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On the uniform perfectness of the groups of diffeomorphisms of even-dimensional manifolds

by

Takashi Tsuboi

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

ON THE UNIFORM PERFECTNESS OF THE GROUPS OF DIFFEOMORPHISMS OF EVEN-DIMENSIONAL MANIFOLDS

TAKASHI TSUBOI

Abstract. We show that the identity component $\mathrm{Diff}^r(M^{2m})_0$ of the group of C^r diffeomorphisms of a compact $(2m)$-dimensional manifold M^{2m} ($1 \leq r \leq \infty$, $r \neq 2m+1$) is uniformly perfect for $2m \geq 6$, i.e., any element of $\mathrm{Diff}^r(M^{2m})_0$ can be written as a product of a bounded number of commutators. It is also shown that for a compact connected manifold M^{2m} ($2m \geq 6$), the identity component $\mathrm{Diff}^r(M^{2m})_0$ of the group of C^r diffeomorphisms of M^{2m} ($1 \leq r \leq \infty$, $r \neq 2m+1$) is uniformly simple, i.e., for elements f and g of $\mathrm{Diff}^r(M^{2m})_0 \setminus \{\mathrm{id}\}$, f can be written as a product of a bounded number of conjugates of g or g^{-1}.

1. Introduction

For an n-dimensional manifold M^n, let $\mathrm{Diff}^r_c(M^n)$ denote the group of C^r diffeomorphisms of M^n with compact support ($1 \leq r \leq \infty$). Here, the *support* of a diffeomorphism f of M^n is defined to be the *closure* of $\{x \in M \mid f(x) \neq x\}$. For a compact manifold M^n, $\mathrm{Diff}^r_c(M^n)$ coincides with the group $\mathrm{Diff}^r(M^n)$ of C^r diffeomorphisms of M^n. Let $\mathrm{Diff}^r_c(M^n)_0$ denote the identity component of $\mathrm{Diff}^r_c(M^n)$. Here $\mathrm{Diff}^r_c(M^n)$ is equipped with the C^r topology ([11], [16]). By the results of Herman, Mather and Thurston ([7], [9], [11], [16], [2]), for an n-dimensional manifold M^n, $\mathrm{Diff}^r_c(M^n)_0$ is a perfect group if $r = 0$ or $1 \leq r \leq \infty$ and $r \neq n+1$. Here, a group is said to be *perfect* if it coincides with its commutator subgroup. In other words, a group is perfect if any element can be written as a product of commutators.

We say that a group is *uniformly perfect* if any element can be written as a product of a *bounded* number of commutators. The following results are shown in [3], [22] and [23].

Theorem 1.1 (Burago-Ivanov-Polterovich [3], Tsuboi [22, 23])**.**

(1) *For the interior M^n of a compact n-dimensional manifold which admits a handle decomposition only with handles of indices not greater than $(n-1)/2$, any element of $\mathrm{Diff}^r_c(M^n)_0$ ($1 \leq r \leq \infty$, $r \neq n+1$) can be written as a product of two commutators.*

(2) *For a compact even-dimensional manifold M^{2m} which has a handle decomposition without handles of the middle index m, any element of*

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$\mathrm{Diff}^r(M^{2m})_0$ $(1 \leq r \leq \infty,\ r \neq 2m+1)$ *can be written as a product of four commutators.*

(3) *For a compact odd-dimensional manifold* M^{2m+1}*, any element of* $\mathrm{Diff}^r(M^{2m+1})_0$ $(1 \leq r \leq \infty,\ r \neq 2m+2)$ *can be written as a product of five commutators.*

Now the result of this paper concerns the remaining cases.

Theorem 1.2. *The identity component* $\mathrm{Diff}^r(M^{2m})_0$ *of the group of* C^r *diffeomorphisms* $\mathrm{Diff}^r(M^{2m})$ *of the compact* $(2m)$*-dimensional manifold* M^{2m} $(1 \leq r \leq \infty,$ $r \neq 2m+1)$ *is uniformly perfect for* $2m \geq 6$*, i.e., any element of* $\mathrm{Diff}^r(M^{2m})_0$ *can be written as a product of a bounded number of commutators.*

Here the bound for the number of commutators may depend on manifolds. For the manifolds of dimensions 2 and 4, the problem of uniform perfectness of the identity component of the group of diffeomorphisms is still open.

The argument deducing the simplicity of $\mathrm{Diff}^r(M^n)_0$ from the proof of its perfectness ([4], [16], [2]) applies to showing the uniform simplicity from the proof of its uniformly perfectness ([23]). We say that a group G is *uniformly simple* if, for elements f and g of $G \setminus \{\mathbf{1}\}$, f can be written as a product of a bounded number of conjugates of g or g^{-1}.

Corollary 1.3. *For a compact connected* $(2m)$*-dimensional manifold* M^{2m} $(2m \geq 6)$*, the identity component* $\mathrm{Diff}^r(M^{2m})_0$ *of the group* $\mathrm{Diff}^r(M^{2m})$ *of* C^r *diffeomorphisms of* M^{2m} $(1 \leq r \leq \infty,\ r \neq 2m+1)$ *is uniformly simple.*

The main part of the proof of Theorem 1.2 is a decomposition of an isotopy into a bounded number of isotopies with controlled support. Then the theorem follows from Theorem 1.1 (1) in a way similar to the proof of Theorem 1.1 (2) and (3) in [22] and in [23]. For the decomposition, we give a technique to find the Whitney disks which guide to separate two subcomplexes of the middle dimension m. The condition $2m \geq 6$ on the dimension implies that the Whitney disks can be disjointly embedded in the manifold and enables us to show Theorem 1.2.

We review the proof of Theorem 1.1 in Section 2 and we give the proof of Theorem 1.2 in Section 3. The proof of lemmas used in Section 3 is given in Section 4. We show Corollary 1.3 in Section 5.

2. Decomposition of isotopies

The proof of our Theorem 1.2 relies on the general position argument for differentiable maps from cellular complexes to a manifold which we used in [22] and [23] and we review several necessary results.

An n-dimensional finite cellular complex X is given by a filtration

$$X = X^{(n)} \supset X^{(n-1)} \supset \cdots \supset X^{(1)} \supset X^{(0)},$$

where

$$X^{(k)} = X^{(k-1)} \cup_{\varphi^{(k)}} (\bigsqcup_i D_i^k) \quad (k = 1, \ldots, n),$$

$\varphi^{(k)} : \bigsqcup_i \partial D_i^k \longrightarrow X^{(k-1)}$ is the attaching map and $X^{(k)}$ is obtained from the disjoint union of $X^{(k-1)}$ and finitely many k-dimensional disks D_i^k $(i = 1, \ldots, \ell_k)$ by identifying ∂D_i^k and its image under $\varphi^{(k)}$.

A *differentiable map* from the finite cellular complex X to a manifold M is a continuous map $X \longrightarrow M$ which is differentiable on each open cell $\mathrm{Int}(D_i^k)$.

We showed in [22] and [23] the following lemma.

Lemma 2.1 ([22], [23]). *Let M^n be a compact n-dimensional manifold. Let K^k and L^ℓ be k-dimensional and ℓ-dimensional finite cellular complexes, respectively. Let $f : K^k \longrightarrow M^n$ and $g : L^\ell \longrightarrow M^n$ be differentiable maps and assume that f is an embedding. If $k + \ell + 1 \leq n$, then there is an isotopy $\{\Phi_t : M^n \longrightarrow M^n\}_{t\in[0,1]}$ ($\Phi_0 = \mathrm{id}$) such that $\Phi_1(f(K^k)) \cap g(L^\ell) = \emptyset$.*

Then this lemma is used to show the following theorem.

Theorem 2.2 ([22], [23]). *Let M^n be a compact n-dimensional manifold. Let P^p and Q^q be p-dimensional and q-dimensional finite cellular complexes differentiably embedded in M^n, respectively. Assume that $p + q + 2 \leq n$ and that $P^p \cap Q^q = \emptyset$. Then any element $f \in \mathrm{Diff}^r(M^n)_0$ $(1 \leq r \leq \infty)$ can be written as a product $f = g \circ h$ such that $g \in \mathrm{Diff}^r_c(M^n \setminus k(Q^q))_0$ and $h \in \mathrm{Diff}^r_c(M^n \setminus P^p)_0$, where $k \in \mathrm{Diff}^r_0(M^n \setminus P^p)_0$, and $\mathrm{Diff}^r_c(M^n \setminus k(Q^q))_0$ and $\mathrm{Diff}^r_c(M^n \setminus P^p)_0$ are considered as subgroups of $\mathrm{Diff}^r(M^n)_0$, respectively.*

The statement of Theorem 2.2 means that the diffeomorphism g of M^n obtained in Theorem 2.2 is isotopic to the identity by an isotopy which is identity on a neighborhood of $k(Q^q)$, and h is isotopic to the identity by an isotopy which is identity on a neighborhood of P^p.

To use Theorem 2.2, we looked at the p-dimensional skeleton of the cellular decomposition associated with a handle decomposition of a compact manifold and the q-dimensional skeleton of that associated with the dual handle decomposition. Then, for an even-dimensional compact manifold M^{2m} which has a handle decomposition without handles of the middle index m, Theorem 2.2 together with Theorem 1.1 (1) implies Theorem 1.1 (2) (see [22]).

For the decomposition of an isotopy on an odd dimensional manifold, we used the following lemma ([22, Remark 4.4]).

Lemma 2.3. *In Lemma 2.1, if $k + \ell = n$, then there is an isotopy $\{\Phi_t : M^n \longrightarrow M^n\}_{t\in[0,1]}$ ($\Phi_0 = \mathrm{id}$) such that $\Phi_1(f(K^{(k-1)}))\cap g(L^\ell) = \emptyset$ for the $(k-1)$-dimensional skeleton $K^{(k-1)}$ of K^k, $\Phi_1(f(K^k)) \cap g(L^{(\ell-1)}) = \emptyset$ for the $(\ell - 1)$-dimensional skeleton $L^{(\ell-1)}$ of L^ℓ and the intersection $\Phi_1(f(\sigma^k)) \cap g(\tau^\ell)$ is transverse for each k-dimensional cell σ^k of K^k and each ℓ-dimensional cell τ^ℓ of L^ℓ.*

Remark 2.4. In fact, we can show the following for finite cellular complexes K and L and differentiable maps $f : K \longrightarrow M^n$ and $g : L \longrightarrow M^n$. Let $K^{(i)}$ and $L^{(j)}$ denote the i-dimensional skeleton and the j-dimensional skeleton of K and L, respectively. If f is an embedding, there exists an isotopy $\{\Phi_t\}_{t\in[0,1]}$ ($\Phi_0 = \mathrm{id}$) with support in a neighborhood of $f(K)$ such that $\Phi_1(f(K^{(i)})) \cap g(L^{(j)}) = \emptyset$ for $i + j + 1 = n$, and the intersection $\Phi_1(f(\sigma^i)) \cap g(\tau^j)$ is transverse for each i-dimensional cell σ^i of K and each j-dimensional cell τ^j of L for $i + j = n$.

Then we proceeded as follows (see [22, Lemma 6.3]).

Lemma 2.5. *Let M^n be a compact n-dimensional manifold. Let P^p and Q^q be p-dimensional and q-dimensional finite cellular complexes differentiably embedded in M^n, respectively. Assume that $p+q+1 = n$ and that $P^p \cap Q^q = \emptyset$. Let $P^{(p-1)}$ and $Q^{(q-1)}$ be the $(p-1)$-dimensional skeleton and the $(q-1)$-dimensional skeleton of P^p and Q^q, respectively. Then any element $f \in \mathrm{Diff}^r(M^n)_0$ can be written as a product $f = g \circ h$ such that $g \in \mathrm{Diff}^r_c(M^n \setminus k(Q^q))_0$ and $h \in \mathrm{Diff}^r_c(M^n \setminus P^{(p-1)})_0$, where $k \in \mathrm{Diff}^r_0(M^n \setminus P^p)_0$. Moreover there is an isotopy $\{h_t\}_{t\in[0,1]}$ such that $h_0 = \mathrm{id}$,*

$h_1 = h$, h_t is the identity on a neighborhood of $P^{(p-1)}$, and for $H(t,x) = h_t(x)$, $H([0,1] \times P^p) \cap k(Q^{(q-1)}) = \emptyset$ and, for each p-dimensional cell σ^p of P^p and each q-dimensional cell τ^q of Q^q, the intersection $H([0,1] \times \sigma^p) \cap k(\tau^q)$ is transverse. Thus $H([0,1] \times P^p) \cap k(Q^q)$ is a finite set.

For an odd dimensional compact manifold M^{2m+1}, we considered a handle decomposition of M^{2m+1} in [22], and we took the m-dimensional skeleton P^m of the associated cell decomposition and the m-dimensional skeleton Q^m of the cell decomposition associated with the dual handle decomposition.

Lemma 2.6. *Let $\{h_t\}_{t\in[0,1]}$ ($h_0 = \mathrm{id}$) be an isotopy which is the identity on a neighborhood of $P^{(m-1)}$ and $H([0,1] \times P^m) \cap k(Q^{(m-1)}) = \emptyset$ for $H(t,x) = h_t(x)$. Let $V^m \subset P^m$ be the complement of a neighborhood of $P^{(m-1)}$ where $h_t = \mathrm{id}$. Then there is an isotopy $\{\overline{h}_t\}_{t\in[0,1]}$ ($\overline{h}_0 = \mathrm{id}$) fixing a neighborhood of $P^{(m-1)}$ such that its trace $\overline{H} : [0,1] \times M^{2m+1} \longrightarrow M^{2m+1}$ is close to $H : [0,1] \times M^{2m+1} \longrightarrow M^{2m+1}$ and $\overline{H}|[0,1] \times V^m$ is an immersion outside of a finite subset. Moreover the image*

$$\overline{H}([0,1] \times V^m) \subset M^{2m+1} \setminus (P^{(m-1)} \cup k(Q^{(m-1)}))$$

has finitely many double point curves which is in general position with respect to the curves $\overline{H}([0,1] \times \{v\})$ ($v \in V^m$). If $m \geq 2$ these double point curves are disjoint, and if $m = 1$, there are at most finitely many triple points and cusps.

Then, using the idea of Burago, Ivanov and Polterovich ([3]), we constructed an isotopy $\{a_t\}_{t\in[0,1]}$ ($a_0 = \mathrm{id}$) with support in a union of disjointly embedded $(2m+1)$-dimensional open balls embedded in M^{2m+1} such that $(a_t \circ \overline{h}_t)(P^m) \cap k(Q^m) = \emptyset$ ($t \in [0,1]$), and we showed the following lemma ([22, Lemma 6.5]).

Lemma 2.7. *For the generic diffeomorphism $\overline{h} = \overline{h}_1 \in \mathrm{Diff}^r_c(M^{2m+1} \setminus P^{(m-1)})_0$ given by Lemma 2.6, $\overline{h}$ can be decomposed as $\overline{h} = a \circ \overline{g} \circ \overline{h}'$, where $a \in \mathrm{Diff}^r_c(\bigsqcup_i U_i)_0$, $\bigsqcup_i U_i$ is a union of disjointly embedded $(2m+1)$-dimensional open balls U_i embedded in M^{2m+1}, $\overline{g} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k(Q^m))_0$ and $\overline{h}' \in \mathrm{Diff}^r_c(M^{2m+1} \setminus P^m)_0$.*

Note that the element $\overline{h}^{-1} \circ h$ is close to the identity and it can be decomposed as $\overline{h}^{-1} \circ h = \widehat{h} \circ \widehat{g}$ with $\widehat{h} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus P^m)_0$ and $\widehat{g} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k(Q^m))_0$ ([22, Remark 5.4]). Then by Lemmas 2.5 and 2.7,

$$\begin{aligned} f &= g \circ h = g \circ \overline{h} \circ (\overline{h}^{-1} \circ h) \\ &= g \circ a \circ \overline{g} \circ \overline{h}' \circ \widehat{h} \circ \widehat{g} \\ &= (g \circ a \circ g^{-1}) \circ (g \circ \overline{g} \circ \widehat{g}) \circ (\widehat{g}^{-1} \circ \overline{h}' \circ \widehat{h} \circ \widehat{g}) \end{aligned}$$

and $g \circ a \circ g^{-1} \in \mathrm{Diff}^r_c(g(\bigsqcup_i U_i))_0$, $g \circ \overline{g} \circ \widehat{g} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus k(Q^m))_0$ and $\widehat{g}^{-1} \circ \overline{h}' \circ \widehat{h} \circ \widehat{g} \in \mathrm{Diff}^r_c(M^{2m+1} \setminus \widehat{g}^{-1}(P^m))_0$. Noticing that a can be taken as a commutator with support in $\bigsqcup_i U_i$, Theorem 1.1 (1) implies Theorem 1.1 (3) (see [22]).

It is worth noticing again that, for any compact manifold M^n, there is a neighborhood of the identity of $\mathrm{Diff}^r(M^n)_0$ ($1 \leq r \leq \infty$, $r \neq n+1$) whose element can be written as a product of four or six commutators([22, Remark 5.4]).

Remark 2.8. For a compact manifold M we have a handle decomposition. For a compact odd-dimensional manifold M^{2m+1}, M^{2m+1} is covered by two open sets U_1 and U_2 which are neighborhoods of the union of handles of indices not greater than m and the union of dual handles of indices not greater than m. Then by the fragmentation lemma ([2]), there is a neighborhood $\mathcal{N}$ of the identity in $\mathrm{Diff}^r(M^{2m+1})_0$

such that any element f of $\mathcal{N}$ can be written as a product $f = g \circ h$, where $g \in \mathrm{Diff}^r_c(U_1)_0$ and $h \in \mathrm{Diff}^r_c(U_2)_0$. Hence by Theorem 1.1 (1), any element f of $\mathcal{N}$ can be written as a product of four commutators of elements of $\mathrm{Diff}^r(M^{2m+1})_0$ $(1 \leq r \leq \infty,\ r \neq 2m+2)$. For a compact even-dimensional manifold M^{2m}, M^{2m} is covered by three open sets U_1, U_2 and U_3. Here, U_1 and U_2 are neighborhoods of the union of handles of indices less than m and the union of dual handles of indices less than m, and U_3 is a union of disjointly embedded open balls which is a neighborhood of the union of m handles. Then by the fragmentation lemma, there is a neighborhood $\mathcal{N}$ of the identity in $\mathrm{Diff}^r(M^{2m})_0$ such that any element f of $\mathcal{N}$ can be written as a product $f = a \circ g \circ h$, where $g \in \mathrm{Diff}^r_c(U_1)_0$, $h \in \mathrm{Diff}^r_c(U_2)_0$ and $a \in \mathrm{Diff}^r_c(U_3)_0$. Hence by Theorem 1.1 (1), any element f of $\mathcal{N}$ can be written as a product of six commutators of elements of $\mathrm{Diff}^r(M^{2m})_0$ $(1 \leq r \leq \infty,\ r \neq 2m+1)$.

3. Proof of the main theorem

For an even dimensional compact manifold M^{2m}, we proceed as follows to prove Theorem 1.2. The proof of lemmas is given in the next section.

For the manifold M^{2m}, we consider its triangulation P and let $P^{(k)}$ denote the k-dimensional skeleton of P. Then the $(m-1)$-dimensional skeleton $P^{(m-1)}$ of the triangulation P has the following property:

> For each m-dimensional simplex σ^m of $P^{(m)}$, let $(P^{(m-1)} \cup \sigma^m)/\sigma^m$ denote the $(m-1)$-dimensional cell complex obtained from $P^{(m-1)} \cup \sigma^m$ by identifying σ^m to a point. Then there is an embedding ι of $(P^{(m-1)} \cup \sigma^m)/\sigma^m$ in M^{2m} such that, for any neighborhood U of $\iota((P^{(m-1)} \cup \sigma^m)/\sigma^m)$, there is a diffeomorphism of M^{2m} isotopic to the identity which maps $P^{(m-1)} \cup \sigma^m$ into U.

Remark 3.1. We may use the cellular complex associated with a handle decomposition of M^{2m} if it has this property for each m-dimensional cell σ^m. The number N of the m-dimensional cells of such a cellular decomposition of M^{2m} appears in the estimate of the bound for the number of commutators at the end of the proof of Theorem 1.2.

For the manifold M^{2m}, the statement of Lemma 2.5 is written as follows.

Lemma 3.2. *Let P^m denote the m-dimensional skeleton of a triangulation of a $(2m)$-dimensional manifold M^{2m}, and Q^m, the m-dimensional skeleton of the dual cell decomposition. Let $P^{(i)}$ and $Q^{(i)}$ denote the i-dimensional skeletons ($i = m-2$, $m-1$) of P^m and Q^m, respectively. Then any element $f \in \mathrm{Diff}^r(M^{2m})_0$ can be written as a product $f = g \circ h$ such that $g \in \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m))_0$ and $h \in \mathrm{Diff}^r_c(M^{2m} \setminus P^{(m-2)})_0$, where $k \in \mathrm{Diff}^r_0(M^{2m} \setminus P^m)_0$. Moreover there is an isotopy $\{h_t\}_{t\in[0,1]}$ which has the following properties:*

1. *$h_0 = \mathrm{id}$, $h_1 = h$, and h_t is the identity on a neighborhood of $P^{(m-2)}$.*
2. *For $H(t,x) = h_t(x)$,*
   $H([0,1] \times P^{(m-1)}) \cap k(Q^{(m-1)}) = \emptyset$ and $H([0,1] \times P^m) \cap k(Q^{(m-2)}) = \emptyset$.
3. *For each $(m-1)$-dimensional simplex σ^{m-1} of $P^{(m-1)}$ and each m-dimensional cell τ^m of Q^m, the intersection $H([0,1] \times \sigma^{m-1}) \cap k(\tau^m)$ is transverse. Thus $H([0,1] \times P^{(m-1)}) \cap k(Q^m)$ is a finite set.*

Then, if $2m \geq 4$, we can separate the image $H([0,1] \times P^{(m-1)})$ from $k(Q^m)$ by an argument similar to the proof of Lemmas 2.6 and 2.7

First, we approximate the isotopy H by a generic one $\overline{H}$. Let

$$\{\overline{h}_t\}_{t\in[0,1]} \subset \mathrm{Diff}^\infty_c(M^{2m} \setminus P^{(m-2)}) \quad (\overline{h}_0 = \mathrm{id})$$

be a C^∞ approximation of $\{h_t\}_{t\in[0,1]} \subset \mathrm{Diff}^r_c(M^{2m} \setminus P^{(m-2)})$ generic with respect to P^m and $k(Q^m)$ such that $\overline{h}_t$ is the identity on a neighborhood of $P^{(m-2)}$. Then $\overline{H}(t,x) = \overline{h}_t(x)$ has the following properties:

(0) $\overline{H} : [0,1] \times M^{2m} \longrightarrow M^{2m}$ is close to $H : [0,1] \times M^{2m} \longrightarrow M^{2m}$ and $\overline{h}_t$ is the identity on a neighborhood of $P^{(m-2)}$.

(1) The restriction

$$\overline{H}|[0,1] \times V^{m-1} : [0,1] \times V^{m-1} \longrightarrow M^{2m}$$

is an immersion, where $V^{m-1}(\subset P^{(m-1)})$ is the complement of a neighborhood of $P^{(m-2)} \subset P^{(m-1)}$ where $\overline{h}_t$ is the identity.

(2) $\overline{H}([0,1] \times P^{(m-1)}) \cap k(Q^{(m-1)}) = \emptyset$ and $\overline{H}([0,1] \times P^m) \cap k(Q^{(m-2)}) = \emptyset$.

(3) $\overline{H}([0,1] \times P^{(m-1)}) \cap k(Q^m)$ is a finite set;

$$\overline{H}([0,1] \times P^{(m-1)}) \cap k(Q^m) = \{\overline{H}(s_i, v_i) \mid i = 1, \ldots, r\},$$

(4) $\overline{H}([0,1] \times \{v_i\}) \cap k(Q^m) = \overline{H}(s_i, v_i)$ $(i = 1, \ldots, r)$,

(5) $\overline{H}([0,1] \times \{v_i\})$ does not contain double points of $\overline{H}([0,1] \times P^{(m-1)})$ $(i = 1, \ldots, r)$,

(6) $\overline{H}|[0,1] \times P^{(m-1)}$ restricted to a neighborhood of $[0,1] \times \{v_i\}$ in $[0,1] \times P^{(m-1)}$ is an embedding $(i = 1, \ldots, r)$, and

(7) $\overline{H}([s_i,1] \times \{v_i\})$ $(i = 1, \ldots, r)$ are disjoint.

Here, the statements (1)–(7) hold for generic $\overline{H}$ (or the properties (1)–(7) are generic in the space of isotopies). In particular, the statement (5) holds because the inverse image of the double point set of $\overline{H}([0,1] \times P^{(m-1)})$ is a finite set which is in general position with respect to $[0,1] \times \{v_i\}$ $(i = 1, \ldots, r)$ and $2m \geq 4$.

For the proof of uniform perfectness, we can approximate the diffeomorphism for a bounded number of times. In fact in this case, $f_1 = g_1 \circ h_1 = g_1 \circ \overline{h}_1 \circ (\overline{h}_1{}^{-1} \circ h_1)$ and $\overline{h}_1{}^{-1} \circ h_1 \in \mathrm{Diff}^r(M^{2m})$ is close to the identity. By Remark 2.8, $\overline{h}_1{}^{-1} \circ h_1$ is written as a product of six commutators.

Lemma 3.3. *For the above generic isotopy $\{\overline{h}_t\}_{t\in[0,1]}$, there is a neighborhood U_i $(i = 1, \ldots, r)$ of the curve $\overline{H}([s_i,1] \times \{v_i\}) \subset M^{2m}$ diffeomorphic to a $(2m)$-dimensional ball such that U_i are disjoint and there is an isotopy $\{a_t\}_{t\in[0,1]}$ $(a_0 = \mathrm{id})$ with support in $\bigsqcup_{i=1}^r U_i$ such that, for $h'_t = a_t \circ \overline{h}_t$,*

$$h'_t(P^{(m-1)}) \cap k(Q^m) = \emptyset \quad (t \in [0,1]).$$

Note that $a_t \in \mathrm{Diff}^r_c(\bigsqcup_{i=1}^r U_i)_0$ can be taken as one commutator with support in $\bigsqcup_{i=1}^r U_i$ (see [23]).

Since $h'_t(P^{(m-1)}) \cap k(Q^m) = \emptyset$ $(t \in [0,1])$, there are isotopies $\{g'_t\}_{t\in[0,1]} \subset \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m))$ and $\{h''_t\}_{t\in[0,1]} \subset \mathrm{Diff}^r_c(M^{2m} \setminus P^{(m-1)})$ such that $h'_1 = g'_1 \circ h''_1$. In other words, g'_t and h''_t $(t \in [0,1])$ are the identity on neighborhoods of $k(Q^m)$ and $P^{(m-1)}$, respectively. Note that, by taking h''_t generically on P^m, $h''_t(P^m) \cap k(Q^{(m-2)}) = \emptyset$.

Here we used the following lemma which is a part of Theorem 2.2 ([22, Theorem 5.1]). We state the lemma in a form convenient for the proof of Theorem 1.2.

Lemma 3.4. *Let M^n be a compact n-dimensional manifold. Let P^p and Q^q be p-dimensional and q-dimensional finite cellular complexes differentiably embedded in M^n, respectively. Let P_0 be a subset of P^p. Let $\{f_t\} \subset \mathrm{Diff}^r(M^n)_0$ ($f_0 = \mathrm{id}$) be an isotopy which is the identity on a neighborhood of P_0. Assume that $f_t(P^p \setminus P_0) \cap Q^q = \emptyset$ ($t \in [0,1]$). Then $f_1 \in \mathrm{Diff}^r(M^n)_0$ can be written as a product $f_1 = g_1 \circ h_1$, where $\{g_t\}_{t\in[0,1]} \subset \mathrm{Diff}^r_c(M^n \setminus Q^q)_0$ ($g_0 = \mathrm{id}$) and $\{h_t\}_{t\in[0,1]} \subset \mathrm{Diff}^r_c(M^n \setminus P^p)_0$ ($h_0 = \mathrm{id}$).*

Put $h_t^{(0)} = h''_t$. Then $h_t^{(0)}$ is the identity on a neighborhood of $P^{(m-1)}$ and $h_t^{(0)}(P^m) \cap k(Q^{(m-2)}) = \emptyset$ ($t \in [0,1]$).

We look at the intersection $h_t^{(0)}(P^m) \cap k(Q^m)$. We assume $2m \geq 6$ and we are going to simplify the intersection, simplex by simplex. This is the main part of the proof of our Theorem 1.2. Let σ_j^m ($j = 1, \ldots, N$) be the m-dimensional simplices of P^m. For each simplex σ_j^m ($j = 1, \ldots, N$), we remove the intersection of the image of the isotopy of σ_j^m and $k(Q^{(m-1)})$ in a way similar to Lemma 3.3, and then we remove the intersection of the resultant isotopy of σ_j^m and $k(Q^m \setminus \sigma_j^{m*})$, where σ_j^{m*} is the m-dimensional cell of Q^m dual to σ_j^m. For the latter process, we will find the Whitney disks which guide the construction of isotopy to reduce the order of the intersection point set.

More precisely, we construct the isotopies inductively. As we wrote, let σ_i^m ($i = 1, \ldots, N$) be the m-dimensional simplices of P^m. For $0 \leq j \leq N$, assume that we have an isotopy

$$\{h_t^{(j)}\}_{t\in[0,1]} \subset \mathrm{Diff}^r(M^{2m})_0 \quad (h_0^{(j)} = \mathrm{id})$$

such that $h_t^{(j)}$ is the identity on a neighborhood of $P^{(m-1)} \cup \bigcup_{i=1}^j \sigma_i^m$. Let $\overline{h}_t^{(j)}$ be a C^∞ approximation of $h_t^{(j)}$ generic with respect to P^m and $k(Q^m)$ such that $\overline{h}_t^{(j)}$ is the identity on a neighborhood of $P^{(m-1)} \cup \bigcup_{i=1}^j \sigma_i^m$. Then $\overline{H}^{(j)}(t,x) = \overline{h}_t^{(j)}(x)$ has the following properties:

(0) $\overline{H}^{(j)} : [0,1] \times M^{2m} \longrightarrow M^{2m}$ is close to $H^{(j)} : [0,1] \times M^{2m} \longrightarrow M^{2m}$ defined by $H^{(j)}(t,x) = h_t^{(j)}(x)$ and $\overline{h}_t^{(j)}$ is the identity on a neighborhood of $P^{(m-1)} \cup \bigcup_{i=1}^j \sigma_i^m$.

(1) The restriction

$$\overline{H}^{(j)}|[0,1] \times V^{m(j)} : [0,1] \times V^{m(j)} \longrightarrow M^{2m}$$

is an immersion outside of a 1-dimensional subset (a codimension m subset) of $[0,1] \times V^{m(j)}$, where $V^{m(j)} (\subset P^m)$ is the complement of a neighborhood of $P^{(m-1)}$ in P^m where $\overline{h}_t^{(j)}$ is the identity.

(2) $\overline{H}^{(j)}([0,1] \times P^{(m-1)}) \cap k(Q^{(m-1)}) = \emptyset$ and $\overline{H}^{(j)}([0,1] \times P^m) \cap k(Q^{(m-2)}) = \emptyset$.

(3) $\overline{H}^{(j)}([0,1] \times P^m) \cap k(Q^{(m-1)})$ is a finite set;

$$\overline{H}^{(j)}([0,1] \times P^m) \cap k(Q^{(m-1)}) = \{\overline{H}^{(j)}(s_i^{(j)}, v_i^{(j)}) \mid i = 1, \ldots, r^{(j)}\},$$

(4) $\overline{H}^{(j)}([0,1] \times \{v_i^{(j)}\}) \cap k(Q^{(m-1)}) = \overline{H}^{(j)}(s_i^{(j)}, v_i^{(j)})$ ($i = 1, \ldots, r^{(j)}$),

(5) $\overline{H}^{(j)}([0,1] \times \{v_i^{(j)}\})$ does not contain double points of $\overline{H}^{(j)}([0,1] \times P^m)$ ($i = 1, \ldots, r^{(j)}$),

(6) $\overline{H}^{(j)}|[0,1]\times P^m$ restricted to a neighborhood of $[0,1]\times\{v_i^{(j)}\}$ in $[0,1]\times P^m$ is an embedding ($i=1,\ldots,r^{(j)}$), and
(7) $\overline{H}^{(j)}([s_i^{(j)},1]\times\{v_i^{(j)}\})$ are disjoint.

Here, the statements (1)–(7) hold for generic $\overline{H}^{(j)}$. In particular, for the statement (1), we notice that the set of rank m matrices in the space of $(m+1)\times(2m)$ matrices is codimension m ([15]). The statement (6) holds because the inverse image of the double point set of $\overline{H}^{(j)}([0,1]\times P^m)$ is 2-dimensional in $[0,1]\times P^m$ which is in general position with respect to $[0,1]\times\{v_i^{(j)}\}$ ($i=1,\ldots,r^{(j)}$) and $2m\geq 6$.

Lemma 3.5. *For the above generic isotopy $\{\overline{h}_t^{(j)}\}_{t\in[0,1]}$, there is a neighborhood $U_i^{(j)}$ ($i=1,\ldots,r^{(j)}$) of the curve $\overline{H}^{(j)}([s_i^{(j)},1]\times\{v_i^{(j)}\})\subset M^{2m}$ diffeomorphic to a $(2m)$-dimensional ball such that $U_i^{(j)}$ are disjoint and there is an isotopy $\{a_t^{(j+1)}\}_{t\in[0,1]}$ ($a_0^{(j+1)}=\mathrm{id}$) with support in $\bigsqcup_{i=1}^{r^{(j)}}U_i^{(j)}$ such that, for $h'^{(j)}_t=a_t^{(j+1)}\circ\overline{h}_t^{(j)}$,*

$$h'^{(j)}_t(P^m)\cap k(Q^{(m-1)})=\emptyset\quad(t\in[0,1]).$$

Note again that $a_t^{(j+1)}\in\mathrm{Diff}_c^r(\bigsqcup_{i=1}^{r^{(j)}}U_i^{(j)})_0$ can be taken as one commutator with support in $\bigsqcup_{i=1}^{r^{(j)}}U_i^{(j)}$ (see [23]).

The isotopy $h'^{(j)}_t$ given by Lemma 3.5 has the following properties.

(0) $h'^{(j)}_t$ is the identity on a neighborhood of $P^{(m-1)}\cup\bigcup_{i=1}^{j}\sigma_i^m$.
(1) $H'^{(j)}([0,1]\times P^m)\cap k(Q^{(m-1)})=\emptyset$.
(2) $h'^{(j)}_t$ is generic with respect to P^m and $k(Q^m)$.

Now we look at the intersection $h'^{(j)}_t(P^m)\cap k(Q^m)$. Since $h'^{(j)}_t$ is the identity on a neighborhood of $P^{(m-1)}\cup\bigcup_{i=1}^{j}\sigma_i^m$, the intersection $h'^{(j)}_t(\sigma_i^m)\cap k(Q^m)$ for $i\leq j$ is always the one point set $\sigma_i^m\cap k(\sigma_i^{m*})$, where σ_i^{m*} is the m-dimensional cell of Q^m dual to σ_i^m ($i\leq j$). For the simplex σ_{j+1}^m, the intersection $h'^{(j)}_t(\sigma_{j+1}^m)\cap k(Q^m)$ is a finite set which vary with respect to the parameter t. If $2m\geq 6$, we can find the Whitney disks which guide to reduce the order of intersection point set $h'^{(j)}_t(\sigma_{j+1}^m)\cap k(Q^m\setminus\sigma_{j+1}^{m*})$, where σ_{j+1}^{m*} is the m-dimensional cell of Q^m dual to σ_{j+1}^m as we explain now.

For the m-dimensional simplex σ_{j+1}^m of P^m, the intersection of σ_{j+1}^m and $k(Q^m)$ is just one point which is the intersection of σ_{j+1}^m and $k(\sigma_{j+1}^{m*})$, Then the behavior of the intersection $h'^{(j)}_t(\sigma_{j+1}^m)\cap k(\sigma_{j+1}^{m*})$ it rather complicated. Hence we look at $H'^{(j)}([0,1]\times\sigma_{j+1}^m)\cap k(Q^m\setminus\sigma_{j+1}^{m*})$ or $h'^{(j)}_t(\sigma_{j+1}^m)\cap k(Q^m\setminus\sigma_{j+1}^{m*})$. First, note that $h'^{(j)}_t(\sigma_{j+1}^m)\cap k(Q^m\setminus\sigma_{j+1}^{m*})$ is the empty set for small t, and since $h'^{(j)}_t(\sigma_{j+1}^m)\cap k(Q^{(m-1)})=\emptyset$ (and $h'^{(j)}_t(P^{(m-1)})\cap k(Q^m)=\emptyset$), the algebraic intersection number of the two m-dimensional cells $h'^{(j)}_t(\sigma_{j+1}^m)$ and $k(\tau^m)$ ($t\in[0,1]$) is always 0 for each m-dimensional cell τ^m of the dual cell complex Q^m other than σ_{j+1}^{m*} .

If we look at the movement of the intersection $h'^{(j)}_t(\sigma_{j+1}^m)\cap k(\tau^m)$ with respect to the parameter t, there happen a finite number of generations of pairs of intersection points and cancellations of pairs of intersection points. For generic $H'^{(j)}$ or $h'^{(j)}_t$, the values of the parameters t of generations and cancellations are different. This genericity argument follows from the following well known lemma.

Lemma 3.6. *Consider the space of C^r maps ($r \geq 2$) $F : \boldsymbol{R} \times \boldsymbol{R}^m \longrightarrow \boldsymbol{R}^m$. Then, for generic F, the inverse image of a generic point $y \in \boldsymbol{R}^m$ consists of regular points and fold points for $F_t = F(t, \bullet)$. At a fold point x for F_t, by changing the coordinates of $\boldsymbol{R}^m$ (both of the second factor of $\boldsymbol{R} \times \boldsymbol{R}^m$ and the target $\boldsymbol{R}^m$), F_t is locally written as*

$$F_t(x_1, \dots, x_m) = (x_1, \dots, x_{m-1}, y_m(t, x_1, \dots, x_m)),$$

where $\dfrac{\partial y_m}{\partial x_m} = 0$, $\dfrac{\partial y_m}{\partial t} \neq 0$ and $\dfrac{\partial^2 y_m}{\partial {x_m}^2} \neq 0$ at x. The fold points are discrete in $F^{-1}(y)$ and correspond to the generations or cancellations of pairs of intersection points.

We use this Lemma 3.6 in the following way. We take a tubular neighborhood of $k(\tau^m)$ and the projection $p_{k(\tau^m)}$ to the fiber which is an m-dimensional disk, and look at the map $p_{k(\tau^m)} \circ (H'^{(j)}|[0,1] \times \sigma^m_{j+1})$. Then for generic $H'^{(j)}$, by using Lemma 3.6, there are only finitely many generations and cancellations of pairs of intersections in the family $\{h'^{(j)}_t(\sigma^m_{j+1}) \cap k(\tau^m)\}_{t \in [0,1]}$.

We are going to construct the disks associated with the intersection $H'^{(j)}([0,1] \times \sigma^m_{j+1}) \cap k(\tau^m)$ for an m-dimensional cell τ^m of Q^m other than σ^{m*}_{j+1}.

For a generation of a pair of intersection points, the intersection points near the generation point are written as $h'^{(j)}_t(x_t)$ and $h'^{(j)}_t(y_t)$ ($t \in [t_0, t_0 + \varepsilon_0)$), where $h'^{(j)}_{t_0}(x_{t_0}) = {h'_{t_0}}^{(j)}(y_{t_0})$ is the generation point. Here, x_t and y_t are continuous functions written as $x_t = (c_1, \dots, c_{m-1}, \sqrt{t - t_0})$ and $y_t = (c_1, \dots, c_{m-1}, -\sqrt{t - t_0})$, respectively, for a suitable choice of coordinate around $(t_0, x_{t_0}) = (t_0, y_{t_0}) \in [0,1] \times \sigma^m_{j+1}$, where $c_1, \dots, c_{m-1}$ are constants.

We take a flat metric on the m-dimensional simplex σ^m_{j+1} and we draw the geodesic segment $\overline{x_t y_t}$ in σ^m_{j+1} joining the intersection points x_t and y_t ($t \in [t_0, t_0 + \varepsilon_0)$).

Once we choose the pair of intersection points to be joined by the geodesic segment, we continue joining them as the parameter t increses unless one of these intersection points meets a cancellation point.

For a cancellation of a pair of intersections, the intersection points near the cancellation point are written as $h'^{(j)}_t(x_t)$ and $h'^{(j)}_t(y_t)$ ($t \in (t_0 - \varepsilon_0, t_0]$), where $h'^{(j)}_{t_0}(x_{t_0}) = h'^{(j)}_{t_0}(y_{t_0})$ is the cancellation point. Here, x_t and y_t are continuous functions written as $x_t = (c_1, \dots, c_{m-1}, \sqrt{-t + t_0})$ and $y_t = (c_1, \dots, c_{m-1}, -\sqrt{-t + t_0})$, respectively, for a suitable choice of coordinate around $(t_0, x_{t_0}) = (t_0, y_{t_0}) \in [0,1] \times \sigma^m_{j+1}$, where $c_1, \dots, c_{m-1}$ are constants.

Assume that we have chosen geodesic segments for the intersection points such that $t < t_0$. Let x'_t ($t \in (t_0 - \varepsilon_0, t_0)$) be the other endpoint of the geodesic segment containing x_t, and y'_t ($t \in (t_0 - \varepsilon_0, t_0)$) be the other endpoint of the geodesic segment containing y_t. There are two cases. In the case where $x'_{t_0} \neq y'_{t_0}$, that is, if it is a cancellation of intersection points belonging to different geodesic segments $\overline{x_t x'_t}$ and $\overline{y_t y'_t}$ in $\{t\} \times \sigma^m_{j+1}$ ($t \in (t_0 - \varepsilon_0, t_0)$), we draw the geodesic triangle joining the

3 points $x_{t_0} = y_{t_0}$, x'_{t_0} and y'_{t_0} in $\{t_0\} \times \sigma^m_{j+1}$, and continue to draw the geodesic segment $\overline{x'_t y'_t}$ joining x'_t and y'_t in $\{t\} \times \sigma^m_{j+1}$ $(t \in (t_0, t_0 + \varepsilon_0))$. In the case where $x'_{t_0} = y'_{t_0}$, that is, if it is a cancellation of intersection points of the same geodesic segment $\overline{x_t y_t}$ in $\{t\} \times \sigma^m_{j+1}$ $(t \in (t_0 - \varepsilon_0, t_0)$, $x'_t = y_t$ and $y'_t = x_t)$, we add the auxiliary band

$$\bigcup_{t \in [t_0 - \varepsilon, t_0]} [t, 1] \times \{x_t\} \cup \bigcup_{t \in [t_0 - \varepsilon, t_0]} [t, 1] \times \{y_t\},$$

which contains the curve $[t_0, 1] \times \{x_{t_0}\} = [t_0, 1] \times \{y_{t_0}\}$, where ε $(< \varepsilon_0)$ is a small positive real number. Note that the image of the auxiliary band does not contain double points of $H'^{(j)}([0,1] \times \sigma^m_{j+1})$ for generic $H'^{(j)}$, and hence $H'^{(j)}$ restricted to the auxiliary band is an embedding into $M^{2m} \setminus k(Q^{(m-1)})$.

Now we have a family of geodesic segments in σ^m_{j+1} moving with respect to the parameter t and there are only finitely many times t_i $(i = 1, \ldots, \overline{r}^{(j)})$ when there appear geodesic triangles.

We are assuming that $2m \geq 6$, and for generic $h'^{(j)}_t$, the family of geodesic segments satisfies the following properties because the preimage of the double points of $h'^{(j)}_t(P^m)$ is 1-dimensional in $[0,1] \times \sigma^m_{j+1}$.

(1) The geodesic segments in σ^m_{j+1} joining the pairs of intersection points in $(h'^{(j)}_t)^{-1}(k(\tau^m))$ never contain the preimage of double points of $(h'^{(j)}_t)(P^m)$.
(2) The geodesic triangles never contain the preimage of double points of $(h'^{(j)}_t)(P^m)$.

For t_i $(i = 1, \ldots, \overline{r}^{(j)})$, let Y be the union of the geodesic triangle with the three vertices $x_{t_i} = y_{t_i}$, x'_{t_i} and y'_{t_i} in $\{t_i\} \times \sigma^m_{j+1}$, the geodesic segments $\overline{x_t x'_t}$ and $\overline{y_t y'_t}$ in $\{t\} \times \sigma^m_{j+1}$, $(t \in (t_i - \varepsilon_i, t_i))$ and the geodesic segments $\overline{x'_t y'_t}$ in $\{t\} \times \sigma^m_{j+1}$ $(t \in (t_i, t_i + \varepsilon_i))$:

$$\begin{aligned} Y = {} & \Big(\bigcup_{t \in (t_i - \varepsilon_i, t_i)} \{t\} \times \overline{x_t x'_t}\Big) \cup \Big(\bigcup_{t \in (t_i - \varepsilon_i, t_i)} \{t\} \times \overline{y_t y'_t}\Big) \\ & \cup \big(\{t_i\} \times \triangle x_{t_i} x'_{t_i} y'_{t_i}\big) \cup \Big(\bigcup_{t \in (t_i, t_i + \varepsilon_i)} \{t\} \times \overline{x'_t y'_t}\Big) \\ \subset {} & (t_i - \varepsilon_i, t_i + \varepsilon_i) \times \sigma^m_{j+1}. \end{aligned}$$

We deform it to obtain a 2-dimensional manifold Y' embedded in $(t_i - \varepsilon_i, t_i + \varepsilon_i) \times \sigma^m_{j+1}$ such that

$$\begin{aligned} \partial Y' = \partial Y = {} & \{(t, x'_t)\}_{t \in (t_i - \varepsilon_i, t_i + \varepsilon_i)} \cup \{(t, y'_t)\}_{t \in (t_i - \varepsilon_i, t_i + \varepsilon_i)} \\ & \cup \{(t, x_t)\}_{t \in (t_i - \varepsilon_i, t_i]} \cup \{(t, y_t)\}_{t \in (t_i - \varepsilon_i, t_i]} \\ & \subset (t_i - \varepsilon_i, t_i + \varepsilon_i) \times \sigma^m_{j+1}, \end{aligned}$$

and Y' coincides with Y for $|t - t_i| \geq \varepsilon_i/2$ and the intersection of Y' and $\{t\} \times \sigma^m_{j+1}$ is a union of two disjoint differentiable curves near the original geodesic segments for $t \in [t_i - \varepsilon_i/2, t_i)$ and is one differentiable curve near the geodesic triangle for $t \in [t_i, t_i + \varepsilon_i/2]$.

Now we look at the union Z of geodesic segments which are not modified by the above operation and the manifolds Y' for all t_i $(i = 1, \ldots, \overline{r}^{(j)})$. If there are auxiliary bands we add them to Z and modify it to make Z an embedded 2-dimensional manifold with boundary in $[0,1] \times \sigma^m_{j+1}$.

For a generic choice of the isotopy $H'^{(j)}$ and manifolds Y', if $2m \geq 8$, Z is a union of disjointly embedded 2-dimensional disks in $[0,1] \times \sigma^m_{j+1}$. If $2m = 6$, the 2-dimensional disks may intersect in $[0,1] \times \sigma^3_{j+1}$ creating finitely many double points.

For $2m \geq 8$, the fact that a connected component of the union Z is diffeomorphic to a 2-dimensional disk can be seen as follows: Consider the space obtained from Z by identifying the points in each connected component of $Z \cap (\{t\} \times \sigma^m_{j+1})$. Then it is a graph with vertices corresponding to the generation points and cancellation points. The generation points correspond to the vertices of valency 1 and the cancellation points correspond to the vertices of valency 3 except the cancellation points with auxiliary bands. For the cancellation points with auxiliary bands, the auxiliary bands become edges ending at $\{1\} \times \sigma^m_{j+1}$. Thus each connected component of the graph is a tree rooted at time $t = 1$ which grows in the negative direction in t. Hence each connected component of Z is a 2-dimensional disk.

In the case where $2m = 6$, we see in a similar way that $Z \subset [0,1] \times \sigma^3_{j+1}$ is an immersed image of 2-dimensional disks which has generically a finite number of double points. That is, the curves joining the pairs of intersection points in $(h'^{(j)}_t)^{-1}(k(\tau^3))$ may intersect at finitely many points $(\widehat{t}_\ell, \widehat{x}_\ell)$ $(\ell = 1, \ldots, \widehat{r}^{(j)})$. Then for generic $H'^{(j)}$, $\widehat{t}_\ell$ are not the time of generations or cancellations. When two geodesic curves $\gamma_1^{(t)}$ and $\gamma_2^{(t)}$ intersect at the time $\widehat{t}_\ell$, we modify one of the family $\{\gamma_2^{(t)}\}$ of geodesic curves near $\widehat{t}_\ell$ by a family $\{\gamma_2'^{(t)}\}$ of curves which does not intersect $\{\gamma_1^{(t)}\}$ near $\widehat{t}_\ell$.

More concretely, for a small positive real number $\widehat{\varepsilon}_\ell$, we can find a neighborhood of $\gamma_1^{(\widehat{t}_\ell)} \cup \gamma_2^{(\widehat{t}_\ell)} \subset [0,1] \times \sigma^m$ which is diffeomorphic to $(\widehat{t}_\ell - \widehat{\varepsilon}_\ell, \widehat{t}_\ell + \widehat{\varepsilon}_\ell) \times X$, where X is a neighborhood of $[-1,1] \times \{0\} \times \{0\} \cup \{0\} \times [-1,1] \times \{0\}$ in $\boldsymbol{R}^3$,

$$\gamma_1^{(\widehat{t}_\ell)} = \{\widehat{t}_\ell\} \times [-1,1] \times \{0\} \times \{0\} \quad \text{and}$$
$$\gamma_2^{(\widehat{t}_\ell)} = \{\widehat{t}_\ell\} \times \{0\} \times [-1,1] \times \{0\}.$$

We can choose the parametrization in this neighborhood so that

$$\gamma_1^{(\widehat{t}_\ell + s)}(u) = (\widehat{t}_\ell + s, u, 0, s) \qquad \text{and}$$
$$\gamma_2^{(\widehat{t}_\ell + s)}(u) = (\widehat{t}_\ell + s, v_1 s, u + v_2 s, v_3 s)$$

for a vector $(v_1, v_2, v_3) \in \boldsymbol{R}^3$ $(v_3 \neq 1)$. By using a smooth bump function $\mu : [-1,1] \longrightarrow [0,1]$ such that $\mu(x) = \mu(-x)$, $\mu|[0,1/3] = 1$, and $\mu|[2/3,1] = 0$, we modify $\gamma_2^{(t)}$. Put

$$\gamma_2'^{(\widehat{t}_\ell + s)}(u) = (\widehat{t}_\ell + s, (1 + c_\ell)\mu(s/\widehat{\varepsilon}_\ell)\mu(u/\delta_\ell) + v_1 s, u + v_2 s, v_3 s),$$

where c_ℓ and δ_ℓ are small positive real numbers such that the image of $\gamma_2'^{(\widehat{t}_\ell + s)}$ is contained in our neighborhood X. Then the curves $\gamma_1^{(t)}$ and $\gamma_2'^{(t)}$ $(t \in (\widehat{t}_\ell - \widehat{\varepsilon}_\ell, \widehat{t}_\ell + \widehat{\varepsilon}_\ell))$ do not intersect in σ^m_{j+1}.

Thus for $2m \geq 6$, using the above family of curves if necessary, we have the union Z' of a finite number of disjointly embedded 2-dimensional disks in $[0,1] \times \sigma^m_{j+1}$ such that

$$(H'^{(j)}|[0,1] \times \sigma^m_{j+1})^{-1}(k(\tau^m)) \subset Z'.$$

Since $2m \geq 6$, the images under generic $H'^{(j)}$ of these 2-dimensional disks are disjointly embedded in $M^{2m} \setminus k(Q^{(m-1)})$. The images of these disks are called the Whitney disks.

We have been looking at the intersection point set ${h'}_t^{(j)}(\sigma_{j+1}^m) \cap k(\tau^m)$ for one m-dimensional cell τ^m of Q^m other than σ^{m*}. These considerations can be applied to the intersection point sets ${h'}_t^{(j)}(\sigma_{j+1}^m) \cap k(\tau^m)$ for all (finitely many) m-dimensional cells τ^m of Q^m other than σ^{m*} simultaneously. This is because, if $2m \geq 8$, the embedded 2-dimensional disks Z' are disjoint for different τ^m for generic $H'^{(j)}$, and if $2m = 6$, we can remove the intersection of the embedded 2-dimensional disks Z' for different τ^m in a way similar to what we did for the intersection of Z for the same τ^m. Thus we obtained the union Z' of a finite number of disjointly embedded 2-dimensional disks in $[0,1] \times \sigma_{j+1}^m$ such that

$$(H'^{(j)}|[0,1] \times \sigma_{j+1}^m)^{-1}(k(Q^m \setminus \sigma_{j+1}^{m*})) \subset Z',$$

and $H'^{(j)}|Z'$ is an embedding.

If $2m \geq 8$, then the Whitney disks $H'^{(j)}(Z')$ do not contain double points of $H'^{(j)}([0,1] \times P^m)$ for generic $H'^{(j)}$. This is because the inverse image of the double point set of $H'^{(j)}([0,1] \times P^m)$ is 2-dimensional in $[0,1] \times P^m$ and $m + 1 \geq 5$.

If $2m = 6$, then the Whitney disks $H'^{(j)}(Z')$ may intersect the double point set of $H'^{(j)}([0,1] \times P^3)$. Then, for generic $H'^{(j)}$, the intersection is a finite set and we pick up the points of Whitney disks which are in the image of ${h'}_t^{(j)}(P^3)$ with larger t;

$$H'^{(j)}(t_i^{(j)}, w_i^{(j)}) = H'^{(j)}({t'}_i^{(j)}, {w'}_i^{(j)}) \quad (i = 1, \ldots, r'^{(j)}),$$

where $(t_i^{(j)}, w_i^{(j)})$ is a point $Z' \subset [0,1] \times \sigma_{j+1}^m$, $({t'}_i^{(j)}, {w'}_i^{(j)}) \in [0,1] \times P^3$ and $t_i^{(j)} < {t'}_i^{(j)}$. Then, for generic $H'^{(j)}$, the curve $H'^{(j)}([{t'}_i^{(j)}, 1] \times \{{w'}_i^{(j)}\})$ is embedded in $M^{2m} \setminus k(Q^m)$ and does not contain double points of $H'^{(j)}([0,1] \times P^3)$ other than $H'^{(j)}({t'}_i^{(j)}, {w'}_i^{(j)})$. Hence if $2m = 6$, we have the Whitney disks $H'^{(j)}(Z')$ and the curves $H'^{(j)}([{t'}_i^{(j)}, 1] \times \{{w'}_i^{(j)}\})$ $(i = 1, \ldots, r'^{(j)})$.

Using the Whitney disks $H'^{(j)}(Z')$ and curves $H'^{(j)}([{t'}_i^{(j)}, 1] \times \{{w'}_i^{(j)}\})$ $(i = 1, \ldots, r'^{(j)})$, we prove the following lemmas in the next section.

Lemma 3.7. *For ${h'}_t^{(j)}$, there is an isotopy $\{b_t^{(j+1)}\}_{t \in [0,1]}$ $(b_0^{(j+1)} = \mathrm{id})$ with support in a union of disjointly embedded open balls such that for $h_t''^{(j)} = b_t^{(j+1)} \circ {h'}_t^{(j)}$, $h_t''^{(j)}$ is the identity on a neighborhood of $P^{(m-1)} \cup \bigcup_{i=1}^{j} \sigma_i^m$ and $h_t''^{(j)}(\sigma_{j+1}^m) \cap k(Q^m \setminus \sigma_{j+1}^{m*}) = \emptyset$.*

Lemma 3.8. *For $h_t''^{(j)}$ given by Lemma 3.7, there are isotopies*

$$\{g_t^{(j+1)}\}_{t \in [0,1]} \subset \mathrm{Diff}_c^r(M^{2m} \setminus k(Q^m \setminus \sigma_{j+1}^{m*})) \quad (g_0^{(j+1)} = \mathrm{id}) \text{ and}$$
$$\{h_t^{(j+1)}\}_{t \in [0,1]} \subset \mathrm{Diff}_c^r(M^{2m} \setminus (P^{(m-1)} \cup \textstyle\bigcup_{i=1}^{j+1} \sigma_i^m)) \quad (h_0^{(j+1)} = \mathrm{id})$$

such that $h_t''^{(j)} = g_t^{(j+1)} \circ h_t^{(j+1)}$.

Now we complete the proof of our main Theorem 1.2.

Proof of Theorem 1.2. Let f be an element of $\mathrm{Diff}^r(M^{2m})_0$. By Lemma 3.2, there are $g \in \mathrm{Diff}_c^r(M^{2m} \setminus k(Q^m))_0$ and $h \in \mathrm{Diff}_c^r(M^{2m} \setminus P^{(m-2)})_0$ such that $f = g \circ h$.

Then by using the approximation $\overline{h}$ of h,

$$f = g \circ \overline{h} \circ (\overline{h}^{-1} \circ h).$$

By Lemmas 3.3 and 3.4, there are a diffeomorphism a with support in a union of disjointly embedded open balls, $g' \in \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m))_0$ and $h'' \in \mathrm{Diff}^r_c(M^{2m} \setminus P^{(m-1)})_0$ such that

$$\overline{h} = a^{-1} \circ (a \circ \overline{h}) = a^{-1} \circ g' \circ h''.$$

Put $h^{(0)} = h'' \in \mathrm{Diff}^r_c(M^{2m} \setminus P^{(m-1)})_0$, and for $h^{(j)} \in \mathrm{Diff}^r_c(M^{2m} \setminus (P^{(m-1)} \cup \bigcup_{i=1}^{j} \sigma_i^m))_0$ $(j = 0, \ldots, N-1)$, we use its approximation $\overline{h}^{(j)}$ and by Lemmas 3.5, 3.7 and 3.8, there are diffeomorphisms $a^{(j+1)}$ and $b^{(j+1)}$ with support in unions of disjointly embedded open balls, $g^{(j+1)} \in \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m \setminus \sigma^{m*}_{j+1}))_0$ and $h^{(j+1)} \in \mathrm{Diff}^r_c(M^{2m} \setminus (P^{(m-1)} \cup \bigcup_{i=1}^{j+1} \sigma_i^m))_0$ such that

$$\begin{aligned}
h^{(j)} &= \overline{h}^{(j)} \circ ((\overline{h}^{(j)})^{-1} \circ h^{(j)}) \\
&= (a^{(j+1)})^{-1} \circ (a^{(j+1)} \circ \overline{h}^{(j)}) \circ ((\overline{h}^{(j)})^{-1} \circ h^{(j)}) \\
&= (a^{(j+1)})^{-1} \circ (b^{(j+1)})^{-1} \circ g^{(j+1)} \circ h^{(j+1)} \circ ((\overline{h}^{(j)})^{-1} \circ h^{(j)}).
\end{aligned}$$

Hence,

$$\begin{aligned}
f &= g \circ \overline{h} \circ (\overline{h}^{-1} \circ h) \\
&= g \circ a^{-1} \circ g' \circ h^{(0)} \circ (\overline{h}^{-1} \circ h) \\
&= g \circ a^{-1} \circ g' \circ (a^{(1)})^{-1} \circ (b^{(1)})^{-1} \circ g^{(1)} \circ h^{(1)} \circ ((\overline{h}^{(0)})^{-1} \circ h^{(0)}) \circ (\overline{h}^{-1} \circ h) \\
&= g \circ a^{-1} \circ g' \circ (a^{(1)})^{-1} \circ (b^{(1)})^{-1} \circ g^{(1)} \circ \cdots \circ (a^{(N)})^{-1} \circ (b^{(N)})^{-1} \circ g^{(N)} \\
&\qquad \circ h^{(N)} \circ ((\overline{h}^{(N-1)})^{-1} \circ h^{(N-1)}) \circ \cdots \circ ((\overline{h}^{(0)})^{-1} \circ h^{(0)}) \circ (\overline{h}^{-1} \circ h).
\end{aligned}$$

Here, note that

$$h^{(N)} \in \mathrm{Diff}^r_c(M^{2m} \setminus (P^{(m-1)} \cup \bigcup_{i=1}^{N} \sigma_i^m))_0 = \mathrm{Diff}^r_c(M^{2m} \setminus P^m).$$

Since

$$((\overline{h}^{(N-1)})^{-1} \circ h^{(N-1)}) \circ \cdots \circ ((\overline{h}^{(0)})^{-1} \circ h^{(0)}) \circ (\overline{h}^{-1} \circ h) \in \mathrm{Diff}^r(M^{2m})$$

is close to the identity, by Remark 2.8, it is written as $\widehat{h} \circ \widehat{a} \circ \widehat{g}$, where $\widehat{h} \in \mathrm{Diff}^r(M^{2m} \setminus P^m)_0$, $\widehat{g} \in \mathrm{Diff}^r(M^{2m} \setminus k(Q^m))_0$ and $\widehat{a}$ is with support in a union of disjointly embedded open balls which is a neighborhood of the union of m handles. Thus

$$\begin{aligned}
f = g \circ a^{-1} \circ g' \circ (a^{(1)})^{-1} \circ (b^{(1)})^{-1} \circ g^{(1)} \circ \\
\cdots \circ (a^{(N)})^{-1} \circ (b^{(N)})^{-1} \circ g^{(N)} \circ h^{(N)} \circ \widehat{h} \circ \widehat{a} \circ \widehat{g}.
\end{aligned}$$

Now by the construction, each of a^{-1}, $(a^{(1)})^{-1}, \ldots, (a^{(N)})^{-1}$, $(b^{(1)})^{-1}, \ldots, (b^{(N)})^{-1}$ can be written as one commutator with support in a union of disjointly embedded open balls. The diffeomorphism $\widehat{a}$ can be written as a product of two commutators by Theorem 1.1 (1). The diffeomorphism $h^{(N)} \circ \widehat{h} \in \mathrm{Diff}^r(M^{2m} \setminus P^m)_0$ is written as a product of two commutators in $\mathrm{Diff}^r(M^{2m} \setminus P^m)_0$ by Theorem 1.1 (1). Each of the diffeomorphisms g, g' and $\widehat{g} \in \mathrm{Diff}^r(M^{2m} \setminus k(Q^m))_0$ is also written as a product of two commutators in $\mathrm{Diff}^r(M^{2m} \setminus k(Q^m))_0$ by Theorem 1.1 (1). By the property of the triangulation, the diffeomorphism $g^{(j)} \in \mathrm{Diff}^r_c(M^{2m} \setminus k(Q^m \setminus \sigma^{m*}_{j+1}))_0$ is supported on an open set which can be deformed in a neighborhood of the embedded $(m-1)$-dimensional complex $\iota((P^{(m-1)} \cup \sigma_j^m)/\sigma_j^m)$, and hence $g^{(j)}$ can be written as a product of two commutators in $\mathrm{Diff}^r(M^{2m} \setminus k(Q^m \setminus \sigma^{m*}_{j+1}))_0$ by Theorem 1.1 (1). Thus f is written as a product of $4N + 11$ commutators. $\square$

4. Proof of lemmas

We give the proof of lemmas we used in the previous section to show Theorem 1.2.

Proof of Lemma 3.2. This follows from Lemma 2.5 and Remark 2.4. □

Proof of Lemma 3.3. The construction of a_t is essentially due to Burago, Ivanov and Polterovich ([3]) and we wrote it in the proof of Lemma 2.7 which is [22, Lemma 6.5]. However, we write it again here, for, we use this argument later again.

For $\overline{H}(s_i, v_i)$, we take a small neighborhood U_i of $\overline{H}([s_i, 1] \times \{v_i\})$ diffeomorphic to the $(2m)$-dimensional ball. We can take these U_i to be disjoint.

The intersection of U_i and $\overline{H}([0, 1] \times P^{(m-1)})$ or $k(Q^m)$ is described as follows. We put a coordinate

$$(x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_{2m}) \in (-2, 2)^{2m}$$

on U_i such that, for $\varepsilon_i > 0$,

$$\begin{aligned}
&k(Q^m) \cap U_i = \{0\} \times \{0\}^{m-1} \times (-2, 2)^m, \\
&\overline{H}((s_i - 2\varepsilon_i(1 - s_i), 1] \times \{v_i\}) \cap U_i = (-2, 1] \times \{0\}^{2m-1}, \quad \text{and} \\
&\overline{h}_{s_i+t(1-s_i)}(P^{(m-1)}) \cap U_i = \{t\} \times (-2, 2)^{m-1} \times \{0\}^m \quad (t \in [-\varepsilon_i, 1]).
\end{aligned}$$

Take an isotopy $\{a_t\}_{t\in[0,1]}$ with support in $\bigsqcup_{i=1}^r U_i$ such that, on each U_i, $a_0 = \mathrm{id}$ and, for $(x_1, x_2, \dots, x_{2m}) \in [-\varepsilon_i, 1] \times [-1, 1]^{2m-1} \subset (-2, 2)^{2m}$,

$$a_t(x_1, x_2, \dots, x_{2m}) = (x_1 - (1 + \varepsilon_i)t, x_2, \dots, x_{2m}).$$

Now $(a_1 \circ \overline{h}_1)(P^{(m-1)}) \cap k(Q^m) = \emptyset$. Moreover, by changing the time parameter of the above a_t, we obtain an isotopy a_t ($a_0 = \mathrm{id}$) with support in $\bigsqcup_{i=1}^r U_i$ such that for $h'_t = a_t \circ \overline{h}_t$,

$$h'_t(P^{(m-1)}) \cap k(Q^m) = \emptyset \quad (t \in [0, 1]).$$

In fact, if we put

$$t = s_i + u_i(1 - s_i) \in [s_i - \varepsilon_i(1 - s_i), 1], \text{ i.e., } u_i \in [-\varepsilon_i, 1],$$

and look at $a_{(u_i+\varepsilon_i)/(1+\varepsilon_i)} \circ \overline{h}_{s_i+u_i(1-s_i)}$, then on U_i,

$$\begin{aligned}
&(a_{(u_i+\varepsilon_i)/(1+\varepsilon_i)} \circ \overline{h}_{s_i+u_i(1-s_i)})(\{-\varepsilon_i\} \times [-1, 1]^{m-1} \times \{0\}^m) \\
&= a_{(u_i+\varepsilon_i)/(1+\varepsilon_i)}(\{u_i\} \times [-1, 1]^{m-1} \times \{0\}^m) \\
&= \{u_i - (u_i + \varepsilon_i)\} \times [-1, 1]^{m-1} \times \{0\}^m \\
&= \{-\varepsilon_i\} \times [-1, 1]^{m-1} \times \{0\}^m.
\end{aligned}$$

Hence by using the above a_t with appropriate time change, we obtain the desired isotopy a_t.

Note that $a_1 \in \mathrm{Diff}^r_c(\bigsqcup_{i=1}^r U_i)_0$ can be taken as one commutator with support in $\bigsqcup_{i=1}^r U_i$ ([23]). □

Proof of Lemma 3.4. Let $F : [0, 1] \times M^n \longrightarrow M^n$ be the trace of the isotopy: $F(t, x) = f_t(x)$.

Let W be a neighborhood of P_0 in M^n where f_t is the identity. Let U be a neighborhood of $F([0, 1] \times (P^p \setminus W \cap P^p))$ and V be a neighborhood of Q^q such that $U \cap V = \emptyset$.

Let ξ be the vector field on $[0,1]\times M^n$ given by

$$\frac{\partial}{\partial t}+\big(\frac{\mathrm{d}f_{t+s}(x)}{\mathrm{d}s}\big)_{s=0}$$

at $(t,f_t(x))$. This ξ generates the isotopy f_t. Let η be a vector field on $[0,1]\times M^n$ with support in $[0,1]\times U$ such that $\eta=\xi$ on a neighborhood of

$$\{(t,f_t(x_0))\bigm| x_0\in P^p\setminus W\cap P^p,\ t\in[0,1]\}.$$

Then $\eta=\partial/\partial t$ on $[0,1]\times(V\cup W)$ which is a neighborhood of $[0,1]\times(Q^q\cup P_0)$. Then η generates an isotopy $\{g_t\}_{t\in[0,1]}$ such that g_t is the identity on the neighborhood $V\cup W$ of $Q^q\cup P_0$ and $g_t(x)=f_t(x)$ for x in a neighborhood of $P^p=(P^p\setminus W\cap P^p)\cup(W\cap P^p)$. Here for $x\in W$, $g_t(x)=x=f_t(x)$.

Put $h={g_1}^{-1}f_1$, then h is the identity on a neighborhood of P^p, and it is isotopic to the identity as an element of $\mathrm{Diff}^r(M^n)$. For, put $h_t={g_t}^{-1}\circ f_t$. Then h_t is the identity on a neighborhood of P^p.

Thus we can write as $f=g\circ h$, where $g\in\mathrm{Diff}^r_c(M^n\setminus Q^q)_0$ and $h\in\mathrm{Diff}^r_c(M^n\setminus P^p)_0$. □

Proof of Lemma 3.5. The proof is similar to that of Lemma 3.3.

For $\overline{H}^{(j)}(s_i^{(j)},v_i^{(j)})$, we take a small neighborhood $U_i^{(j)}$ of $\overline{H}([s_i^{(j)},1]\times\{v_i^{(j)}\})$ diffeomorphic to the $(2m)$-dimensional ball. We can take these $U_i^{(j)}$ to be disjoint.

The intersection of $U_i^{(j)}$ and $\overline{H}^{(j)}([0,1]\times P^m)$ or $k(Q^{(m-1)})$ is described as follows. We put a coordinate

$$(x_1,x_2,\dots,x_{m+1},x_{m+2},\dots,x_{2m})\in(-2,2)^{2m}$$

on $U_i^{(j)}$ such that, for $\varepsilon_i^{(j)}>0$,

$$\begin{aligned}
&k(Q^{(m-1)})\cap U_i^{(j)}=\{0\}\times\{0\}^m\times(-2,2)^{m-1},\\
&\overline{H}((s_i^{(j)}-2\varepsilon_i^{(j)}(1-s_i^{(j)}),1]\times\{v_i^{(j)}\})\cap U_i^{(j)}=(-2,1]\times\{0\}^{2m-1},\quad\text{and}\\
&\overline{h}^{(j)}_{s_i^{(j)}+t(1-s_i^{(j)})}(P^m)\cap U_i^{(j)}=\{t\}\times(-2,2)^m\times\{0\}^{m-1}\quad(t\in[-\varepsilon_i^{(j)},1]).
\end{aligned}$$

Take an isotopy $\{a_t^{(j+1)}\}_{t\in[0,1]}$ with support in $\bigsqcup_{i=1}^{r^{(j)}}U_i^{(j)}$ such that, on each $U_i^{(j)}$, $a_0^{(j+1)}=\mathrm{id}$ and, for $(x_1,x_2,\dots,x_{2m})\in[-\varepsilon_i^{(j)},1]\times[-1,1]^{2m-1}\subset(-2,2)^{2m}$,

$$a_t^{(j+1)}(x_1,x_2,\dots,x_{2m})=(x_1-(1+\varepsilon_i^{(j)})t,x_2,\dots,x_{2m}).$$

Now $(a_1^{(j+1)}\circ\overline{h}_1^{(j)})(P^m)\cap k(Q^{(m-1)})=\emptyset$. Moreover, by changing the time parameter, we obtain an isotopy $a_t^{(j+1)}$ ($a_0^{(j+1)}=\mathrm{id}$) with support in $\bigsqcup_{i=1}^{r}U_i^{(j)}$ such that, for ${h'}_t^{(j)}=a_t^{(j+1)}\circ\overline{h}_t^{(j)}$,

$${h'}_t^{(j)}(P^m)\cap k(Q^{(m-1)})=\emptyset\quad(t\in[0,1]).$$

In fact, if we put

$$t=s_i^{(j)}+u_i^{(j)}(1-s_i^{(j)})\in[s_i^{(j)}-\varepsilon_i^{(j)}(1-s_i^{(j)}),1],\ \text{i.e.,}\ u_i^{(j)}\in[-\varepsilon_i^{(j)},1],$$

and look at $a^{(j+1)}_{(u_i^{(j)}+\varepsilon_i^{(j)})/(1+\varepsilon_i^{(j)})} \circ \overline{h}^{(j)}_{s_i^{(j)}+u_i^{(j)}(1-s_i^{(j)})}$, then on $U_i^{(j)}$,

$$\begin{aligned}
&(a^{(j+1)}_{(u_i+\varepsilon_i)/(1+\varepsilon_i)} \circ \overline{h}^{(j)}_{s_i^{(j)}+u_i^{(j)}(1-s_i^{(j)})})(\{-\varepsilon_i^{(j)}\} \times [-1,1]^m \times \{0\}^{m-1})\\
&= a^{(j+1)}_{(u_i^{(j)}+\varepsilon_i^{(j)})/(1+\varepsilon_i^{(j)})}(\{u_i^{(j)}\} \times [-1,1]^m \times \{0\}^{m-1})\\
&= \{u_i^{(j)} - (u_i^{(j)} + \varepsilon_i^{(j)})\} \times [-1,1]^m \times \{0\}^{m-1}\\
&= \{-\varepsilon_i^{(j)}\} \times [-1,1]^m \times \{0\}^{m-1}.
\end{aligned}$$

Hence by using the above $a_t^{(j+1)}$ with appropriate time change, we obtain the desired isotopy $a_t^{(j+1)}$.

Note that $a_1^{(j+1)} \in \mathrm{Diff}_c^r(\bigsqcup_{i=1}^{r^{(j)}} U_i^{(j)})_0$ can be taken as one commutator with support in $\bigsqcup_{i=1}^{r^{(j)}} U_i^{(j)}$ ([23]). □

Proof of Lemma 3.6. For

$$F(t, x_1, \dots, x_m) = (f_1(t, x_1, \dots, x_m), \dots, f_m(t, x_1, \dots, x_m)),$$

put

$$\frac{\partial F}{\partial t} = \begin{pmatrix} \frac{\partial f_1}{\partial t} \\ \vdots \\ \frac{\partial f_m}{\partial t} \end{pmatrix} \quad \text{and} \quad \frac{\partial F}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix}.$$

On the 2-jet bundle $J^2(\boldsymbol{R} \times \boldsymbol{R}^m, \boldsymbol{R}^m)$, we consider the subbundle E_1 defined by $\mathrm{rank}\begin{pmatrix} \frac{\partial F}{\partial t} & \frac{\partial F}{\partial x} \end{pmatrix} = m-1$ and the subbundle E_2 defined by the two equations, $\mathrm{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \end{pmatrix} = m-1$ and $\mathrm{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial}{\partial x}\det\frac{\partial F}{\partial x} \end{pmatrix} = m-1$, where

$$\frac{\partial}{\partial x}\det\frac{\partial F}{\partial x} = \begin{pmatrix} \frac{\partial}{\partial x_1}\det\frac{\partial F}{\partial x} & \cdots & \frac{\partial}{\partial x_n}\det\frac{\partial F}{\partial x} \end{pmatrix}.$$

Then E_1 and E_2 are codimension 2 subbundles. The closures of these subbundles are the set determined by the inequalities expressing the ranks are not greater than $m-1$.

By the jet transversality theorem, the jet of a generic map F intersects these subbundles transversely. Hence the set

$$\{(t, x) \mid J^2_{(t,x)}F \in E_1 \cup E_2\}$$

is an $(m-1)$-dimensional subset and its image in $\boldsymbol{R}^m$ is nowhere dense. We take a point y in $\boldsymbol{R}^m$ in the complement of this image and consider its inverse image $F^{-1}(y)$. Then for a point $x \in F^{-1}(y)$, either $\mathrm{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \end{pmatrix} = m$ holds or the three equations $\mathrm{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \end{pmatrix} = m-1$, $\mathrm{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \end{pmatrix} = m$ and $\mathrm{rank}\begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial}{\partial x}\det\frac{\partial F}{\partial x} \end{pmatrix} = m$ hold.

If rank $\left(\dfrac{\partial F}{\partial x}\right) = m$ at x, then x is a regular point of $F_t = F(t, \bullet)$ and the inverse image is locally a 1-dimensional manifold transverse to $\{t\} \times \boldsymbol{R}^m$.

Assume that the three equations hold. Since rank $\left(\dfrac{\partial F}{\partial x}\right) = m - 1$, by the implicit function theorem, we can change the local coordinate $(x_1, \ldots, x_m)$ of the second factor of the source to $(x'_1, \ldots, x'_m)$ and that $(y_1, \ldots, y_m)$ of the target to $(y'_1, \ldots, y'_m)$ so that

$$F(t, x'_1, \ldots, x'_m) = (x'_1, \ldots, x'_{m-1}, y'_m(t, x'_1, \ldots, x'_m)).$$

Then $\det\left(\dfrac{\partial F}{\partial x}\right) = \dfrac{\partial y'_m}{\partial x'_m}$ and the matrix $\begin{pmatrix} \dfrac{\partial F}{\partial x} \\ \dfrac{\partial}{\partial x}\det\dfrac{\partial F}{\partial x} \end{pmatrix}$ with respect to these coordinates is written as

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ \frac{\partial y'_m}{\partial x'_1} & \cdots & \cdots & \frac{\partial y'_m}{\partial x'_{m-1}} & \frac{\partial y'_m}{\partial x'_m} \\ \frac{\partial^2 y'_m}{\partial x'_m \partial x'_1} & \cdots & \cdots & \frac{\partial^2 y'_m}{\partial x'_m \partial x'_{m-1}} & \frac{\partial^2 y'_m}{\partial {x'_m}^2} \end{pmatrix}$$

and the matrix $\begin{pmatrix} \dfrac{\partial F}{\partial t} & \dfrac{\partial F}{\partial x} \end{pmatrix}$ with respect to these coordinates is written as

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \frac{\partial y'_m}{\partial t} & \frac{\partial y'_m}{\partial x'_1} & \cdots & \cdots & \frac{\partial y'_m}{\partial x'_{m-1}} & \frac{\partial y'_m}{\partial x'_m} \end{pmatrix}.$$

Hence, $\dfrac{\partial y'_m}{\partial x'_m} = 0$, $\dfrac{\partial y'_m}{\partial t} \neq 0$ and $\dfrac{\partial^2 y'_m}{\partial {x'_m}^2} \neq 0$ at x.

Thus at $x \in F^{-1}(y)$, either $\det\left(\dfrac{\partial F}{\partial x}\right) \neq 0$ or F is locally written as

$$F(t, x'_1, \ldots, x'_m) = (x'_1, \ldots, x'_{m-1}, y'_m(t, x'_1, \ldots, x'_m)),$$

where $\dfrac{\partial y'_m}{\partial x'_m} = 0$, $\dfrac{\partial y'_m}{\partial t} \neq 0$ and $\dfrac{\partial^2 y'_m}{\partial {x'_m}^2} \neq 0$. □

The proof of Lemma 3.7 is divided into two cases.

Proof of Lemma 3.7 *in the case where* $2m \geq 8$. If $2m \geq 8$, the Whitney disks guide the way to construct the isotopy $b_t^{(j+1)}$ with support in a union of disjoint open balls. In fact, the support of $b_t^{(j+1)}$ is in a neighborhood of the union of the Whitney disks. The construction of the isotopy $b_t^{(j+1)}$ is possible because the neighborhood of one of the Whitney disks can be considered as a neighborhood of a tree growing in the negative direction in t in $[0,1]\times\sigma^m_{j+1}$.

The construction of $b_t^{(j+1)}$ is as follows. Take a vector field of the form $\dfrac{\partial}{\partial t}+\zeta(t,v)$ on the union of disks $Z'\subset[0,1]\times\sigma^m_{j+1}$ which is tangent to Z' and transverse to the boundary $\partial Z'\subset Z'$, where $\zeta(t,v)$ is a vector field in the direction of σ^m_{j+1}. Such a vector field $\dfrac{\partial}{\partial t}+\zeta(t,v)$ exists because Z' deforms to a tree which grows in the negative direction in t by shrinking the connected components of $Z'\cap(\{t\}\times\sigma^m_{j+1})$ to a point. We extend $\zeta(t,\bullet)$ on σ^m_{j+1} so that the support is contained in a small neighborhood of Z'. Let ${b'}_t^{(j+1)}$ denote the isotopy generated by $\dfrac{\partial}{\partial t}+\zeta(t,v)$. Then the support of ${b'}_t^{(j+1)}$ is contained in a neighborhood $U'^{(j)}$ of the union of the Whitney disks $H'^{(j)}(Z')$. Since $H'^{(j)}(Z')$ does not contain double points of $H'^{(j)}([0,1]\times P^m)$, the support of ${b'}_t^{(j+1)}$ intersects $H'^{(j)}([0,1]\times P^m)$ only in $U'^{(j)}$. Here, $U'^{(j)}$ is a union of disjointly embedded open balls in M^{2m}. Moreover, $({h'}_t^{(j)})_*\zeta(t,\bullet)$ is tangent to the union of the Whitney disks $H'^{(j+1)}(Z')$ in M^{2m} and

$$({b'}_t^{(j+1)})^{-1}({h'}_t^{(j)}(\sigma^m_{j+1}))\cap k(Q^m\setminus\sigma^{m*}_{j+1})=\emptyset\quad(t\in[0,1]).$$

Put $b_t^{(j+1)}=({b'}_t^{(j+1)})^{-1}$, then

$$(b_t^{(j+1)}\circ{h'}_t^{(j)})(\sigma^m_{j+1})\cap k(Q^m\setminus\sigma^{m*}_{j+1})=\emptyset\quad(t\in[0,1]).$$

Note that $b_1^{(j+1)}\in\mathrm{Diff}^r_c(U'^{(j)})_0$ can be taken as one commutator with support in $U'^{(j)}$ ([23]). □

Proof of Lemma 3.7 *in the case where* $2m=6$. If $2m=6$, then we also consider the curves $H'^{(j)}([{t'}_i^{(j)},1]\times\{{w'}_i^{(j)}\})$ $(i=1,\ldots,r'^{(j)})$.

First take a small neighborhood $U'^{(j)}$ of the union of the Whitney disks which is a union of disjointly embedded open balls in M^6, and construct $b_t^{(j+1)}$ as in the case where $2m\geq 8$. Then we modify it by using an isotopy.

We take a small neighborhood $U_i'^{(j)}$ of the curve $H'^{(j)}([{t'}_i^{(j)},1]\times\{{w'}_i^{(j)}\})$ $(i=1,\ldots,r'^{(j)})$. We put a coordinate

$$(x_1,x_2,x_3,x_4,x_5,x_6)\in(-2,3)\times(-2,2)^5$$

on ${U'}_i^{(j)}$ such that, for ${\varepsilon'}_i^{(j)}>0$,

$$\begin{aligned}&H'^{(j)}(({t'}_i^{(j)}-2{\varepsilon'}_i^{(j)}(1-{t'}_i^{(j)}),1]\times\{{w'}_i^{(j)}\})\cap{U'}_i^{(j)}=(-2,1]\times\{0\}^5,\quad\text{and}\\&h'_{{t'}_i^{(j)}-2{\varepsilon'}_i^{(j)}(1-{t'}_i^{(j)})}(P^3)\cap{U'}_i^{(j)}=\{t\}\times(-2,2)^3\times\{0\}^2\quad(t\in[-{\varepsilon'}_i^{(j)},1]).\end{aligned}$$

We take an isotopy $\{{a'}_t^{(j+1),i}\}_{t\in[0,1]}$ with support in ${U'}_i^{(j)}$ such that ${a'}_0^{(j+1),i}=\mathrm{id}$ and, for $(x_1,x_2,x_3,x_4,x_5,x_6)\in[-{\varepsilon'}_i^{(j)},1]\times[-1,1]^5\subset(-2,3)\times(-2,2)^5$,

$${a'}_t^{(j+1),i}(x_1,x_2,x_3,x_4,x_5,x_6)=(x_1+t(1+{\varepsilon'}_i^{(j)}),x_2,x_3,x_4,x_5,x_6).$$

Put $\overline{a}=\displaystyle\prod_{i=1}^{r'^{(j)}}{a'}_1^{(j+1),i}$. Then $\overline{a}\circ b_1^{(j+1)}\circ\overline{a}^{-1}$ is isotopic to the identity by the isotopy with support in the union of disjoint 6-dimensional open balls $\overline{a}(U'^{(j)})$. By the construction,

$$((\overline{a} \circ b_1^{(j+1)} \circ \overline{a}^{-1}) \circ \overline{h}_1)(\sigma_{j+1}^3) \cap k(Q^3 \setminus \sigma_{j+1}^{3*}) = \emptyset.$$

Moreover, by an appropriate change of time parameter on each ${U'}_i^{(j)}$, we obtain an isotopy $\overline{a}_t$ $(t \in [0,1])$ such that

$$((\overline{a}_t \circ b_t^{(j+1)} \circ \overline{a}_t^{-1}) \circ \overline{h}_t)(\sigma_{j+1}^3) \cap k(Q^3 \setminus \sigma_{j+1}^{3*}) = \emptyset$$

and the support of the isotopy $\overline{a}_t \circ b_t^{(j+1)} \circ \overline{a}_t^{-1}$ is contained in $U'^{(j)} \cup \bigsqcup_{i=1}^{r'(j)} {U'}_i^{(j)}$ which is a union of disjointly embedded open balls in M^{2m}. Thus we obtained the desired isotopy.

Note that $\overline{a} \circ b_1^{(j+1)} \circ \overline{a}^{-1}$ can be taken as one commutator with support in a union of disjointly embedded open balls. □

Proof of Lemma 3.8. This follows from Lemmas 3.7 and 3.4. □

5. Uniform simplicity

We prove Corollary 1.3. In [23, Theorem 2.2], we showed the following theorem.

Theorem 5.1 ([23]). *Let M^n be the interior of a compact n-dimensional manifold with handle decomposition with handles of indices not greater than $(n-1)/2$. Let c be the order of the set of indices appearing in the handle decomposition. Then any element of $\mathrm{Diff}_c^r(M^n)_0$ $(1 \leq r \leq \infty,\ r \neq n+1)$ can be written as a product of two commutators. Moreover, if M^n is connected, any element of $\mathrm{Diff}_c^r(M^n)_0$ can be written as a product of $4c+1$ commutators with support in embedded open balls.*

In Section 3, we showed that any element $f \in \mathrm{Diff}^r(M^{2m})_0$ can be written as

$$\begin{aligned} f = {} & g \circ a^{-1} \circ g' \circ (a^{(1)})^{-1} \circ (b^{(1)})^{-1} \circ g^{(1)} \circ \\ & \cdots \circ (a^{(N)})^{-1} \circ (b^{(N)})^{-1} \circ g^{(N)} \circ h^{(N)} \circ \widehat{h} \circ \widehat{a} \circ \widehat{g}. \end{aligned}$$

Since a compact subset of a union of disjointly embedded open balls is contained in a larger embedded open ball, each of diffeomorphisms a^{-1}, $(a^{(1)})^{-1}, \ldots, (a^{(N)})^{-1}$, $(b^{(1)})^{-1}, \ldots, (b^{(N)})^{-1}$ can be written as one commutator with support in an embedded open ball and the diffeomorphism $\widehat{a}$ can be written as a product of two commutators with support in an embedded open ball. Now by Theorem 5.1, each of the diffeomorphisms $h^{(N)} \circ \widehat{h} \in \mathrm{Diff}^r(M^{2m} \setminus P^m)_0$, g, g' and $\widehat{g} \in \mathrm{Diff}^r(M^{2m} \setminus k(Q^m))_0$, $g^{(j)} \in \mathrm{Diff}_c^r(M^{2m} \setminus k(Q^m \setminus \sigma_{j+1}^{m*}))_0$ is written as a product of $4m+1$ commutators with support in embedded open balls. Hence f is written as a product of $4(N+4)m + 3N + 7$ commutators with support in embedded open balls.

Now Corollary 1.3 follows from the following lemma ([23, Lemma 3.1]).

Lemma 5.2 ([23]). *Let M^n be a connected n-dimensional manifold. Let g be a nontrivial element of $\mathrm{Diff}_c^r(M^n)_0$. Assume that $f \in \mathrm{Diff}_c^r(M^n)_0$ is written as a product of commutators $[a_i, b_i]$ $(i = 1, \ldots, k)$; $f = [a_1, b_1] \cdots [a_k, b_k]$, where a_i and b_i are with support in an embedded open ball $U_i \subset \overline{U_i} \subset M^n$. Then f can be written as a product of $4k$ conjugates of g or g^{-1}.*

Proof of Corollary 1.3. Let g be a nontrivial element of $\mathrm{Diff}^r(M^{2m})_0$ $(1 \leq r \leq \infty$, $r \neq 2m+1)$. Since any element f of $\mathrm{Diff}^r(M^{2m})_0$ can be written as a product of $4(N+4)m+3N+7$ commutators with support in embedded open balls, by Lemma

5.2, f can be written as a product of $16(N+4)m+12N+28$ conjugates of g or g^{-1}. □

Remark 5.3. We showed in [23] that, for a compact connected n-dimensional manifold M^n with handle decomposition without handles of the middle index $n/2$, for any elements f and g of $\mathrm{Diff}^r(M^n)_0 \setminus \{\mathrm{id}\}$, f can be written as a product of at most $16n+28$ conjugates of g or g^{-1}. For such manifolds, the bound for the number of conjugates depends only on the dimension n. In Corollary 1.3, however, the bound for the number of conjugates may depend on the topology of M^{2m}.

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Graduate School of Mathematical Sciences, University of Tokyo, Komaba Meguro, Tokyo 153-8914, Japan
E-mail address: tsuboi@ms.u-tokyo.ac.jp

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012