,T)|^2 + |v(x,0)|^2)dx - C_1\int_\Omega (|\nabla v(x,T)|^2 + |\nabla v(x,0)|^2)dx.$$

We have to estimate $\int_Q |\nabla v|^2 dxdt$. For it, we argue similarly to step (3) in section 3, and we consider $\int_Q (Pv)v dxdt$:

$$\int_Q (Pv)v dxdt = \int_Q (\partial_t v)v dxdt - \int_Q s\lambda\varphi v^2 dxdt - \int_Q \sum_{i,j=1}^n v\partial_i(a_{ij}\partial_j v)dxdt$$

$$= J_1 + J_2 + J_3.$$

We have

$$|J_1| = \left|\int_Q (\partial_t v) v dxdt\right| = \left|\frac{1}{2}\int_Q \partial_t(v^2) dxdt\right|$$
$$= \left|\frac{1}{2}\int_\Omega \left[|v(x,t)|^2\right]_{t=0}^{t=T} dx\right| \le \frac{1}{2}\int_\Omega (|v(x,T)|^2 + |v(x,0)|^2) dx.$$

Next

$$|J_2| = \left|-\int_Q s\lambda\varphi v^2 dxdt\right| \le C_2 \int_Q s\lambda\varphi v^2 dxdt$$

and

$$J_3 = -\sum_{i,j=1}^n \int_Q \partial_i(a_{ij}\partial_j v) v dxdt = \sum_{i,j=1}^n \int_Q a_{ij}(\partial_i v)\partial_j v dxdt$$
$$\ge \sigma_1 \int_Q |\nabla v|^2 dxdt \tag{9.10}$$

by (1.2). Hence

$$\int_Q \lambda(Pv) v dxdt \ge \sigma_1 \int_Q \lambda|\nabla v|^2 dxdt - C_2 \int_Q s\lambda^2\varphi v^2 dxdt$$
$$-\frac{1}{2}\lambda \int_\Omega (|v(x,T)|^2 + |v(x,0)|^2) dx. \tag{9.11}$$

On the other hand,

$$\lambda|(Pv, v)_{L^2(Q)}| \le \|Pv\|_{L^2(Q)}(\lambda\|v\|_{L^2(Q)}) \le \frac{1}{2}\|Pv\|^2_{L^2(Q)} + \frac{\lambda^2}{2}\|v\|^2_{L^2(Q)}$$
$$\le \frac{1}{2}\|fe^{s\varphi}\|^2_{L^2(Q)} + \frac{\lambda^2}{2}\|v\|^2_{L^2(Q)}.$$

Hence (9.11) yields

$$\sigma_1 \int_Q \lambda|\nabla v|^2 dxdt \le C_2 \int_Q s\lambda^2\varphi v^2 dxdt$$
$$+\frac{1}{2}\int_Q |fe^{s\varphi}|^2 dxdt + \frac{\lambda^2}{2}\int_Q v^2 dxdt + \frac{1}{2}\lambda \int_\Omega (|v(x,T)|^2 + |v(x,0)|^2) dx.$$

Estimating the first term on the right-hand side by (9.9), we obtain

$$
\begin{aligned}
&\sigma_1 \int_Q \lambda|\nabla v|^2 dxdt \\
\leq& C_3 \int_Q |fe^{s\varphi}|^2 dxdt + C_3 \int_Q |\nabla v|^2 dxdt + C_3 \int_Q \lambda^2 v^2 dxdt \\
&+ C_3\lambda(\|v(\cdot,T)\|^2_{L^2(\Omega)} + \|v(\cdot,0)\|^2_{L^2(\Omega)}) + C_3(\|\nabla v(\cdot,T)\|^2_{L^2(\Omega)} + \|\nabla v(\cdot,0)\|^2_{L^2(\Omega)}) \\
&+ C_3 s\lambda(e^{\lambda T}\|v(\cdot,T)\|^2_{L^2(\Omega)} + \|v(\cdot,0)\|^2_{L^2(\Omega)}).
\end{aligned}
\tag{9.12}
$$

Adding (9.12) and (9.9), we have

$$
\begin{aligned}
&\int_Q s\lambda^2\varphi v^2 dxdt + \sigma_1 \int_Q \lambda|\nabla v|^2 dxdt \\
\leq& C_4 \int_Q |fe^{s\varphi}|^2 dxdt + C_4 \int_Q |\nabla v|^2 dxdt + C_4 \int_Q \lambda^2 v^2 dxdt \\
&+ C_4(\|\nabla v(\cdot,T)\|^2_{L^2(\Omega)} + \|\nabla v(\cdot,0)\|^2_{L^2(\Omega)}) + C_4 s\lambda(e^{\lambda T}\|v(\cdot,T)\|^2_{L^2(\Omega)} + \|v(\cdot,0)\|^2_{L^2(\Omega)}).
\end{aligned}
$$

By noting $\varphi = e^{\lambda t} \geq \lambda t$, we take $s > 0$ and $\lambda > 0$ large to absorb the second and the third terms on the right-hand side into the left-hand side, and we obtain

$$
\begin{aligned}
&\int_Q s\lambda^2\varphi v^2 dxdt + \int_Q \lambda|\nabla v|^2 dxdt \leq C_5 \int_Q |fe^{s\varphi}|^2 dxdt \\
&+ C_5 e^{C(\lambda)s}(\|u(\cdot,0)\|^2_{H^1(\Omega)} + \|u(\cdot,T)\|^2_{H^1(\Omega)}).
\end{aligned}
\tag{9.13}
$$

Hence

$$
\begin{aligned}
&\int_Q (s\lambda^2\varphi u^2 + \lambda|\nabla u|^2) e^{2s\varphi} dxdt \leq C_5 \int_Q |fe^{s\varphi}|^2 dxdt \\
&+ C_5 e^{C(\lambda)s}(\|u(\cdot,0)\|^2_{H^1(\Omega)} + \|u(\cdot,T)\|^2_{H^1(\Omega)}).
\end{aligned}
\tag{9.14}
$$

for all large $s > 0$ and $\lambda > 0$.

Next we will estimate $|\partial_t v|^2$. Since $u = e^{-s\varphi}v$, we have $\partial_t u = -s\lambda\varphi e^{-s\varphi}v + e^{-s\varphi}\partial_t v$, we have

$$
\frac{1}{s\varphi}|\partial_t u|^2 e^{2s\varphi} \leq 2s\lambda^2\varphi v^2 + \frac{2}{s\varphi}|\partial_t v|^2.
$$

For all large $s > 0$ and $\lambda > 0$, we have

$$\int_Q \frac{1}{s\varphi}|\partial_t u|^2 e^{2s\varphi} dxdt \le \int_Q 2s\lambda^2\varphi v^2 dxdt + \int_Q \frac{2}{s\varphi}|\partial_t v|^2 dxdt$$
$$\le C\int_Q s\lambda^2\varphi v^2 dxdt + C\int_Q |\partial_t v|^2 dxdt$$
$$\le C\int_Q s\lambda^2\varphi v^2 dxdt + C\int_Q f^2 e^{2s\varphi} dxdt + C|I_1 + I_2|$$

by (9.6). By (9.7), (9.8) and (9.14), we have

$$\int_Q \frac{1}{s\varphi}|\partial_t u|^2 e^{2s\varphi} dxdt \le C\int_Q s\lambda^2\varphi v^2 dxdt + C\int_Q f^2 e^{2s\varphi} dxdt + C\int_Q |\nabla v|^2 dxdt$$
$$+ Ce^{C(\lambda)s}(\|u(\cdot,0)\|^2_{H^1(\Omega)} + \|u(\cdot,T)\|^2_{H^1(\Omega)})$$
$$\le C\int_Q f^2 e^{2s\varphi} dxdt + Ce^{C(\lambda)s}(\|u(\cdot,0)\|^2_{H^1(\Omega)} + \|u(\cdot,T)\|^2_{H^1(\Omega)}).$$

Moreover, in place of (9.2), we can consider other boundary condition:

$$\frac{\partial u}{\partial \nu_A} + p(x)u = 0, \quad x \in \partial\Omega,\ 0 < t < T \tag{9.15}$$

where $p \in C(\partial\Omega)$ and $p \ge 0$ on $\partial\Omega$. In the above arguments, only in (9.7) and (9.10), we need to modify: That is,

$$I_1 = -2\int_Q (\partial_t v)\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j v) dxdt$$
$$= 2\int_Q \sum_{i,j=1}^n (\partial_i\partial_t v)a_{ij}\partial_j v - 2\int_{\partial\Omega\times(0,T)} \sum_{i,j=1}^n a_{ij}(\partial_j v)\nu_i(\partial_t v) dSdt$$
$$= 2\int_Q a_{ij}(\partial_i\partial_t v)\partial_j v dxdt - 2\int_{\partial\Omega\times(0,T)} \frac{\partial v}{\partial\nu_A}(\partial_t v) dSdt.$$

By (9.15), we obtain

$$-2\int_{\partial\Omega\times(0,T)} \frac{\partial v}{\partial\nu_A}(\partial_t v) dSdt = 2\int_{\partial\Omega\times(0,T)} pv(\partial_t v) dSdt$$
$$= \int_{\partial\Omega\times(0,T)} p\partial_t(v^2) dSdt = -\int_{\partial\Omega} \left[pv^2\right]_{t=0}^{t=T} dSdt.$$

Thus also in the case (9.15), we have

$$I_1 = 2\int_Q \sum_{i,j=1}^n (\partial_i\partial_t v)a_{ij}\partial_j v dxdt - \int_{\partial\Omega} \left[pv^2\right]_{t=0}^{t=T} dSdt$$

and by the trace theorem: $\|u\|_{L^2(\partial\Omega)} \le C\|u\|_{H^1(\Omega)}$, we can obtain the same estimate

$$|I_1| \le C_1 \int_Q |\nabla v|^2 dxdt + C_1(\|u(\cdot,0)\|^2_{H^1(\Omega)} + \|u(\cdot,T)\|^2_{H^1(\Omega)}).$$

On the other hand, for J_3 in (9.10), by (9.15) we have

$$\begin{aligned} J_3 =& \sum_{i,j=1}^n \int_Q a_{ij}(\partial_i v)(\partial_j v)dx - \int_{\partial\Omega\times(0,T)} \frac{\partial v}{\partial \nu_A} vdSdt \\ =& \sum_{i,j=1}^n \int_Q a_{ij}(\partial_i v)(\partial_j v)dx + \int_{\partial\Omega\times(0,T)} pv^2 dSdt \\ \ge& \sigma_1 \int_Q |\nabla v|^2 dxdt \end{aligned}$$

by $p \ge 0$.

Finally we apply an a priori estimate for the boundary value problem for

$$-\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j u(\cdot,t)) = f(\cdot,t) - \partial_t u(\cdot,t) \quad \text{in } \Omega$$

for $t \in [0,T]$, we prove the following:

Theorem 9.1. *We set*

$$\varphi(t) = e^{\lambda t}.$$

Then there exists $\lambda_0 > 0$ such that for any arbitrary $\lambda \ge \lambda_0$ we can choose a constant $s_0(\lambda) > 0$ satisfying: there exists a constant $C = C(s_0, \lambda_0) > 0$ such that

$$\begin{aligned} &\int_Q \left\{ \frac{1}{s\varphi}\left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i\partial_j u|^2 \right) + \lambda|\nabla u|^2 + s\lambda^2\varphi u^2 \right\} e^{2s\varphi} dxdt \\ \le& C\int_Q |Lu|^2 e^{2s\varphi} dxdt \\ &+ Ce^{C(\lambda)s}(\|u(\cdot,0)\|^2_{H^1(\Omega)} + \|u(\cdot,T)\|^2_{H^1(\Omega)}) \end{aligned}$$

for all $s > s_0$ and all $u \in H^{2,1}(Q)$ satisfying

$$u = 0 \qquad \text{on } \partial\Omega \times (0,T) \tag{9.16}$$

or

$$\frac{\partial u}{\partial \nu_A} + p(x)u = 0 \qquad on\ \partial\Omega \times (0,T) \tag{9.17}$$

with $p \in C(\partial\Omega)$ *and* $p \ge 0$ *on* $\partial\Omega$.

Next we apply Theorem 9.1 to the backward parabolic problem and establish a conditional stability estimate. Let $a_{ij} \in C^1(\overline{Q})$, $1 \le i, j \le n$, satisfy (1.1) and (1.2), $b_i, c \in L^\infty(Q)$. We consider

$$(Lu)(x,t) \equiv \partial_t u(x,t) - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j u(x,t)$$

$$-\sum_{i=1}^n b_i(x,t)\partial_i u(x,t) - c(x,t)u = 0, \quad (x,t) \in Q \tag{9.18}$$

with (9.16) or (9.17).

Parabolic equation backward in time. Ler $0 \le t_0 < T$. Determine $u(x,t_0)$, $x \in \Omega$ from $u(x,T)$, $x \in \Omega$.

This is a determination problem of the history of a heat process, by means of values at a final time T, and there are definitely many applications in the mathematical physics (e.g., Ames and Straughan [4], Payne [118]). Moreover there are numerical treatments (e.g., Baumeister [8], Eldén [44], Ewing [50], Lattés and Lions [106]). In Escauriaza, Seregin and Šverák [47], [48], a sharp backward uniqueness result is proved: Let $|\partial_t u + \Delta u| \le C(|\nabla u| + |u|)$ in $\{|x| > R\} \times (0,T)$ and let $|u(x,t)| \le Ce^{C|x|^2}$. If $u(x,0) = 0$, $|x| \ge R$, then $u \equiv 0$ in $\{|x| > R\} \times (0,T)$.

As is well known, the backward parabolic equation is ill-posed: small errors in data may cause huge deviations in solutions $u(\cdot,t_0)$. However, if we assume an a priori bound for $u(\cdot,0)$, then we can restore the stability. The conditional stability can be formulated as follows. Let $t_0 \in [0,T)$ and let us set

$$\mathcal{U}_{M,\zeta} = \{a \in H_0^\zeta(\Omega);\ \|a\|_{H_0^\zeta(\Omega)} \le M\}$$

with $M > 0$ and $\zeta \ge 0$. Then by the conditional stability we mean: can we choose a function $\omega \in C[0,\infty)$ such that $\omega \ge 0$, ω is strictly monotone increasing and $\lim_{\eta\downarrow 0}\omega(\eta) = 0$ satisfying

$$\|u(\cdot,t_0)\|_{L^2(\Omega)} \le \omega(\|u(\cdot,T)\|_{L^2(\Omega)})$$

provided that $u(\cdot,0) \in \mathcal{U}_{M,\zeta}$?

There are several methods yielding the conditional stability in cases of

(i) $t_0 > 0$ and $\zeta = 0$.

(ii) $t_0 = 0$ and $\zeta > 0$.

In case (i), it is known that

$$\omega(\eta) = O\left(\eta^{\frac{\theta}{T}}\right) \quad \text{in case (i)}$$

(e.g., Lavrent'ev, Romanov and Shishat·skiĭ [107]). In particular, in Theorem 1 (p.99) in [107], it is proved that

$$\|u\|_{L^2(t_0,T;L^2(\Omega))} \le C\|u(\cdot,T)\|^{\theta}_{L^2(\Omega)}$$

with $\theta \in (0,1)$, under an a priori boundedness condition on $u(\cdot,0)$ in $\mathcal{U}_{M,1}$. On the other hand, in case (ii),

$$\omega(\eta) = O\left(\left(\frac{1}{\log\frac{1}{\eta}}\right)^{\theta}\right) \quad \text{with } \theta \in (0,1)$$

(e.g., Exercise 3.1.2 (p.44) in Isakov [86] for $\zeta = 2$ and Theorems 3 and 4 in Klibanov [97] for $\zeta = 1$).

Among the methods for the conditional stability for the backward parabolic equation, we shall firstly refer to the logarithmic convexity and see Agmon and Nirenberg [2], Ames and Straughan [4], Carasso [28], [29] and Colton [35], Chapter

7, Payne [118] and the references therein. See also pp.44-49 in Isakov [86] as for an argument for the stability in the backward problem which was shown in Agmon and Nirenberg [2], but that method does not work for a parabolic inequality $|Lu| \le |f|$ in Q. Our method on the basis of the Carleman estimate is more effective than the logarithmic convexity: for example, we can treat also semilinear equations directly.

Secondly we refer to the method by the t-analyticity by Kreĭn and Prozorovskaya [103]. Hence the method requires the t-analyticity of solution $u(\cdot, t)$ which implies that we have to assume that the coefficients a_{ij}, b_j and c are t-analytic.

Third we can refer to a weighted energy method or a method by a Carleman estimate and see Ames and Straughan [4], Klibanov [97], Lavrent'ev, Romanov and Shishat·skĭi [107], Lees and Protter [109], Murray and Protter [117], Payne [118], for example.

Now we apply Theorem 9.1 to establish a conditional stability estimate for $\|u(\cdot, t_0)\|_{L^2(\Omega)}$ in the case of $\zeta = 0$ and $t_0 > 0$.

Theorem 9.2. *Let $u \in H^{2,1}(Q)$ satisfy (9.18). For any $t_0 \in (0, T)$, there exist constants $\theta \in (0, 1)$ and $C > 0$ depending on t_0, $\max_{1\le i,j\le n} \|a_{ij}\|_{C^1(\overline{Q})}$, $\max_{1\le i\le n} \|b_i\|_{L^\infty(Q)}$, $\|c\|_{L^\infty(Q)}$, T and Ω such that*

$$(9.19) \qquad \|u(\cdot, t_0)\|_{L^2(\Omega)} \le C\|u\|^{1-\theta}_{L^2(Q)} \|u(\cdot, T)\|^{\theta}_{H^1(\Omega)}.$$

Thanks to the large parameters s, λ in Theorem 9.1, we can argue similarly also for a semilinear parabolic equation $Lu = F(u, \nabla u, x, t)$ with suitable F and an a priori boundedness assumption.

Remark. As is seen from the proof, we can clarify the dependency of θ on t_0 (see

(9.25)), and after integrating (9.19) with respect to t_0 from 0 to T, we can see:

$$\|u\|_{L^2(\Omega\times(0,T))} \le C(1+\|u(\cdot,0)\|_{L^2(\Omega)}) \left(\log \frac{1}{\|u(\cdot,T)\|_{H^1(\Omega)}}\right)^{-\frac{1}{2}}. \tag{9.20}$$

In particular, (9.20) implies

$$\|u\|_{L^2(\Omega\times(0,T))} = O\left(\log \frac{1}{\|u(\cdot,T)\|_{H^1(\Omega)}}\right)^{-\frac{1}{2}}$$

as $\|u(\cdot,T)\|_{H^1(\Omega)} \longrightarrow 0$, provided that $\|u(\cdot,0)\|_{L^2(\Omega)} \le M$.

By suitable regularity of a_{ij}, b_i, c, we apply the a priori estimate for the initial value/boundary value problem (e.g., Friedman [56], Ladyženskaja, Solonnikov and Ural'ceva [104], Pazy [119]), we can have

$$\|u\|_{L^2(Q)} \le C\|u(\cdot,0)\|_{L^2(\Omega)}, \quad \|u(\cdot,T)\|_{H^2(\Omega)} \le C\|u(\cdot,0)\|_{L^2(\Omega)}.$$

Hence thanks to the interpolation inequality:

$$\|u(\cdot,T)\|_{H^1(\Omega)} \le C\|u(\cdot,T)\|^{\frac{1}{2}}_{H^2(\Omega)}\|u(\cdot,T)\|^{\frac{1}{2}}_{L^2(\Omega)},$$

we can improve (9.19) as follows: for any $\mu \in (0,2)$, there exists a constant $C_\mu > 0$ such that

$$\|u(\cdot,t_0)\|_{H^\mu(\Omega)} \le C_\mu \|u(\cdot,0)\|^{1-\frac{\theta(2-\mu)}{4}}_{L^2(\Omega)} \|u(\cdot,T)\|^{\frac{\theta(2-\mu)}{4}}_{L^2(\Omega)}. \tag{9.21}$$

Remark. Our result proves the backward uniqueness if $a_{ij} : [0,T] \longrightarrow L^\infty(\Omega)$, $1 \le i,j \le n$, are Lipschitz continuous on $t \in [0,T]$. This regularity in t is the best possible because in Miller [115], there is a counter-example constructed for the non-uniqueness in a backward problem for a parabolic equation with Hölder continuous coefficient in t.

Proof. In Theorem 9.1, we fix λ and we choose t_1, t_2 such that $0 < t_2 < t_1 < t_0$. We set $\delta_k = e^{\lambda t_k}$, $k = 0, 1, 2$. Since Theorem 9.1 requires $u(\cdot, 0)$ as data on the right-hand side, we need a cut-off function, which is a quite common technique. Let $\chi \in C^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$ and

$$\chi(t) = \begin{cases} 1, & t > t_1, \\ 0, & t < t_2. \end{cases} \tag{9.22}$$

Setting $w = \chi u$, we have $w(\cdot, 0) = 0$ and $Lw = \chi'(t)u$ in Q, and $w = 0$ or $\frac{\partial w}{\partial \nu_A} + pw = 0$ on $\partial\Omega \times (0, T)$. Applying Theorem 9.1 and we obtain

$$\begin{aligned} &\int_Q \frac{1}{s}\left\{\left(|\partial_t w|^2 + \sum_{i,j=1}^n |\partial_i \partial_j w|^2\right) + |\nabla w|^2 + s|w|^2\right\} e^{2s\varphi} dxdt \\ \le& C\int_Q |\chi'(t)|^2 |u|^2 e^{2s\varphi} dxdt + Ce^{Cs}\|w(\cdot, T)\|^2_{H^1(\Omega)} \end{aligned}$$

for all large $s > 0$. By (9.22), we see that

$$\int_Q |\chi'(t)|^2 |u|^2 e^{2s\varphi} dxdt \le C\int_{t_2}^{t_1}\int_\Omega |u|^2 e^{2s\varphi} dxdt \le Ce^{2s\delta_1}\|u\|^2_{L^2(Q)},$$

so that

$$\begin{aligned} &e^{2s\delta_0}\int_{t_0}^T \int_\Omega \left(\frac{1}{s}|\partial_t u|^2 + s|u|^2\right) dxdt \\ \le& C\int_0^T \int_\Omega \left(\frac{1}{s}|\partial_t w|^2 + s|w|^2\right) e^{2s\varphi} dxdt \\ \le& Ce^{2s\delta_1}\|u\|^2_{L^2(Q)} + Ce^{Cs}\|u(\cdot, T)\|^2_{H^1(\Omega)} \end{aligned}$$

for all $s \ge s_0$. Consequently

$$\begin{aligned} &\int_{t_0}^T \int_\Omega \left(\frac{1}{s}|\partial_t u|^2 + s|u|^2\right) dxdt \\ \le& Ce^{2s(\delta_1 - \delta_0)}\|u\|^2_{L^2(Q)} + Ce^{Cs}\|u(\cdot, T)\|^2_{H^1(\Omega)} \end{aligned} \tag{9.23}$$

for all $s \ge s_0$. Therefore we have

$$\begin{aligned}&\int_\Omega |u(x,t_0)|^2 dx\\ =& -\int_{t_0}^T \partial_t \left(\int_\Omega |u(x,t)|^2 dx\right) dt + \int_\Omega |u(x,T)|^2 dx\\ =& -\int_{t_0}^T \int_\Omega 2|u(x,t)||\partial_t u(x,t)| dxdt + \|u(\cdot,T)\|^2_{L^2(\Omega)}.\end{aligned}$$

Hence (9.23) and the Cauchy-Schwarz inequality yield

$$\begin{aligned}\|u(\cdot,t_0)\|^2_{L^2(\Omega)} \le& \int_{t_0}^T \int_\Omega \left(\frac{1}{s}|\partial_t u|^2 + s|u|^2\right) dxdt + \|u(\cdot,T)\|^2_{L^2(\Omega)}\\ \le& Ce^{-2s(\delta_0-\delta_1)}\|u\|^2_{L^2(Q)} + Ce^{Cs}\|u(\cdot,T)\|^2_{H^1(\Omega)}\end{aligned} \tag{9.24}$$

for all $s \ge s_0$. Replacing C by Ce^{Cs_0}, we have (9.24) for all $s \ge 0$. Choosing $s \ge 0$ minimizing the right-hand side of (9.24), we obtain

$$\|u(\cdot,t_0)\|_{L^2(\Omega)} \le C\|u\|^{\frac{C}{C+2(\delta_0-\delta_1)}}_{L^2(Q)} \|u(\cdot,T)\|^{\frac{2(\delta_0-\delta_1)}{C+2(\delta_0-\delta_1)}}_{H^1(\Omega)}. \tag{9.25}$$

Thus the proof of (9.19) is completed.

§9.2. Carleman estimate for a forward problem in time and an application to an inverse source problem.

For simplicity, we assume that a_{ij}, $1 \le i,j \le n$ are t-independent. In the argument in proving Theorem 9.1, we choose

$$\varphi(t) = e^{-\lambda t}$$

with $\lambda > 0$. Then we can similarly prove

Theorem 9.3. *We set*

$$\varphi(t) = e^{-\lambda t}.$$

Then there exists $\lambda_0 > 0$ such that for any arbitrary $\lambda \ge \lambda_0$ we can choose a constant $s_0(\lambda) > 0$ satisfying: there exists a constant $C = C(s_0, \lambda_0) > 0$ such that

$$\int_Q \left\{ \frac{1}{s\varphi} \left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right) + \lambda |\nabla u|^2 + s\lambda^2 \varphi u^2 \right\} e^{2s\varphi} dxdt$$
$$\le C \int_Q |Lu|^2 e^{2s\varphi} dxdt$$
$$+ Ce^{C(\lambda)s} (\|u(\cdot,0)\|^2_{L^2(\Omega)} + \|\nabla u(\cdot,0)\|^2_{L^2(\Omega)}) + C \int_{\partial\Omega\times(0,T)} s\lambda(|u| + |\partial_t u|) \left| \frac{\partial u}{\partial \nu_A} \right| e^{2s\varphi} dSdt$$

for all $s > s_0$ and all $u \in H^{2,1}(Q)$ satisfying

$$u(\cdot, T) = 0 \qquad \textit{in } \Omega.$$

This Carleman estimate can yield a conditional stability estimate or $u(\cdot,t)$ with $t > 0$ which is a solution to the initial value/ boundary value problem, but of course, such an application of Theorem 9.3 is not interesting because other classical methods for the well-posed forward problem produce better and well-known results.

Our interest here is the application of Theorem 9.3 to an inverse problem. For convenience, we conside a simple case but the treatment for a general L is similar.

Let $x = (x_1, x') \in \mathbb{R}^n$ and $x' = (x_2, ..., x_n) \in \mathbb{R}^{n-1}$, $\Omega = (0, \ell) \times D'$, $D' \subset \mathbb{R}^{n-1}$ be a bounded domain with smooth boundary $\partial D'$. We consider

$$\partial_t u(x,t) = \Delta u(x,t) + f(x',t) R(x,t), \quad x \in \Omega,\ t > 0 \tag{9.26}$$

$$u(x,0) = 0, \qquad x \in \Omega \tag{9.27}$$

$$\frac{\partial u}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega,\ t > 0. \tag{9.28}$$

Inverse heat source problem. Let R be given and let $t_0 > 0$. Determine $f(x', t)$, $x' \in D'$, $0 < t < t_0$ by $u|_{\partial\Omega\times(0,t_0)}$.

Although we can discuss the conditional stability, we consider only the uniqueness:

Theorem 9.4. *We assume that* $u, \partial_1 u \in H^{2,1}(Q)$ *and*

$$R \in C^2(\overline{Q}), \quad |R(x,t)| \ne 0, \quad x \in \overline{\Omega},\ 0 \le t \le t_0. \tag{9.29}$$

If $u|_{\partial\Omega\times(0,t_0)} = 0$*, then* $f(x', t) = 0$*,* $x' \in D'$*,* $0 \le t \le t_0$*.*

Proof. For arbitrary small $\varepsilon > 0$, we choose t_1, t_2 such that $0 < t_0 - \varepsilon < t_1 < t_2 < t_0$. We set $\delta_k = e^{-\lambda t_k}$, $k = 0, 1, 2$. Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function such that $0 \le \chi \le 1$ and

$$\chi(t) = \begin{cases} 1, & t < t_1, \\ 0, & t \ge t_2. \end{cases} \tag{9.30}$$

Setting $u = Rw$ on $\overline{\Omega} \times [0, t_2]$, we have

$$\begin{aligned} &\partial_t w - \Delta w + \frac{\partial_t R}{R} w - \frac{2\nabla R}{R} \cdot \nabla w \\ &- \frac{\Delta R}{R} w = f(x', t), \quad x \in \Omega,\ 0 < t < t_2, \end{aligned} \tag{9.31}$$

and

$$w|_{\partial\Omega\times(0,t_0)} = \frac{\partial w}{\partial \nu}|_{\partial\Omega\times(0,t_0)} = 0.$$

Differentiating the both sides of (9.31) with respect to x_1 and setting $y = \partial_1 w$, we obtain

$$\begin{cases} \partial_t y - \Delta y + \dfrac{\partial_t R}{R} y - \dfrac{2\nabla R}{R} \cdot \nabla y - \dfrac{\Delta R}{R} y \\ + \partial_1 \left(\dfrac{\partial_t R}{R} \right) w - \partial_1 \left(\dfrac{2\nabla R}{R} \right) \cdot \nabla w - \partial_1 \left(\dfrac{\Delta R}{R} \right) w = 0, \quad x \in \Omega,\ 0 < t < t_0, \\ y|_{\partial\Omega\times(0,t_0)} = 0. \end{cases}$$

Setting $v = \chi y$, we have

$$\begin{aligned}(9.32)\quad &\partial_t v - \Delta v + \frac{\partial_t R}{R} v - \frac{2\nabla R}{R}\cdot\nabla v - \frac{\Delta R}{R} v\\ =&\chi'(t) y - \partial_1\left(\frac{\partial_t R}{R}\right)\chi w + \partial_1\left(\frac{2\nabla R}{R}\right)\cdot\chi\nabla w\\ &+\partial_1\left(\frac{\Delta R}{R}\right)\chi w, \quad x\in\Omega,\ 0<t<t_0\end{aligned}$$

$$(9.33)\qquad v|_{\partial\Omega\times(0,t_0)} = 0.$$

By $y = \partial_1 w$, and $u(0, x', t) = 0$, $x' \in D'$, $0 < t < t_0$, by the assumption, we see that

$$(9.34)\qquad w(x_1, x', t) = \int_0^{x_1} y(\eta, x', t)d\eta.$$

Hence (9.32) implies

$$\begin{aligned}(9.35)\quad &\partial_t v - \Delta v + \frac{\partial_t R}{R} v - \frac{2\nabla R}{R}\cdot\nabla v - \frac{\Delta R}{R} v\\ =&\chi' y - \partial_1\left(\frac{\partial_t R}{R}\right)\int_0^{x_1} v(\eta, x', t)d\eta + \partial_1\left(\frac{2\nabla R}{R}\right)\cdot\int_0^{x_1}\nabla v(\eta, x', t)d\eta\\ &+\partial_1\left(\frac{\Delta R}{R}\right)\int_0^{x_1} v(\eta, x', t)d\eta, \quad x\in\Omega,\ 0<t<t_0\end{aligned}$$

$$(9.36)\qquad v(x, t_0) = v(x, 0) = 0, \qquad x\in\Omega.$$

Apply Theorem 9.3 to (9.35), in terms of (9.33) and (9.36), we obtain

$$\begin{aligned}(9.37)\quad &\int_0^{t_0}\int_\Omega (\lambda|\nabla v|^2 + s\lambda^2\varphi|v|^2)e^{2s\varphi}dxdt\\ \le& C\int_0^{t_0}\int_\Omega |\chi' y|^2 e^{2s\varphi}dxdt + C\int_0^{t_0}\int_\Omega\left|\int_0^{x_1} v(\eta, x', t)d\eta\right|^2 e^{2s\varphi}dxdt\\ &+C\int_0^{t_0}\int_\Omega\left|\int_0^{x_1}\nabla v(\eta, x', t)d\eta\right|^2 e^{2s\varphi}dxdt.\end{aligned}$$

Since

$$\left|\int_0^{x_1}\nabla^j v(\eta, x', t)d\eta\right|^2 \le \ell\int_0^{\ell}|\nabla^j v(\eta, x', t)|^2 d\eta$$

for $j = 0, 1$, we have

$$\begin{aligned}
&\int_0^{t_0}\int_\Omega \left|\int_0^{x_1} \nabla^j v(\eta, x', t)d\eta\right|^2 e^{2s\varphi(t)}dxdt \\
\le& \ell \int_0^\ell dx_1 \int_0^{t_0}\int_{D'}\left(\int_0^\ell |\nabla^j v(\eta, x', t)|^2 d\eta\right) e^{2s\varphi(t)}dx'dt \\
\le& \ell^2 \int_0^{t_0}\int_\Omega |\nabla^j v(\eta, x', t)|^2 e^{2s\varphi(t)}d\eta dx'dt.
\end{aligned}$$

Hence, taking $s > 0$ and $\lambda > 0$ large, we can absorb the second and the third terms on the right-hand side of (9.37) into the left-hand side:

$$\int_0^{t_0}\int_\Omega (\lambda|\nabla v|^2 + s\lambda^2\varphi|v|^2)e^{2s\varphi}dxdt \le C\int_0^{t_2}\int_\Omega |\chi' y|^2 e^{2s\varphi}dxdt. \tag{9.38}$$

for all large $s > 0$ and $\lambda > 0$. We fix $\lambda > 0$. By (9.30), we see that

$$\int_0^{t_0}\int_\Omega |\chi' y|^2 e^{2s\varphi}dxdt \le e^{2s\delta_1}\int_Q |y|^2 dxdt \le CM^2 e^{2s\delta_1},$$

where we set $M = \|\partial_1 u\|_{L^2(Q)}$. Hence with (9.38), we obtain

$$\begin{aligned}
&e^{2s\delta_\varepsilon}\int_0^{t_0-\varepsilon}\int_\Omega (|\nabla v|^2 + s|v|^2)dxdt \le \int_0^{t_0-\varepsilon}\int_\Omega (|\nabla v|^2 + s|v|^2)e^{2s\varphi}dxdt \\
\le& \int_0^{t_0}\int_\Omega (|\nabla v|^2 + s|v|^2)e^{2s\varphi}dxdt \le CM^2 e^{2s\delta_1}
\end{aligned}$$

for all large $s > 0$, where we set $\delta_\varepsilon = e^{-\lambda(t_0-\varepsilon)}$. Consequently

$$\|v\|_{L^2(0,t_0-\varepsilon;H^1(\Omega))} \le CM^2 e^{-2s(\delta_\varepsilon - \delta_1)}$$

for all large $s > 0$. Since $\delta_\varepsilon - \delta_1 = e^{-\lambda(t_0-\varepsilon)} - e^{-\lambda t_1} > 0$ by $t_0 - \varepsilon < t_1$, letting $s \to \infty$, we obtain $v = 0$ in $\Omega \times (0, t_0 - \varepsilon)$. Equation (9.34) implies $w = 0$ in $\Omega \times (0, t_0 - \varepsilon)$, with which (9.31) means $f = 0$ in $D' \times (0, t_0 - \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, the proof of the theorem is completed.

Remark. For the proof, the original argument in Bukhgeim and Klibanov [20] works but the argument in section 6.2 is not applicable.

We conclude this section with strongly coupled parabolic systems: We assume that $a_{ij}^{k\ell} \in C^1(\overline{Q})$, $1 \le i,j \le n$, $1 \le k,\ell \le N$ satisfy (7.19) - (7.21) and that $b_i^{k\ell}, c^{k\ell} \in L^\infty(Q)$.

Then we consider a strongly coupled parabolic system whose principal parts are different:

$$
\begin{aligned}
&[L\mathbf{u}]_\ell(x,t) \equiv \partial_t u_\ell(x,t) - \sum_{k=1}^N \sum_{i,j=1}^n \partial_i(a_{ij}^{k\ell}(x,t)\partial_j u_k(x,t)) \\
&- \sum_{k=1}^N \sum_{i=1}^n b_i^{k\ell}(x,t)\partial_i u_k(x,t) - \sum_{k=1}^N c^{k\ell}(x,t)u_k(x,t) = f(x,t), \\
&(x,t) \in Q,\ 1 \le \ell \le N.
\end{aligned} \tag{9.39}
$$

Since the strong coupling occurs only for x and for the Carleman estimates discussed in this section, the weight functions are independent of x, we can follow the arguments for Theorems 9.1 and 9.3, so that we can prove Carleman estimates for the strongly coupled system (9.39).

Acknowledgements. The author thanks Professor J.-M. Coron (Université de Paris VI), Professor V. Isakov (Wichita State University), Professor M.V. Klibanov (University of North Carolina at Charlotte), Professor J. Le Rousseau (Université d'Orleans), Professor L. Rosier (Université de Nancy) and Professor X. Zhang (Academy of Mathematics and System Sciences) for information of references and comments.

References

1. R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
2. S. Agmon and L. Nirenberg, *Properties of solutions of ordinary differential equations in Banach spaces*, Comm. Pure Appl. Math. **16** (1963), 121-239.

3. O. M. Alifanov, *Inverse Heat Transfer Problems*, Springer-Verlag, Berlin, 1994.
4. K. A. Ames and B. Straughan, *Non-standard and Improperly Posed Problems*, Academic Press, San Diego, 1997.
5. A. K. Amirov, *Integral Geometry and Inverse Problems for Kinetic Equations*, VSP, Utrecht, 2001.
6. F. Ammar-Khodja, A. Benabdallah and C. Dupaix, *Null controllability of some reaction-diffusion systems with one control force*, J. Math. Anal. Appl. **320** (2006), 928-943.
7. F. Ammar-Khodja, A. Benabdallah, C. Dupaix and M. González-Burgos, *A Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems*, to appear in J. Evolution Equations (2009).
8. J. Baumeister, *Stable Solution of Inverse Problems*, Vieweg, Braunschweig, 1987.
9. L. Beilina and M.V. Klibanov, *A globally convergent numerical method for a coefficient inverse problem*, SIAM J. Sci. Compt. **31** (2008), 478-509.
10. L. Beilina and M.V. Klibanov, *A globally convergent numerical method and adaptivity for a hyperbolic coefficient inverse problem*, preprint, http://www.ma.utexas.edu/mp_arc.
11. M. Bellassoued and M. Yamamoto, *Inverse source problem for a transmission problem for a parabolic equation*, J. Inverse Ill-Posed Probl. **14** (2006), 47–56.
12. A. Benabdallah, M. Cristofol, P. Gaitan and M. Yamamoto, *Inverse problem for a parabolic system with two components by measurements of one component*, to appear, Appl. Anal. (2009).
13. A. Benabdallah, Y. Dermenjian and J. Le Rousseau, *Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient, and applications*, C. R. Mech. **334** (2006), 582-586.
14. A. Benabdallah, Y. Dermenjian and J. Le Rousseau, *On the controllability of linear parabolic equations with an arbitrary control location for stratified media*, C. R. Acad. Sci. Paris Ser. I **344** (2007), 357-362.
15. A. Benabdallah, Y. Dermenjian and J. Le Rousseau, *Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications to controllability and an inverse problem*, J. Math. Anal. Appl. **336** (2007), 865-887.
16. A. Benabdallah, P. Gaitan and J. Le Rousseau, *Stability of discontinuous diffusion coefficients and initial conditions in an inverse problem for the heat equation*, SIAM J. Control Optim. **46** (2007), 1849-1881.
17. F. Bloom, *Ill-posed Problems for Integrodifferential Equations in Mechanics and Electromagnetic Theory*, SIAM, Philadelphia, 1981.
18. M. Boulakia, C. Grandmont and A. Osses, *Some inverse stability results for the bistable reaction-diffusion equation using Carleman inequalities*, C. R. Acad. Sci. Paris Ser. I **347** (2009), 619-622.
19. A.L. Bukhgeim, *Introduction to the Theory of Inverse Problems*, VSP, Utrecht, 2000.
20. A.L. Bukhgeim and M.V. Klibanov, *Global uniqueness of a class of multidimensional inverse problems*, Soviet Math. Dokl. **24** (1981), 244–247.
21. P. Cannarsa, G. Fragnelli and D. Rocchetti, *Null controllability of degenerate parabolic operators with drift*, Networks Hetero. Media **2** (2007), 695-715.

22. P. Cannarsa, G. Fragnelli and D. Rocchetti, *Controllability results for a class of one-dimensional degenerate parabolic problems in nondivergence form*, J. Evolutions Equations **8** (2008), 1-34.
23. P. Cannarsa, P. Martinez and J. Vancostenoble, *Null controllability of degenerate heat equations*, Adv. Differential Equations **10** (2005), 153-190.
24. P. Cannarsa, P. Martinez and J. Vancostenoble, *Carleman estimates for a class of degenerate parabolic operators*, SIAM J. Control Optim. **47** (2008), 1-19.
25. P. Cannarsa and M. Yamamoto, *Determination of source terms in a degenerate parabolic equation*, preprint.
26. J.R. Cannon, *The One-dimensional Heat Equation*, Addison-Wesley, Reading, 1984.
27. B. Canuto, E. Rosset and S. Vessella, *Quantitative estimates of unique continuation for parabolic equations and inverse initial-boundary value problems with unknown boundaries*, Trans. Amer. Math. Soc. **354** (2001), 491-535.
28. A.S. Carasso, *Overcoming Hölder continuity in ill-posed continuation problems*, SIAM J. Numer. Anal. **31** (1994), 1535-1557.
29. A.S. Carasso, *Logarithmic convexity and the "slow evolution" constraints in ill-posed initial value problems*, SIAM J. Math. Anal. **30** (1999), 479-496.
30. T. Carleman, *Sur un problème d'unicité pour les systèmes d'équations aux derivées partielles à deux variables independentes*, Ark. Mat. Astr. Fys. **2B** (1939), 1-9.
31. C. Cavaterra, A. Lorenzi and M. Yamamoto, *A stability result via Carleman estimates for an inverse source problem related to a hyperbolic integro-differential equation*, Comput. Appl. Math. **25** (2006), 229–250.
32. D. Chae, O.Y. Imanuvilov and S.M. Kim, *Exact controllability for semilinear parabolic equations with Neumann boundary conditions*, J. Dynamical and Control Systems **2** (1996), 449–483.
33. M. Choulli, *Une Introduction aux Problèmes Inverses Elliptiques et Paraboliques*, Springer-Verlag, Berlin, 2009.
34. M. Choulli, O.Y. Imanuvilov and M. Yamamoto, *Inverse source problem for the Navier-Stokes equations*, Gradute School of Mathematical Sciences, The University of Tokyo **UTMS 2006-3** (2006), http://kyokan.ms.u-tokyo.ac.jp/users/preprint/pdf/2006-3.pdf.
35. D. L. Colton, *Analytic Theory of Partial Differential Equations*, Pitman, Boston, 1980.
36. J.-M. Coron and S. Guerrero, *Singular optimal control: a linear 1-D parabolic-hyperbolic example*, Asymptotic Analysis **44** (2005), 237–257.
37. J.-M. Coron and S. Guerrero, *Null controllability of the N-dimensional Stokes system with N-1 scalar controls*, J. Differential Equations **246** (2009), 2908-2921.
38. M. Cristofol, P. Gaitan and H. Ramoul, *Inverse problems for a 2X2 reaction-diffusion system using a Carleman estimate with one observation*, Inverse Problems **22** (2006), 1561-1573.
39. M. Cristofol, P. Gaitan, H. Ramoul and M. Yamamoto, *Identification of two independent coefficients with one observation for a nonlinear parabolic system*, preprint.
40. M. Di Cristo, L. Rondi and S. Vessella, *Stability properties of an inverse*

parabolic problem with unknown boundaries, Ann. Mat. **185(2)** (2006), 223-255.

41. A. Doubova, A. Osses and J.-P. Puel, *Exact controllability to trajectories for semilinear heat equations with discontinuous diffusion coefficients*, ESAIM : Control, Optimisation and Calculus of Variations **8** (2002), 621-661.
42. H. Egger, H.W. Engl and M.V. Klibanov, *Global uniqueness and Hölder stability for recovering a nonlinear source term in a parabolic equation*, Inverse Problems **21** (2005), 271-290.
43. Y. V. Egorov, *Linear Differential Equations of Principal Type*, Consultants Bureau, New York, 1986.
44. L. Eldén, *Time discretization in the backward solution of parabolic equations I*, Math. Compt. **39** (1982), 53-84.
45. M.M. Eller and V. Isakov, *Carleman estimates with two large parameters and applications*, Contemp. Math. **268** (2000), 117–136.
46. M.M. Eller and M. Yamamoto, *A Carleman inequality for the stationary anisotropic Maxwell system*, J. Math. Pures Appl. **86** (2006), 449–462.
47. L. Escauriaza, G. Seregin and V. Šverák, *Backward uniqueness for parabolic equations*, Arch. Rat. Mech. Anal. **169** (2003), 147-157.
48. L. Escauriaza, G. Seregin and V. Šverák, *Backward uniqueness for the heat operator in a half-space*, St. Petersburg Math. J. **15** (2004), 139-148.
49. L. Escauriaza and S. Vessella, *Optimal three cylinder inequalities for solutions to parabolic equations with Lipschitz leading coefficients*, Compt. Math. **333** (2003), 79-87.
50. R.E. Ewing, *The approximation of certain parabolic equation backward in time by Sobolev equations*, SIAM J. Math. Anal. **6** (1975), 283-294.
51. C. Fabre and G. Lebeau, *Prolongement unique des solutions de l'équation de Stokes*, Comm. Partial Differential Equations **21** (1996), 573-596.
52. H.O. Fattorini and D. L. Russell, *Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations*, Quart. Appl. Math. **32** (1974), 45-69.
53. E. Fernández-Cara and L. de Teresa, *Null controllability of a cascade system of parabolic-hyperbolic equations*, Discrete Conti. Dynmical Systems **11** (2004), 699-714.
54. E. Fernández-Cara and S. Guerrero, *Global Carleman inequalities for parabolic systems and applications to controllability*, SIAM J. Control Optim. **45** (2006), 1399–1446.
55. E. Fernández-Cara, S. Guerrero, O. Y. Imanuvilov and J.-P. Puel, *Local exact controllability of the Navier-Stokes system*, Journal Math. Pures et Appl. **83** (2004), 1501-1542.
56. A. Friedman, *Partial Differential Equations of Parabolic Type*, Krieger, Malabar, Florida, 1983.
57. X. Fu, *A weight identity for partial differential operators of second order and its applications*, C. R. Acad. Sci. Paris Ser. I **342** (2006), 579-584.
58. A. V. Fursikov and O.Y. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes Series 34, Seoul National University, Korea, 1996.
59. M. Giaquinta, *Introduction to Regularity Theory for Nonlinear Ellitpic Systems*, Birkhäuser-Verlag, Basel, 1993.
60. D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 2001.

61. O. Glass and S. Guerrero, *On the uniform controllability of the Burgers equation*, SIAM J. Control and Optim. **46** (2007), 1211-1238.
62. M. González-Burgos and L. de Teresa, *Some results on controllability for linear and nonlinear heat equations in unbounded domains*, Adv. Differential Equations **12** (2007), 1201-1240.
63. S. Guerrero, *Controllability of systems of Stokes equations with one control force: existence of insensitizing controls*, Ann. I. H. Poincaré - AN **24** (2007), 1029-1054.
64. L.F. Ho, *Observabilité frontière de l'équations des ondes*, C. R. Acad. Sci. Paris Ser. I **302** (1986), 443-446.
65. L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.
66. L. Hörmander, *The Analysis of Linear Partial Differential Operators I - IV*, Springer-Verlag, Berlin, 1985.
67. O. Y. Imanuvilov, *Exact boundary controllability of the parabolic equation*, Russian Math. Surveys **48** (1993), 211-212.
68. O. Y. Immanuvilov Controllability of parabolic equations, Sbornik Math. **186** (1995), 879–900.
69. O. Y. Imanuvilov, *On exact controllability for the Navier-Stokes equations*, ESAIM : Control, Optimisation and Calculus of Variations **3** (1998), 97-131.
70. O. Y. Imanuvilov, *Remarks on exact controllability for the Navier-Stokes equations*, ESAIM : Control, Optimisation and Calculus of Variations **6** (2001), 39-72.
71. O. Y. Imanuvilov, *On Carleman estimates for hyperbolic equations*, Asymptotic Analysis **32** (2002), 185-220.
72. O. Y. Imanuvilov, V. Isakov and M. Yamamoto, *An inverse problem for the dynamical Lamé system with two sets of boundary data*, Comm. Pure Appl. Math. **56** (2003), 1366-1382.
73. O. Y. Imanuvilov, J.-P. Puel and M. Yamamoto, *Carleman estimates for parabolic equations with nonhomogeneous boundary conditions*, to appear in Chinese Ann. Math. (2009).
74. O. Y. Imanuvilov and M. Yamamoto, *Lipschitz stability in inverse parabolic problems by the Carleman estimate*, Inverse Problems **14** (1998), 1229–1245.
75. O. Y. Imanuvilov and M. Yamamoto, *Inverse source problem for the Stokes system*, in "Direct and Inverse Problems of Mathematical Physics" (2000), Kluwer, Dordrecht, 441–451.
76. O. Y. Imanuvilov and M. Yamamoto, *Carleman estimate for a parabolic equation in a Sobolev space of negative order and its applications*, in "Control of Nonlinear Distributed Parameter Systems" (2001), Marcel Dekker, New York, 113–137.
77. O. Yu. Imanuvilov and M. Yamamoto, *Global uniqueness and stability in determining coefficients of wave equations*, Comm. Partial Differential Equations **26** (2001), 1409–1425.
78. O. Y. Imanuvilov and M. Yamamoto, *Global Lipschitz stability in an inverse hyperbolic problem by interior observations*, Inverse Problems **17** (2001), 717–728.
79. O. Y. Imanuvilov and M. Yamamoto, *Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for*

semilinear parabolic equations, Publ. Res. Inst. Math. Sci. **39** (2003), 227–274.
80. V. Isakov, *A nonhyperbolic Cauchy problem for $\square_b \square_c$ and its applications to elasticity theory*, Comm. Pure and Appl. Math. **39** (1986), 747–767.
81. V. Isakov, *Inverse Source Problems*, American Mathematical Society, Providence, Rhode Island, 1990.
82. V. Isakov, *Carleman type estimates in an anisotropic case and applications*, J. Differential Equations **105** (1993), 217–238.
83. V. Isakov, *Some inverse problems for the diffusion equation*, Inverse Problems **15** (1999), 3-10.
84. V. Isakov, *Carleman type estimates and their applications*, in "New Analytic and Geometric Methods in Inverse Problems" (2004), Springer-Verlag, Berlin, 93-125.
85. V. Isakov, *Carleman estimates and applications to inverse problems*, Milan J. Math. **72** (2004), 249-271.
86. V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, Berlin, 1998, 2006.
87. V. Isakov and N. Kim, *Carleman estimates with second large parameter for second order operators*, in "Some Applications of Sobolev Spaces to PDE" (2008), Springer-Verlag, Berlin, 135-160.
88. V. Isakov and N. Kim, *Carleman estimates with second large parameter and applications to elasticity with residual stress*, Applicationes Mathematicae **35** (2008), 447-465.
89. F. John, *Continuous dependence on data for solutions of partial differential equations with a prescribed bound*, Comm. Pure Appl. Math. **13** (1960), 551-585.
90. B. Kaltenbacher and M.V. Klibanov, *An inverse problem for a nonlinear parabolic equation with applications in population dynamics and magnetics*, SIAM J. Math. Anal. **39** (2008), 1863.
91. M. Kazemi and M.V. Klibanov, *Stability estimates for ill-posed Cauchy problem involving hyperbolic equations and inequalities*, Applicable Analysis **50** (1993), 93-102.
92. A. Khaĭdarov, *On stability estimates in multidimensional inverse problems for differential equations*, Soviet Math. Dokl. **38** (1989), 614–617.
93. M.V. Klibanov, *Inverse problems in the "large" and Carleman bounds*, Differential Equations **20** (1984), 755-760.
94. M.V. Klibanov, *A class of inverse problems for nonlinear parabolic equations*, Siberian Math. J. **27** (1987), 698-707.
95. M.V. Klibanov, *Inverse problems and Carleman estimates*, Inverse Problems **8** (1992), 575–596.
96. M.V. Klibanov, *Global uniqueness of a multidimensionnal inverse problem for a nonlinear parabolic equation by a Carleman estimate*, Inverse Problems **20** (2004), 1003-1032.
97. M.V. Klibanov, *Estimates of initial conditions of parabolic equations and inequalities via lateral Cauchy data*, Inverse Problems **22** (2006), 495-514.
98. M.V. Klibanov and J. Malinsky, *Newton-Kantorovich method for 3-dimensional potential inverse scattering problem and stability of the hyperbolic Cauchy problem with time dependent data*, Inverse Problems **7** (1991), 577-595.

99. M.V. Klibanov and A.V. Tikhonravov, *Estimates of initial conditions of parabolic equations and inequalities in infinite domains via lateral Cauchy data*, J. Differential Equations **237** (2007), 198-224.
100. M.V. Klibanov and A.A. Timonov, *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications*, VSP, Utrecht, 2004.
101. M.V. Klibanov and M. Yamamoto, *Lipschitz stability of an inverse problem for an accoustic equation*, Applicable Analysis **85** (2006), 515–538.
102. V. Komornik, *Exact Controllability and Stabilization*, John Wiley, Chichester, 1994.
103. S.G. Kreĭn and O.I. Prozorovskaya, *Analytic semigroups and incorrect problems for evolutionary equations*, Soviet Math. Dokl. **133** (1960), 35-38.
104. O. A. Ladyženskaja, V.A. Solonnikov and N.N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, Rhode Island, 1968.
105. E.M. Landis, *Some questions in the qualitative theory of elliptic and parabolic equations*, AMS Transl. Series 2 **20** (1962), 173-238.
106. R. Lattés and J.-L. Lions, *The Method of Quasireversibility: Applications to Partial Differential Equations*, Elsevier, New York, 1969.
107. M.M. Lavrent'ev, V.G. Romanov and S.P. Shishat·skiĭ, *Ill-posed Problems of Mathematical Physics and Analysis*, English translation, American Mathematical Society, Providence, Rhode Island, 1986.
108. G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Comm. Partial Differential Equations **20** (1995), 335–356.
109. M. Lees and M.H. Protter, *Unique continuation for parabolic differential equations and inequalities*, Duke Math. J. **28** (1961), 369-382.
110. J. Le Rousseau, *Carleman estimates and controllability results for the one-dimensional heat equation with BV coefficients*, J. Differential Equations **233** (2007), 417-447.
111. J. Le Rousseau and L. Robbiano, *Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations*, to appear in Arch. Rat. Mech. (2009).
112. J. Le Rousseau and L. Robbiano, *Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces*, preprint.
113. J. Li, M. Yamamoto and J. Zou, *Conditional stability and numerical reconstruction of initial temperature*, Comm. Pure Appl. Anal. **8** (2009), 361–382.
114. J.-L. Lions, *Contrôlabilité Exacte Perturbations et Stabilisation de Systèmes Distribués*, Masson, Paris, 1988.
115. K. Miller, *Nonunique continuation for uniformly parabolic and elliptic equations in self-adjoint divergence form with Hölder continuous coefficients*, Arch. Rat. Mech. Anal. **54** (1974), 105-117.
116. S. Mizohata, *Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques*, Mem. College Sci. Univ. Kyoto **A31** (1958), 219-239.
117. A.C. Murray and M.H. Protter, *The asymptotic behavior of solutions of seconfd order systems of partial differential equations*, J. Differential Equations **13** (1973), 57-80.
118. L. E. Payne, *Improperly Posed Problems in Partial Differential Equations*, SIAM, Philadelphia, 1975.

119. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, 1983.
120. O. Poisson, *Carleman estimates for the heat equation with discontinuous diffusion coefficients*, Appl. Analysis **87** (2008), 1129-1144.
121. L. Rosier and B.Y. Zhang, *Null controllability of the complex Ginzburg-Landau equation*, Ann. I. H. Poincaré AN **26** (2009), 649-673.
122. D. L. Russell, *A unified boundary controllability theory*, Studies Appl. Math. **52** (1973), 189-211.
123. D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions*, SIAM Review **20** (1978), 639-739.
124. J.-C. Saut and B. Scheurer, *Unique continuation for some evolution equations*, J. Differential Equations **66** (1987), 118-139.
125. C.D. Sogge, *A unique continuation theorem for second order parabolic differential operators*, Ark. Mat. **28** (1990), 159-182.
126. T. Takeuchi and M. Yamamoto, *Tikhonov regularization by a reproducing kernel Hilbert space for the Cauchy problem for an elliptic equation*, SIAM J. Sci. Compt. **31** (2008), 112–142.
127. D. Tataru, *Boundary controllability for conservative P.D.E.*, Appl. Math. Optim. **31** (1995), 257-295.
128. D. Tataru, *Carleman estimates and unique continuation for solutions to boundary value problems*, J. Math. Pures Appl. **75** (1996), 367–408.
129. M. Taylor, *Pseudodifferential Operators*, Princeton University Press, Princeton, New Jersey, 1981.
130. F. Trèves, *Linear Partial Differential Equations*, Gordon and Breach, New York, 1970.
131. S. Vessella, *Stability estimates in an inverse problem for a three-dimensional heat equation*, SIAM J. Math. Anal. **28** (1997), 1354-1370.
132. S. Vessella, *Carleman estimates, optimal three cylinder inequality, and unique continuation properties for solutions to parabolic equations*, Comm. Partial Differential Equations **28** (2003), 637-676.
133. S. Vessella, *Quantitative estimates of unique continuation for parabolic equations, determination of unknown time-varying boundaries and optimal stability estimates*, 023001 (81pp), Inverse Problems **24** (2008).
134. D. Xu and M. Yamamoto, *Stability estimates in state-estimation for a heat process*, in "Proceedings of the Second ISAAC Congress, Vol. 1 (Fukuoka, 1999) (2000), Kluwer, Dordrecht, 193–198.
135. M. Yamamoto and J. Zou, *Simultaneous reconstruction of the initial temperature and heat radiative coefficient*, Inverse Problems **17** (2001), 1181–1202.
136. G. Yuan and M. Yamamoto, *Lipshitz stability in the determination of the principal parts of a parabolic equation by boundary measurements*, to appear in ESAIM : Control, Optimisation and Calculus of Variations (2009).
137. X. Zhang, *Explicit observability estimate for the wave equation with potential and its application*, R. Soc. London Proc. Ser. A. Math. Phys. Eng. Sci. **456** (2000), 1101-1115.
138. C. Zuily, *Uniqueness and non-uniqueness in the Cauchy problem*, Birkhäuser, Boston, 1983.

UTMS

2009–2 Takashi Tsuboi: *On the uniform perfectness of the groups of diffeomorphisms of even-dimensional manifolds.*

2009–3 Hitoshi Kitada: *An implication of Gödel's incompleteness theorem.*

2009–4 Jin Cheng, Junichi Nakagawa, Masahiro Yamamoto and Tomohiro Yamazaki: *Uniqueness in an inverse problem for one-dimensional fractional diffusion equation.*

2009–5 Y. B. Wang, J. Cheng, J. Nakagawa, and M. Yamamoto : *A numerical method for solving the inverse heat conduction problem without initial value.*

2009–6 Dietmar Hömberg, Nataliya Togobytska, Masahiro Yamamoto: *On the evaluation of dilatometer experiments.*

2009–7 Toshio Oshima and Nobukazu Shimeno: *Heckman-Opdam hypergeometric functions and their specializations.*

2009–8 Atsushi Yamashita: *Compactification of the homeomorphism group of a graph.*

2009–9 Jingzhi Li, Masahiro Yamamoto, and Jun Zou: *Conditional stability and numerical reconstruction of initial temperature.*

2009–10 Taku Ishii and Takayuki Oda: *Calculus of principal series Whittaker functions on* $SL(n, \mathbb{R})$.

2009–11 Atsushi Nitanda: *The growth of the Nevanlinna proximity function.*

2009–12 Paola Loreti and Daniela Sforza: *Reachability problems for a class of integro-differential equations.*

2009–13 Masahiro Yamamoto: *Carleman estimates for parabolic equations and applications.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012