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Torus fibrations and localization of index II
- Local index for acyclic compatible system -

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Abstract

We give a framework of localization for the index of a Dirac-type operator on an open manifold. Suppose the open manifold has a compact subset whose complement is covered by a finitely many open subset, each of which has a structure of the total space of a torus bundle. Under a certain compatibility condition and acyclicity we show that the index of the Dirac-type operator is localized on the compact set.

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1 Introduction

This paper is the second of the series concerning a localization of index of elliptic operator.

For a linear elliptic operator on a closed manifold, its Fredholm index is sometimes determined by the information on a specific subset under appropriate geometric condition. Such a phenomenon is called *localization of* *index.* A typical example is Hopf's theorem identifying the index of de Rham operator with the number of zeros of a vector field counted with sign and multiplicity. In this case the geometric condition is given by the vector field, and the index is localized around the zeros of the vector field. Another typical example is Atiyah-Segal's Lefschetz formula for the equivariant index under torus action, when the geometric condition is given by the torus action. The index is localized around its fixed point set, and the localization is understood in terms of an algebraic localization of equivariant K-group. In particular when the manifold is symplectic and the elliptic operator is a Dirac type operator, the localization is extensively investigated using the relation between the algebraic localizations of the equivariant K-group and that of the equivariant ordinary cohomology group.

In the previous paper [4], we dealt with closed symplectic manifold equipped with a prequantizing line bundle and a structure of Lagrangian fibration, and described a localization of the index of Dirac-type operators, twisted by the prequantizing line bundle, on the subset consisting of Bohr-Sommerfeld fibers and singular fibers. A novel feature of our method is that we do not use a global group action but use only a structure of torus bundle on an open subset of the manifold.

In the present paper we generalize our method to deal with the case when we do not have a global torus bundle on the open subset, but we just have a structure of torus bundle on a neighborhood of each points, which gives a family of torus bundles satisfying some compatibility condition. The various torus bundles may have tori of various dimensions as their fibers. This generalization enables us to describe the localization phenomenon more precisely. Even for the case in the previous paper, we could replace the subset on which the index is localized with a smaller subset. A typical example of our generalization is the localization of index for prequantized toric manifold, for which we would need an orbifold version of our formulation. Moreover we can deal with some prequantized singular Lagrangian fibration without global toric action (Section 6). In our subsequent paper we will use the localization to give an approach to V. Guillemin and S. Sternberg's conjecture concerning "quantization commutes with reduction" in the case of torus actions. Though our motivating example is the index of a prequantized symplectic manifold, the localization of index is formulated for more general cases. In fact we first establish a general framework to formulate the index of elliptic operator on a complete manifold (Section 3). This section is independent of the other sections and the framework may be interesting of its own.

The mechanism of our localization is explained as a version of Witten deformation, where the potential term itself is a first order differential operator. Our geometric input data is a family of torus bundles. Roughly speaking we deform the operator like an *adiabatic limit* shrinking the various fiber directions at the same time in a compatible manner. The *potential term* corresponds to some average of the de Rham operators along the various fiber directions.

Formally our localization is formulated as a property for the *index* of the elliptic operator on an open manifold: let D be an elliptic operator on a (possibly non-compact) manifold X, and V is an open subset of X whose complement $X \smallsetminus V$ is compact. Suppose V has a certain geometric structure s, by which we can modify D to construct a Fredholm operator. The index of the Fredholm operator depends on the data (X, V, s, D). Suppose the index satisfies the following properties. Firstly the index is deformation invariant. Secondly if X' is an open subset of X containing $X \smallsetminus V$, and hence $X' \smallsetminus V$ is compact. Let D' be the restriction of D on X'. We assume that the structure s has its *restriction* s' on $V' = X' \cap V$. Then we have the index of the Fredholm operator constructed from the data (X', V', s', D'). The required excision property is the equality between the two indices. We will construct Fredholm operators which satisfies the above type of excision property. The structure s on V is not extended on the whole X. In this sense $X \smallsetminus V$ is regarded as *singular locus* of the structure. The index is *localized* on the singular locus $X \smallsetminus V$, and we call it the *local index* of the data (X, V, s, D). When $X \smallsetminus V$ is of the form of the disjoint union of finitely many compact subsets, the *localized* index is equal to the sum of the contributions from the compact subsets.

Our first main result is the construction of the local index when the structure s is the *strongly acyclic compatible system* defined in Section 2. Our second main result is a few basic properties of the local index, in particular a *product formula* of the local index.

The organization of this paper is as follows. In Section 2 we define the notion of strongly acyclic compatible system, which we use as the geometric structure s in the above explanation. In Section 3 we give a formulation of index for elliptic operators on complete Riemannian manifolds. This formulation is a generalization of the one given in Section 5 of [5]. This section is independent of the other sections. In Section 4 we define the index of elliptic operator using the framework of Section 3 under the assumption that a strongly acyclic compatible system is given on an end of the base manifold. In Section 5 we show a product formula for the index defined in Section 4. In Section 6 we give an example of our formulation using some 4-dimensional Lagrangian fibration with singular fibers. In Appendix we give proofs of the technical lemmas used in the main part.

1.1 Conventions/Notations

- **Tangent bundle along fibers**

When π is a projection of a fiber bundle over a smooth manifold M we denote by $T[\pi]$ the vector bundle over M consisting of tangent vectors along fibers of π.

- **Tensor products of $\mathbb{Z}/2$-graded algebras and modules.**

Let $A = A^0 \oplus A^1$ and $B = B^0 \oplus B^1$ be two $\mathbb{Z}/2$-graded algebras. Then we define a structure of a $\mathbb{Z}/2$-graded algebra on the tensor product $A \otimes B$ as follows. The $\mathbb{Z}/2$-grading is defined as the one for vector spaces,

$$A \otimes B = ((A^0 \otimes B^0) \oplus (A^1 \otimes B^1)) \oplus ((A^0 \otimes B^1) \oplus (A^1 \otimes B^0)).$$

The multiplicative structure is defined by

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{\deg b \deg a'}(aa') \otimes (bb'),$$

where $a, a' \in A^0 \cup A^1$ and $b, b' \in B^0 \cup B^1$.

Now let R_A and R_B be $\mathbb{Z}/2$-graded A and B modules respectively. Then we define a structure of a $\mathbb{Z}/2$-graded $A \otimes B$-module on the tensor product $R_A \otimes R_B$ by the following formula.

$$(a \otimes b) \cdot (r_A \otimes r_B) = (-1)^{\deg b \deg r_A}(ar_A \otimes br_B),$$

where $a \in A^0 \cup A^1$, $b \in B^0 \cup B^1$, $r_A \in R_A^0 \cup R_A^1$ and $r_B \in R_B^0 \cup R_B^1$.

Note that under this convention there is a natural isomorphism between $\mathbb{Z}/2$-graded Clifford algebras

$$Cl(T_1 \oplus T_2) \cong Cl(T_1) \otimes Cl(T_2)$$

for any Hermitian vector spaces T_1 and T_2.

- **Complex structure on vector bundles**

If we denote $(\mathbb{R}^{2n})_{\mathbb{C}}$, then we consider $\mathbb{R}^{2n}$ as the complex vector space with the standard complex structure. Let E be a real vector bundle over a topological space. We denote by $E^{\mathbb{C}}$ the complex vector space $E \otimes_{\mathbb{R}} \mathbb{C}$. If E is equipped with a complex structure J, then we denote $E_{\mathbb{C}}$ by the complex vector space with $\sqrt{-1} := J : E \to E$. In addition for such E and J, we denote the anti-holomorphic part by $E^{0,1} = E_{\mathbb{C}}^{0,1}$, i.e., $E^{0,1}$ is the complex vector bundle consisting of eigenvectors of $J : E^{\mathbb{C}} \to E^{\mathbb{C}}$ with eigenvalue $-\sqrt{-1}$.

2 Compatible fibrations and acyclic compatible system

2.1 Compatible fibrations

Let M be a manifold.

Definition 2.1 A *compatible fibration on* M is a collection of data $\{\pi_\alpha \colon V_\alpha \to U_\alpha \mid \alpha \in A\}$ satisfying the following properties.

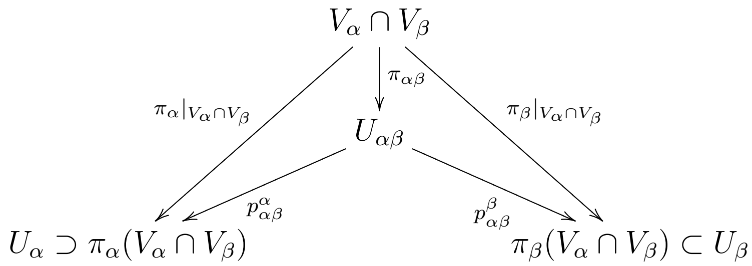
1. $M = \cup_{\alpha \in A} V_\alpha$ is an open covering.

2. Each $\pi_\alpha \colon V_\alpha \to U_\alpha$ is a fiber bundle whose fiber is a closed manifold.

3. For each α and β, we have

$$V_\alpha \cap V_\beta = \pi_\alpha^{-1}(\pi_\alpha(V_\alpha \cap V_\beta)) = \pi_\beta^{-1}(\pi_\beta(V_\alpha \cap V_\beta)).$$

4. If $V_\alpha \cap V_\beta \neq \emptyset$ and $\alpha \neq \beta$, then there exist a smooth manifold $U_{\alpha\beta}$, fiber bundles $\pi_{\alpha\beta} : V_\alpha \cap V_\beta \to U_{\alpha\beta}$ and $p_{\alpha\beta}^\alpha : U_{\alpha\beta} \to \pi_\alpha(V_\alpha \cap V_\beta)$ such that fibers are closed manifolds and the following diagram commutes;



Let $\{\pi_\alpha \colon V_\alpha \to U_\alpha \mid \alpha \in A\}$ be a compatible fibration on M. We often denote it by $\{\pi_\alpha\}$ for simplicity.

Definition 2.2 For $\alpha \in A$ and $x \in V_\alpha$, we define $A_{\subset\alpha}$ and $A(\alpha; x)$ as follows.

1. $A_{\subset\alpha} := \{\beta \in A \,|V_\alpha \cap V_\beta \neq \emptyset,\ p_{\alpha\beta}^\beta : U_{\alpha\beta} \to \pi_\beta(V_\alpha \cap V_\beta)$ is a diffeomorphism.$\}$.

2. $A(\alpha; x) := \{\beta \in A \mid x \in V_\alpha \cap V_\beta,\ \beta \in A_{\subset\alpha}\}$.

Remark 2.3 Note that if $\beta \in A(\alpha, x)$, then we have $\pi_\beta^{-1}\pi_\beta(x) \subset \pi_\alpha^{-1}\pi_\alpha(x)$.

Definition 2.4 A subset C of M is *admissible* if for each α, we have

$$C \cap V_\alpha = \pi_\alpha^{-1}(\pi_\alpha(C \cap V_\alpha)).$$

Example 2.5 Each V_α is admissible.

Proposition 2.6 *Let C be an admissible open subset of M. Then $\{\pi_\alpha|_{C\cap V_\alpha} : C\cap V_\alpha \to \pi_\alpha(C\cap V_\alpha)\}$ is a compatible fibration on C.*

Proof. It is sufficient to show the following three equalities

1. $\pi_\alpha^{-1}\pi_\alpha(C\cap V_\alpha\cap V_\beta) = C\cap V_\alpha\cap V_\beta$,
2. $\pi_{\alpha\beta}^{-1}\pi_{\alpha\beta}(C\cap V_\alpha\cap V_\beta) = C\cap V_\alpha\cap V_\beta$, and
3. ${p_{\alpha\beta}^{\alpha}}^{-1}p_{\alpha\beta}^{\alpha}(\pi_{\alpha\beta}(C\cap V_\alpha\cap V_\beta)) = \pi_{\alpha\beta}(C\cap V_\alpha\cap V_\beta)$.

First let us show the right facing inclusion $\subset$ for 1. For each $z\in\pi_\alpha^{-1}\pi_\alpha(C\cap V_\alpha\cap V_\beta)$ there exists $x\in C\cap V_\alpha\cap V_\beta$ such that $\pi_\alpha(z)=\pi_\alpha(x)$. Then $z\in\pi_\alpha^{-1}\pi_\alpha(x)\subset\pi_\alpha^{-1}\pi_\alpha(C\cap V_\alpha)\cap\pi_\alpha^{-1}\pi_\alpha(V_\alpha\cap V_\beta)=C\cap V_\alpha\cap V_\beta$.

Next we show the right facing inclusion $\subset$ for 2. For each $z\in\pi_{\alpha\beta}^{-1}\pi_{\alpha\beta}(C\cap V_\alpha\cap V_\beta)$ there exists $x\in C\cap V_\alpha\cap V_\beta$ such that $\pi_{\alpha\beta}(z)=\pi_{\alpha\beta}(x)$. Then $\pi_\alpha(x)=p_{\alpha\beta}^{\alpha}\circ\pi_{\alpha\beta}(x)=p_{\alpha\beta}^{\alpha}\circ\pi_\alpha(z)=\pi_\alpha(z)$. In particular $z\in\pi_\alpha^{-1}\pi_\alpha(x)\subset\pi_\alpha^{-1}\pi_\alpha(C\cap V_\alpha\cap V_\beta)=C\cap V_\alpha\cap V_\beta$.

Finally we show the right facing inclusion $\subset$ for 3. For each $z\in{p_{\alpha\beta}^{\alpha}}^{-1}p_{\alpha\beta}^{\alpha}(\pi_{\alpha\beta}(C\cap V_\alpha\cap V_\beta))$ there exists $x\in C\cap V_\alpha\cap V_\beta$ such that $p_{\alpha\beta}^{\alpha}(z)=\pi_\alpha(x)$. We show that $\pi_{\alpha\beta}^{-1}(z)\subset C$. For $w\in\pi_{\alpha\beta}^{-1}(z)$ we have $\pi_\alpha(w)=p_{\alpha\beta}^{\alpha}\circ\pi_{\alpha\beta}(w)=p_{\alpha\beta}^{\alpha}(z)=\pi_\alpha(x)$. Then $w\in\pi_\alpha^{-1}\pi_\alpha(x)\subset C$. This shows $\pi_{\alpha\beta}^{-1}(z)\subset C$. Hence $z\in\pi_{\alpha\beta}(\pi_{\alpha\beta}^{-1}(z))\subset\pi_{\alpha\beta}(C\cap V_\alpha\cap V_\beta)$. □

Definition 2.7 Let $f : M\to\mathbb{R}$ be a function. If there exists an admissible open covering $\{V'_\alpha\}_{\alpha\in A}$ of M such that f is constant along fibers of $\pi_\alpha|_{V'_\alpha}$ for all $\alpha\in A$, then we call f an *admissible function*.

In this article we impose the following technical assumptions for a compatible fibration $\{\pi_\alpha : V_\alpha\to U_\alpha\}_{\alpha\in A}$.

Assumption 2.8

1. The index set A is a finite set.
2. Each π_α has a continuous extension as a fiber bundle to the closure of V_α with the condition
$$V_\alpha\cap\overline{V_\beta}=\pi_\beta^{-1}\pi_\beta(V_\alpha\cap\overline{V_\beta})$$
for all $\beta\in A$.
3. There is an averaging operation $I : C^\infty(M)\to C^\infty(M)$ whose definition is given below in Definition 2.9.

Definition 2.9 If a linear map $I : C^\infty(M) \to C^\infty(M)$ satisfies the following properties, then we call I an *averaging operation*.

1. $I(f)$ is an admissible function for all $f \in C^\infty(M)$.
2. If f is a constant function, then $I(f)$ is also a constant function with the same value of f.
3. If f is a non-negative function, then $I(f)$ is so.
4. For all $f \in C^\infty(M)$ and $x \in M$ we have
$$\min_{y \in \pi_\alpha^{-1}\pi_\alpha(x)} f(y) \le I(f)(x) \le \max_{y \in \pi_\beta^{-1}\pi_\beta(x)} f(y),$$
for some $\alpha, \beta \in \{\alpha' \in A \mid x \in \overline{V_{\alpha'}}\}$.
5. Let $f : M \to \mathbb{R}$ be a function and C an admissible subset of M. If *supp*f is contained in C then *supp*$I(f)$ is also contained in C.

Using the averaging operation we can construct an *admissible partition of unity* as in the following.

Lemma 2.10 (Existence of admissible partition of unity) *Let V be an open subset of M with a compatible fibration $\{\pi_\alpha\}$. There is a smooth partition of unity $\{\rho_\alpha^2\}$ of the open covering $V = \cup_\alpha V_\alpha$ which is constant along each fiber of $\pi_{\alpha'}$ for every $\alpha' \in A$.*

Proof. Take any partition of unity $\{\phi_\alpha\}_\alpha$ of $V = \cup_\alpha V_\alpha$. Applying the averaging operation we have a family of admissible functions $\{I(\phi_\alpha)\}_\alpha$. Note that it is an another partition of unity of $\{V_\alpha\}$ because of the Property 2,3 and 5 of the averaging operation. We put $\rho_\alpha := I(\phi_\alpha)/\sqrt{\sum_\beta I(\phi_\beta)^2}$. Then $\{\rho_\alpha^2\}_\alpha$ is a required admissible partition of unity. $\square$

We give a sufficient condition for Assumption 2.8.

Definition 2.11 (Good compatible fibration) If a compatible fibration $\{\pi_\alpha : V_\alpha \to U_\alpha\}$ over V satisfies 1 and 2 in Assumption 2.8 together with the following 5', then we call $\{\pi_\alpha : V_\alpha \to U_\alpha\}$ a *good compatible fibration*.

5'. If $V_\alpha \cap V_\beta \neq \emptyset$, then we have $\alpha \in A(\beta; x)$ or $\beta \in A(\alpha; x)$ for all $x \in V_\alpha \cap V_\beta$.

For a good compatible fibration we denote by $p^\alpha_{\alpha\beta} = p_{\alpha\beta}$ for α, $\beta \in A$ with $\beta \in A_{\subset\alpha}$.

We show the following proposition in Appendix A.

Proposition 2.12 *If $\{\pi_\alpha\}$ is a good compatible fibration, then there exists an averaging operation $I : C^\infty(M) \to C^\infty(M)$ such that for all $f \in C^\infty(M)$ and $x \in M$ we have*

$$\min_{y \in \pi^{-1}_{\overline{\alpha}_x}\pi_{\overline{\alpha}_x}(x)} f(y) \le I(f)(x) \le \max_{y \in \pi^{-1}_{\overline{\alpha}_x}\pi_{\overline{\alpha}_x}(x)} f(y),$$

where $\pi^{-1}_{\overline{\alpha}_x}\pi_{\overline{\alpha}_x}(x) \subset \overline{V}_{\overline{\alpha}_x}$ is the maximal fiber which contains x.

Now we define an appropriate notion of Riemannian metric and connection for a compatible fibration. We first note that there exist following four types of short exact sequences for $i = \alpha, \beta$

$$0 \to T[\pi_i] \to TV_i \to \pi_i^* TU_i \to 0, \tag{1}$$
$$0 \to T[\pi_{\alpha\beta}] \to T(V_\alpha \cap V_\beta) \to \pi_{\alpha\beta}^* TU_{\alpha\beta} \to 0, \tag{2}$$
$$0 \to T[\pi_{\alpha\beta}] \to T[\pi_i]|_{V_\alpha \cap V_\beta} \to \pi_{\alpha\beta}^* T[p^i_{\alpha\beta}] \to 0, \tag{3}$$
$$0 \to T[p^i_{\alpha\beta}] \to TU_{\alpha\beta} \to {p^i_{\alpha\beta}}^* T\pi_i(V_\alpha \cap V_\beta) \to 0.. \tag{4}$$

Definition 2.13 Let E^0, E^1 and E^2 be smooth vector bundles with metrics. A short exact sequence $0 \to E^0 \to E^1 \to E^2 \to 0$ is *orthogonally split* if the isomorphism $E^1 \cong E^0 \oplus E^2$ defined by the orthogonal splitting with respect to the metric on E^1 is isometric with respect to the metrics on E^0, E^1 and E^2.

Definition 2.14 A *compatible Riemannian metric of a compatible fibration* is a collection of metrics on the vector bundles $T[\pi_i], TU_i$, $T[\pi_{\alpha\beta}]$, $TU_{\alpha\beta}$, and $T[p^i_{\alpha\beta}]$ such that the exact sequences (3) and (4) are orthogonally split with respect to these metrics.

From the definition, we have a canonical isometric isomorphism

$$(T[\pi_\alpha] \oplus \pi_\alpha^* TU_\alpha)|_{V_\alpha \cap V_\beta} \cong (T[\pi_{\alpha\beta}] \oplus \pi_{\alpha\beta}^* TU_{\alpha\beta}) \cong (T[\pi_\beta] \oplus \pi_\beta^* TU_\beta)|_{V_\alpha \cap V_\beta} \tag{5}$$

over $V_\alpha \cap V_\beta$.

Definition 2.15 Suppose we have a compatible fibration with a compatible Riemannian metric. A *compatible connection* is a collection of the splittings of of the short exact sequences (1) and (2) such that the isomorphism (5) is equal to the composition of the isomorphisms

$$(T[\pi_i] \oplus \pi_i^* TU_i)|_{V_\alpha \cap V_\beta} \cong T(V_\alpha \cap V_\beta) \cong (T[\pi_{\alpha\beta}] \oplus \pi_{\alpha\beta}^* TU_{\alpha\beta})$$

induced from the splittings.

2.2 Acyclic compatible system

Definition 2.16 Suppose we have a compatible fibration $\{\pi_\alpha\}$ on M with a compatible Riemannian metric. A bundle W over M is a *compatible Clifford module bundle* if we have the following structures.

1. W has a structure of a $\mathbb{Z}/2$-graded $Cl(T[\pi_\alpha] \oplus \pi_\alpha^* TU_\alpha)$-module bundle over V_α.

2. Over $V_\alpha \cap V_\beta$, the above module structures on V_α and V_β are compatible with the isomorphism (5).

The next lemma follows immediately from the definitions of compatible metric, compatible connection and compatible Clifford module bundle.

Lemma 2.17 *Suppose we have a compatible metric and compatible connection. Then we have a well-defined Riemannian metric on M. Moreover if we have a compatible Clifford module bundle, then it has a structure of Clifford module with respect to the well-defined Riemannian metric on M.*

Let $\{\pi_\alpha\}$ be a compatible fibration on M with compatible Riemannian metric and $W \to M$ a compatible Clifford module bundle.

Definition 2.18 (Compatible system of Dirac-type operators) A *compatible system* on $(\{\pi_\alpha\}, W)$ is a data $\{D_\alpha\}$ satisfying the following properties.

1. $D_\alpha \colon \Gamma(W|_{V_\alpha}) \to \Gamma(W|_{V_\alpha})$ is an order-one formally self-adjoint differential operator of degree-one.

2. D_α contains only the derivatives along fibers of $\pi_\alpha \colon V_\alpha \to U_\alpha$, i.e. D_α commutes with multiplication of the pullback of smooth functions on U_α.

3. The principal symbol $\sigma(D_\alpha)$ of D_α is given by $\sigma(D_\alpha) = c \circ p_\alpha \circ \iota_\alpha^* \colon T^*V_\alpha \to \mathrm{End}(W|_{V_\alpha})$, where $\iota_\alpha \colon T[\pi_\alpha] \to TV_\alpha$ is the natural inclusion, $p_\alpha \colon T^*[\pi_\alpha] \to T[\pi_\alpha]$ is the isomorphism induced by the Riemannian metric and $c \colon T[\pi_\alpha] \to \mathrm{End}(W|_{V_\alpha})$ is the Clifford multiplication.

4. For $b \in U_\alpha$ and $u \in T_bU_\alpha$, let $\widetilde{u} \in \Gamma(\pi_\alpha^* TU_\alpha|_{\pi_\alpha^{-1}(b)})$ be the section naturally induced by u. $\widetilde{u}$ acts on $W|_{\pi_\alpha^{-1}(b)}$ by the Clifford multiplication $c(\widetilde{u})$. Then D_α and $c(\widetilde{u})$ anti-commute each other, i.e.
$$0 = \{D_\alpha, c(\widetilde{u})\} := D_\alpha \circ c(\widetilde{u}) + c(\widetilde{u}) \circ D_\alpha$$
for all $b \in U_\alpha$ and $u \in T_bU_\alpha$.

5. If $V_\alpha \cap V_\beta \neq \emptyset$, then the anti-commutator $\{D_\alpha, D_\beta\} := D_\alpha \circ D_\beta + D_\beta \circ D_\alpha$ is a differential operator along fibers of $\pi_{\alpha\beta}$ of order at most 2.

The properties 1, 2, and 3 in Definition 2.18 imply that D_α is of Dirac-type when restricted to each fiber of π_α.

We call a compatible system of Dirac-type operators $\{D_\alpha\}$ a compatible system for short.

Definition 2.19 (Acyclic compatible system) A compatible system $\{D_\alpha\}_{\alpha \in A}$ is *acyclic* if for all $\alpha \in A$, $x \in V_\alpha$ and a family of non-negative numbers $\{t_\beta\}_{\beta \in A(\alpha;x)}$ satisfying $t_\beta > 0$ for some β, the operator $\sum_{\beta \in A(\alpha;x)} t_\beta D_\beta \colon \Gamma(W|_{\pi_\alpha^{-1}(\pi_\alpha(x))}) \to \Gamma(W|_{\pi_\alpha^{-1}(\pi_\alpha(x))})$ has zero kernel. Note that the above operator is well-defined because of Remark 2.3.

Definition 2.20 (strongly acyclic compatible system) A compatible system $\{D_\alpha\}$ is *strongly acyclic* if it satisfies the following conditions.

1. For each α and $b \in U_\alpha$ $D_\alpha|_{\pi_\alpha^{-1}(b)}$ has zero kernel.
2. If $V_\alpha \cap V_\beta \neq \emptyset$, then the anti-commutator $\{D_\alpha, D_\beta\}$ is a non-negative operator over $V_\alpha \cap V_\beta$.

We first note that the following lemma.

Lemma 2.21 *A strongly acyclic compatible system is acyclic.*

Proof. If $\{D_\alpha\}$ is strongly acyclic compatible system, then we have

$$\left(D_\alpha + \sum_{\beta \in A(\alpha;x)} \tau_\beta D_\beta\right)^2 = D_\alpha^2 + \sum \tau_\beta \{D_\alpha, D_\beta\} + \left(\sum_\beta \tau_\beta D_\beta\right)^2 \geq D_\alpha^2$$

for any family of non-negative numbers $\{\tau_\beta\}$. Suppose $\left(\sum_{\beta \in A(\alpha;x)} t_\beta D_\beta\right) s = 0$ for $s \in \Gamma(W|_{\pi_\alpha^{-1}(\pi_\alpha(x))})$. Take $\alpha_0 \in A(\alpha;x)$ so that t_{α_0} is not 0. Then we have

$$\left(D_{\alpha_0} + \sum_{\beta \in A(\alpha;x) \smallsetminus \{\alpha_0\}} (t_\beta / t_{\alpha_0}) D_\beta\right) s = 0$$

and $s = 0$ by the above inequality and the first condition in Definition 2.20. □

Remark 2.22 It is obvious that we have an orbifold version of the definitions of compatible fibration and compatible system, for which Lemma 2.21 also holds.

Example 2.23 Let M be $\mathbb{R}\times S^1$ with the standard Riemannian metric and (t,θ) its standard coordinate. We introduce a compatible fibration on M by

$$\pi_\alpha\colon V_\alpha := (-\infty,1)\times S^1 \to U_\alpha := (-\infty,1)$$
$$\pi_\beta\colon V_\beta := (-1,\infty)\times S^1 \to U_\beta := (-1,\infty).$$

Let W be the trivial rank 2 Hermitian vector bundle $M\times\mathbb{C}^2$ on M with $\mathbb{Z}/2$-grading

$$W^0 := M\times(\mathbb{C}\times 0),\ W^1 := M\times(0\times\mathbb{C}).$$

We define the Clifford multiplication of $Cl(TM)$ on W by

$$c(\partial_\theta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ c(\partial_t) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

For smooth functions $f_\alpha\colon V_\alpha\to\mathbb{R}$, $f_\beta : V_\beta\to\mathbb{R}$, let D_α, D_β be differential operators on $\Gamma(W|_{V_\alpha})$, $\Gamma(W|_{V_\beta})$ which are defined by

$$D_\alpha := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\partial_\theta + f_\alpha(t,\theta)\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$$
$$D_\beta := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\partial_\theta + f_\beta(t,\theta)\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}.$$

They are order-one formally self-adjoint differential operators of degree-one. Then, it is easy to see that the data $\{D_\alpha, D_\beta\}$ is an acyclic compatible system if and only if f_α and f_β satisfy the following properties.

1. $f_\alpha(t,\theta)\notin\mathbb{Z}$ for any $(t,\theta)\in V_\alpha$. The same property also holds for f_β.

2. $\frac{t_\alpha f_\alpha(t,\theta)+t_\beta f_\beta(t,\theta)}{t_\alpha+t_\beta}\notin\mathbb{Z}$ for any $(t,\theta)\in V_\alpha\cap V_\beta$ and any non-negative real numbers t_α, t_β which satisfy $t_\alpha+t_\beta\neq 0$.

Example 2.24 For non-negative integers m and n satisfying $n\le m$ let M be $\mathbb{R}^{2m-n}\times T^n$, where we regard T^n as $\mathbb{R}^n/(2\pi\mathbb{Z})^n$. Let A be an ordered set, $\{V'_\alpha\}_{\alpha\in A}$ a finite open covering of $\mathbb{R}^{2m-n}$, and $\{R_\alpha\}_{\alpha\in A}$ a family of subspaces of $\mathbb{R}^n$ spanned by rational vectors. We assume that if $\alpha<\beta$, then $R_\alpha\subset R_\beta$. We put $V_\alpha := V'_\alpha\times T^n$ and $T_\alpha := R_\alpha/R_\alpha\cap(2\pi\mathbb{Z})^n$. Define U_α to be $V'_\alpha\times T^n/T_\alpha$ and $\pi_\alpha : V_\alpha\to U_\alpha$ to be the natural projection. Then these data define a good compatible fibration on M.

Let (g,J) be the pair of the Riemannian metric and the almost complex structure on M which is defined by

$$g_x\left(\sum_{i=1}^{2m-n} a_i^1\partial_{y_i} + \sum_{i=1}^{n} b_i^1\partial_{\theta_i},\ \sum_{j=1}^{2m-n} a_j^2\partial_{y_j} + \sum_{j=1}^{n} b_j^2\partial_{\theta_j}\right) = \sum_{i=1}^{2m-n} a_i^1 a_i^2 + \sum_{i=1}^{n} b_i^1 b_i^2,$$

$$J(\partial_{y_i}) = \begin{cases} \partial_{\theta_i} & 1 \leq i \leq n \\ \partial_{y_{m-n+i}} & n+1 \leq i \leq m \\ -\partial_{y_{i-m+n}} & m+1 \leq i \leq 2m-n \end{cases}$$

for $x = (y, \theta) \in M$. Note that since g is invariant under J, (g, J) defines the Hermitian metric on M by

$$h_x(u, v) = g_x(u, v) + \sqrt{-1} g_x(u, Jv)$$

for $u, v \in T_x M$. By using the horizontal lift $\pi_\alpha^* TU_\alpha \to TV_\alpha$ with respect to g, it is obvious that $\{\pi_\alpha\}$ is equipped with a compatible Riemannian metric and a compatible connection.

Next we define a compatible Clifford module bundle W and a strongly acyclic compatible system $\{D_\alpha\}_{\alpha \in A}$ in the following way. Take a Hermitian line bundle (L, ∇^L) with Hermitian connection on M whose restriction to $\pi_\alpha^{-1}(b)$ is a flat connection for each $\alpha \in A$ and $b \in U_\alpha$. We assume the following condition.

(∗) For all α and $b \in U_\alpha$ the restriction $\nabla^L|_{\pi_\alpha^{-1}(b)}$ is not trivially flat connection, i.e., its holonomy representation is non-trivial.

We define a Hermitian vector bundle W on M by

$$W := \wedge_{\mathbb{C}}^{\bullet} TM_{\mathbb{C}} \otimes L.$$

A Clifford module structure $c\colon Cl(TM) \to \mathrm{End}(W)$ is defined by

$$c(u)(\varphi) = u \wedge \varphi - u \llcorner \varphi \tag{6}$$

for $u \in TM$, $\varphi \in W$, where $\llcorner$ is the interior product with respect to h, namely,

$$v \llcorner (v_1 \wedge v_2 \wedge \cdots \wedge v_k) := \sum_{i=1}^{k} (-1)^{i-1} h(v_i, v) v_1 \wedge \hat{v}_i \wedge \cdots \wedge v_k,$$

$v_1, \ldots v_k \in TM$.

Let $\nabla^{\wedge_{\mathbb{C}}^{\bullet} TM_{\mathbb{C}}}$ be the Hermitian connection on $\wedge_{\mathbb{C}}^{\bullet} TM_{\mathbb{C}}$ which is induced from the Levi-Civita connection on TM with respect to g. Two connections $\nabla^{\wedge_{\mathbb{C}}^{\bullet} TM_{\mathbb{C}}}$ and ∇^L define the Hermitian connection on W by

$$\nabla := \nabla^{\wedge_{\mathbb{C}}^{\bullet} TM_{\mathbb{C}}} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^L.$$

Then we define D_α by

$$\begin{aligned} D_\alpha := c \circ \iota_\alpha^* \circ \nabla|_{V_\alpha} : \Gamma\left(W|_{V_\alpha}\right) &\xrightarrow{\nabla|_{V_\alpha}} \Gamma\left(T^*V_\alpha \otimes W|_{V_\alpha}\right) \\ &\xrightarrow{\iota_\alpha^*} \Gamma\left(T^*[\pi_\alpha]\right) \otimes W|_{V_\alpha}) \\ &\xrightarrow{c} \Gamma\left(W|_{V_\alpha}\right), \end{aligned}$$

where $\iota_\alpha \colon T[\pi_\alpha] \to TV_\alpha$ is the natural inclusion.

By the construction it is obvious that $\{D_\alpha\}$ satisfies the condition 1, 2, and 3 in Definition 2.18. The condition 4 in Definition 2.18 follows from the fact that g restricted to each fiber of π_α is flat. We can show that for each α and $b \in U_\alpha$ the kernel of $D_\alpha|_{\pi_\alpha^{-1}(b)}$ vanishes. It follows from Property $(*)$ and the following lemma.

Lemma 2.25 *Let $(E, \nabla^E) \to T$ be a flat Hermitian line bundle on a flat n-torus. If the degree zero cohomology $H^0(T; E)$ with local system (E, ∇^E) vanishes, then all cohomologies $H^\bullet(T; E)$ vanish.*

Proof. Take and fix harmonic 1-forms $\{\omega_1, \cdots, \omega_n\}$ which represent a basis of $H^1(T; \mathbb{R})$. Note that harmonic forms on a flat torus are parallel forms and they induce a trivialization of T^*T. Let ω be a d_E-closed form in $\Omega^k(T; E) = \Gamma(E \otimes \wedge^k T^*T)$, where d_E is the covariant derivative induced by ∇^E. Using the parallel basis and the trivialization of T^*T we can describe ω as

$$\omega = \sum_{i_1, \cdots, i_k} s_{i_1 \cdots i_k} \omega_{i_1} \wedge \cdots \wedge \omega_{i_k},$$

where $s_{i_1 \cdots i_k}$ is a section of E. Since each ω_i is harmonic and ω is d_E-closed we have

$$0 = d_E^* d_E \omega = \sum_{i_1, \cdots, i_k} \left((\nabla^E)^* \nabla^E s_{i_1 \cdots i_k} \right) \omega_{i_1} \wedge \cdots \wedge \omega_{i_k},$$

and hence $(\nabla^E)^* \nabla^E s_{i_1 \cdots i_k} = 0$. It implies $s_{i_1 \cdots i_k}$ is a parallel section. Since $H^0(M; E) = 0$ we have $s_{i_1 \cdots i_k} = 0$. □

These facts and Proposition 2.29, which will be shown in the next subsection, imply that $\{D_\alpha\}$ is a strongly acyclic compatible system.

2.3 Example from torus action

Suppose an n-dimensional torus G acts on a manifold M smoothly. Let A be the set of all the subgroups of G which appear as stabilizers

$$G_x := \{g \in G \mid gx = x\}$$

at some points $x \in M$. Note that A has a partial order with respect to the inclusion. In this subsection we assume that A is a finite set.

The following lemma is useful to construct a good compatible fibration satisfying a convenient property for some cases with torus actions. We give a proof in Appendix B.

Lemma 2.26 (Existence of a good open covering) *There exists an open covering $\{V_H\}_{H \in A}$ of M parameterized by A satisfying the following properties.*

1. *Each V_H is G-invariant.*
2. *For each $x \in V_H$ we have $G_x \subset H$.*
3. *If $V_H \cap V_{H'} \neq \emptyset$, then we have $H \subset H'$ or $H \supset H'$.*

Remark 2.27 Using a good covering over M we can construct a good compatible fibration as follows. We endow G with a rational flat Riemannian metric. Precisely speaking we take an Euclidian metric on the Lie algebra of G such that the intersection of the integral lattice and the lattice generated by some orthonormal basis has rank n. We extend it on the whole G.

For a subgroup H of G let $H^\perp$ be the orthogonal complement of H defined as the image of the orthogonal complement of the Lie algebra of H by the exponential map. Since the metric is rational $H^\perp$ is well-defined as a compact subgroup of G and it has finitely many intersection points $H \cap H^\perp$.

Let $\{V_H\}_{H \in A}$ be the open covering of M in Lemma 2.26. For each $H \in A$ we define U_H to be $V_H/H^\perp$ and $\pi_H : V_H \to U_H$ to be the natural projection. Then the data $\{\pi_H : V_H \to U_H \mid H \in A\}$ define the good compatible fibration because of the property of the good covering.

Remark 2.28 We will show in Section 6 that there is an example that has a good compatible fibration, but does not have a global torus action.

2.3.1 Family of flat torus bundles

Let $\{\pi_\alpha\}_{\alpha \in A}$ be a compatible fibration on V with a compatible Riemannian metric and W a compatible Clifford module bundle on V. We show the following.

Proposition 2.29 *Suppose that an acyclic compatible system $\{\pi_\alpha, W, D_\alpha\}$ satisfies the following three conditions.*

- *$\{\pi_\alpha\}_{\alpha \in A}$ is a good compatible fibration.*

- *$\pi_\alpha : V_\alpha \to U_\alpha$ is a flat torus bundle for all α.*
- *There is a Clifford connection ∇ on W such that the restriction of ∇ on each fiber of π_α is a flat connection for all α.*
- *D_α is the Dirac operator along fibers of π_α defined by $\nabla|_{V_\alpha}$ for all $\alpha \in A$.*

Then for all $\alpha, \beta \in A$ such that $\beta \in A_{\subset\alpha}$ the anti-commutator $D_\alpha \circ D_\beta + D_\beta \circ D_\alpha$ is a non-negative operator along fibers of π_β. In particular if $\ker D_\alpha = 0$ *for all $\alpha \in A$, then $\{\pi_\alpha, W, D_\alpha\}$ is strongly acyclic.*

Since to show this proposition it is enough to show the non-negativity of the anti-commutator along fibers, we consider the following setting.

- E : Euclidian space
- Γ : maximal lattice of E
- $F := E/\Gamma$: flat torus
- $W \to F$: $Cl(TF)$-module bundle
- $\nabla : \Gamma(W) \to \Gamma(TF \otimes W)$: flat Clifford connection of W
- $c : TF \otimes W \to W$: Clifford action of TF
- A : finite set
- $\{E_\alpha\}_{\alpha \in A}$: family of subspaces of E
- $\{p_\alpha\}$: family of orthogonal projections to $\{E_\alpha\}$
- We assume $p_\alpha p_\beta = p_\beta p_\alpha$ for all $\alpha, \beta \in A$.

Note that the last condition implies that the Proposition 2.29 holds for a compatible fibration which is not necessarily good. See Remark 2.32. Using the metric we have the identification $TF = T^*F = F \times E$. For a symmetric endmorphism $S : E \to E$ let $\hat{S} : F \times E \to F \times E$ be the induced bundle map on the (co)tangent bundle. We define a differential operator D_S by the composition

$$D_S := c \circ \hat{S} \circ \nabla : \Gamma(W) \to \Gamma(W).$$

Since S is symmetric D_S is a self-adjoint operator.

Proposition 2.30 *Let S_1 and S_2 be symmetric endmorphisms which commute each other. Then we have*

$$D_{S_1} \circ D_{S_2} + D_{S_2} \circ D_{S_1} = 2\nabla^* \circ \hat{S}_1 \circ \hat{S}_2 \circ \nabla,$$

where $\nabla^* : \Gamma(TF \otimes W) \to \Gamma(W)$ is the adjoint operator of ∇.

Proof. The equality can be checked by the direct computation using the orthonormal basis of E consisting of simultaneously eigenvectors of S_1 and S_2. □

When we put $S_1 := p_\alpha$ and $S_2 := p_\beta$ in Proposition 2.30, we have the following.

Corollary 2.31 $D_\alpha D_\beta + D_\beta D_\alpha = 2D_{\alpha\beta}^2$, *where $D_{\alpha\beta}$ is the self-adjoint operator $c \circ \hat{p}_{\alpha\beta} \circ \nabla$ defined by the projection $p_{\alpha\beta}$ to the intersection $E_\alpha \cap E_\beta$.*

Proof of Proposition 2.29. Since $\{\pi_\alpha\}$ is a good compatible fibration we have a family of tori at each point on V which comes from a family of subspaces whose projections commute each other. Then the claim follows from Corollary 2.31. □

Remark 2.32 Note that a product of good compatible fibrations is not a good compatible fibration. But Proposition 2.29 still holds for products of good compatible fibrations. Since such compatible fibrations satisfy the last condition in the setting of Proposition 2.30.

2.3.2 Symplectic manifold with a torus action

Let (M, ω) be a $2m$-dimensional symplectic manifold equipped with a Hamiltonian action of an n-dimensional torus G. In this case each orbit is an affine isotropic torus in M. Suppose that there is a G-equivariant prequantizing line bundle (L, ∇) on (M, ω), i.e., L is a Hermitian line bundle over M with a Hermitian connection ∇ whose curvature form is equal to $-2\pi\sqrt{-1}\omega$, and all these data are G-equivariant. Since an orbit is isotropic the restriction of (L, ∇) on each orbit is a flat line bundle. According to Lemma 2.26, and Remark 2.27, we have a good compatible fibration on M using a good open covering $\{V_H\}$ parameterized by the set of isotropy subgroups $A = \{H\}$. We show the following.

Proposition 2.33 *If the restriction of L on each G-orbit has no nontrivial parallel sections, then M is equipped with a strongly acyclic compatible system $\{D_H\}_{H \in A}$.*

Proof. Fix a G-invariant ω-compatible almost complex structure J on M. Then using the associated G-invariant metric g_J we can construct a compatible Riemannian metric on M as follows. It is sufficient to construct metrics of $T[\pi_H]$ for each $H \in A$ and $T[p_{HK}]$ for a sequence $H \subset K$, and check the compatibility coming from (2) and (3). The metric on $T[\pi_H]$ is defined as the restriction of g_J, and the metric on $TU_H = T(V_H/H^\perp)$ is defined as the quotient metric of g_J. Note that the fiber of p_{HK} is the quotient of the $H^\perp$-orbit by the $K^\perp$-orbit. Then the metric on $T[p_{HK}]$ is defined as the quotient of the restriction of the metric g_J on $H^\perp$-orbit by the $K^\perp$-action. It is straightforward to check the compatibilities. We remark that the metric g_J restricted to each orbit is a flat affine metric because it is G-invariant.

Let W be the $\mathbb{Z}/2$-graded compatible Clifford module bundle

$$W = \wedge^\bullet TM_{\mathbb{C}} \otimes L$$

with Clifford module structure $c\colon Cl(TM) \to \mathrm{End}(W)$ defined by (6). We show that W is a compatible Clifford module bundle with some more additional structures. For $H \in A$, let $(T[\pi_H] \oplus JT[\pi_H])^\perp$ be the orthogonal complement of $T[\pi_H] \oplus JT[\pi_H]$ with respect to g_J. Since (g_J, J) is G-invariant $H^\perp$-action preserves $(T[\pi_H] \oplus JT[\pi_H])^\perp$ and $JT[\pi_H]$. Then we define

$$\begin{aligned} E_H &:= (T[\pi_H] \oplus JT[\pi_H])^\perp / H^\perp, \\ F_H &:= JT[\pi_H]/H^\perp. \end{aligned}$$

It is obvious that TU_H has the natural isometry $TU_H \cong E_H \oplus F_H$.

Since g_J is invariant under J, J preserves $(T[\pi_H] \oplus JT[\pi_H])^\perp$. In particular $T[\pi_H] \oplus JT[\pi_H]$ and $(T[\pi_H] \oplus JT[\pi_H])^\perp$ have the structures of Hermitian vector bundles with respect to the restriction of (g_J, J) to them. Moreover (g_J, J) is G-invariant it descends to the Hermitian structure on E_H. Then, we define $W_{1,H}$ and $W_{2,H}$ by

$$\begin{aligned} W_{1,H} &:= \wedge^\bullet_{\mathbb{C}} (T[\pi_H] \oplus JT[\pi_H])_{\mathbb{C}} \otimes L, \\ W_{2,H} &:= \wedge^\bullet_{\mathbb{C}} (E_H)_{\mathbb{C}} \otimes L, \end{aligned}$$

and define the Clifford module structures

$$\begin{aligned} c_{1,H} &\colon Cl(T[\pi_H] \oplus \pi_H^* F_H) \to \mathrm{End}\left(W_{1,H}\right), \\ c_{2,H} &\colon Cl\left(E_H\right) \to \mathrm{End}\left(W_{2,H}\right) \end{aligned}$$

by the same formula as in (6). By definition, TV_H has a decomposition

$$TV_H = (T[\pi_H] \oplus \pi_H^* F_H) \oplus \pi_H^* E_H$$

as Hermitian vector bundles. With respect to this decomposition there are the following isomorphisms

$$\begin{aligned} Cl(TM)|_{V_H} &\cong Cl(T[\pi_H] \oplus \pi_H^* F_H) \otimes Cl(\pi_H^* E_H), \\ &\cong Cl(T[\pi_H] \oplus \pi_H^* TU_H) \\ W|_{V_H} &\cong W_{1,H} \otimes \pi_H^* W_{2,H}. \end{aligned}$$

Then by the direct calculation one can check $c = c_{1,H} \otimes c_{2,H}$ under the above identifications.

Now we define a strongly acyclic compatible system $\{D_H\}$ on W. Let $\nabla^{T[\pi_H]} \colon \Gamma(TV_H) \to \Gamma(T^*[\pi_H] \otimes TV_H)$ be the the family of Levi-Civita connections along fibers of π_H, namely,

$$\nabla^{T[\pi_H]} = \iota_H^* \otimes q_H \circ \nabla^{TM} \circ q_H,$$

where $\iota_H \colon T[\pi_H] \to TV_H$ is the natural inclusion, ∇^{TM} is the Levi-Civita connection on TM with respect to g_J, and $q_H \colon TV_H \to TV_H$ is the orthogonal projection to $T[\pi_H]$ with respect to g_J. $\nabla^{T[\pi_H]}$ induces the family of Hermitian connections on $\wedge^\bullet TM_{\mathbb{C}}|_{V_H}$ along fibers of π_H, which is denoted by $\nabla^{\wedge^\bullet TM_{\mathbb{C}}|_{V_H}}$. We define the family of Hermitian connections ∇^H on $W|_{V_H}$ along fibers of π_H by

$$\nabla^H := \nabla^{\wedge^\bullet TM_{\mathbb{C}}|_{V_H}} \otimes \mathrm{id} + \mathrm{id} \otimes \left(\iota_H^* \otimes \mathrm{id} \circ \nabla^L\right) : \Gamma(W|_{V_H}) \to \Gamma(T^*[\pi_H] \otimes W|_{V_H}).$$

Then we define $D_H \colon \Gamma(W|_{V_H}) \to \Gamma(W|_{V_H})$ to be the family of de Rham operators along fibers of π_H which is defined by

$$D_H := c_{1,H} \circ p_H \circ \nabla^H,$$

where $p_H \colon T^*[\pi_H] \to T[\pi_H]$ is the isomorphism via g_J.

Since the restriction of L on each G-orbit has no non-trivial parallel sections, Lemma 2.25 implies $\ker D_H = 0$. Moreover since the collection of data $\{\pi_H : V_H \to U_H, W, D_H\}$ satisfies the assumptions in Proposition 2.29, $\{D_H\}$ is strongly acyclic. $\square$

3 An index theory for complete Riemannian manifolds

3.1 Formulation of index on complete manifolds

Suppose M is a complete Riemannian manifold, W is a $\mathbb{Z}/2$-graded Hermitian vector bundle, and $\sigma : TM \to \mathrm{End}(W)$ is a homomorphism such that $\sigma(v)$ is a skew-Hermitian isomorphism of degree-one for each $v \in TM \setminus \{0\}$. Let D be a degree-one formally self-adjoint first-order elliptic differential operator on W with principal symbol σ. We assume that σ and the coefficients of D are smooth. We formulate an index theory on M under the following assumption.

Assumption 3.1

- D has finite propagation speed: there exists a positive real number C_0 satisfying $|\sigma| \leq C_0$ uniformly on M,
- There exist a positive real number $\lambda_0 > 0$ and an open subset V of M with its complement $M \setminus V$ compact such that

$$\lambda_0 ||s||_V^2 \leq ||Ds||_V^2$$

  for any smooth compactly-supported section s of W with support contained in V.

It is known that the finite propagation speed implies that D is essentially self-adjoint [2]. We will give a direct proof of the following theorem.

Theorem 3.2 *D is essentially self-adjoint as an operator on L^2-sections of W and its spectrum is discrete in $(-\sqrt{\lambda_0}, \sqrt{\lambda_0})$.* □

The proof of the first part is given as Lemma 3.11 in Section 3.2. The rest of the statement follows from Proposition 3.14 in Section 3.3.

Definition 3.3 E_λ is the vector space of smooth sections s of W such that s is L^2-bounded and satisfies $D^2 s = \lambda s$.

Theorem 3.2 implies that E_λ is zero for $\lambda < 0$, and E_λ is finite dimensional for $\lambda < \lambda_0$. Moreover there are only discrete values $\lambda < \lambda_0$ for which E_λ is non-zero. Note that the super dimension of E_λ is zero for $0 < \lambda < \lambda_0$, and hence the super dimension of $\oplus_{\lambda<\lambda_1} E_\lambda$ is constant for $0 < \lambda_1 < \lambda_0$.

Definition 3.4 $\operatorname{ind} D$ is the super dimension of E_0, or the super dimension of $\oplus_{\lambda<\lambda_1} E_\lambda$ for $0 < \lambda_1 < \lambda_0$.

The index has the following deformation invariance. Let $\{D_t\}$ $(|t| < \epsilon)$ be a one-parameter family of degree-one formally self-adjoint first-order elliptic differential operators on W with principal symbols $\{\sigma_t\}$.

Assumption 3.5

- Each D_t and σ_t satisfy Assumption 3.1 for common λ_0 and V.

- On each compact subset of M the coefficients of D_t are C^∞ convergent to those of D_0 as $t \to 0$.

We do not assume that the propagation speed is uniform with respect to t.

We will show the following theorem in Section 3.4.

Theorem 3.6 *Under Assumption 3.5* $\operatorname{ind} D_t$ *is constant with respect to* t.

Remark 3.7 So far we are fixing M and W. In Section 3.5 we will formulate a deformation for which M and W can vary. We give a proof of Theorem 3.6 so that it can be directly generalized to this case. The generalization immediately implies an excision property of index for complete Riemannian manifolds.

3.2 Partial integration

We need two partial integration formulas. In general let W be a Hermitian vector bundle over a complete Riemannian manifold M, and $D_\tau : \Gamma(W) \to \Gamma(W)$ be a first order partial differential operator on W with smooth coefficients whose principal symbol is τ. We assume that D_τ has finite propagation speed, i.e., τ is a smooth L^∞-bounded section of $TM \otimes \operatorname{End}(W)$.

Lemma 3.8 *Let* $s \in \Gamma(W)$ *is an* L^2*-bounded section such that* $D_\tau^* D_\tau s$ *is also* L^2*-bounded. Then* $D_\tau s$ *is also* L^2*-bounded and we have*

$$\int_M (D_\tau^* D_\tau s, s) = \int_M |D_\tau s|^2.$$

Lemma 3.9 *Suppose* s_0 *and* s_1 *are* L^2*-bounded sections of* W *such that* $D_\tau s_0$ *and* $D_\tau^* s_1$ *are also* L^2*-bounded. Then we have*

$$\int_M (D_\tau s_0, s_1) = \int_M (s_0, D_\tau^* s_1).$$

We follow the argument in [7] using a family of cut-off functions:

Lemma 3.10 *Let* M *be a complete Riemmanian manifold.*

1. *There is a smooth proper function* $f : M \to \mathbb{R}$ *such that* $|df|$ *is bounded and* $f^{-1}((-\infty, c])$ *is compact for any* c.
2. *There is a constant* $C > 0$ *such that for each* $\epsilon > 0$ *and* $a \in \mathbb{R}$, *we have a compact supported function* $\rho_{a,\epsilon} : M \to [0, 1]$ *which is equal to* 1 *on* $f^{-1}((-\infty, a])$, *and satisfies* $|d\rho_{a,\epsilon}| < C\epsilon$.

A proof of the above lemma is given in [6]. For completeness we give a detailed construction in Appendix C. The existence of such a function in 1 of Lemma 3.10 is equivalent to the completeness of M. For mored details see [6].

If we choose a Hermitian connection ∇ on W, we can describe $D_\tau s = \tau\cdot\nabla s + Bs$, where B is a smooth section of $\mathrm{End}(W)$ and $\cdot$ is the combination of the product $\mathrm{End}(W)\otimes W \to W$ and the contraction $T^*M\otimes TM \to \underline{\mathbb{R}}$. We do not assume that B is bounded.

Proof of Lemma 3.8. We first assume that s is smooth. We follow Gromov's proof of [7, Lemma 1.1 B]. From the equality

$$\begin{aligned}\int_M (D_\tau^* D_\tau s,\, \rho_{a,\epsilon}^2 s) &= \int_M (D_\tau s,\, D_\tau(\rho_{a,\epsilon}^2 s)) \\ &= \int_M (D_\tau s,\, \rho_{a,\epsilon}^2 D_\tau s) + \int_M (D_\tau s,\, 2\rho_{a,\epsilon}\tau(d\rho_{a,\epsilon})s), \qquad (7)\end{aligned}$$

there is a constant C independent of s, a, ϵ such that

$$||D_\tau^* D_\tau s||_2 ||s||_2 \geq ||\rho_{a,\epsilon} D_\tau s||_2^2 - C\epsilon ||\rho_{a,\epsilon} D_\tau s||_2.$$

It implies that, as a increases, $||\rho_{a,\epsilon} D_\tau s||_2$ is bounded, i.e., $D_\tau s$ is L^2-bounded. Using (7) again we have

$$\int_M (D_\tau^* D_\tau s, s) = ||D_\tau s||_2^2 + I, \qquad |I| \leq C\epsilon ||D_\tau s||_2 ||s||_2.$$

Taking $\epsilon \to 0$, we obtain the required equality.

When s is not smooth, take a smooth compactly supported section which approximate s in L_2^2-norm on the support of $\rho_{a,\epsilon}$. Then we can reduce the argument to the smooth case.

□

Proof of Lemma 3.9. We first assume that s is smooth. We have

$$0 = \int_M (D_\tau(\rho_{a,\epsilon} s_0), s_1) - \int_M (s_0, D_\tau^*(\rho_{a,\epsilon} s_1)) = \int_M (D_\tau s_0, s_1) - \int_M (s_0, D_\tau^* s_1) + I'$$

with an error term I' satisfying $|I'| \leq C\epsilon ||s_0||_2 ||s_1||_2$, which implies the required equality. When s is not smooth, we can reduce the argument to the smooth case as in the proof of Lemma 3.8. □

Using the cut off function and a standard argument we can also show:

Lemma 3.11 *Under Assumption 3.1 D is essentially self-adjoint.*

Proof of Lemma 3.11. Suppose L^2-sections u and v satisfies $Du = v$ weakly. We show that for any $\epsilon > 0$ there is a compactly supported smooth section u_ϵ satisfying $||u_\epsilon - u||_M \le 2\epsilon$ and $||Du_\epsilon - v||_M \le 4\epsilon$. Take a compact subset K such that $||u||_{M \smallsetminus K}, ||v||_{M \smallsetminus K} < \epsilon$. Using Lemma 3.10 choose a smooth compactly-supported function $\rho : M \to [0,1]$ satisfying $\rho = 1$ on K and $|d\rho| < 1$. Let K' be the compact support of ρ. Since D is elliptic, the weak equality $Du = v$ and the regularity theorem imply that u is locally L^2_1-bounded and there is a smooth sections u' satisfying $||u' - u||_{K'} < \epsilon$ and $||Du' - v||_{K'} < \epsilon$. Then we have

$$||\rho u' - u||_M \le ||\rho(u' - u)||_{K'} + ||(1-\rho)u||_{M \smallsetminus K} \le 2\epsilon$$

and

$$\begin{aligned} ||D(\rho u') - v||_M &= ||d\rho \cdot (u' - u) + d\rho \cdot u + \rho(Du - v) - (1-\rho)v||_M \\ &\le ||u' - u||_{K' \smallsetminus K} + ||u||_{K' \smallsetminus K} + ||Du' - v||_{K'} + ||v||_{M \smallsetminus K} \\ &\le 4\epsilon. \end{aligned}$$

It implies that D is essentially self-adjoint. □

3.3 Min-max principle

In this section we use Assumption 3.1 for a single operator D, and Assumption 3.5 for a one-parameter family $\{D_t\}$.

Lemma 3.12 *For any compact subset K containing $M \smallsetminus V$ there is a compact set K' containing K such that if s and Ds are L^2-bounded, then we have the estimate*

$$\lambda_0^{1/2}||s||_{M \smallsetminus K} - 2\lambda_0^{1/2}||s||_{K' \smallsetminus K} \le ||Ds||_{M \smallsetminus K}.$$

Moreover if the coefficients of D_t are C^∞-convergent to those of $D_0 = D$ on any compact set as $t \to 0$, then we can choose K' so that the above estimate is valid for any t sufficiently close to 0.

Proof. Lemma 3.11 implies that we can assume that s is smooth and compactly supported without loss of generality. From Lemma 3.10, for any $\epsilon > 0$, there is a compact set K' containing K and a smooth non-negative function $\rho : M \to \mathbb{R}$ such that $\rho = 1$ on $M \smallsetminus K'$, $\rho = 0$ on K and $|d\rho| \le \epsilon$. Then the above estimate follows from the next two inequalities

$$\begin{aligned} ||D(\rho s)||_M &\ge \lambda_0^{1/2}||\rho s||_V \ge \lambda_0^{1/2}||s||_{M \smallsetminus K'} \ge \lambda_0^{1/2}||s||_{M \smallsetminus K} - \lambda_0^{1/2}||s||_{K' \smallsetminus K}, \\ ||D(\rho s)||_M &\le ||\rho D s||_M + ||\sigma \cdot ((d\rho) \otimes s)||_M \le ||Ds||_{M \smallsetminus K} + C(D, K')\epsilon||s||_{K' \smallsetminus K}, \end{aligned}$$

where $C(D, K') := \max_{K'}|\sigma|$. The last statement of the lemma follows from the fact that $C(D_t, K')$ is continuous with respect to t. □

Proposition 3.13 *Suppose* $0 \leq \lambda_1 < \lambda_0$. *Let* $\{s_i\}$ *be a sequence of* L^2-*sections of* W *satisfying* $||s_i||_M = 1$, *and* $\{t_i\}$ *is a sequence convergent to* 0. *Suppose each* $D_{t_i}s_i$ *is* L^2-*bounded and satisfies* $||D_{t_i}s_i||_M^2 \leq \lambda_1$. *Then there is a subsequence* $\{s_{i'}\}$ *which is weakly convergent to some non-zero* $s_\infty \neq 0$ *such that* $D_0 s_\infty$ *is* L^2-*bounded and satisfies*

$$||D_0 s_\infty||_M^2 \leq \lambda_1 ||s_\infty||_M^2 \tag{8}$$

Proof. Take a subsequence $\{s_{i'}\}$ so that $\{s_{i'}\}$ and $\{D_{t_{i'}}s_{i'}\}$ are weakly convergent to some s_∞ and u_∞ in $L^2(M, W)$ respectively. Since D_t is a smooth family, for each smooth compactly-supported section ϕ the sequence $D_{t_{i'}}\phi$ is strongly convergent to $D_0\phi$. The equality $\int_M (D_{t_{i'}}\phi, s_{i'}) = \int_M (\phi, D_{t_{i'}}s_{i'})$ implies $\int_M (D_0\phi, s_\infty) = \int_M (\phi, u_\infty)$, i.e., $D_0 s_\infty = u_\infty$ weakly.

Since $\{D_{t_{i'}}s_{i'}\}$ is L^2-bounded, Assumption 3.5 and a priori estimate imply that on any compact set $s_{i'}$ is strongly L^2-convergent to s_∞.

On the other hand for any compact set K containing $M \setminus V$ there exists a compact set K' such that

$$\lambda_0^{1/2} ||s_{i'}||_{M \setminus K} - 2\lambda_0^{1/2} ||s_{i'}||_{K' \setminus K} \leq ||D_{t_{i'}} s_{i'}||_{M \setminus K} \leq \lambda_1^{1/2}$$

by Lemma 3.12. If s_∞ is 0, then we have $||s_{i'}||_{K' \setminus K}$ converges to 0, which contradicts to $||s_{i'}||_M = 1$ and $\lambda_1 < \lambda_0$.

Suppose the estimate (8) does not hold. Then for any $\epsilon > 0$ and any sufficiently small $\epsilon' > 0$ there exists a compact set K containing $M \setminus V$ satisfying $||s_\infty||_{M \setminus K} < \epsilon$ and $\lambda_1 ||s_\infty||_K^2 + \epsilon' < ||D_0 s_\infty||_K^2$. We choose ϵ and ϵ' so that they satisfy $8\epsilon\lambda_0(1 + 2\epsilon) < \epsilon'/2$. Note that the weak convergence implies $||D_0 s_\infty||_K^2 \leq \liminf_{i' \to \infty} ||D_{t_{i'}} s_{i'}||_K^2$. Since $s_{i'}$ is strongly L^2-convergent to s_∞ on the compact set K, we have

$$\lambda_1 ||s_{i'}||_K^2 + \frac{\epsilon'}{2} < ||D_{i'} s_{i'}||_K^2 \tag{9}$$

for sufficiently large i'. Let K' be the compact set containing K which gives the estimate in Lemma 3.12 for sufficiently small t. Since , $s_{i'}$ is strongly L^2-convergent to s_∞ on the pre-compact set $K' \setminus K$, we have $||s_{i'}||_{K' \setminus K} < 2\epsilon$ for sufficiently large i'. The estimate in Lemma 3.12 implies that we have $\lambda_0^{1/2} ||s_{i'}||_{M \setminus K} \leq ||Ds_{i'}||_{M \setminus K} + 4\epsilon\lambda_0^{1/2}$ for sufficiently large i'. Taking square, and using $\lambda_1 < \lambda_0$ and $||D_{i'} s_{i'}||_{M \setminus K} \leq \lambda_1^{1/2}$, we obtain

$$\lambda_1 ||s_{i'}||_{M \setminus K}^2 \leq ||D_{i'} s_{i'}||_{M \setminus K}^2 + 8\epsilon\lambda_0(1 + 2\epsilon)$$

Adding with (9) we have $\lambda_1 ||s_{i'}||_M^2 < ||D_{i'} s_{i'}||_M^2$ for sufficiently large i', which contradicts our assumption. □

For a single operator D we have

Proposition 3.14 *1. Suppose $\lambda < \lambda' < \lambda_0$.*

(a) If $s \in E_\lambda$, then Ds is L^2-bounded and $||Ds||_M^2 = \lambda ||s||_M^2$.

(b) E_λ and $E_{\lambda'}$ are L^2-orthogonal to each other.

2. Suppose $0 \leq \lambda_1 < \lambda_0$.

(a) $\dim \oplus_{\lambda \leq \lambda_1} E_\lambda(D) < \infty$.

(b) Let R_{λ_1} be the set of L^2-bounded sections s satisfying $||s||_M = 1$ and $||Ds||_M^2 < \lambda_0$ such that s is L^2-orthogonal to $\oplus_{\lambda \leq \lambda_1} E_\lambda(D)$. If R_{λ_1} is not empty, then the functional $I_{\lambda_1} : R_{\lambda_1} \to [0, \lambda_0)$, $I_{\lambda_1}(s) = ||Ds||_M^2$ attains its minimum value.

(c) Let $\lambda_2 = ||Ds_0||_M^2$ be the minimum value of I_{λ_1} at a minimum s_0. Then we have $\lambda_1 < \lambda_2 < \lambda_0$ and $s_0 \in E_{\lambda_2}$

Proof. The first statement for $\lambda < \lambda' < \lambda_0$ follows from the partial integration formulas Lemma 3.8 and Lemma 3.9.

Suppose $\lambda_1 < \lambda_0$. If $\oplus_{\lambda \leq \lambda_1} E_\lambda(D)$ is not finite dimensional, then we have a sequence e_i in the infinite space with $||e_i||_M = 1$ and mutually L^2-orthogonal each other. Proposition 3.13 implies that we have a weakly convergent limit for a subsequence with non zero limit, which is a contradiction.

Suppose s_i is a sequence in R_{λ_1} such that $I_{\lambda_1}(s_i)$ convergent to the infimum of I_{λ_1}. Proposition 3.13 implies that we have a weakly convergent limit $s_\infty \neq 0$ for a subsequence such that $s_0 := s_\infty / ||s_\infty||$ is an element of R_λ which attains the infimum. For any compactly-supported smooth section s', let s'' be the L^2-orthogonal projection of s' to $\oplus_{\lambda \leq \lambda_1} E_\lambda(D)$ and put $s''' = s' - s''$. Since s_0 attains the minimum of I_{λ_1}, the derivative of $(s_0 + ts''')/||s_0 + ts'''||_M$ at 0 vanishes, and we obtain

$$\int_M (Ds_0, Ds''') = \lambda_2 \int_M (s_0, s''').$$

Since $Ds''' = Ds' - Ds''$ and $D^2 s''' = D^2 s' - D^2 s''$ is L^2-bounded, Lemma 3.9 implies

$$\int_M (s_0, D^2 s''') = \lambda_2 \int_M (s_0, s''').$$

On the other hand we have

$$\int_M (s_0, D^2 s'') = \int_M (s_0, s'') = 0.$$

These relations imply

$$\int_M (s_0, D^2 s') = \lambda_2 \int_M (s_0, s'),$$

i.e., $D^2 s_0 = \lambda_2 s_0$ weakly. Lemma 3.8 implies that Ds_0 is L^2-bounded and $||Ds_0||_M^2 = \lambda_2 ||s_0||_M = \lambda_2$. Since s_0 is L^2-orthogonal to $\oplus_{\lambda \le \lambda_1} E_\lambda$, we have $\lambda_1 < \lambda_2$. The regularity theorem implies s_0 is smooth and hence $s_0 \in E_{\lambda_2}$. □

Corollary 3.15 *Suppose $\lambda_1 < \lambda_0$. Let E be a $\mathbb{Z}/2$ graded subspace of $L^2(M, W)$ such that such that Ds is in $L^2(M, W)$ and $||Ds||_M^2 \le \lambda_1 ||s||_M^2$ for any $s \in E$. Then E is finite dimensional and*

$$\dim \oplus_{\lambda \le \lambda_1} E_\lambda(D) \ge \dim E.$$

Moreover the above inequality holds for each degree of $\mathbb{Z}/2$.

3.4 Deformation invariance of index

For a family $\{D_t\}$ we have:

Proposition 3.16 *Suppose $\lambda_1 < \lambda_0$. Let $\{t_i\}$ be a sequence convergent to 0 and $E^{(i)}$ be a $\mathbb{Z}/2$ graded subspace of $L^2(M, W)$ such that $D_{t_i} s$ is in $L^2(M, W)$ and $||D_{t_i} s||_M^2 \le \lambda_1 ||s||_M^2$ for any i and $s \in E_i$. Then each $E^{(i)}$ is finite dimensional and*

$$\dim \oplus_{\lambda \le \lambda_1} E_\lambda(D_0) \ge \limsup_{i \to \infty} \dim E^{(i)}.$$

Moreover the above inequality holds for each degree of $\mathbb{Z}/2$.

Proof. Suppose $\dim \oplus_{\lambda \le \lambda_1} E_\lambda(D_0) < \dim E^{(i')}$ for a subsequence $\{i'\}$. Let $s_{i'}$ be an element of $E^{(i')}$ with $||s_{i'}||_M = 1$ which is L^2-orthogonal to $\oplus_{\lambda \le \lambda_1} E_\lambda(D_0)$. Let s_∞ be the L^2-bounded section given by Proposition 3.13. Then the weak limit s_∞ is also L^2 orthogonal to $\oplus_{\lambda \le \lambda_1} E_\lambda(D_0)$, which contradicts Proposition 3.14. □

Remark 3.17 In the above proof the choice of $s_{i'}$ can be generalized as follows: Fix an L^2-orthonormal basis $e_1, e_2, \ldots, e_N$ of the finite dimensional space $\oplus_{\lambda \le \lambda_1} E_\lambda(D_0)$. Fix any sequence $\{e_k^{(i')}\}_{i'}$ for each $1 \le k \le N$ which is strongly L^2-convergent to e_k as $i' \to \infty$. Let $s_{i'}$ be an element of $E^{(i')}$ with $||s_{i'}||_M = 1$ which is L^2-orthogonal to all $e_k^{(i')}$ $(1 \le k \le N)$. Then the rest of the proof remains valid.

Corollary 3.18 *For each degree of $\mathbb{Z}/2$ we have the inequality*

$$\dim \oplus_{\lambda \le \lambda_1} E_\lambda(D_0) \ge \limsup_{t\to 0} \dim \oplus_{\lambda \le \lambda_1} E_\lambda(D_t)$$

Proposition 3.19 *Suppose $\lambda_1 < \lambda_0$. For each degree of $\mathbb{Z}/2$ we have the inequality*

$$\dim \oplus_{\lambda < \lambda_1} E_\lambda(D_0) \le \liminf_{t\to 0} \dim \oplus_{\lambda < \lambda_1} E_\lambda(D_t)$$

Proof. Let $\epsilon_0 > 0$ be a sufficiently small number, which we fix later. Let $e_1, e_2, \dots, e_N$ be an L^2-orthonormal basis of the finite dimensional space $E := \oplus_{\lambda<\lambda_1} E_\lambda(D_0)$ consisting of eigenvectors of D_0 with eigenvalues $\mu_1, \mu_2, \dots, \mu_N$ respectively. For a sufficiently large a and sufficiently small $\epsilon > 0$, the truncated sections $e_i' = \rho_{a,\epsilon} e_i$ satisfy

$$\left|\delta_{ij} - \int_M (e_i', e_j') -\right| < \epsilon_0, \quad \left|\mu_i\mu_j\delta_{ij} - \int_M (D_0 e_i', D_0 e_j')\right| < \epsilon_0$$

for every $1 \le i, j \le N$ as in the proof of Lemma 3.8 or Lemma 3.9. Here δ_{ij} is Kronecker's delta. Let E' be the vector space spanned by e_i'. Since the support of all the e_i' are contained in the compact support of $\rho_{a,\epsilon}$, Assumption 3.5 implies that

$$\left|\mu_i\mu_j\delta_{ij} - \int_M (D_t e_i', D_t e_j')\right| < \epsilon_0.$$

for every t sufficiently closed to 0. Let E' be the vector space spanned by e_i'. Then if ϵ_0 is sufficiently small, each element s' of E' satisfies $||D_t s'||_M^2 \le \lambda_1 ||s'||_M^2$. Corollary 3.15 implies $\dim E' \le \dim \oplus_{\lambda<\lambda_1} E_\lambda(D_t)$. It is easy to check that the above inequality holds for each degree of $\mathbb{Z}/2$. □

Proof of Theorem 3.6. From Proposition 3.14 there is $0 < \lambda_1 < \lambda_0$ such that $E_{\lambda_1} = 0$. Then Corollary 3.18 and Proposition 3.19 imply

$$\dim \oplus_{\lambda \le \lambda_1} E_\lambda(D_0) = \lim_{t\to 0} \dim \oplus_{\lambda \le \lambda_1} E_\lambda(D_t)$$

and the equality holds for each degree, from which the claim follows. □

3.5 Gluing formula

In this subsection we generalize Theorem 3.6. We first need to generalize Assumption 3.5.

Let (M, W, σ, D, V) be as in Section 3.1. For $i = 1, 2, \dots$ we further suppose that the data $(M_i, W_i, \sigma_i, D_i, V_i)$ satisfies the setting in Section 3.1.

Suppose M has a sequence of compact open subsets $\{K_i\}$ $(i = 1, 2, \ldots)$ satisfying $K_1 \cup V = M$, $K_i \subset int(K_{i+1})$ and $M = \cup_i K_i$. Suppose there exists an isometric open embedding $\iota_i : int(K_i) \to M_i$ and an isomorphism $\widetilde{\iota}_i : W|_{int(K_i)} \cong \iota_i^* W_i$ as $\mathbb{Z}/2$-graded Hermitian vector bundle over $int(K_i)$ for each i.

If s is a compactly supported section of W, then $\widetilde{\iota}_i s|_{int(K_i)}$ makes sense as a compactly supported section of W_i for large i with $supp\, s \subset int(K_i)$ by extending as 0 outside $\iota_i(supp\, s)$. We simply write $\widetilde{\iota}_i s$ for this section on W_i for large i.

If K is a compact subset of M and s_i is a section of W_i for large i with $K \subset int(K_i)$, then $\widetilde{\iota}_i^{-1}(s_i|_{\iota(K_i)})|_K$ is a section of $W|_K$. We simply write $\widetilde{\iota}_i^{-1} s_i|_K$ for this section.

Then $s \mapsto \{\widetilde{\iota}_i^{-1}(D_i \widetilde{\iota}_i s)|_K\}$ is a differential operator on K for large i. We write $\iota_i^* D_i$ for this operator on K.

Assumption 3.20

1. $\iota_i(M \setminus V) = M_i \setminus V_i$.
2. The data $(M_i, W_i, \sigma_i, D_i, V_i)$ satisfy Assumption 3.1 with the same constant λ_0 for (M, W, σ, D, V).
3. On each compact subset K of M the coefficients of the differential operator $\iota_i^* D_i$ is C^∞-convergent to those of D on K as $i \to \infty$.

We do not assume that the propagation speed of $\{D_i\}$ is uniformly bounded with respect to i.

Theorem 3.21 *Under Assumption 3.20,* ind D_i *is equal to* ind D *for large* i.

Proof. The most of the arguments in Sections 3.3 and 3.4 go through.

The statement and proof of Proposition 3.13 is straightforwardly generalized with replacement of D_{t_i} by D_i.

The statement and proof of Proposition 3.19 is also straightforwardly generalized.

To generalize Proposition 3.16 we need the construction in Remark 3.17. As for the statement we replace D_{t_i} with D_i, and let $E^{(i)}$ be a $\mathbb{Z}/2$ graded subspace of $L^2(M_i, W_i)$. As for the proof take $e_k^{(i')}$ with support contained in $int(K_i)$. Let $s_{i'}$ be an element of $E^{(i')}$ with $||s_{i'}||_{M_i} = 1$ which is L^2-orthogonal to all $\widetilde{\iota}_i e_k^{(i')}$ $(1 \le k \le N)$. Then the rest of the proof remain valid.

Then the argument of Section 3.4 can be straightforwardly generalized to show Theorem 3.21. □

When M has two (or more) connected components while each M_i' is connected, Theorem 3.21 is regarded as a gluing formula of index as explained below.

Proposition 3.22 *Let (M, W, σ, D, V) be a data satisfying Assumption 3.1. Suppose M is the disjoint union of M' and M''. Let (W', σ', D', V') and $(W'', \sigma'', D'', V'')$ be the restrictions of (W, σ, D, V) to M' and M'' respectively. Then we have*

$$\text{ind } D_i = \text{ind } D' + \text{ind } D'',$$

for sufficiently large i.

Proof. Since ind D = ind D' + ind D'', Theorem 3.21 implies the required gluing formula. □

The following vanishing lemma follows from the partial integration formula Lemma 3.8 and the second inequality in Assumption 3.1.

Lemma 3.23 *Let (M, W, σ, D, V) be a data satisfying Assumption 3.1. If $M = V$, then we have* $\ker D \cap L^2(W) = 0$.

Using the gluing formula Proposition 3.22 and the above Lemma 3.23, we have the following excision formula of index.

Proposition 3.24 *Let (M, W, σ, D, V) be a data satisfying Assumption 3.1. Suppose M is the disjoint union of M' and M'', and M'' is contained in V. Let (W', σ', D', V') be the restriction of (W, σ, D, V) to M'. Then we have*

$$\text{ind } D_i = \text{ind } D'$$

for sufficiently large i.

3.6 Product formula

Following Atiyah and Singer [1], we formulate a product formula for elliptic operators. Except that we need Lemma 3.8 for partial integration on complete Riemannian manifolds, the argument is exactly the same as in [1]. The main purpose of this subsection is to formulate Assumption 3.26 below, which is crucial for the case of complete manifolds. To apply the product formula it is necessary to check the assumption for specific operators, which is our another task and is carried out in Section 5.

For $k = 0, 1$ let M_k be a complete Riemannian manifold, W_k a $\mathbb{Z}/2$-graded Hermitian vector bundle over M_k, and $D_k : \Gamma(W_k) \to \Gamma(W_k)$ a degree-one formally self-adjoint order-one elliptic operator with principal symbol σ_k.

Let G be a compact Lie group and $P \to M_0$ a principal G-bundle. Suppose G acts on M_1 isometrically, W_1 is G-equivariant $\mathbb{Z}/2$-graded Hermitian vector bundle, and D_k is a G-invariant operator.

Then $M = P \times_G M_1$ is a fiber bundle over M_0 with fiber M_1. We write $\pi : M \to M_0$ for the projection map. Let $\widetilde{W_0}$ and $\widetilde{W_1}$ be the vector bundles over M defined by $\widetilde{W_0} = \pi^* W_0$ and $\widetilde{W_1} = P \times_G W_1$, and we put $W = \widetilde{W_0} \times \widetilde{W_1}$.

We would like to lift D_0 and D_1 as operators on W. The lift of D_1 is given straightforward: Defining the operator $\widetilde{D}_1$ on $\Gamma(\widetilde{W_0} \otimes \widetilde{W_1})$ by $\epsilon \otimes D_1$ on each fiber of $\pi : M \to M_0$, where $\epsilon : W_0 \to W_0$ is equal to $+id$ on the degree 0 part of W_0, and to $-$ id on the degree 1 part of W_1.

We next construct $\widetilde{D}_0 : \Gamma(W) \to \Gamma(W)$. Let $\{V_\alpha\}$ be an open covering of M_0 and $\{\rho_\alpha^2\}$ a partition of unity. Suppose we have local trivializations $P|_{V_\alpha} \cong V_\alpha \times G$ with transition functions $g_{\alpha\beta}$. Using the local trivialization on V_α we have the identifications $\pi^{-1}(V_\alpha) \cong V_\alpha \times M_1$ and $W|_{\pi^{-1}(V_\alpha)} \cong W_0|_{V_\alpha} \times W_1$. Let $\widetilde{D}_{0,\alpha}$ be the operator on $W|_{\pi^{-1}(V_\alpha)}$ defined by D_0 using the product structure. We put $\widetilde{D}_0 := \sum_\alpha \rho_\alpha \widetilde{D}_{0,\alpha} \rho_\alpha$.

Lemma 3.25 $\widetilde{D}_0 \widetilde{D}_1 + \widetilde{D}_1 \widetilde{D}_0 = 0$.

Proof. It follows from $\widetilde{D}_1 \widetilde{D}_{0,\alpha} + \widetilde{D}_{0,\alpha} \widetilde{D}_1 = 0$ and $\widetilde{D}_1 \rho_\alpha - \rho_\alpha \widetilde{D}_1 = 0$. □

Let R be a G-invariant $\mathbb{Z}/2$-graded finite dimensional subspace of $\Gamma(M_1, W_1)$, and $\widetilde{R}$ the fiber bundle $P \times_G R$ over M_0. Then we have an embedding

$$\Gamma(M_0, W_0 \otimes \widetilde{R}) \to \Gamma(M, \widetilde{W}) \tag{10}$$

which is preserved by the action of $\widetilde{D}_0$. Let $\widetilde{D}_R$ be the restriction of $\widetilde{D}_0$ on $\Gamma(M_0, W_0 \otimes \widetilde{R})$. Then $\widetilde{D}_R$ is a differential operator on $W_0 \otimes \widetilde{R}$ with principal symbol $\sigma_0 \otimes \mathrm{id}_{\widetilde{R}}$.

Assumption 3.26

1. D_0 has finite propagation speed, i.e., σ_0 is L^∞-bounded.
2. The data (M_1, W_1, D_1) satisfies Assumption 3.1.
3. $R = E_0(D_1)$.
4. The data $(M, W, \widetilde{D}_0 + \widetilde{D}_1)$ satisfies Assumption 3.1.

We do not assume the second condition of Assumption 3.1 for the data (M_0, W_0, D_0).

Recall that $\widetilde{D}_R$ is given by $\widetilde{D}_0$ via the embedding (10). Since $\widetilde{D}_1 = 0$ on the image of the embedding, Assumption 3.26 implies that the data $(M_0, W_0 \otimes \widetilde{R}, \widetilde{D}_R$ satisfies Assumption 3.1 as well.

Theorem 3.27 (product formula) $\mathrm{ind}(\widetilde{D}_0 + \widetilde{D}_1) = \mathrm{ind}\, \widetilde{D}_R$

Proof. We show that the embedding (10) gives the isomorphism $E_0(\widetilde{D}_R) \cong E_0(\widetilde{D}_0 + \widetilde{D}_1)$. If s is in the image of $E_0(D_R)$, then the construction of D_R implies that s is obviously in $E_0(\widetilde{D}_0 + \widetilde{D}_1)$. From Lemma 3.8and Lemma 3.25 if s is an element of $E_0(\widetilde{D}_0 + \widetilde{D}_1)$ we have

$$0 = \int_M ((\widetilde{D}_0 + \widetilde{D}_1)^2 s, s) = \int_M (\widetilde{D}_0^2 s + \widetilde{D}_1^2 s, s) = ||\widetilde{D}_0 s||^2 + ||\widetilde{D}_1 s||^2,$$

i.e., $\widetilde{D}_0 s = \widetilde{D}_1 s = 0$. In particular $\widetilde{D}_1 s = 0$ implies that s is in the image of the embedding (10). Moreover $\widetilde{D}_0 s = 0$ implies s is in the image of $E_0(D_R)$. □

4 Local index

In this section we first define a class of Riemannian manifolds and compatible fibrations (resp. compatible systems) on them. Using such a class we will define the local index of a strongly acyclic compatible system in this section and prove the product formula in the next section.

Definition 4.1 Let M be a Riemannian manifold. If there exists an open subset V of M which satisfies the following properties, then we call M a *manifold with the Euclidian end* V.

1. The complement $M \smallsetminus V$ is compact.
2. V contains an open subset V' with the pre-compact complement $V \smallsetminus V'$.
3. There exist a closed Riemannian manifold N and positive integers $\{m_i\}_{i=1}^k$ such that V' is isometric to the product Riemannian manifold $N \times \prod_i \mathbb{R}_0^{m_i}$, where $\mathbb{R}_0^{m_i}$ denotes the complement of a compact subset of $\mathbb{R}^{m_i}$ for $m_i > 1$ and $\mathbb{R}_+$ for $m_i = 1$.

A typical example of a manifold with Euclidian end is a manifold with cylindrical end $V' = N \times \mathbb{R}_+$. Products of such manifolds twisted by principal bundles are another examples, which we need to formulate the product formula in the next section.

Definition 4.2 Let M be a manifold with the Euclidian end V and $\{\pi_\alpha : V_\alpha \to U_\alpha\}_{\alpha\in A}$ (resp. $\{W, D_\alpha\}$) a compatible fibration (resp. compatible system) on V. If there exists a subset $\hat{A}$ of A which satisfies the following conditions, then we call $\{\pi_\alpha : V_\alpha \to U_\alpha\}_{\alpha\in A}$ (resp. $\{W, D_\alpha\}$) a *translationally invariant compatible fibration (resp. compatible system)* on V.

1. $V'(\cong N \times \prod_i \mathbb{R}_0^{m_i})$ is an admissible open subset of V.
2. There exists an open covering $N = \cup_{\alpha\in\hat{A}} N_\alpha$ such that $V_\alpha \cap V'$ is isometric to $N_\alpha \times \prod_i \mathbb{R}_0^{m_i}$ for all $\alpha \in \hat{A}$.
3. If $\alpha \in A \setminus \hat{A}$, then V_α is pre-compact.
4. $\bigcup_{\alpha\in A\setminus\hat{A}} V_\alpha$ is an open covering of $V \setminus V'$.
5. There exists a family of compatible fibrations (resp. compatible systems) $\{\hat{\pi}_\alpha : N_\alpha \to \hat{U}_\alpha\}_{\alpha\in\hat{A}}$ (resp. $\{\hat{W}, \hat{D}_\alpha\}$) on N such that $\{\pi_\alpha|_{V_\alpha\cap V'} : V_\alpha \cap V' \to \pi_\alpha(V_\alpha \cap V')\}_{\alpha\in\hat{A}}$ (resp. $\{W|_{V_\alpha\cap V'}, D_\alpha|_{V_\alpha\cap V'}\}$) is the product of $\{\hat{\pi}_\alpha : N_\alpha \to \hat{U}_\alpha\}_{\alpha\in\hat{A}}$ (resp. $\{\hat{W}, \hat{D}_\alpha\}$) and the trivial one over $\prod_i \mathbb{R}_0^{m_i}$.

Let M be a manifold with Euclidian end V. Suppose that there is a translationally invariant compatible fibration $\{\pi_\alpha : V_\alpha \to U_\alpha\}_{\alpha\in A}$ on V. Take and fix a pre-compact open neighborhood V_∞ of $M \setminus V$. Put $\widetilde{A} := A \cup \{\infty\}$. For later convenience we think $M = \cup_{\alpha\in\widetilde{A}} V_\alpha$ is equipped with a compatible fibration such that $\pi_\infty = \mathrm{id} : V_\infty \to V_\infty$. Let $\{\rho_\alpha^2\}_{\alpha\in\widetilde{A}}$ be an admissible partition of unity of $M = \cup_{\alpha\in\widetilde{A}} V_\alpha$ constructed in Lemma 2.10. By retaking V' we may assume that $V_\infty \cap V' = \emptyset$ and $\{\rho_\alpha^2\}_{\alpha\in\hat{A}}$ is translationally invariant on $V'(= N \times \prod_i \mathbb{R}_0^{m_i})$. Namely there exists an admissible partition of unity $\{\hat{\rho}_\alpha^2\}_{\alpha\in\hat{A}}$ of $N = \cup_{\alpha\in\hat{A}} N_\alpha$ such that $\rho_\alpha|_{N_\alpha\times\prod_i \mathbb{R}_0^{m_i}}$ is equal to the pull back of $\hat{\rho}_\alpha$ via the projection $N_\alpha \times \prod_i \times\mathbb{R}_0^{m_i} \to \prod_i \mathbb{R}_0^{m_i}$. We first show the following technical lemma which is used to show Theorem 4.7.

Lemma 4.3 *There exists an admissible partition of unity $\{\chi_\alpha^2\}_{\alpha\in\widetilde{A}}$ of $M = \cup_{\alpha\in\widetilde{A}} V_\alpha$ which is translationally invariant on the end and satisfies supp $\chi_\alpha \subsetneqq$ supp ρ_α for each $\alpha \in A$.*

Proof. Since A is a finite set it is enough to show that if we fix $\alpha \in A$, then there exists an admissible partition of unity $\{\chi_\beta^2\}_{\beta\in\widetilde{A}}$ of $M = \cup_{\beta\in\widetilde{A}} V_\beta$ which is translationally invariant on the end and satisfies *supp* $\chi_\alpha \subsetneqq$ *supp* ρ_α and *supp* $\chi_\beta =$ *supp* ρ_β for all $\beta \in \widetilde{A} \setminus \{\alpha\}$. To construct χ_α we first put

$$K_\alpha := V_\alpha \setminus \bigcup_{\beta\in\widetilde{A}\setminus\{\alpha\}} \rho_\beta^{-1}(\mathbb{R}_+).$$

and show that the minimum $m_\alpha := \min(\rho_\alpha|_{K_\alpha})$ exists. If $\alpha \in \widetilde{A} \setminus \hat{A}$ then it is true because K_α is compact. If $\alpha \in \hat{A}$, then we consider a decomposition of K_α into K_α^1 and K_α^2;

$$K_\alpha^1 := (V_\alpha \setminus V') \setminus \bigcup_{\beta \in \widetilde{A} \setminus \{\alpha\}} \rho_\beta^{-1}(\mathbb{R}_+), \quad K_\alpha^2 := (V_\alpha \cap V') \setminus \bigcup_{\beta \in \widetilde{A} \setminus \{\alpha\}} \rho_\beta^{-1}(\mathbb{R}_+)$$

Since K_α is a closed subset in M we have $M \supset K_\alpha = \overline{K_\alpha} = \overline{K_\alpha^1} \cup \overline{K_\alpha^2} = \overline{K_\alpha^1} \cup K_\alpha^2$. Note that $\overline{K_\alpha^1} (\subset V \setminus V')$ is compact. On the other hand since $V_\infty \cap V' = \emptyset$, we have

$$\begin{aligned} K_\alpha^2 &= N_\alpha \times \prod_i \mathbb{R}_0^{m_i} \setminus \bigcup_{\beta \in \hat{A} \setminus \{\alpha\}} \rho_\beta^{-1}(\mathbb{R}_+) \\ &= \hat{K}_\alpha^2 \times \prod_i \mathbb{R}_0^{m_i}, \end{aligned}$$

where $\hat{K}_\alpha^2$ is the compact set define by $\hat{K}_\alpha^2 := N_\alpha \setminus \bigcup_{\beta \in \hat{A} \setminus \{\alpha\}} \hat{\rho}_\beta^{-1}(\mathbb{R}_{>0})$. Then there exist minimums;

- $m_\alpha^1 := \min(\rho_\alpha|_{\overline{K_\alpha^1}})$
- $m_\alpha^2 := \min(\rho_\alpha|_{K_\alpha^2}) = \min(\hat{\rho}_\alpha|_{\hat{K}_\alpha^2})$,

and hence $m_\alpha = \min\{m_\alpha^1, m_\alpha^2\}$ does exists.

Take and fix a non-decreasing function $\varphi_\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that

$$\varphi_\alpha(r) = \begin{cases} 0 & (0 \leq r < m_\alpha/2) \\ r & (m_\alpha \leq r), \end{cases}$$

and define $\rho'_\alpha : M \to \mathbb{R}$ by the composition $\rho'_\alpha := \varphi_\alpha \circ \rho_\alpha$. Then this ρ'_α is an admissible and translationally invariant on the end, and we have

$$\rho'_\alpha(x) + \sum_{\beta \in \widetilde{A} \setminus \{\alpha\}} \rho_\beta(x) > 0$$

for all $x \in M$. By normalizing the family of functions $\{\rho'_\alpha\} \cup \{\rho_\beta\}_{\beta \in \widetilde{A} \setminus \{\alpha\}}$ we obtained the required family of functions $\{\chi_\beta^2\}_{\beta \in \widetilde{A}}$. □

4.1 Vanishing theorem

Let M be a Riemannian manifold and V an open subset of M. Suppose that there exist a compatible fibration $\{\pi_\alpha : V_\alpha \to U_\alpha\}_{\alpha \in A}$ and a compatible system $\{W, D_\alpha\}_{\alpha \in A}$ on V. Using an admissible partition of unity $\{\rho_\alpha^2\}_{\alpha \in \widetilde{A}}$ of $M = \cup_{\alpha \in \widetilde{A}} V_\alpha$ we put $D'_\alpha := \rho_\alpha D_\alpha \rho_\alpha$ for $\alpha \in A$. Take any Dirac-type operator D on $\Gamma(W)$ and a positive real number t. We define the operator acting on $\Gamma(W)$ by $D_t := D + t \sum_{\alpha \in A} D'_\alpha$.

Remark 4.4 Note that D_α and ρ_β commute each other because ρ_β is a pull-back of a function on U_β and D_α contains only the derivatives along fibers.

We show the following fundamental lemma in our argument.

Lemma 4.5 *For each α the anti-commutator $DD'_\alpha + D'_\alpha D$ is a differential operator along fibers of π_α of order at most 2.*

Proof. Recall that, for each α the principal symbol of horizontal direction D^{H_α} of D with respect to π_α anti-commutes not only with the symbol of D'_α, but also with the whole operator D'_α. The statement follows from this property. It is straightforward to check it using local description. Instead of giving the detail of the local calculation, however, we here give an alternative formal explanation for the above lemma. For $b \in U_\alpha$ let $\mathcal{W}_b$ be the sections of the restriction of W on the fiber $\pi_\alpha^{-1}(b)$. Then $\mathcal{W} = \coprod \mathcal{W}_b$ is formally an infinite dimensional vector bundle over U_α. We can regard D'_α as an endmorphism on $\mathcal{W}$. Then D'_α is a order-zero differential operator on $\mathcal{W}$ whose principal symbol is equal to D'_α itself. Then, as a differential operator on $\mathcal{W}$, the anti-commutator $D'_\alpha D^{H_\alpha} + D^{H_\alpha} D'_\alpha$ is an (at most) order-one operator whose principal symbol is given by the anti-commutator between the Clifford action by TU_α and D'_α. This principal symbol vanishes, which implies that the anti-commutator is order-zero as a differential operator on $\mathcal{W}$, i.e., it does not contain derivatives of U_α-direction. □

For an operator appearing in the above lemma we have the following a priori estimate.

Lemma 4.6 *For each fiber F of π_α and arbitrary differential operator Q of order at most 2 along F, there exists a constant C_Q such that the inequality*

$$\left| \int_F (s_F, Q s_F) \right| \le C_Q \int_F |D_\alpha s_F|^2$$

holds for all sections s_F.

The following vanishing theorem is a main theorem in this subsection.

Theorem 4.7 *Let M be a closed manifold or a manifold with Euclidian end V. Suppose that M is equipped with a translationally invariant strongly acyclic compatible system $\{\{\pi_\alpha\}, W, \{D_\alpha\}\}_{\alpha \in A}$. Let D be a Dirac-type operator acting on $\Gamma(W)$ which is translationally invariant on the end. Put $D'_\alpha := \rho_\alpha D_\alpha \rho_\alpha$ and $D_t := D + t \sum_\alpha D'_\alpha$ for a positive number $t > 0$. Then the space of L^2-solutions of the equation $D_t s = 0$ is trivial for all $t \gg 1$.*

To show this theorem we make several preparations. Let $\{\chi_\alpha^2\}_{\alpha\in\widetilde{A}}$ be an admissible partition of unity constructed in Lemma 4.3 and put $K_\alpha := \mathit{supp}\ \chi_\alpha$.

Lemma 4.8 *1. D'_α is an elliptic operator on each fiber of π_α which is contained in K_α.*

2. If Q is an differential operator along fibers of order at most 2, then there exists an constant C_Q such that for each section s_α satisfying supp $s_\alpha \subset K_\alpha$ we have an estimate

$$\left|\int_M (s_\alpha, Qs_\alpha)\right| \le C_Q \int_M |D'_\alpha s_\alpha|^2.$$

Proof. The first statement follows from the fact that ρ_α takes positive values on K_α. Note that K_α is compact or has Euclidian end and each D_α is translationally invariant, and hence by the similar argument in the proof of Lemma 4.3 we can choose the constants C_Q in Lemma 4.6 uniformly for all fibers of π_α contained in K_α. The second statement follows from this fact. □

Lemma 4.9 *There exists an operator Z which does not contain any differential terms and satisfying*

$$D_t^2 = \sum_\alpha \chi_\alpha D_t^2 \chi_\alpha + Z.$$

Moreover Z does not depend on t.

Proof. If χ is an admissible function then it commutes with D_α and hence we have $[D_t, \chi] = [D, \chi]$. Using this equality and the fact $[D, \chi]$ does not contain any differential operators we have

$$\begin{aligned}
[[D_t^2, \chi], \chi] &= [(D_t[D_t, \chi] + [D_t, \chi]D_t), \chi] \\
&= [(D_t[D, \chi] + [D, \chi]D_t), \chi] \\
&= [D_t, \chi][D, \chi] + [D, \chi][D_t, \chi] \\
&= 2[D, \chi]^2.
\end{aligned}$$

Put $\chi := \chi_\alpha$ and take summation for all α we have

$$2D_t^2 - 2\sum_\alpha \chi_\alpha D_t^2 \chi_\alpha = \sum_\alpha [[D_t^2, \chi], \chi] = 2\sum_\alpha [D, \chi_\alpha]^2.$$

Then $Z := \sum_\alpha [D, \chi_\alpha]^2$ is the required operator of order 0. □

Proposition 4.10 *If s_α is an L^2-bounded section of W such that Ds_α and $D'_\alpha s_\alpha$ are also L^2-bounded section for all $\alpha \in A$ and supp $s_\alpha \subset K_\alpha$, then we have the inequality*

$$\int_M |D_t s_\alpha|^2 \geq \left| \int_M (Zs_\alpha, s_\alpha) \right| + \int_M |s_\alpha|^2$$

for all $t \gg 1$.

We use the above proposition and lemmas to show Theorem 4.7 as follows.

Proof of Theorem 4.7 assuming Proposition 4.10. We take $t \gg 1$ so that the inequality in Proposition 4.10 holds. We first note that since M is a closed manifold or a manifold with Euclidian end, an L^2-bounded section s which satisfies $D_t s = 0$ is an element in the Sobolev space $L^2_k(W)$ for arbitrary $k \in \mathbb{N}$ by the elliptic estimate, and hence $s_\alpha = \chi_\alpha s$ is. Moreover since D_t is translationally invariant it has the bounded extension $D_t : L^2_1(W) \to L^2(W)$. Then we can use the partial integration formula in Lemma 3.8 for $D_t s_\alpha$, and we have $s = 0$ as in the following;

$$\begin{aligned}
0 &= \int_M (D_t^2 s, s) \\
&= \sum_\alpha \int_M (\chi_\alpha D_t^2 \chi_\alpha s, s) + \int_M (Zs, s) \qquad \text{(Lemma 4.9.1)} \\
&= \sum_\alpha \left(\int_M |D_t s_\alpha|^2 + \int_M (Zs_\alpha, s_\alpha) \right) \qquad (s_\alpha := \chi_\alpha s) \\
&\geq \sum_\alpha \left(\int_M |(Zs_\alpha, s_\alpha)| + \int_M |s_\alpha^2| + \int_M (Zs_\alpha, s_\alpha) \right) \qquad \text{(Proposition 4.10)} \\
&\geq \sum_\alpha \int_M |s_\alpha^2| = \int_M |s|^2.
\end{aligned}$$

□

4.1.1 Proof of Proposition 4.10.

For each fixed $\alpha \in A$, we can write D_t as

$$D_t = D_{\neq\alpha} + tD'_\alpha$$

on V_α, where we put

$$D_{\neq\alpha} := D + t \sum_{\beta\neq\alpha} D'_\beta.$$

Using these notations together with $Q_\alpha := DD'_\alpha + D'_\alpha D$ and $Q_{\beta\alpha} := D'_\beta D'_\alpha + D'_\alpha D'_\beta$, we have

$$D_t^2 = D_{\neq\alpha}^2 + tQ_\alpha + t^2 \sum_{\beta\neq\alpha} Q_{\beta\alpha} + t^2 D'^2_\alpha,$$

and since $\int_M (D_{\neq\alpha}^2 s_\alpha, s_\alpha)$ and $\int_M (Q_{\beta\alpha} s_\alpha, s_\alpha)$ are non-negative for an L^2-section s_α satisfying the assumptions we also have

$$\int_M |D_t s_\alpha|^2 \geq t^2 \int_M |D'_\alpha s_\alpha|^2 - t \left| \int_M (Q_\alpha s_\alpha, s_\alpha) \right|.$$

From Lemma 4.5 Q_α is a differential operator along fibers of π_α of order at most 2. Then from Lemma 4.8.2 there exist a constant C' such that

$$\left| \int_M (Q_\alpha s_\alpha, s_\alpha) \right| \leq C' \int_M |D'_\alpha s_\alpha|^2.$$

Combining these inequalities we have

$$\int_M |D_t s_\alpha|^2 \geq (t^2 - C't) \int_M |D'_\alpha s_\alpha|^2.$$

On the other hand using Lemma 4.8.2 again there exists a constant C'' such that

$$\left| \int_M (Z s_\alpha, s_\alpha) \right| + \int_M |s_\alpha|^2 \leq C'' \int_M |D'_\alpha s_\alpha|^2,$$

and hence if we take $t \gg 1$ so that $t^2 - C't \geq C''$, then we have

$$\int_M |D_t s_\alpha|^2 \geq \left| \int_M (Z s_\alpha, s_\alpha) \right| + \int_M |s_\alpha|^2.$$

Note that since A is a finite set we may assume that C' and C'' do not depend on α, and we complete the proof.

4.2 Definition of the local index - Euclidian end case

In this subsection we give the definition of the local index of a strongly acyclic compatible system on a manifold with Euclidian end. Let M be a manifold with Euclidian end V. Let W be a $Cl(TM)$-module bundle. Assume that there is a translationally invariant strongly acyclic compatible system $\{\{\pi_\alpha\}, \{V_\alpha\}, \{D_\alpha\}\}_{\alpha\in A}$ on V. Take any Dirac-type operator D acting on $\Gamma(W)$ which is translationally invariant on the end. For an admissible partition of unity $\{\rho_\alpha^2\}_{\alpha\in\widetilde{A}}$ and a positive number $t > 0$ we put $D_t := D + t\sum_\alpha \rho_\alpha D_\alpha \rho_\alpha$.

Lemma 4.11 *If t is large enough so that the inequality in Proposition 4.10 holds, then D_t satisfies the Assumption 3.1.*

Proof. Since the principal symbol of D_t is given by a linear combination of the Clifford multiplication of TM and that of fiber directions of $\{\pi_\alpha\}_{\alpha\in A}$, (i) of Assumption 3.1 is satisfied. We show the condition (ii) of Assumption 3.1. Let s be a smooth compactly-supported section of W with *supp* $s \subset V$. Let $\{\chi_\alpha^2\}_{\alpha\in\widetilde{A}}$ be the admissible partition of unity constructed in Lemma 4.3. For each $s_\alpha := \chi_\alpha s$ we can apply Proposition 4.10, and hence, we have $||D_ts||_V^2 \geq ||s||_V^2$ as in the same way in the proof of Theorem 4.7. □

Results in Section. 3 imply the following.

Proposition 4.12 *If t is large enough so that the inequality in Proposition 4.10 holds, then the space of L^2-solutions of $D_ts = 0$ is finite dimensional and its super-dimension is independent for $t \gg 1$ and any other continuous deformations of data.*

Definition 4.13 We define the local index $\mathrm{ind}(M, V, W)$ as the index of D_t in the sense of Section 3.

In the case of cylindrical end we have the following sum formula of local indices.

Lemma 4.14 *For $i = 1, 2$ let M_i be manifolds with cylindrical ends $V_i = N_i \times \mathbb{R}_{>0}$ and N_i^0 be connected components of N_i^0. Suppose that there is an isometry $\phi : N_1^0 \to N_2^0$, and for some $R > 0$ the map $\phi : N_1^0 \times (0, R) \to N_2^0 \times (0, R)$ given by $(x, r) \mapsto (\phi(x), R - r)$ induces the isomorphism between the strongly acyclic compatible systems on them. Then we can glue $M_1 \smallsetminus (N_1^0 \times [R, \infty))$ and $M_2 \smallsetminus (N_2^0 \times [R, \infty))$ to obtain a new manifold $\hat{M}$ with cylindrical end $\hat{V} = \hat{N} \times (0, \infty)$ for $\hat{N} = (N_1 \smallsetminus N_1^0) \cup (N_2 \smallsetminus N_2^0)$, and we also have a Clifford module bundle $\hat{W}$ obtained by gluing W and W' on $N_1^0 \times (0, R) \cong N_2^0 \times (0, R)$. Then we have*

$$\mathrm{ind}(\hat{M}, \hat{V}, \hat{W}) = \mathrm{ind}(M_1, V_1, W_1) + \mathrm{ind}(M_2, V_2, W_2).$$

4.3 Definition of the local index - general case

Let V be an open subset of M such that $M \smallsetminus V$ is compact. Assume that V has a strongly acyclic compatible system. We would like to define the local index for such a general case. The way to define it is almost same in [4]. To verify the construction we have to check the following.

Proposition 4.15 *For given (M, V, W) and the strongly acyclic compatible system $(V_\alpha \overset{\pi_\alpha}{\to} U_\alpha, D_\alpha)$ on V we can deform them to (M', V', W') so that it has a cylindrical end with a translationally invariant strongly acyclic compatible system.*

To prove the proposition, it is enough to show the following lemma.

Lemma 4.16 *There exists a smooth admissible function $f : M \to \mathbb{R}$ and a regular value c of f such that $f^{-1}(-\infty, c]$ is a compact subset containing $M \smallsetminus V$.*

Proof. For any subset D of M, let $K(D)$ be

$$K(D) = \cup_\alpha \pi_\alpha^{-1}\pi_\alpha(D \cap \overline{V_\alpha}).$$

Since π_α is a proper map, if D is compact, then $K(D)$ is again a compact subset.

Let $f_0 : M \to \mathbb{R}$ be the distance function from the compact subset $K(M \smallsetminus V)$. Take a real number $r > 0$ so that $f_0^{-1}[0, r]$ is a compact neighborhood of $K(M \smallsetminus V)$. Let $\epsilon > 0$ be a positive real number satisfying $2\epsilon < r$. Let $h : M \to \mathbb{R}$ be a smooth function such that $|f_0(x) - h(x)| < \epsilon$ for all $x \in M$. Put $f := I(h)$, where $I : C^\infty(M) \to C^\infty(M)$ is the averaging operation in Definition 2.9. Note that using the property 4 in Definition 2.9 one can check that for all subset D of M and a connected interval $J \subset \mathbb{R}$, if $K(D) \subset h^{-1}(J)$, then we have $K(D) \subset f^{-1}(J)$. Let c be a regular value of f satisfying $\epsilon < c < r - \epsilon$. Then we have

$$K(M \smallsetminus V) = f_0^{-1}(0) \subset h^{-1}(-\epsilon, \epsilon).$$

It implies

$$K(M \smallsetminus V) \subset f^{-1}(-\epsilon, \epsilon).$$

In particular we have

$$M \smallsetminus V \subset f^{-1}(-\infty, c].$$

On the other hand if $x \notin K(h^{-1}(-\infty, c])$, then we have $K(\{x\}) \subset h^{-1}(c, \infty)$, and hence, $f(x) > c$. Then

$$f^{-1}(-\infty, c] \subset K(h^{-1}(-\infty, c]) \subset K(f_0^{-1}(-\infty, c + \epsilon]) \subset K(f_0^{-1}[0, r]).$$

In particular $f^{-1}(-\infty, c]$ is compact. □

Definition 4.17 We define the local index $\mathrm{ind}(M, V, W)$ to be the local index for the deformed data (M', V', W').

Note that the local index $\mathrm{ind}(M, V, W)$ is well-defined, i.e, it does not depend on various choice of the construction. It follows from the sum formula (Lemma 4.14) and Theorem 4.7 as in the same way in [4]. The well-definedness means the excision property of local index.

Theorem 4.18 *Let M be a Riemannian manifold and V an open subset such that $M \smallsetminus V$ is compact. Let W be a $Cl(TM)$-module bundle on M and suppose that the metric on V is a compatible metric and V has a strongly acyclic compatible system. Let V' be an admissible open subset of M such that $M \smallsetminus V'$ is a compact neighborhood of $M \smallsetminus V$. Put $M' := M \smallsetminus V'$. Then we have*

$$\mathrm{ind}(M, V, W) = \mathrm{ind}(M', V \smallsetminus V', W|_{M'}).$$

Note that if M is closed, then $\mathrm{ind}(M, V, W)$ is equal to the index of the Dirac-type operator D because of the homotopy invariance of indices. Using the excision property, additivity for disjoint unions and vanishing theorems we have the localization theorem.

Theorem 4.19 *Let M be a closed Riemannian manifold and V an open subset. Let W be a $Cl(TM)$-module bundle on M and D a Dirac-type operator acting on $\Gamma(W)$. Suppose that V has a strongly acyclic compatible system. Let $\cup_{i=1}^{N} V_i$ be an open neighborhood of $M \smallsetminus V$ such that $V_i \cap V_j = \emptyset$ if $i \neq j$. Then we have the following equality.*

$$\mathrm{ind}\, D = \sum_{i=1}^{N} \mathrm{ind}(V_i, V_i \cap V, W|_{V_i \cap V}).$$

Remark 4.20 The arguments in this section are valid in orbifold category.

5 Product formula of local indices

In this section we formulate the product of acyclic compatible systems. Once we have an appropriate formulation of the product, then we obtain the product formula of local indices of the strongly acyclic compatible systems by results in Section 3.

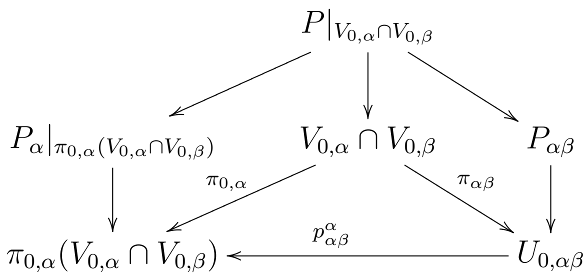
5.1 Product of compatible fibrations

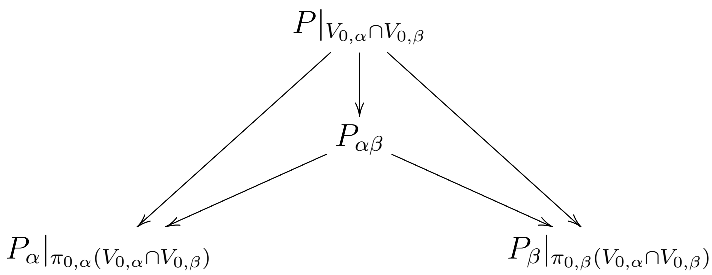
In this subsection we formulate a product of compatible fibrations. The product is defined for the following collection of data for $i = 0, 1$ which satisfy the Assumption 5.1.

1. M_i : a manifold.
2. V_i : an open set of M_i.
3. $\{\pi_{i,\alpha} : V_{i,\alpha} \to U_{i,\alpha},\ U_{i,\alpha\beta} \mid \alpha, \beta \in A_i\}$: a compatible fibration on V_i.
4. G : a compact Lie group which acts smoothly on M_1.
5. $\pi_P : P \to M_0$: a principal G-bundle over M_0.

Assumption 5.1 (1) V_1 is G-invariant and the fibrations $V_{1,\alpha} \to U_{1,\alpha}$, $V_{1,\alpha} \cap V_{1,\beta} \to U_{1,\alpha\beta}$ and $U_{1,\alpha\beta} \to \pi_{1,\alpha}(V_{1,\alpha} \cap V_{1,\beta})$ are G-equivariant fiber bundles for all $\alpha, \beta \in A_1$.

(2) there exist principal G-bundles $P_\alpha \to U_{0,\alpha}$, $P_{\alpha\beta} \to U_{0,\alpha\beta}$ and bundle maps $P|_{V_{0,\alpha}\cap V_{0,\beta}} \to P_\alpha|_{\pi_{0,\alpha}(V_{0,\alpha}\cap V_{0,\beta})}$, $P|_{V_{0,\alpha}\cap V_{0,\beta}} \to P_{\alpha\beta}$, $P_{\alpha\beta} \to P_\alpha|_{\pi_{0,\alpha}(V_{0,\alpha}\cap V_{0,\beta})}$ for all $\alpha, \beta \in A_0$ such that the following diagrams commute;





For later convenience we take an open neighborhood $V_{i,\infty}$ of $M_i \setminus V_i$ and consider the trivial fiber bundle structure $\pi_{i,\infty} : V_{i,\infty} \to V_{i,\infty}$. In other words we consider a compatible fibration $\{\pi_{i,\alpha} : V_{i,\alpha} \to U_{i,\alpha} \mid \alpha \in A_i \cup \{\infty\}\}$ on $M_i = V_{i,\infty} \cup (\cup_\alpha V_{i,\alpha})$. Let M be the quotient manifold by the diagonal action of G on $P \times M_1$. Then the natural map $\pi : M \to M_0$ is a fiber bundle whose fiber is equal to M_1. To define a structure of compatible fibration on M we first prepare several notations for $i = 0, 1$.

- $\widetilde{A}_i := A_i \cup \{\infty\}$.

- $\widetilde{A} := \widetilde{A}_0 \times \widetilde{A}_1$.
- $A := \widetilde{A} \setminus (\infty, \infty)$.
- $V_{\alpha_0,\alpha_1} := P|_{V_{0,\alpha_0}} \times_G V_{1,\alpha_1}$ for $\alpha \in \widetilde{A}_i$.
- $U_{\alpha_0,\alpha_1} := P_{\alpha_0} \times_G U_{1,\alpha_1}$ for $\alpha \in \widetilde{A}_i$.
- $U_{(\alpha_0,\alpha_1)(\beta_0,\beta_1)} := P_{\alpha_0\beta_0} \times_G U_{1,\alpha_1\beta_1}$.
- $V := \bigcup_{(\alpha_0,\alpha_1)\in A} V_{\alpha_0,\alpha_1}$.

Then we have the following.

Proposition 5.2 *A collection of data*

$$\{\pi_{\alpha_0,\alpha_1} : V_{\alpha_0,\alpha_1} \to U_{\alpha_0,\alpha_1},\ U_{(\alpha_0,\alpha_1)(\beta_0,\beta_1)} \mid (\alpha_0,\alpha_1),(\beta_0,\beta_1) \in A\}$$

is a compatible fibration on $V = \bigcup_{(\alpha_0,\alpha_1)\in A} V_{\alpha_0,\alpha_1}$.

5.2 Product of acyclic compatible systems

In this subsection we define a product of acyclic compatible systems. To define the product we consider the following data together with the data 1, 2, 3, 4 and 5 in Subsection 5.1.

6. a compatible Riemannian metric on M_i.
7. W_i : a compatible $Cl(TM_i)$-module bundle over M_i.
8. $\{D_{i,\alpha} \mid \alpha \in A_i\}$: strongly acyclic compatible system over $V_i = \cup_{\alpha\in A_i} V_{i,\alpha}$.

Together with Assumption 5.1 we assume the following.

Assumption 5.3 The metric on M_1 is G-invariant, and $W_1 \to M_1$ and $\{D_{1,\alpha}\}$ are G-equivariant.

From the Assumption 5.1 (2). the restrictions of P at each fibers of $\pi_{0,\alpha}$ are trivial. Moreover we have the following.

Lemma 5.4 *There exists a connection on P which is trivial flat over each fibers of $\pi_{0,\alpha}$ for all $\alpha \in A_0$.*

Proof. Take connections $\bar{\nabla}_\alpha$ for each $P_\alpha \to U_{0,\alpha}$. Let ∇_α be the pull-back connections of them to $P|_{V_{0,\alpha}} \to V_{0,\alpha}$ by $\pi_{0,\alpha}$. Define a connection ∇ on P by patching $\{\nabla_\alpha\}_\alpha$ by an admissible partition of unity $\{\rho_\alpha^2\}_\alpha$, which satisfies the required property. □

Using this connection on P and the compatible metric on M_0 we have the metric on P, and hence the metric on M. Moreover since the connection is trivial along fibers of $\{\pi_{0,\alpha}\}$ it induces a family of connections of $\{P_\alpha\}$ and $\{P_{\alpha\beta}\}$, and hence a family of metrics on them. Combining them with the G-invariant compatible metric on M_1 we have a family of metrics of $\{T[\pi_{\alpha_0\alpha_1}]\}$, $\{TU_{\alpha_0\alpha_1}\}$ and so on. It defines a compatible metric of a compatible fibration $M = \cup_{(\alpha_0,\alpha_1)} V_{\alpha_0,\alpha_1}$. We put $\widetilde{W_0} := \pi^* W_0 = \pi_P^* W_0 \times_G M_1$, $\widetilde{W_1} := P \times_G W_1$ and $W := \widetilde{W_0} \otimes \widetilde{W_1}$. Then $W \to M$ is a compatible Clifford module bundle of $M = \cup_{(\alpha_0,\alpha_1)} V_{\alpha_0,\alpha_1}$ with respect to the above induced compatible metric. Now we define differential operators $\widetilde{D}_{0,\alpha_0}$ and $\widetilde{D}_{1,\alpha_1}$ for each $\alpha_0 \in A_0$ and $\alpha_1 \in A_1$ which act on $\Gamma(\widetilde{W_0}|_{V_{\alpha_0,\alpha_1}})$ and $\Gamma(\widetilde{W_1}|_{V_{\alpha_0,\alpha_1}})$ respectively. The operator $\widetilde{D}_{1,\alpha_1}$ is the one induced from the G-equivariant operator D_{1,α_1} on $\Gamma(W_1|_{V_{1,\alpha_1}})$. On the other hand $\widetilde{D}_{0,\alpha_0}$ is the operator defined as follows: Since D_{0,α_0} is a differential operator along fibers of π_{0,α_0} and P is trivial at each fiber of π_{0,α_0}, we can define the operator acting on the restriction $\Gamma(\pi_P^* W_0|_{\text{fiber}} \times_G M_1)$ using $D_{0,\alpha_0}|_{\text{fiber}}$ and a trivialization of $P|_{\text{fiber}}$. Since such operators along fibers do not depend on trivialization we have a differential operator $\widetilde{D}_{0,\alpha_0}$ acting on $\Gamma(\widetilde{W_0}|_{V_{\alpha_0,\alpha_1}})$. Using these operators we define an operator acting on $\Gamma(W|_V)$ by $D_{\alpha_0,\alpha_1} := \widetilde{D}_{0,\alpha_0} \otimes \mathrm{id}_{\widetilde{W_1}} + \epsilon_{\widetilde{W_0}} \otimes \widetilde{D}_{1,\alpha_1}$, where $\epsilon_{\widetilde{W_0}}$ is a map on $\widetilde{W_0}$ defined by $\epsilon_{\widetilde{W_0}}(v) := (-1)^{\deg v} v$. For later convenience we put $\widetilde{D}_{\alpha_0,\infty} = \widetilde{D}_{\infty,\alpha_1} = 0$.

Proposition 5.5 *A collection of differential operators $\{D_{\alpha_0,\alpha_1} \mid (\alpha_0,\alpha_1) \in A\}$ is a strongly acyclic compatible system on $(V, W|_V)$.*

Proof. Since $\widetilde{D}_{0,\alpha_0} \otimes \mathrm{id}_{\widetilde{W_1}}$ and $\epsilon_{\widetilde{W_0}} \otimes \widetilde{D}_{1,\alpha_1}$ anti-commute each other we have

$$\left(\sum t_{\alpha_0,\alpha_1} D_{\alpha_0,\alpha_1}\right)^2 = \left(\sum_{\alpha_0} \left(\sum_{\alpha_1} t_{\alpha_0,\alpha_1}\right) \widetilde{D}_{0,\alpha_0}\right)^2 \otimes \mathrm{id}_{\widetilde{W_1}} + \mathrm{id}_{\widetilde{W_0}} \otimes \left(\sum_{\alpha_1} \left(\sum_{\alpha_0} t_{\alpha_0,\alpha_1}\right) \widetilde{D}_{1,\alpha_1}\right)^2$$

for any family of non-negative numbers (t_{α_0,α_1}), and the equality among anti-commutators

$$\{D_{\alpha_0,\alpha_1}, D_{\alpha_0',\alpha_1'}\} = \{\widetilde{D}_{0,\alpha_0}, \widetilde{D}_{0,\alpha_0'}\} \otimes \mathrm{id}_{\widetilde{W_1}} + \mathrm{id}_{\widetilde{W_0}} \otimes \{\widetilde{D}_{1,\alpha_1}, \widetilde{D}_{1,\alpha_1'}\}.$$

These equalities imply that if $\{D_{i,\alpha_i}\}_{\alpha_i}$ are strongly acyclic compatible systems, then $\{D_{\alpha_0,\alpha_1}\}_{(\alpha_0,\alpha_1)}$ is so. □

5.3 Product formula

To apply results in Subsection 4.1 and 4.2, we have to deform the end of M_i and M as in the following way; As we showed in Subsection 4.3, we can deform M_i into $\hat{M}_i$ together with their strongly acyclic compatible systems so that $\hat{M}_i$ has the cylindrical end structure. In addition we can deform P into $\hat{P}$ so that it has the cylindrical end and may assume that the deformation $\hat{M}_1$ for M_1 is G-equivariant. Then one can check that the product $\hat{M} = \hat{P} \times_G \hat{M}_1$ has the Euclidian end structure. Because of the excision property of the local index, we have that M_i (resp. M) and $\hat{M}_i$ (resp. $\hat{M}$) have the same local index. So hereafter we assume that M_i and M has the Euclidian end structure.

Let $\{\rho_{i,\alpha}^2\}_{\alpha \in \widetilde{A}_i}$ be admissible partition of unities of M_i. We may assume that $\{\rho_{1,\alpha}^2\}$ is G-invariant. Using these partition of unities we have an admissible partition of unity $\{\rho_{\alpha_0,\alpha_1}^2\}_{(\alpha_0,\alpha_1)\in \widetilde{A}}$ on $M = \cup_{(\alpha_0,\alpha_1)} V_{\alpha_0,\alpha_1}$ which is defined by $\rho_{\alpha_0,\alpha_1}([u,y]) := \rho_{0,\alpha_0}(\pi(u))\rho_{1,\alpha_1}(y)$ for $[u,y] \in M$.

For any translationally invariant Dirac-type operators D_i on $\Gamma(W_i)$, using a local trivialization of P we have their lifts $\widetilde{D}_0 \otimes \mathrm{id}_{\widetilde{W}_1}$ and $\epsilon_{\widetilde{W}_0} \otimes \widetilde{D}_1$ on $\Gamma(W)$ as in Subsection 3.6. Note that $D := \widetilde{D}_0 \otimes \mathrm{id}_{\widetilde{W}_1} + \epsilon_{\widetilde{W}_0} \otimes \widetilde{D}_1$ is a translationally invariant Dirac-type operator on $\Gamma(W)$.

Because of Lemma 5.5 if we take a positive number t large enough, then the inequality in Proposition 4.10 holds for deformed operators $D_{i,t}$ and D_t on M_i and M. On the other hand we have a decomposition $D_t = D_t^B + D_t^F$, where

$$\begin{aligned} D_t^B &:= \left(\widetilde{D}_0 + t\sum_{\alpha_0}\left(\sum_{\alpha_1}\rho_{\alpha_1}^2\right)\pi^*\rho_{0,\alpha_0}\widetilde{D}_{0,\alpha_0}\pi^*\rho_{0,\alpha_0}\right)\otimes \mathrm{id}_{\widetilde{W}_1} \\ &= \left(\widetilde{D}_0 + t\sum_{\alpha_0}\pi^*\rho_{0,\alpha_0}\widetilde{D}_{0,\alpha_0}\pi^*\rho_{0,\alpha_0}\right)\otimes \mathrm{id}_{\widetilde{W}_1} \\ D_t^F &:= \epsilon_{\widetilde{W}_0}\otimes\left(\widetilde{D}_1 + t\sum_{\alpha_1}\left(\sum_{\alpha_0}\pi^*\rho_{\alpha_0}^2\right)\rho_{1,\alpha_1}\widetilde{D}_{1,\alpha_1}\rho_{1,\alpha_1}\right) \\ &= \epsilon_{\widetilde{W}_0}\otimes\left(\widetilde{D}_1 + t\sum_{\alpha_1}\rho_{1,\alpha_1}\widetilde{D}_{1,\alpha_1}\rho_{1,\alpha_1}\right). \end{aligned}$$

Note that D_t^F is a differential operator along fibers of $\pi : M \to M_0$. They anti-commutes each other. Namely,

Lemma 5.6 $D_t^B D_t^F + D_t^F D_t^B = 0.$

Moreover since M_1 and M have the Euclidian end structure, we have the following by Lemma 4.2.

Lemma 5.7 *D_t and D_t^F satisfy Assumption 3.26.*

When we write $\ker D_{1,t} = E^0 \oplus E^1$ as the G-equivariant $\mathbb{Z}/2$-graded vector space, G-equivariant local index of (M_1, V_1, W_1) can be written as $\mathrm{ind}_G(M_1, V_1, W_1) = [E^0] - [E^1] \in R(G)$. Let $\underline{E}^i$ be the vector bundle over M_0 defined by $\underline{E}^i = P \times_G E^i$. Then the strongly acyclic compatible system on (M_0, V_0, W_0) induces another strongly acyclic compatible systems on $(M_0, V_0, W_0 \otimes \underline{E}^i)$ via $\{D_{0,\alpha} \otimes \mathrm{id}_{E^i}\}$ for $i = 0, 1$. Lemma 5.6, Lemma 5.7 and the product formula in Section 3 imply the following product formula of local indices.

Theorem 5.8 *We have the following product formula.*

$$\mathrm{ind}(M_0, V_0, W_0 \otimes \underline{E}^0) - \mathrm{ind}(M_0, V_0, W_0 \otimes \underline{E}^1) = \mathrm{ind}(M, V, W) \in \mathbb{Z}.$$

6 Four-dimensional case

6.1 Local indices for elliptic singularities

A critical point of a $2n$-dimensional singular Lagrangian fibration $\mu\colon (M, \omega) \to B$ is called a *nondegenerate elliptic singular point of rank k* ($\leq n$) if there exists a symplectic coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that in these coordinate, μ is written as $\mu = (x_1, \ldots, x_k, x_{k+1}^2 + y_{k+1}^2, \ldots, x_n^2 + y_n^2)$. See [10, 9, 8]. In this subsection we calculate local indices for elliptic singularities in four-dimensional case.

6.1.1 Definition of $RR_0(a_1, a_2)$

Let $D := \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc in $\mathbb{C}$. Let X_0 be the product of two copies of D with symplectic structure

$$\omega_0 := \frac{\sqrt{-1}}{2\pi} \sum_{k=1}^{2} dz_k \wedge d\overline{z}_k,$$

and (L_0, ∇^{L_0}) a prequantizing line bundle on (X_0, ω_0).

Let us consider the structure of a singular Lagrangian fibration $\mu_0\colon (X_0, \omega_0) \to [0,1) \times [0,1)$ on X_0 which is defined by

$$\mu_0(z) := \left(|z_1|^2, |z_2|^2\right).$$

We put the following assumption.

Assumption 6.1 The cohomology groups $H^*\left(\mu_0^{-1}(b); (L_0, \nabla_0)|_{\mu_0^{-1}(b)}\right)$ vanish for all points $b \in [0,1) \times [0,1)$ except for $b = (0,0)$.

Let a_1 and $a_2 \in \mathbb{Z}$ be arbitrary integers. We define a good compatible fibration on $X_0 \smallsetminus \{(0,0)\}$ consisting of three quotient maps of the torus actions by

$$\begin{aligned}
&\pi_0^0 \colon V_0^0 := X_0 \cap (\mathbb{C}^* \times \mathbb{C}^*) \to U_0^0 := V_0^0/T^2,\\
&\pi_1^0 \colon V_1^0 := \{(z_1, z_2) \in X_0 \mid |z_1| > |z_2|\} \to U_1^0 := V_1^0/S^1,\\
&\pi_2^0 \colon V_2^0 := \{(z_1, z_2) \in X_0 \mid |z_1| < |z_2|\} \to U_2^0 := V_2^0/S^1,
\end{aligned}$$

where the T^2-action on V_0^0 is the standard one, the S^1-action on V_1^0 is defined by

$$t(z_1, z_2) := (tz_1, t^{a_1} z_2),$$

and the S^1-action on V_2^0 is defined by

$$t(z_1, z_2) := (t^{a_2} z_1, tz_2).$$

We take and fix an arbitrary Hermitian structure (g_0, J_0) invariant under the standard T^2-action on X_0 and compatible with ω_0. Since g_0 is T^2-invariant g_0 induces a compatible Riemannian metric of this compatible fibration.

Let W_0 be the Hermitian vector bundle on X_0 which is defined by

$$W_0 := \bigwedge^\bullet (TX_0)_{\mathbb{C}} \otimes_{\mathbb{C}} L_0.$$

W_0 is a $\mathbb{Z}_2$-graded Clifford module bundle with respect to the Clifford module structure (6). We take a compatible system $\{D_i\}_{i=0,1,2}$ to be the family of de Rham operators along fibers of π_i^0 $(i = 0, 1, 2)$ which is defined by the same way as in Example 2.24. Assumption 6.1 implies that the kernel of all D_i vanish. Hence $\{D_i\}$ is strongly acyclic.

Definition 6.2 Let D be a Dirac-type operator on W_0. We define $RR_0(a_1, a_2)$ to be the local index in the sense of Definition 4.17 with respect to D and the above data.

Remark 6.3 $RR_0(a_1, a_2)$ does not depend on the choice of a compatible Hermitian structure (g_0, J_0) and a connection ∇^{L_0} of the prequantizing line bundle which satisfies Assumption 6.1 since it is deformation invariant.

6.1.2 Definition of $RR_1(a_+, a_-)$

Let $X_1 := (0,1)\times S^1 \times D$ be the product of $(0,1)\times S^1$ and D with symplectic structure

$$\omega_1 := dr\wedge d\theta + \frac{\sqrt{-1}}{2\pi} dz\wedge d\bar{z}$$

for $(r, e^{2\pi\sqrt{-1}\theta}, z)\in X_1$, and (L_1, ∇^{L_1}) a prequantizing line bundle on (X_1, ω_1).

Let us consider the structure of singular Lagrangian fibration $\mu_1\colon (X_1,\omega_1)\to (0,1)\times[0,1)$ which is defined by

$$\mu_1(r,u,z) := \left(r, |z|^2\right).$$

We put the following assumption.

Assumption 6.4 For all points $b\in[0,1)\times[0,1)$ $H^*(\mu_1^{-1}(b); (L_1,\nabla_1)|_{\mu_1^{-1}(b)})$ vanish.

Let a_+ and $a_-\in\mathbb{Z}$ be arbitrary integers. We take an element $r_1\in(0,1)$ and fix it. Then, we define a good compatible fibration on $X_1\smallsetminus \mu_1^{-1}(r_1,0)$ consisting of three quotient maps of the torus actions by

$$\begin{aligned}
\pi_0^1\colon V_0^1 &:= (0,1)\times S^1\times (D\smallsetminus\{0\}) \to U_0^1 := V_0^1/T^2,\\
\pi_1^1\colon V_1^1 &:= (r_1,1)\times S^1\times D \to U_1^1 := V_1^1/S^1,\\
\pi_2^1\colon V_2^1 &:= (0,r_1)\times S^1\times D \to U_2^1 := V_2^1/S^1,
\end{aligned}$$

where the T^2-action on V_0^1 is defined by

$$t(r,u,z) := (r, t_1u, t_2z),$$

the S^1-action on V_1^1 is defined by

$$t(r,u,z) := (r, tu, t^{a_+}z),$$

and the S^1-action on V_2^1 is defined by

$$t(r,u,z) := (r, tu, t^{a_-}z).$$

We take an arbitrary Hermitian structure (g_1, J_1) which is invariant under the standard T^2-action on X_1 and compatible with ω_1 and fix it. We define the $\mathbb{Z}_2$-graded Clifford module bundle W_1 and the strongly acyclic compatible system in the same way as in Section 6.1.1.

Definition 6.5 Let D be a Dirac-type operator on W_1. Then, we define $RR_1(a_+, a_-)$ to be the local index in the sense of Definition 4.17 with respect to D and the above data.

Remark 6.6 $RR_1(a_+, a_-)$ does not depend on the choice of a compatible Hermitian structure (g_1, J_1) and a connection ∇^{L_1} of the prequantizing line bundle which satisfies Assumption 6.4 since it is deformation invariant.

6.1.3 Computation

First we can show the following lemma.

Lemma 6.7 *For integers $a, b, c \in \mathbb{Z}$ we have*

$$RR_0(a,b) = RR_0(b,a), \quad RR_1(a,b) = RR_1(a+c, b+c).$$

Proof. We prove the latter equation. The proof of the former equation is similar. Let $\varphi\colon (0,1) \times S^1 \times D \to (0,1) \times S^1 \times D$ be the diffeomorphism which is defined by

$$\varphi(r,u,z) = (r,u,u^c z).$$

On the target space of φ we consider the same compatible fibration as $\{\pi_i^1\}_{i=0,1,2}$ except that a and b are replaced by $a+c$ and $b+c$, respectively. Then φ induces an isomorphism between compatible fibrations.

As the other data on the target space of φ we consider the data which are induced from those on the source space by φ^{-1}. Then the local index for the induced data on the target space is nothing but $RR_1(a,b)$.

On the other hand, the data $(\varphi^{-1})^*\omega_1$ and $(\varphi^{-1})^*\nabla^{L_1}$ can be deformed to ω_1 and ∇^{L_1} by linear deformations. Since the local index is invariant under continuous deformation this implies that the latter equation. □

Moreover, we can also show the following lemma by Theorem 4.18.

Lemma 6.8

$$RR_0(a,b) = RR_0(a',b) + RR_1(a',a), \qquad RR_1(a,c) = RR_1(a,b) + RR_1(b,c).$$

Then we can calculate $RR_0(a_1,a_2)$ and $RR_1(a_+,a_-)$.

Theorem 6.9

$$RR_0(a_1,a_2) = 1, \qquad RR_1(a_+,a_-) = 0.$$

Proof. We show $RR_0(0,1) = 1$ and $RR_0(0,0) = 1$. Then the theorem follows from these equalities and Lemma 6.7 and 6.8.

First we show $RR_0(0,1) = 1$. Let us consider the standard toric action on $\mathbb{C}P^2$ with hyperplane bundle as a prequantizing line bundle. We adopt the moment map μ of this action as a singular Lagrangian fibration. The image B of μ is the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, $(0,1)$, and μ has three Bohr-Sommerfeld fibers which corresponds one-to-one to three fixed points $[1:0:0], [0:1:0], [0:0:1]$ of the toric action.

We construct a compatible fibration on $\mathbb{C}P^2 \smallsetminus \{[1:0:0],[0:1:0],[0:0:1]\}$. For each $k \in \mathbb{Z}/3$ let V_k be a pairwise disjoint T^2-invariant open neighborhood of $\{[z_0:z_1:z_2] \in \mathbb{C}P^2 \mid z_k = 0\} \smallsetminus \{[1:0:0],[0:1:0],[0:0:1]\}$, and G_k the stabilizer of $\{[z_0:z_1:z_2] \in \mathbb{C}P^2 \mid z_k = 0\} \smallsetminus \{[1:0:0],[0:1:0],[0:0:1]\}$. Each G_k is a circle subgroup in T^2 and G_{k-1} acts on V_k freely. Then we put $U_k := V_k/G_{k-1}$ and define $\pi_k \colon V_k \to U_k$ to be the quotient map. We also put $V_4 := U_4 := B \smallsetminus \partial B$ and define $\pi_4 \colon V_4 \to U_4$ to be the identity map. These data define a good compatible fibration on $\mathbb{C}P^2 \smallsetminus \{[1:0:0],[0:1:0],[0:0:1]\}$.

The $\mathbb{Z}_2$-graded Clifford module bundle and the strongly acyclic compatible system are defined by the same way as in Section 6.1.1.

Then by Theorem 4.19 the Riemann-Roch number is localized at $[1:0:0],[0:1:0],[0:0:1]$, and the contribution of each fixed point is equal to $RR_0(0,1)$.

On the other hand it is well-known that the Riemann-Roch number of $\mathbb{C}P^2$ is 3. Thus we obtain $RR_0(0,1) = 1$.

Next we show $RR_0(0,0) = 1$. It is a direct consequence of the product formula 5.8 and the fact $[D^+] = 1$(see [4, Theorem 6.7]).

We can also show $RR_0(0,0) = 1$ in the following way. We consider $\mathbb{C}P^1 \times \mathbb{C}P^1$ with standard toric action. The image of the moment map is a square. By the similar construction as above the Riemann-Roch number is localized at four vertices and the contribution of any vertex is $RR_0(0,0)$. On the other hand the Riemann-Roch number of $\mathbb{C}P^1 \times \mathbb{C}P^1$ is four. This implies $RR_0(0,0) = 1$. □

6.2 Application to locally toric Lagrangian fibrations

In this subsection we apply the localization formula (Theorem 4.19), the product formula (Theorem 5.8), and Theorem 6.9 to show that for a four-dimensional closed locally toric Lagrangian fibration the Riemann-Roch number is equal to the number of Bohr-Sommerfeld fibers (Theorem 6.23).

6.2.1 Locally toric Lagrangian fibrations

Let $\omega_{\mathbb{C}^n}$ be the standard symplectic structure on $\mathbb{C}^n$

$$\omega_{\mathbb{C}^n} := \frac{\sqrt{-1}}{2\pi} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k.$$

The standard action of T^n on $\mathbb{C}^n$ preserves $\omega_{\mathbb{C}^n}$ and the map $\mu_{\mathbb{C}^n} \colon \mathbb{C}^n \to \mathbb{R}^n$ which is defined by

$$\mu_{\mathbb{C}^n}(z) := \left(|z_1|^2, \ldots, |z_n|^2\right)$$

for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ is a moment map of the standard T^n-action. Note that the image of $\mu_{\mathbb{C}^n}$ is the n-dimensional standard positive cone

$$\mathbb{R}^n_+ := \{r = (r_1, \dots, r_n) \in \mathbb{R}^n \colon r_i \geq 0 \ i = 1, \dots, n\}.$$

Let (M, ω) be a $2n$-dimensional symplectic manifold and B an n-dimensional manifold with corners.

Definition 6.10 ([8, 11]) A map $\mu \colon (M, \omega) \to B$ is called a *locally toric Lagrangian fibration* if there exists a system $\{(U_\alpha, \varphi^B_\alpha)\}$ of coordinate neighborhoods of B modeled on $\mathbb{R}^n_+$, and for each α there exists a symplectomorphism $\varphi^M_\alpha \colon (\mu^{-1}(U_\alpha), \omega) \to (\mu^{-1}_{\mathbb{C}^n}(\varphi^B_\alpha(U_\alpha)), \omega_{\mathbb{C}^n})$ such that $\mu_{\mathbb{C}^n} \circ \varphi^M_\alpha = \varphi^B_\alpha \circ \mu$.

Note that a locally toric Lagrangian fibration is a singular Lagrangian fibration that allows only elliptic singularities.

By the definition of a manifold with corners, B is equipped with a natural stratification. We denote by $\mathcal{S}^{(k)}B$ the k-dimensional part of B, namely, $\mathcal{S}^{(k)}B$ consists of those points which have exactly k nonzero components in a local coordinate system. Then, it is easy to see that the fiber of μ at a point in $\mathcal{S}^{(k)}B$ is a k-dimensional torus. In particular, all fibers of μ are smooth.

Example 6.11 (Projective toric variety) The moment map of a nonsingular projective toric variety is a locally toric Lagrangian fibration.

Example 6.12 (Non toric example) Let $c \in \mathbb{N}$ be a positive integer. We consider the diagonal Hamiltonian S^1-action on $(\mathbb{C}^2, \omega_{\mathbb{C}^2})$ with moment map

$$\Phi(z) := \|z\|^2 - c.$$

It is well-known that the symplectic quotient $\left(\Phi^{-1}(0), \omega_{\mathbb{C}^2}|_{\Phi^{-1}(0)}\right)/S^1$ is $\mathbb{C}P^1$ with c times Fubini-Study form ω_{FS}. In the rest of this example we identify $(\mathbb{C}P^1, c\omega_{FS})$ with $\left(\Phi^{-1}(0), \omega_{\mathbb{C}^2}|_{\Phi^{-1}(0)}\right)/S^1$.

Let $\widetilde{\mu} \colon (\widetilde{M}, \widetilde{\omega}) \to \widetilde{B}$ be the singular Lagrangian fibration which is defined by

$$\begin{aligned}
(\widetilde{M}, \widetilde{\omega}) &:= (\mathbb{R} \times S^1 \times \mathbb{C}P^1, dr \wedge d\theta \oplus c\omega_{FS}), \\
\widetilde{B} &:= \mathbb{R} \times [0, c], \\
\widetilde{\mu}(r, u, [z_0 : z_1]) &:= (r, |z_1|^2),
\end{aligned}$$

where we use the coordinate $(r, e^{2\pi\sqrt{-1}\theta}) \in \mathbb{R} \times S^1$. For a negative integer $a \in \mathbb{Z}$ $(a < 0)$ and a positive integer $b \in \mathbb{N}$, we define the $\mathbb{Z}$-actions on $\widetilde{M}$

and $\widetilde{B}$ by

$$n(r,u,[z_0:z_1]) := \bigl(r+n(-a|z_1|^2+b), u, [z_0:u^{na}z_1]\bigr)\,, \tag{11}$$
$$n(r_1,r_2) := (r_1+n(-ar_2+b), r_2). \tag{12}$$

It is easy to see that (11) and (12) are free $\mathbb{Z}$-actions and (11) preserves $\widetilde{\omega}$. Then we put

$$(M,\omega) := (\widetilde{M},\widetilde{\omega})/\mathbb{Z},$$
$$B := \widetilde{B}/\mathbb{Z}.$$

It is also easy to see that $\widetilde{\mu}$ is equivariant with respect to (11) and (12). Hence $\widetilde{\mu}$ induces the map from M to B which we denote by $\mu\colon (M,\omega)\to B$. By construction, B is a cylinder and μ is a locally toric Lagrangian fibration which has singular fibers on ∂B.

Let $\mu\colon (M^{2n},\omega)\to B$ be a locally toric Lagrangian fibration. By definition, for each α there is a symplectomorphism $\varphi_\alpha^M\colon \mu^{-1}(U_\alpha)\to \mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_\alpha))$, and $\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_\alpha))$ has a T^n-action which is obtained by restricting the standard T^n-action on $\mathbb{C}^n$. Then, it is known by [11, Proposition 3.13] that on each nonempty overlap $U_\alpha\cap U_\beta$ there exists an automorphism $\rho_{\alpha\beta}\in \mathrm{Aut}(T^n)$ of T^n such that $\varphi_{\alpha\beta}^M := \varphi_\alpha^M\circ(\varphi_\beta^M)^{-1}$ is $\rho_{\alpha\beta}$-equivariant, namely,

$$\varphi_{\alpha\beta}^M(tx) = \rho_{\alpha\beta}(t)\varphi_{\alpha\beta}^M(x)$$

for $t\in T^2$ and $x\in \mu_{\mathbb{C}^n}^{-1}(\varphi_\beta^B(U_\alpha\cap U_\beta))$. Moreover,we can show that $\rho_{\alpha\beta}$'s form a Čech one-cocycle $\{\rho_{\alpha\beta}\}$ on $\{U_\alpha\}$ with coefficients in $\mathrm{Aut}(T^n)$. Hence it defines an element $[\{\rho_{\alpha\beta}\}]$ in the Čech cohomology $H^1(B;\mathrm{Aut}(T^n))$. Then we have the following lemma.

Lemma 6.13 ([11]) *The Čech cohomology class $[\{\rho_{\alpha\beta}\}]$ is the obstruction class in order that the T^n-actions on $\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_\alpha))$ for all α can be patched together to obtain a global T^n-action on M.*

For more detail see [11].

Let $q_B\colon \widetilde{B}\to B$ be the universal covering of B. Since the Čech cohomology $H^1(B;\mathrm{Aut}(T^n))$ is identified with the moduli space of representations of the fundamental group $\pi_1(B)$ of B to $\mathrm{Aut}(T^n)$, the fiber product $q_B^*M := \{(\widetilde{b},x)\in \widetilde{B}\times M \mid q_B(\widetilde{b})=\mu(x)\}$ admits a T^n-action.

We take a representative $\rho\colon \pi_1(B)\to \mathrm{Aut}(T^n)$ of the equivalence class of representations corresponding to $[\{\rho_{\alpha\beta}\}]$. Then the T^n-action on q_B^*M can be written explicitly. See [12, Lemma 3.1] for the explicit description.

On the other hand, by the construction, $\pi_1(B)$ acts on q_B^*M from the left by the inverse of the deck transformation, and it is shown that the T^n-action and the $\pi_1(B)$-action satisfy the following relationship

$$t(a\widetilde{x}) = a\left(\rho(a^{-1})(t)\widetilde{x}\right) \tag{13}$$

for $t \in T^n$, $a \in \pi_1(B)$, and $\widetilde{x} \in q_B^*M$. Let $T^n \rtimes_\rho \pi_1(B)$ be the semidirect product of T^n and $\pi_1(B)$ with respect to ρ. Then, (13) implies that these actions form an action of $T^n \rtimes_\rho \pi_1(B)$ on q_B^*M. For more details see [12].

Let $q_M\colon q_B^*M \to M$ be the natural projection. Note that $q_M^*\omega$ is $T^n \rtimes_\rho \pi_1(B)$-invariant since ω is invariant under the T^n-action on $\mu^{-1}(U_\alpha)$ induced by the standard T^n-action on $\mathbb{C}^n$ for each α. Now we show the following lemma.

Lemma 6.14 *There exists a Hermitian structure $(\widetilde{g}, \widetilde{J})$ on q_B^*M compatible with $q_M^*\omega$ which is invariant under the action of $T^n \rtimes_\rho \pi_1(B)$.*

Proof. It is sufficient to show that the existence of an invariant Riemannian metric. Let g' be a Riemannian metric on M. We define the Riemannian metric $\widetilde{g}$ on q_B^*M by

$$\widetilde{g}_{\widetilde{x}}(u,v) := \int_{T^n} \left(\varphi_t^*(q_M^*g')\right)_{\widetilde{x}}(u,v)dt,$$

where φ_t implies the T^n-action for $t \in T^n$. It is sufficient to show that $\widetilde{g}$ is $\pi_1(B)$-invariant. For $a \in \pi_1(B)$ we denote the $\pi_1(B)$-action by ϕ_a. Then we have

$$\begin{aligned}
(\phi_a^*\widetilde{g})_{\widetilde{x}}(u,v) &= \int_{T^n} \left(\phi_a^*\left(\varphi_t^*(q_M^*g')\right)\right)_{\widetilde{x}}(u,v)dt \\
&= \int_{T^n} \left(\varphi_{\rho(a^{-1})(t)}^*\left(\phi_a^*(q_M^*g')\right)\right)_{\widetilde{x}}(u,v)dt \\
&= \int_{T^n} \left(\varphi_{\rho(a^{-1})(t)}^*(q_M^*g')\right)_{\widetilde{x}}(u,v)dt \\
&= \det\rho(a^{-1})\int_{T^n} \left(\varphi_{\rho(a^{-1})(t)}^*(q_M^*g')\right)_{\widetilde{x}}(u,v)\rho(a^{-1})^*dt \\
&= \int_{T^n} \left(\varphi_t^*(q_M^*g')\right)_{\widetilde{x}}(u,v)dt. \\
&= \widetilde{g}_{\widetilde{x}}(u,v).
\end{aligned}$$

Here we remark that $\det\rho(a^{-1}) = \pm 1$ since $\rho(a^{-1}) \in \mathrm{Aut}(T^n)$. □

Corollary 6.15 (the existence of an invariant Hermitian structure) *There exists a Hermitian structure (g, J) on M compatible with ω such that on each $\mu^{-1}(U_\alpha)$ (g, J) is invariant under the T^n-action on $\mu^{-1}(U_\alpha)$ which is induced from the T^n-action on $\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_\alpha))$ with the identification φ_α^M.*

Proof. By Lemma 6.14 there is a $T^n \rtimes_\rho \pi_1(B)$-invariant Hermitian structure $(\widetilde{g}, \widetilde{J})$ on q_B^*M compatible with $q_M^*\omega$. In particular, since $(\widetilde{g}, \widetilde{J})$ is $\pi_1(B)$-invariant, $(\widetilde{g}, \widetilde{J})$ induces an ω-compatible Hermitian structure on M which is denoted by (g, J). Then, (g, J) is the required one. □

Lemma 6.16 (The existence of an averaging operation) *Suppose that there exists a compatible fibration $\{\pi_\alpha \colon V_\alpha \to U_\alpha\}$ on M such that for each α a fiber of π_α is contained in that of μ, namely, $\pi_\alpha^{-1}\pi_\alpha(x) \subset \mu^{-1}\mu(x)$ for $x \in V_\alpha$. There exists an averaging operation $I \colon C^\infty(M) \to C^\infty(M)$ with respect to $\{\pi_\alpha \colon V_\alpha \to U_\alpha\}$.*

Proof. For $f \in C^\infty(M)$ let $\widetilde{f} \in C^\infty(q_B^*M)$ be the function on q_B^*M which is defined by

$$\widetilde{f}(\widetilde{x}) := \int_{T^n} (f \circ q_M)(t\widetilde{x})dt.$$

Then, by the similar way to that in the proof of Lemma 6.14, we can show that $\widetilde{f}$ is $T^n \rtimes_\rho \pi_1(B)$-invariant. Hence it descends to the function on M. We denote it by $I(f)$. Then, it is clear that $I(f)$ satisfies the properties in Definition 2.9. □

6.2.2 Bohr-Sommerfeld fibers and the Riemann-Roch number

Let $\mu \colon (M, \omega) \to B$ be a prequantizable locally toric Lagrangian fibration with prequantizing line bundle (L, ∇). Recall that, as described above, all fibers are smooth.

Definition 6.17 A fiber F of μ is said to be *Bohr-Sommerfeld* if the restriction $(L, \nabla)|_F$ is trivially flat. A point b of B is also said to be *Bohr-Sommerfeld* if the fiber $\mu^{-1}(b)$ is Bohr-Sommerfeld.

Remark 6.18 A fiber F of μ is Bohr-Sommerfeld if and only if the cohomology $H^*(F; (L, \nabla)|_F)$ does not vanish, see Lemma 2.25. This is also equivalent to the condition that the de Rham operator on F with coefficients in $(L, \nabla)|_F$ has non zero kernel.

First we specify Bohr-Sommerfeld points for the local model.

Proposition 6.19 *Let (L, ∇) be a prequantizing line bundle on $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$. Then, a point $b \in \mathbb{R}^n_+$ is Bohr-Sommerfeld if and only if $b \in \mathbb{R}^n_+ \cap \mathbb{Z}^n$.*

Proof. Since $\mathbb{C}^n$ is contractible L is trivial as a complex line bundle. Then we can assume that L is of the form $L = \mathbb{C}^n \times \mathbb{C}$ without loss of generality. Then, ∇ can be written as

$$\nabla = d - 2\pi\sqrt{-1}A$$

for some one form on $\mathbb{C}^n$ with $dA = \omega_{\mathbb{C}^n}$. Moreover A is unique up to exact one form since $\mathbb{C}^n$ is contractible. In particular, A is of the form

$$A = \frac{\sqrt{-1}}{4\pi}\sum_{i=1}^{n}(z_i d\bar{z}_i - \bar{z}_i dz_i) + df$$

for some smooth function f on $\mathbb{C}^n$.

By using the polar coordinate $z_i = r_i e^{2\pi\sqrt{-1}\theta_i}$ we can write $\mu_{\mathbb{C}}$ and A in the following forms

$$\mu_{\mathbb{C}^n} = (r_1^2, \ldots, r_n^2), \ A = \sum_i r_i^2 d\theta_i + df.$$

In particular, we see that the tangent space along a nonsingular fiber of $\mu_{\mathbb{C}^n}$ is spanned by ∂_{θ_i}'s. Thus a direct computation shows that a point $b \in \mathbb{R}^n_+$ is Bohr-Sommerfeld if and only if $b \in \mathbb{R}^n_+ \cap \mathbb{Z}^n$. □

By the above proposition and the definition of a locally toric Lagrangian fibration we can obtain the following corollary.

Corollary 6.20 *For a locally toric Lagrangian fibration Bohr-Sommerfeld fibers appear discretely.*

Example 6.21 For a nonsingular projective toric variety it is well-known that Bohr-Sommerfeld fibers correspond one-to-one to the integral points in the moment polytope. For example see [3].

Example 6.22 We consider the locally toric Lagrangian fibration $\mu\colon (M, \omega) \to B$ in Example 6.12. We show that (M, ω) is prequantizable.

Let (H_c, ∇^{H_c}) be the c times tensor power of the hyperplane bundle on $\mathbb{C}P^1$. With the identification of $(\mathbb{C}P^1, c\omega_{FS})$ and the symplectic quotient $\left(\Phi^{-1}(0), \omega_{\mathbb{C}^2}|_{\Phi^{-1}(0)}\right)/S^1$ in Example 6.12 (H_c, ∇^{H_c}) can be written in the following explicit way

$$(H_c, \nabla^{H_c}) = \left(\Phi^{-1}(0) \times \mathbb{C}, d + 1/2\sum_i (z_i d\overline{z}_i - \overline{z}_i dz_i)\right)/S^1,$$

where the S^1-action is defined by

$$t \cdot (z_0, z_1, w) := (tz_0, tz_1, t^c w).$$

Now let us define the prequantizing line bundle $(\widetilde{L}, \widetilde{\nabla})$ on $(\widetilde{M}, \widetilde{\omega})$ by

$$(\widetilde{L}, \widetilde{\nabla}) := \left(\mathrm{pr}_1^*(\mathbb{R} \times S^1 \times \mathbb{C}, d - 2\pi\sqrt{-1}r d\theta) \otimes_{\mathbb{C}} \mathrm{pr}_2^*(H_c, \nabla^{H_c})\right).$$

We also define the lift of the $\mathbb{Z}$-action (11) on $\widetilde{M}$ to $\widetilde{L}$ by

$$n(r, u, [z_0 : z_1, w]) := \left(r + n(-a|z_1|^2 + b), u, [z_0 : u^{na} z_1, u^{nb} w]\right). \tag{14}$$

It is easy to see that (14) preserves $\widetilde{\nabla}$ and the standard Hermitian metric. We put

$$(L, \nabla) := (\widetilde{L}, \widetilde{\nabla})/\mathbb{Z}.$$

Then (L, ∇) is a prequantizing line bundle on (M, ω).

Next we see the Bohr-Sommerfeld fibers of μ with respect to (L, ∇). The direct computation shows that Bohr-Sommerfeld fibers of $\widetilde{\mu}$ correspond one-to-one to the elements in $\widetilde{B} \cap \mathbb{Z}^2$. Let F be a fundamental domain of the $\mathbb{Z}$-action (12) on $\widetilde{B}$ which is defined by

$$F := \{(r_1, r_2) \in \widetilde{B} \mid 0 \le r_2 \le c,\ -1/2 \le r_1 < -ar_2 + b - 1/2\}$$

Then, Bohr-Sommerfeld fibers of μ correspond one-to-one to the elements in $F \cap \mathbb{Z}^2$. See Figure 1.

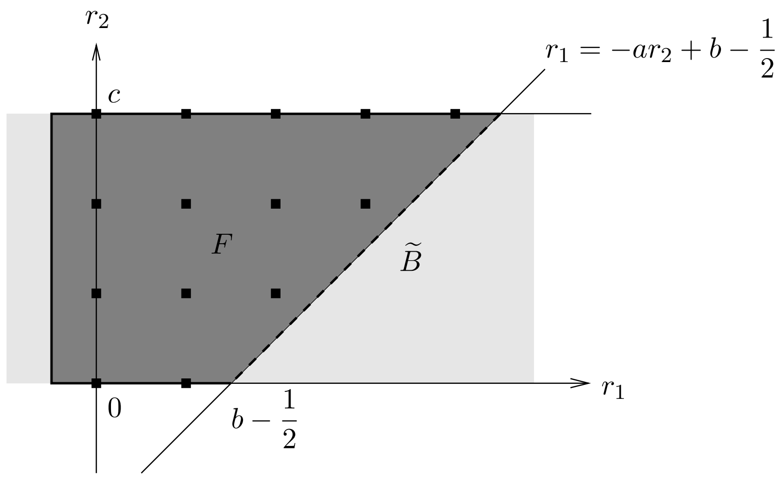


Figure 1: Bohr-Sommerfeld points in Example 6.12

In the rest of this section we assume that M is closed. Let (g, J) be a Hermitian structure on M compatible with ω as in Corollary 6.15. We define the Hermitian vector bundle W on M by

$$W := \bigwedge^{\bullet} TM_{\mathbb{C}} \otimes_{\mathbb{C}} L. \tag{15}$$

W is a $\mathbb{Z}_2$-graded Clifford module bundle with respect to the Clifford module structure (6). Let D be the Dirac-type operator on W. We define the *Riemann-Roch number* to be the index of D.

The purpose of this section is to show the following theorem.

Theorem 6.23 *Let $\mu\colon (M, \omega) \to B$ be a four-dimensional prequantizable locally toric Lagrangian fibration with prequantizing line bundle (L, ∇). Then the Riemann-Roch number is equal to the number of both nonsingular and singular Bohr-Sommerfeld fibers.*

Proof. Let B_{BS} be the set of Bohr-Sommerfeld points of μ in B. We put $V := \mu^{-1}(B \smallsetminus B_{BS})$. In order to prove Theorem 6.23 we define a good compatible fibration on V as follows.

On the regular non Bohr-Sommerfeld points $U_0 := \mathcal{S}^{(2)}B \smallsetminus B_{BS}$ of μ we define the fibration by

$$\pi_0 := \mu|_{V_0} \colon V_0 := \mu^{-1}(U_0) \to U_0.$$

Since B is compact, there are only finitely many Bohr-Sommerfeld points in $\mathcal{S}^{(1)}B$. Suppose we have exactly k Bohr-Sommerfeld points $p_1, \ldots, p_k$ in $\mathcal{S}^{(1)}B$, namely,

$$\{p_1, \ldots, p_k\} = B_{BS} \cap \mathcal{S}^{(1)}B.$$

For each i we take an contractible open neighborhood W_i of p_i in B which satisfies the following properties.

- W_i's are pairwise disjoint, namely, $W_i \cap W_j = \emptyset$ for all $i \neq j$.
- For each i W_i does not intersect $\mathcal{S}^{(0)}B$, namely, $W_i \cap \mathcal{S}^{(0)}B = \emptyset$.
- There exist finitely many non Bohr-Sommerfeld points in $\mathcal{S}^{(1)}B$, say $q_1, \ldots, q_l$, such that we have

$$\bigcup_{i=1}^{k} W_i \cap \mathcal{S}^{(1)}B = \mathcal{S}^{(1)}B \smallsetminus \{q_1, \ldots, q_l\}.$$

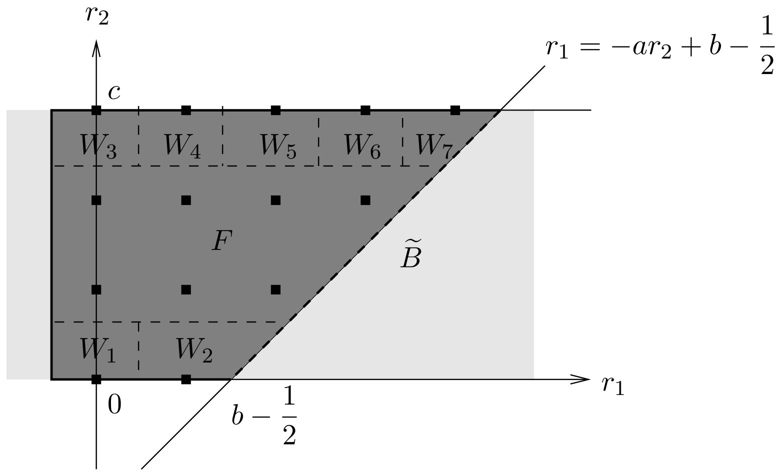


Figure 2: W_i's in Example 6.12

It is possible to take such neighborhoods since a connected component of ∂B is compact.

We put $V_i' := \mu^{-1}(W_i)$. Since W_i is contractible, by [11, Proposition 3.5], there exists a T^2-action on V_i'. Moreover, there exist a coordinate neighborhood $(U_{\alpha_i}, \varphi^B_{\alpha_i})$ of B containing p_i, a diffeomorphism $\varphi^M_{\alpha_i} \colon \mu^{-1}(U_{\alpha_i}) \to \mu^{-1}_{\mathbb{C}^2}(\varphi^B_{\alpha_i}(U_{\alpha_i}))$ in Definition 6.10, and an automorphism $\rho_{\alpha_i} \in \mathrm{Aut}(T^2)$ which satisfy the following properties.

- $\mu_{\mathbb{C}^2} \circ \varphi^M_{\alpha_i} = \varphi^B_{\alpha_i} \circ \mu$.
- On $V_i' \cap \mu^{-1}(U_{\alpha_i})$ $\varphi^M_{\alpha_i}$ is ρ_{α_i}-equivariant with respect to the T^2-action on V_i' and the standard T^2-action on $\mathbb{C}^2$.

Let $\varphi^B_{\alpha_i}(p_i) = (r_1, r_2) \in \mathbb{R}^2_+$. Since $p_i \in \mathcal{S}^{(1)}B$ there exists a unique coordinate r_{j_i} such that $r_{j_i} = 0$. We define the circle subgroup T_i of T^2 by

$$T_i := \rho^{-1}_{\alpha_i}\left(\{t = (t_1, t_2) \in T^2 \mid t_{j_i} = e\}\right).$$

By definition T_i acts on V_i' freely. Then for each i we define the fibration $\pi_i \colon V_i \to U_i$ to be the natural projection

$$\pi_i \colon V_i := V_i' \setminus \mu^{-1}(p_i) \to U_i := V_i / T_i.$$

By the construction $\{\pi_i \colon V_i \to U_i \mid i = 0, \dots, k\}$ is a good compatible fibration on V. Moreover, by Lemma 6.16, there is an averaging operation with respect to $\{\pi_i \colon V_i \to U_i \mid i = 0, \dots, k\}$.

Recall that (g, J) is a Hermitian structure on M compatible with ω as in Corollary 6.15. Then, as in the case of usual torus actions, g defines the compatible Riemannian metric of $\{\pi_i\colon V_i \to U_i \mid i = 0, \dots, k\}$ whose restriction to each fiber of μ is flat, and the $\mathbb{Z}_2$-graded Clifford module bundle W defined by (15) becomes a compatible Clifford module bundle in the sense of Definition 2.16. We define the strongly acyclic compatible system in the same way as in Section 6.1.1. Then by Theorem 4.19, the Riemann-Roch number is localized at Bohr-Sommerfeld fibers and the fibers at $q_1, \dots, q_l$.

We consider their contributions. Since a fiber of μ is connected, by Theorem [4, Theorem 6.11], the contribution of a regular Bohr-Sommerfeld fiber is equal to one.

Next we consider the contributions of singular Bohr-Sommerfeld fibers. By Definition each fiber on $\mathcal{S}^{(0)}B$ is Bohr-Sommerfeld, and its contribution is $RR_0(a_1, a_2)$ for some a_1 and a_2. By Theorem 6.9 it is equal to one.

By the construction of the compatible fibration the local Riemann-Roch number for each singular Bohr-Sommerfeld fiber on $\mathcal{S}^{(1)}B$ is obtained from $[BS^+]$ and $[D^+]$ in [4, Theorem 6.7] by the product formula 5.8. It is also one.

Finally it is easy to see that the contribution of each fibers at $q_1, \dots, q_l$ is equal to $RR_1(a_+, a_-)$ in Section 6.1.2 for some a_+ and a_-. Then by Theorem 6.9 it is zero. This proves Theorem 6.23. □

Example 6.24 Theorem 6.23 recovers Danilov's result [3], which says that for a nonsingular projective toric variety the Riemann-Roch number is equal to the number of the lattice points in the moment polytope, in the four-dimensional case.

Example 6.25 As we described in Example 6.22 the Bohr-Sommerfeld fibers correspond one-to-one to the elements in $F \cap \mathbb{Z}^2$. Then by Theorem 6.23 the Riemann-Roch number of (M, ω) is equal to the number of the elements in $F \cap \mathbb{Z}^2$ which is $(c+1)(2b - ac)/2$.

A Proof of Proposition 2.12

In this appendix we give a proof of Proposition 2.12. That is,

Proposition A.1 *If $\{\pi_\alpha\}$ is a good compatible fibration, then there exists an averaging operation $I : C^\infty(M) \to C^\infty(M)$ such that for all $f \in C^\infty(M)$ and $x \in M$ we have*

$$\min_{y \in \pi_{\overline{\alpha}_x}^{-1}\pi_{\overline{\alpha}_x}(x)} f(y) \le I(f)(x) \le \max_{y \in \pi_{\overline{\alpha}_x}^{-1}\pi_{\overline{\alpha}_x}(x)} f(y),$$

where $\pi_{\overline{\alpha}_x}^{-1}\pi_{\overline{\alpha}_x}(x) \subset \overline{V}_{\overline{\alpha}_x}$ *is the maximal fiber which contains* x*.*

Recall that we assume the following;

Assumption A.2 Each π_α has a continuous extension as a fiber bundle to the closure of V_α with the condition

$$V_\alpha \cap \overline{V_\beta} = \pi_\beta^{-1}\pi_\beta(V_\alpha \cap \overline{V_\beta})$$

for all $\beta \in A$.

We first show the following.

Lemma A.3 *There exist an admissible open covering* $\{V'_\alpha \mid \alpha \in A\}$ *of* M *such that* $\overline{V}'_\alpha \subset V_\alpha$*.*

Proof. Take and fix any open covering $\{W_\alpha\}_{\alpha\in A}$ of V which satisfies $\overline{W}_\alpha \subset V_\alpha$. Fix any total order of $A = \{\alpha_1, \cdots, \alpha_n\}$ so that if $V_{\alpha_i} \cap V_{\alpha_j} \neq \emptyset$ and the dimension of π_{α_i} is bigger than that of π_{α_j} then $i > j$. Fix $\alpha \in A$ and we define an increasing sequence of open sets $V_\alpha^{(k)} \subset V_\alpha$ inductively in the following way:

$$\begin{aligned} V_\alpha^{(0)} &:= W_\alpha \\ &\vdots \\ V_\alpha^{(k)} &:= \pi_{\alpha_k}^{-1}\pi_{\alpha_k}(V_\alpha^{(k-1)} \cap V_{\alpha_k}) \cup V_\alpha^{(k-1)} \\ &\vdots \\ V_\alpha^{(n)} &:= \pi_{\alpha_n}^{-1}\pi_{\alpha_n}(V_\alpha^{(n-1)} \cap V_{\alpha_n}) \cup V_\alpha^{(n-1)}. \end{aligned}$$

By the construction $\{V'_\alpha := V_\alpha^{(n)}\}_{\alpha\in A}$ is an admissible open covering of V. We show $\overline{V}_\alpha^{(k)} \subset V_\alpha$ by induction on k. Suppose that $\{p_i\}_{i\in\mathbb{N}}$ is a sequence in $V_\alpha^{(k)}$ which converges to p_∞ in V. It is enough to show that if $p_i \in \pi_{\alpha_k}^{-1}\pi_{\alpha_k}(V_\alpha^{(k-1)} \cap V_{\alpha_k})$ for all i then $p_\infty \in V_\alpha$. In this case we have $p_\infty \in \overline{\pi_{\alpha_k}^{-1}\pi_{\alpha_k}(V_\alpha^{(k-1)} \cap V_{\alpha_k})}$. On the other hand since the fibers are compact, $\pi_{\alpha_k} : \overline{V}_{\alpha_k} \to \overline{U}_{\alpha_k}$ is a closed map. Using the Assumption A.2 we have

$$\overline{\pi_{\alpha_k}^{-1}\pi_{\alpha_k}(V_\alpha^{(k-1)} \cap V_{\alpha_k})} \subset \pi_{\alpha_k}^{-1}(\overline{\pi_{\alpha_k}(V_\alpha^{(k-1)} \cap V_{\alpha_k})}) \subset \pi_{\alpha_k}^{-1}\pi_{\alpha_k}(\overline{V_\alpha^{(k-1)}} \cap \overline{V_{\alpha_k}}) \subset V_\alpha \cap \overline{V_{\alpha_k}}.$$

In particular we have $p_\infty \in V_\alpha$. □

Remark A.4 Since $\overline{V'_\alpha} \subset V_\alpha$, one can check that $\{V'_\alpha\}_\alpha$ satisfies the same condition as in Assumption A.2, i.e., $V'_\alpha \cap \overline{V'_\beta} = \pi_\beta^{-1}\pi_\beta(V'_\alpha \cap \overline{V'_\beta})$ for all $\alpha, \beta \in A$.

Let $\{V'_\alpha\}$ be an admissible open covering of M obtained in Lemma A.3.

Proof of Proposition A.1. We first take an open covering $\{V''_\alpha\}_\alpha$ of M and a family of smooth functions $\{\tau_\alpha : M \to [0,1] \mid \alpha \in A\}$ which satisfy

- $\overline{V'_\alpha} \subset V''_\alpha$ and $\overline{V''_\alpha} \subset V_\alpha$,
- $\tau_\alpha \equiv 1$ on V'_α and $\tau_\alpha \equiv 0$ on $M \setminus V''_\alpha$.

For each $\alpha \in A$ we define a map $I_\alpha : C^\infty(M) \to C^\infty(M)$ by

$$I_\alpha(f)(x) := (1 - \tau_\alpha(x))f(x) + \tau_\alpha(x) I^0_\alpha(f)(x),$$

where $I^0_\alpha(f)$ is the integration along fibers of $\pi_\alpha : \overline{V_\alpha} \to \overline{U_\alpha}$ with the normalization condition $I^0_\alpha(1) \equiv 1$. Fix any total order of $A = \{\alpha_1, \cdots, \alpha_n\}$ so that if $V_{\alpha_i} \cap V_{\alpha_j} \neq \emptyset$ and the dimension of π_{α_i} is bigger than that of π_{α_j} then $i > j$. Using this total order we can define the map $I : C^\infty(M) \to C^\infty(M)$ with the required properties by

$$I(f) := I_{\alpha_1} \cdots I_{\alpha_n}(f).$$

In fact the first four properties is clear. To show the Property 5 we show that for $f \in C^\infty(M)$ if $supp f \subset C$ for some admissible subset C, then we have $supp I_\beta(f) \subset C$ for all $\beta \in A$. Take $x \in supp I_\beta(f)$ and a sequence $\{x_n\}$ in M which satisfies $I_\beta(f)(x_n) \neq 0$ and converges to x. By definition of $I_\beta(f)$ we have $f(x_n) \neq 0$ or $\tau_\beta(x_n) I^0_\beta(f)(x_n) \neq 0$ for infinitely many n. The former case implies that $x \in supp f \subset C$. In the latter case by taking a subsequence we may assume that $x_n \in V''_\beta$ and $I^0_\beta(f)(x_n) \neq 0$ for all n. In particular we have $x \in \overline{V''_\beta} \subset V_\beta$. Since I^0_β is the integration along fibers there exist a sequence $\{y_n \in \pi_\beta^{-1}\pi_\beta(x_n)\}$ such that $f(y_n) \neq 0$ for all n. By taking a subsequence we may assume $\{y_n\}$ converges to some $y \in \pi_\beta^{-1}\pi_\beta(x) \cap supp f \subset C$. Since C is admissible we have $x \in \pi_\beta^{-1}\pi_\beta(x) \subset C$. □

B Proof of Lemma 2.26

Proof. Let $H_1, H_2, \ldots, H_m$ be the elements of A. Without loss of generality we assume that $H_i \supset H_j$ implies $i \leq j$. We construct a family of open sets $V_i^{(j)}$ $(1 \leq i \leq j \leq m)$ by induction on $1 \leq i \leq m$. For the construction with $i = i_0$ we assume the following properties.

(A1) $V_i^{(j)}$ contains the closure of $V_i^{(j+1)}$ for all $1 \leq i < i_0$ and $i \leq j < m$.

(A2) If $x \in V_i^{(i)}$ for $1 \leq i < i_0$, then we have $G_x \subset H_i$.

(A3) For $x \in M$ with $G_x = H_i$ for some $1 \le i < i_0$, we have

$$x \in \bigcup_{\{j \mid H_j \supset H_i\}} V_j^{(m)}.$$

(A4) If the intersection $V_i^{(j)} \cap V_j^{(j)}$ is not empty for $1 \le i < j < i_0$, then we have $H_i \supset H_j$.

If $i_0 = 1$, then the above is the empty assumption. For $1 \le i_0 \le m$, using the above properties as the assumption of induction, we construct $V_{i_0}^{(j)}$ $(i_0 \le j \le m)$ which satisfy the above properties with replacement of i_0 by $i_0 + 1$.

Suppose $1 \le i_0 < m$ and assume (A1),(A2),(A3) and (A4). Then (A3) implies that the closed set

$$K_{i_0} := M^{H_{i_0}} \setminus \bigcup_{\{k \mid H_k \supsetneq H_{i_0}\}} V_k^{(m)}$$

is contained in $\{x \in M \mid G_x = H_{i_0}\}$, where $M^{H_{i_0}}$ is the fixed point set $M^{H_{i_0}} = \{x \in M \mid G_x \supset H_{i_0}\}$. Hence (A2) implies that K_{i_0} does not intersect with the open set

$$\bigcup_{\{j < i_0 \mid H_{i_0} \not\subset H_j\}} V_j^{(i_0 - 1)}.$$

Let L_{i_0} be the closure of

$$\bigcup_{\{j < i_0 \mid H_{i_0} \not\subset H_j\}} V_j^{(i_0)}.$$

Then (A1) implies $K_{i_0} \cap L_{i_0} = \emptyset$. Since K_{i_0} is a subset of $\{x \in M \mid G_x = H_{i_0}\}$, there is an open neighborhood V of the closed set K_{i_0} in the complement of L_{i_0} such that for each $x \in V$ we have $G_x \subset H_{i_0}$. Now we take a decreasing sequence of open neighborhoods $V_{i_0}^{(j)}$ $(i_0 \le j \le m)$ of K_{i_0} so that $V_{i_0}^{(i_0)} = V$, $V_{i_0}^{(m)} \supset K_{i_0}$ and $V_{i_0}^{(j)}$ contains the closure of $V_{i_0}^{(j+1)}$ for $i_0 \le j < m$. We can choose the decreasing sequence so that the open sets $V_{i_0}^{(j)}$ $(i_0 \le j \le m)$ are G-invariant because the quotient space M/G is a regular space.

Then it is straightforward to check (A1),(A2),(A3) and (A4) are satisfied with i_0 replaced by $i_0 + 1$.

The family of open sets $\{V_{H_i} := V_i^{(m)}\}_{1 \le i \le m}$ is an open covering of M and satisfies the required properties. □

C Proof of Lemma 3.10

Proof of Lemma 3.10. If there is a function f satisfying the property in (1), then $\rho_{\epsilon,a}$ is constructed as follows: For each $\epsilon > 0$ let $\rho_\epsilon : \mathbb{R} \to [0,1]$ be a smooth non-increasing function such that $\rho_\epsilon(l) = 1$ for $l \leq 0$, $\rho_\epsilon(l) = 0$ for $l \geq 2/\epsilon$ and $|d\rho_\epsilon(l)| < \epsilon$ for $i \in \mathbb{R}$. Then the composition $\widetilde{\rho}_{a,\epsilon}(x) = \rho_\epsilon(f(x)-a)$ has the required properties.

Not we construct f by smoothing the length function as follows. Fix a point $x_0 \in M$. Let $f_0 : M \to \mathbb{R}$ be the length from x_0. Then f_0 is a Lipschitz continuous function with Lipschitz constant 1. Since M is complete, f_0 is a proper function such that $f_0^{-1}((-\infty, c])$ is compact for any c. Let $\{int(D_{x_\gamma}(R_\gamma))\}$ be a locally finite open covering of M by open disks centered in x_γ with radius R_γ. Fix an isometry $TM_{x_\gamma} \cong \mathbb{R}^n$. We also assume that the exponential map centered in x_γ gives a coordinate of $M_\gamma = int(D_{x_\gamma}(R_\gamma))$, and the derivative of the exponential map and its inverse at any point has bounded by 2 with respect to operator norm. In particular f_0 has Lipschitz constant 2 for the standard metric on $\mathbb{R}^n$. We use this coordinate in the following local construction. Let $\{\rho_\gamma\}$ be a smooth partition of unity for it. Let $0 < r_\gamma < R_\gamma$ be the radius of the smallest disk centered in x_r containing the image of the support of ρ_γ. Let C_γ be the maximal value of $|d\rho_\gamma|$ for the standard metric on $\mathbb{R}^n$. Let n_γ be the number of open disks in the locally finite covering which intersects $D_{x_\gamma}(r_\gamma)$. Take a smooth function $K : \mathbb{R}^n \to \mathbb{R}$ satisfying $\int K(y)dy = 1$ and $K(y) = 0$ if $|y| > min\{1, (R_\gamma - r_\gamma)/2, 1/(n_\gamma C_\gamma)\}$. Then the smoothing of f defined by $f_\gamma(x) = \int K(x-y) f_0(y)dy$ $(x \in D_{x_\gamma}(r_\gamma))$ is Lipschitz continuous with Lipschitz constant 2 for the standard metric on $\mathbb{R}^n$, and satisfies $|f_\gamma(x) - f_0(x)| < min\{1, 2/(n_\gamma C_\gamma)\}$ for $x \in D_{x_\gamma}(r_\gamma)$. Now define f to be $\sum_\gamma \rho_\gamma f_\gamma$. Then $|f - f_0| \leq 1$. In particular f is also a proper map and $f^{-1}((-\infty, c])$ is compact for any c. Decompose df as follows: $df = (\sum_\gamma \rho_\gamma df_\gamma) + (\sum_\gamma (d\rho_\gamma) f_0) + (\sum_\gamma d\rho_\gamma (f_\gamma - f_0))$ Since the second term is zero, we have $|df| \leq \sum_\gamma \rho_\gamma \cdot |df_\gamma| + \sum_\gamma |d\rho_\gamma||f_\gamma - f_0|$. Both terms are bounded from our construction. □

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