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A numerical method for solving the inverse heat conduction problem without initial value

by

Y. B. Wang, J. Cheng, J. Nakagawa,
and M. Yamamoto

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

A NUMERICAL METHOD FOR SOLVING THE INVERSE HEAT CONDUCTION PROBLEM WITHOUT INITIAL VALUE

Y. B. WANG, J. CHENG, J. NAKAGAWA, AND M. YAMAMOTO

Abstract. We consider the inverse heat conduction problem for the one-dimensional heat equation, where we are requested to determine a boundary value at one end of a spatial interval over a time interval and an initial value by means of Cauchy data at another end. By the existing theory we can prove the uniqueness in determining both a boundary value and an initial value, and our method does not require any initial value. We test our numerical method and show stable numerical reconstruction.

Key words: inverse heat conduction problem, truncated Fourier series, numerical reconstruction

AMS 2000 Mathematical Subject Classification:
35R25, 35R30, 35K05

1. Introduction

The inverse heat conduction problem arises in most thermal manufacturing processes of solids and has recently attracted much attention. In this inverse problem, for the heat equation, one is requested to reconstruct a heat flux or boundary temperature on an inaccessible subboundary. The typical case is the determination of the heat flux on an inaccessible subboundary through measurements on an accessible subboundary. In the real applications, only discrete data with noises at finite points are available. This problem is known to be extremely ill-posed (e.g., [1]), that is, small perturbations in data may cause dramatically large errors in the solution. Therefore for stable numerical reconstruction, we need stabilization or regularization techniques. For general treatment of such techniques, see for example, [5], [7].

As for numerical methods for the inverse heat conduction problem, there are many works and we refer for example, to [1], [2], [4], [6], [8], [9], [10], [11], [16], [19], [20], [21], [22], [23], [24] and the references therein. In most of the existing works, initial data are assumed to be given, although one knows the uniqueness in determining both a boundary value and an initial value by Cauchy data (e.g., [12], [17]). In particular, in [24], the author develops a numerical method for the inverse heat conduction problem without information of initial values and our method is more direct.

In many practical situations such as an on-line testing, we cannot know the initial condition for example because we have to estimate the problem for the heat process which was already started.

The main purpose of this paper is to propose a numerical method which does not need initial data for reconstruction of boundary values and is stable against the intrinsic instability of the inverse heat conduction problem. Our method can be implemented for heat equations with variable coefficients in general spatial dimensions, but here we will exclusively discuss the one-dimensional heat equation with constant coefficients for demonsrating the essence of our method. As for a basic idea for the numerical scheme in such a general case, see Appendix.

As for other kinds of ill-posed problems for the heat equation, we refer to [14], [18], [27].

2. Formulation of the problem and algorithm

We consider the one-dimensional heat equation:

$$\partial_t u(x,t) = \alpha \partial_x^2 u(x,t), \qquad 0 < x < \ell,\ t > 0. \tag{2.1}$$

Here $\alpha > 0$ is a given constant. For (2.1), we discuss

Inverse heat conduction problem. Determine

$$f(t) \equiv \partial_x u(0,t), \quad t > 0 \tag{2.2}$$

and

$$u_0(x) \equiv u(x,0), \qquad 0 < x \le \ell \tag{2.3}$$

from

$$\partial_x u(\ell,t) \equiv g(t), \qquad t > 0 \tag{2.4}$$

and

$$u(\ell, t) \equiv h(t), \qquad t > 0. \tag{2.5}$$

Then we can prove the uniqueness in the heat conduction problem: $g(t), h(t)$, $0 \le t \le T$ uniquely determine $f(t)$, $0 \le t \le T$ and $u_0(x)$, $0 \le x \le 1$ (e.g., Theorem 3.3.10 (p.63) inl [12]) and §1 of Chapter IV in [17]). As for related theoretical results, see [15], [25] for example.

Let $0 < t_1 < ... < t_M$ be given. Our target is to reconstruct the value of $f(t)$ and $u_0(x)$ from discrete noisy values of $g(t_j)$ and $h(t_j)$, $j = 1, 2, \cdots, M$.

Our primary interest is the reconstruction of $f(t)$, and the boundary value is often a more serious influence for controlling the heat process from the practical point of view.

Set

$$\lambda_n = \alpha \frac{n^2 \pi^2}{\ell^2}, \quad n \ge 0$$

and

$$G(t, x, y) = \frac{2}{\ell} \sum_{n=1}^{\infty} e^{-\lambda_n t} \cos \frac{n\pi}{\ell} x \cos \frac{n\pi}{\ell} y + \frac{1}{\ell}.$$

Then we can represent a solution u to (2.1) by

$$u(x,t) = \textstyle\sum_{n=0}^{\infty} A_n(x) e^{-\lambda_n t} - \int_0^t G(t-s, x, 0) f(s) ds + \int_0^t G(t-s, x, \ell) g(s) ds,$$
$$0 \le x \le \ell, \; t > 0,$$

where

$$A_0(x) = \frac{1}{\ell} \int_0^{\ell} u_0(y) dy$$

and

$$A_n(x) = \frac{2}{\ell} \left(\int_0^{\ell} u_0(y) \cos \frac{n\pi}{\ell} y dy \right) \cos \frac{n\pi}{\ell} x.$$

([3], [13], [28]).

The terms $\int_0^t G(t-s, x, 0) f(s) ds$ and $\int_0^t G(t-s, x, \ell) g(s) ds$ cause the instability for the computation. Hence we will try to avoid the integration by solving the following forward problem.

Define

$$
\begin{cases}
\partial_t v = \alpha \partial_x^2 v(x,t), & 0 < x < \ell,\ t > 0,\\
\partial_x v(0,t) = 0,\\
\partial_x v(\ell,t) = g(t),\\
v(x,0) = \dfrac{x^2}{2\ell} g(0)
\end{cases}
\tag{2.6}
$$

and

$$
\omega(x,t) = u(x,t) - v(x,t), \qquad 0 < x < \ell,\ t > 0.
$$

Then we have

$$
\begin{cases}
\partial_t \omega = \alpha \partial_x^2 \omega(x,t), & 0 < x < \ell,\ t > 0,\\
\partial_x \omega(0,t) = f(t),\\
\partial_x \omega(\ell,t) = 0,\\
\omega(\ell,t) = h(t) - v(\ell,t),
\end{cases}
\tag{2.7}
$$

Here $v(\ell,t)$ can be taken by solving the forward problem of (2.6), so $\omega(\ell,t)$ is given.

The solution to (2.7) can be represented by

$$
\omega(x,t) = \sum_{n=0}^{\infty} B_n(x) e^{-\lambda_n t} - \int_0^t G(t-s,x,0) f(s) ds, \qquad 0 \le x \le \ell,\ t > 0,
$$

where

$$
B_0(x) = \frac{1}{\ell} \int_0^{\ell} \omega(y,0) dy
$$

and

$$
B_n(x) = \frac{2}{\ell} \int_0^{\ell} \left(\omega(y,0) \cos \frac{n\pi}{\ell} y dy \right) \cos \frac{n\pi}{\ell} x.
$$

Suppose that $\{0 = t_0 < t_1 < \cdots < t_M = T\}$ is a uniform gird of the time interval $[0,T]$. Then at time t_j, we have the following approximation equation:

$$
\begin{aligned}
\omega(\ell,t_j) &= \sum_{n=0}^{N} B_n(\ell) e^{-\lambda_n t_j} - \int_0^{t_j} G(t_j - s, \ell, 0) f(s) ds\\
&= \sum_{n=0}^{N} B_n(\ell) e^{-\lambda_n t_j} - \sum_{i=1}^{j} f(t_i) \int_{t_{i-1}}^{t_i} G(t_j - s, \ell, 0) ds\\
&= \sum_{n=0}^{N} B_n(\ell) e^{-\lambda_n t_j} - \sum_{i=1}^{j} \left(\int_0^{t_i} G(t_j - s, \ell, 0) ds - \int_0^{t_{i-1}} G(t_j - s, \ell, 0) ds \right) f(t_i)
\end{aligned}
$$

Theorem 2.1. *Define*

$$\widehat{G}_i(x,t) = \begin{cases} -\displaystyle\int_0^t G(t-s,x,0)ds, & 0 < t \le t_i, \\ -\displaystyle\int_0^{t_i} G(t-s,x,0)ds, & t > t_i. \end{cases}$$

Then $\widehat{G}_i(x,t)$ *satisfies*

$$\begin{cases} \partial_t \widehat{G}_i = \alpha \partial_x^2 \widehat{G}_i, & 0 < x < \ell,\ t > 0, \\ \partial_x \widehat{G}_i(0,t) = \phi_i(t), \\ \partial_x \widehat{G}_i(\ell,t) = 0, \\ \widehat{G}_i(x,0) = 0. \end{cases} \tag{2.8}$$

Here

$$\phi_i(t) = \begin{cases} 1, & 0 < t \le t_i, \\ 0, & t_i < t. \end{cases}$$

Proof: In terms of a change of independent variables: $(x,t) \longrightarrow (\widehat{x},\widehat{t})$ defined by $\widehat{x} = \frac{x}{\ell}$ and $\widehat{t} = \frac{\alpha t}{\ell^2}$, we can reduce the heat equation (2.1) to $\frac{\partial u}{\partial \widehat{t}}(\widehat{x},\widehat{t}) = \frac{\partial^2 u}{\partial \widehat{x}^2}(\widehat{x},\widehat{t})$, $0 < \widehat{x} < 1$, $\widehat{t} > 0$, so that we can assume that $\ell = 1$ and $\alpha = 1$ without loss of generality. Here we write x and t in place of $\widehat{x}$ and $\widehat{t}$. Since

$$\widehat{G}_i(x,t) = \begin{cases} -\displaystyle\int_0^t G(s,x,0)ds, & 0 < t \le t_i, \\ -\displaystyle\int_0^{t_i} G(t-s,x,0)ds, & t > t_i \end{cases}$$

and

$$\partial_x^2 G(t,x,0) = \partial_t G(t,x,0), \quad 0 < x < 1,\ t > 0,$$

it is easy to see that

$$\partial_x^2 \widehat{G}_i = \partial_t \widehat{G}_i, \quad 0 < x < 1,\ t > 0.$$

Define

$$G_0(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}, \qquad 0 < x < 1, t > 0$$

and

$$\theta(x,t) = \sum_{n=-\infty}^{\infty} G_0(x+2n,t).$$

We know that

$$\lim_{x\to 0+}\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)d\tau = -\frac{1}{2}, \qquad \lim_{x\to 1-}\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)d\tau = 0$$

(Lemma 6.2.3 and Lemma 6.2.5, p. 60-61, [3]).

Also, since we can prove

$$G(t,x,0) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi^2 t}\cos n\pi x = \sum_{n=-\infty}^{\infty} e^{n\pi x i - n^2\pi^2 t} = 2\theta(x,t)$$

similarly to Theorem 4.1, p. 90, [28], for $t > 0$ we have

$$\lim_{x\to 0+}\int_0^t \partial_x G(t-\tau,x,0)d\tau = 2\lim_{x\to 0+}\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)d\tau = -1$$

and

$$\lim_{x\to 1-}\int_0^t \partial_x G(t-\tau,x,0)d\tau = 2\lim_{x\to 1-}\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)d\tau = 0.$$

Furthermore, when $t > t_i$,

$$\begin{aligned}
&\lim_{x\to 0+}\int_0^{t_i} \partial_x G(t-\tau,x,0)d\tau = 2\lim_{x\to 0+}\int_0^{t_i} \frac{\partial\theta}{\partial x}(x,t-\tau)d\tau \\
= \;& 2\lim_{x\to 0+}\int_0^{t_i}\left\{\frac{\partial G_0}{\partial x}(x,t-\tau) + J(x,t-\tau)\right\}d\tau
\end{aligned}$$

(e.g., p.60, [3]). According to the result of Lemma 6.2.1 on p.60 in [3], we know that

$$\lim_{x\to 0} J(x,t) = 0$$

for $t > 0$. Moreover, since

$$\lim_{x\to 0}\partial_x G_0(x,t) = \lim_{x\to 0}\frac{-xe^{-x^2/4t}}{2t\sqrt{4\pi t}} = 0, \qquad t > 0,$$

we have

$$\lim_{x\to 0+}\int_0^{t_i}\partial_x G(t-\tau,x,0)d\tau = 0, \qquad t > t_i.$$

This completes the proof of the theorem.

This theorem can be proved in general dimensions but here we use a more direct way for the proof in the case of the one-dimensional case.

Let $\widehat{G}_{j,i} = \widehat{G}_i(\ell,t_j)$ for $j \ge i$ and $\widehat{G}_{j,i} = 0$ for $j < i$. Then

$$\omega(\ell,t_j) = \sum_{n=0}^{N} B_n(\ell)e^{-\lambda_n t_j} + \sum_{i=1}^{j}(\widehat{G}_{j,i} - \widehat{G}_{j,i-1})f(t_i).$$

We define an $M \times (M+N+1)$ matrix P by

$$P = \{P_1 | P_2\},$$

where we set

$$P_1 = \begin{pmatrix} e^{-\lambda_0 t_1} & e^{-\lambda_1 t_1} & \dots & e^{-\lambda_N t_1} \\ e^{-\lambda_0 t_2} & e^{-\lambda_1 t_2} & \dots & e^{-\lambda_N t_2} \\ e^{-\lambda_0 t_3} & e^{-\lambda_1 t_3} & \dots & e^{-\lambda_N t_3} \\ \dots & \dots & \dots & \dots \\ e^{-\lambda_0 t_M} & e^{-\lambda_1 t_M} & \dots & e^{-\lambda_N t_M} \end{pmatrix}$$

and

$$P_2 = \begin{pmatrix} \widehat{G}_{1,1} & 0 & 0 & \cdots & 0 \\ \widehat{G}_{2,1} & \widehat{G}_{2,2} - \widehat{G}_{2,1} & 0 & \cdots & 0 \\ \widehat{G}_{3,1} & \widehat{G}_{3,2} - \widehat{G}_{3,1} & \widehat{G}_{3,3} - \widehat{G}_{3,2} & \cdots & 0 \\ \dots & \dots & \dots & \cdots & \dots \\ \widehat{G}_{M,1} & \widehat{G}_{M,2} - \widehat{G}_{M,1} & \widehat{G}_{M,3} - \widehat{G}_{M,2} & \cdots & \widehat{G}_{M,M} - \widehat{G}_{M,M-1} \end{pmatrix}.$$

Setting

$$W = \begin{pmatrix} \omega(\ell, t_1) \\ \omega(\ell, t_2) \\ \omega(\ell, t_3) \\ \dots \\ \omega(\ell, t_M) \end{pmatrix}, \quad X = \begin{pmatrix} B_0(\ell) \\ B_1(\ell) \\ \dots \\ B_N(\ell) \\ f(t_1) \\ f(t_2) \\ \dots \\ f(t_M) \end{pmatrix},$$

we have

$$PX = W.$$

By solving this linear equations, we can obtain values $f(t_i), 1 \le i \le M$. However, since P is an $M \times (N+M+1)$ matrix, it is under-determining. For specifying a solution more stably, we introduce an additional constraint for a solution $f(t_i)$, $i = 1, ..., M$. That is, assume $M = 2M_0$ and $f(t_{2i}) = f(t_{2i-1})$, $i = 1, \cdots, M_0$. Set an $M \times (M_0 + N + 1)$ matrix:

$$\widehat{P} = \{P_1 | \widehat{P}_2\},$$

where we define an $M \times M_0$ matrix $\widehat{P}_2$ and an $(M_0 + N + 1)$-vector $\widehat{X}$ by

$$\widehat{P}_2 = \begin{pmatrix} \widehat{G}_{1,1} & 0 & 0 & \cdots & 0 \\ \widehat{G}_{2,2} & 0 & 0 & \cdots & 0 \\ \widehat{G}_{3,2} & \widehat{G}_{3,3} - \widehat{G}_{3,2} & 0 & \cdots & 0 \\ \widehat{G}_{4,2} & \widehat{G}_{4,4} - \widehat{G}_{4,2} & 0 & \cdots & 0 \\ \widehat{G}_{5,2} & \widehat{G}_{5,4} - \widehat{G}_{5,2} & \widehat{G}_{5,5} - \widehat{G}_{5,4} & \cdots & 0 \\ \widehat{G}_{6,2} & \widehat{G}_{6,4} - \widehat{G}_{6,2} & \widehat{G}_{6,6} - \widehat{G}_{6,4} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \widehat{G}_{M-1,2} & \widehat{G}_{M-1,4} - \widehat{G}_{M-1,2} & \widehat{G}_{M-1,6} - \widehat{G}_{M-1,4} & \cdots & \widehat{G}_{M-1,M-1} - \widehat{G}_{M-1,M-2} \\ \widehat{G}_{M,2} & \widehat{G}_{M,4} - \widehat{G}_{M,2} & \widehat{G}_{M,6} - \widehat{G}_{M,4} & \cdots & \widehat{G}_{M,M} - \widehat{G}_{M,M-2} \end{pmatrix},$$

and

$$\widehat{X} = \begin{pmatrix} B_0(\ell) \\ B_1(\ell) \\ \cdots \\ B_N(\ell) \\ f(t_2) \\ f(t_4) \\ \cdots \\ f(t_M) \end{pmatrix}.$$

Then

$$\widehat{P}\widehat{X} = W$$

and it is an over-determined linear system with the extra constraints $f(t_{2i}) = f(t_{2i-1})$, $i = 1, ..., M_0$. Thanks to the constraint, we can limit the set of solutions and expect better numerical performances, which is verified by numerical tests in Section 3.

3. Numerical Results

In this section we will give some numerical examples to test the algorithm given in the previous section. First we will give an example where the exact solution is known. We add some random noises to measured data and use these data to test our algorithm.

We set

$$u(x,t) = e^{-K^2 t}\cos(Kx) + e^{-t}\sin(x), \qquad 0 < x < 1, 0 < t < 2,$$

and $0 = t_0 < t_1 < \cdots < t_{200} = 2$ is a uniform grid of $[0,2]$, i.e., $t_i = \frac{i}{100}$. $\frac{\partial u}{\partial x}(1,t_i)$, $u(1,t_i)$, $i = 1,2,\cdots,200$, are given and we want to calculate the value of $\frac{\partial u}{\partial x}(0,t)$ and $u(0,t)$. We will try different values of K.

Figures 1a and 1b are the numerical results where the measurement data has no noise. Henceforth, the solid lines in the figures indicate the exact values and dashed lines indicate the numerical results.

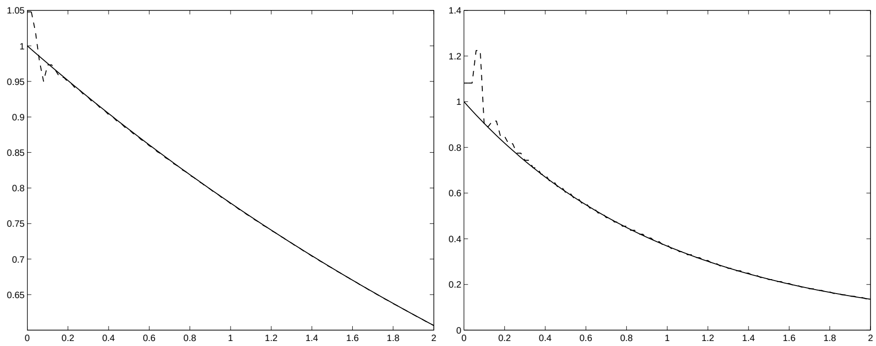

Figure 1a: $u(0,t)$ Figure 1b: $\frac{\partial u}{\partial x}(0,t)$

$K = 0.5$, without noise

From the figures, we can see that for $u(0,t)$ the calculated value is quite close to the exact value and for $\frac{\partial u}{\partial x}(0,t)$, the result is good when t is large enough. For small t, the numerical performances are bad because we do not know initial data.

Next we will give some random noises to the measurement data. Also, since the measurement data with random noise are oscillating, we will use some skills to smooth the measurement data first and then apply our algorithm. Here we use the numerical differentiation method to smooth the measurement data ([26]). Figure 2a-3a-4a and Figures 2b-3b-4b indicate numerical results for $u(0,t)$ and $\frac{\partial u}{\partial x}(0,t)$ in various cases of noise levels.

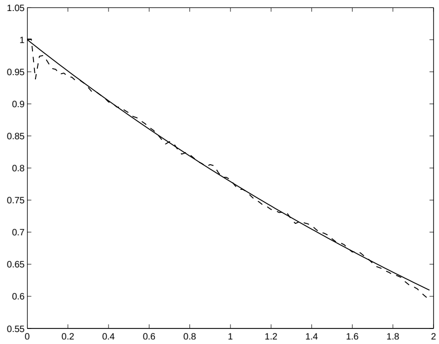

Figure 2a: $u(0,t)$

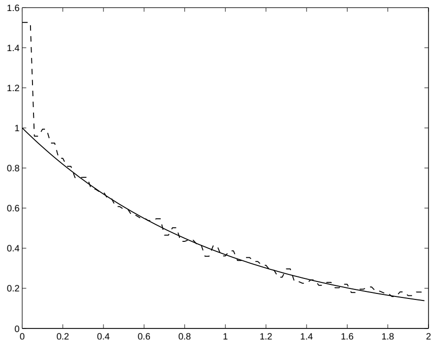

Figure 2b: $\frac{\partial u}{\partial x}(0,t)$

$K = 0.5$, with 0.5% random noise

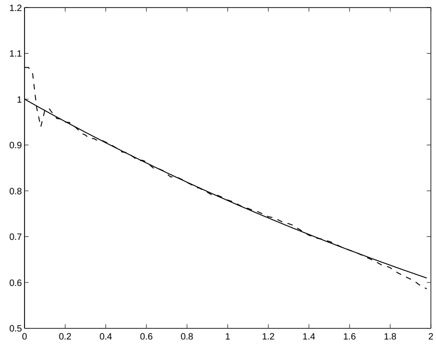

Figure 3a: $u(0,t)$

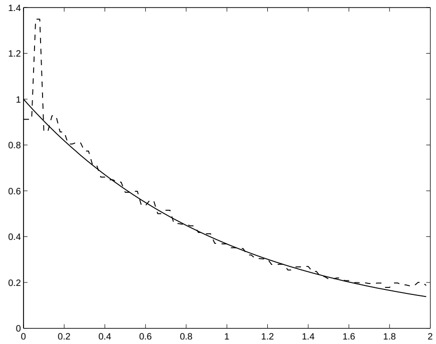

Figure 3b: $\frac{\partial u}{\partial x}(0,t)$

$K = 0.5$, with 1% random noise

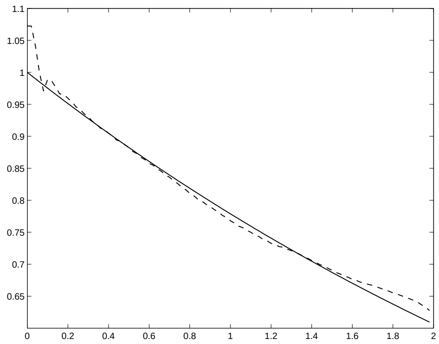

Figure 4a: $u(0,t)$

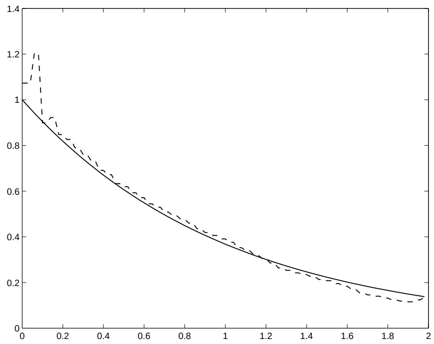

Figure 4b: $\frac{\partial u}{\partial x}(0,t)$

$K = 0.5$, with 3% random noise

Next we will show the numerical results with different values of K.

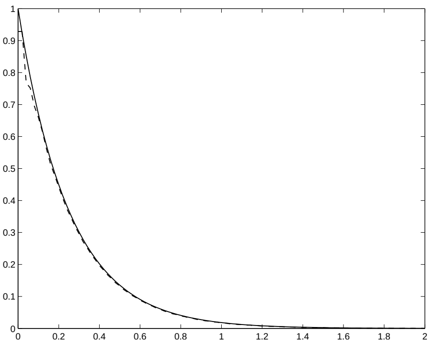

Figure 5a: $u(0,t)$

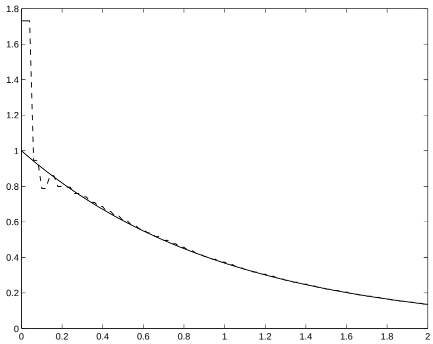

Figure 5b: $\frac{\partial u}{\partial x}(0,t)$

$K=2$, without noise

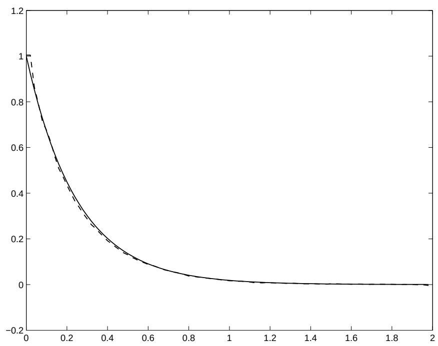

Figure 6a: $u(0,t)$

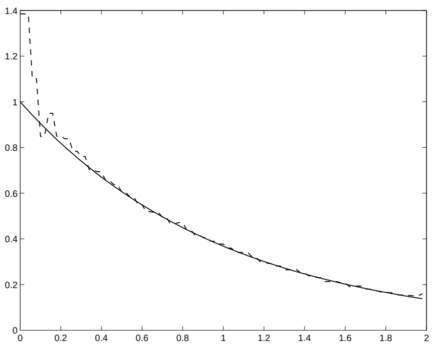

Figure 6b: $\frac{\partial u}{\partial x}(0,t)$

$K=2$, with 0.5% random noise

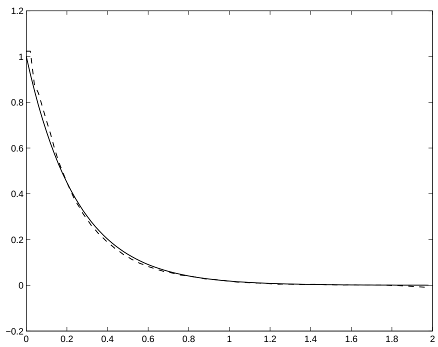

Figure 7a: $u(0,t)$

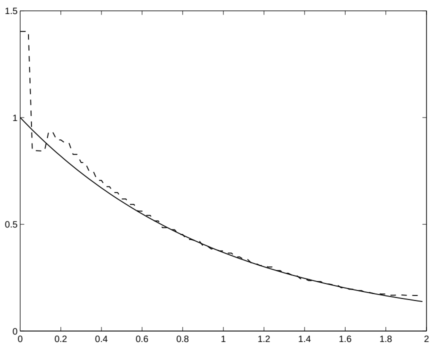

Figure 7b: $\frac{\partial u}{\partial x}(0,t)$

$K=2$, with 1% random noise

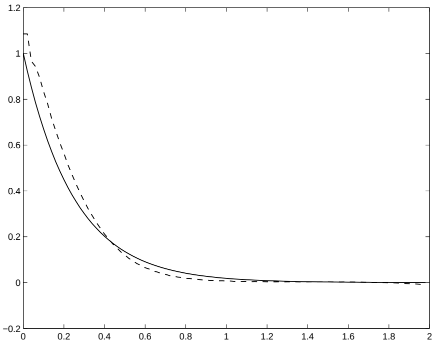

Figure 8a: $u(0,t)$

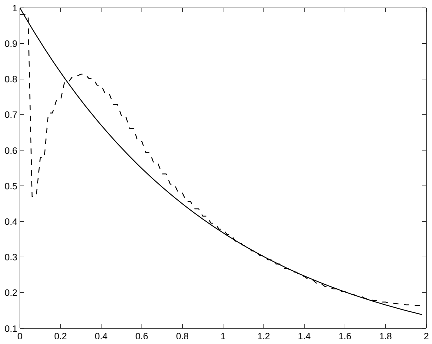

Figure 8b: $\frac{\partial u}{\partial x}(0,t)$

$K = 2$, with 3% random noise

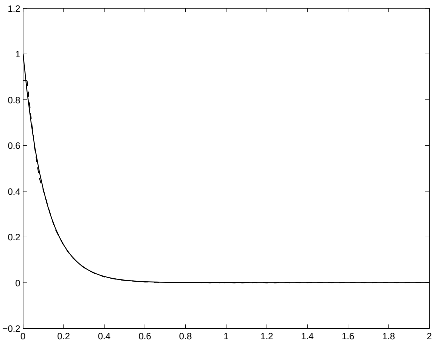

Figure 9a: $u(0,t)$

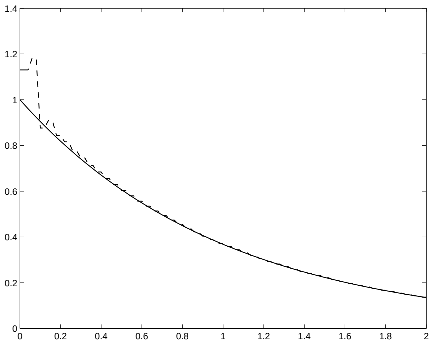

Figure 9b: $\frac{\partial u}{\partial x}(0,t)$

$K = 3$, without noise

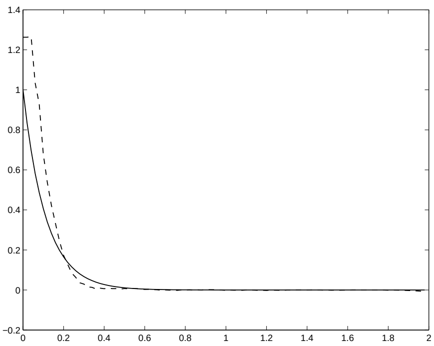

Figure 10a: $u(0,t)$

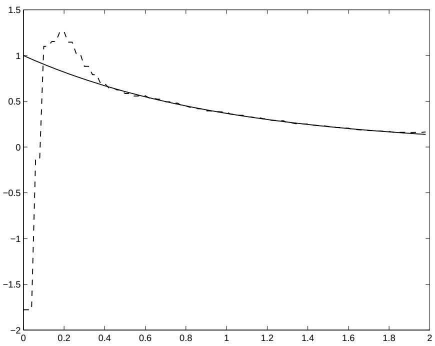

Figure 10b: $\frac{\partial u}{\partial x}(0,t)$

$K = 3$, with 0.5% random noise

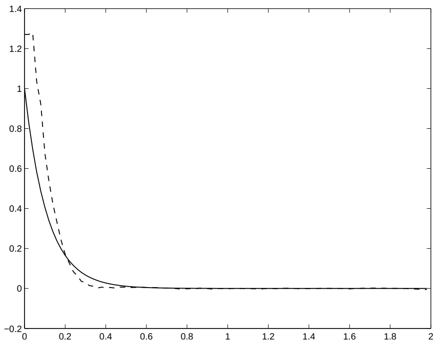

Figure 11a: $u(0,t)$

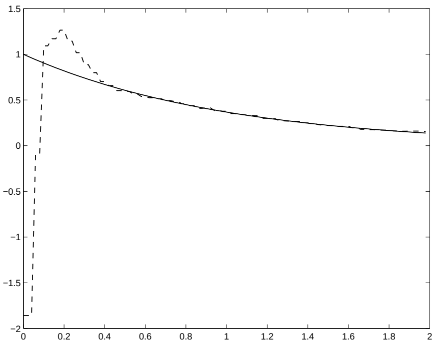

Figure 11b: $\frac{\partial u}{\partial x}(0,t)$

$K = 3$, with 1% random noise

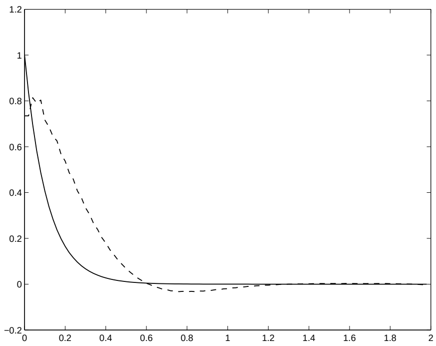

Figure 12a: $u(0,t)$

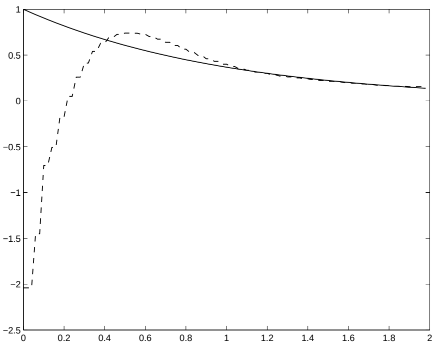

Figure 12b: $\frac{\partial u}{\partial x}(0,t)$

$K = 3$, with 3% random noise

From these numerical tests, we can see that errors are large near $t = 0$, but better when t is large enough; the results for $u(0,t)$ are quite good, and results for $\frac{\partial u}{\partial x}(0,t)$ are acceptable.

We will also show the L^2-norms of the differences between the exact solution and the reconstructed solutions in the interval $(0.2, 2)$.

In the next example, we will show the case in which the exact solution is not given. We will use the finite difference method to solve the forward problem and take data at one side of the boundary. Taking some random noises and applying our method, we can obtain reconstructed values at another side of the boundary. We compare these results with the data given by the finite difference method.

Table 1. Absolute errors with different noise level and K.

K	noise level	errors for $u(0,t)$	errors for $\frac{\partial u}{\partial x}(0,t)$
0.5	0	0.00095	0.0090
0.5	0.5%	0.0058	0.0270
0.5	1%	0.0080	0.0282
0.5	3%	0.0127	0.0296
2	0	0.0035	0.0079
2	0.5%	0.0070	0.0158
2	1%	0.0086	0.0316
2	3%	0.0348	0.0595
3	0	0.0011	0.0091
3	0.5%	0.0150	0.1132
3	1%	0.0157	0.1137
3	3%	0.1342	0.2568

The problem itself satisfies the following equations:

$$\begin{cases} u_t = u_{xx}, \qquad 0 < x < 1,\, 0 < t < 2, \\ u(0,t) = 10(\sin(3t) + 1), \\ u(1,t) = 2(t-1)^2, \end{cases} \tag{3.1}$$

The numerical results are given in the following:

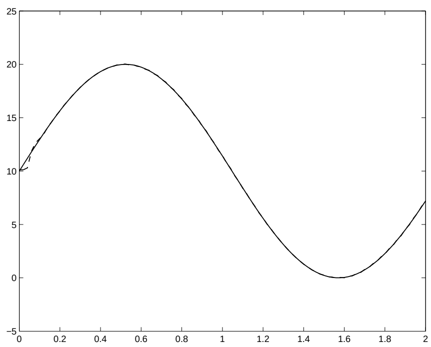

Figure 13a: $u(0,t)$

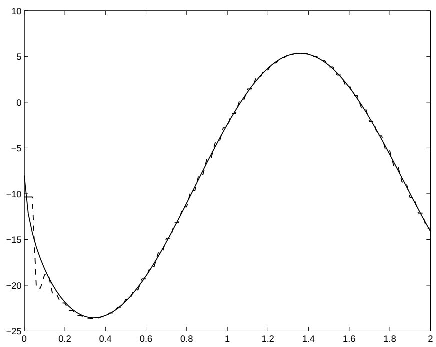

Figure 13b: $\frac{\partial u}{\partial x}(0,t)$

without random noise

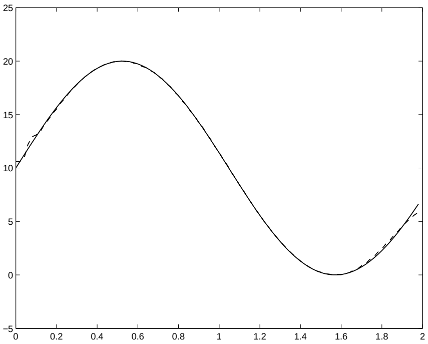

Figure 14a: $u(0,t)$

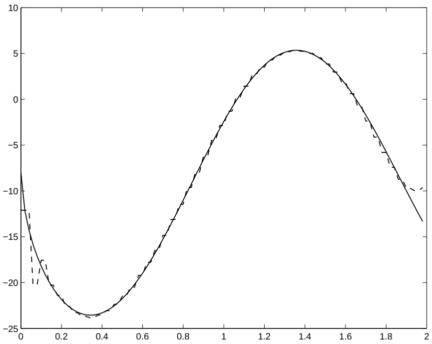

Figure 14b: $\frac{\partial u}{\partial x}(0,t)$

with 0.5% random noise

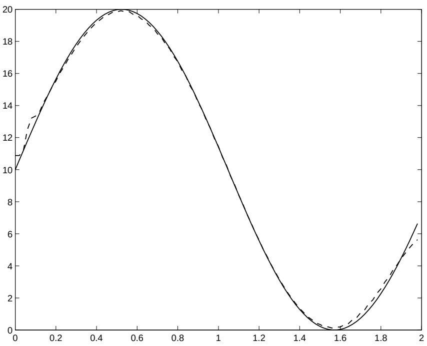

Figure 15a: $u(0,t)$

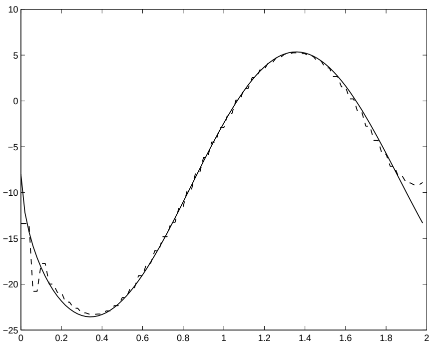

Figure 15b: $\frac{\partial u}{\partial x}(0,t)$

with 1% random noise

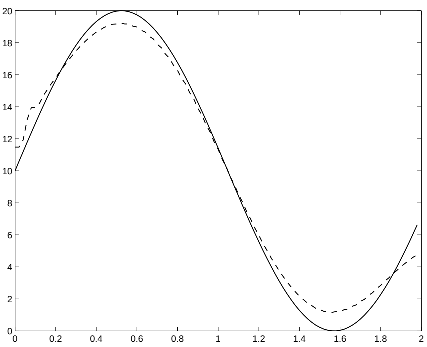

Figure 16a: $u(0,t)$

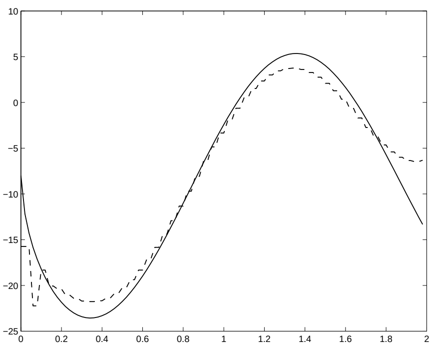

Figure 16b: $\frac{\partial u}{\partial x}(0,t)$

with 3% random noise

The following is the table of the L^2 norm of the errors in the interval $(0.2, 2)$.

Table 2. Absolute errors with different noise level.

noise level	errors for $u(0,t)$	errors for $\frac{\partial u}{\partial x}(0,t)$
0	0.0426	0.4233
0.5%	0.1604	0.8297
1%	0.2675	1.0645
3%	0.9866	2.4189

4. Appendix. Derivation of our numerical scheme for a heat equation with variable coefficients in general dimensions

Let $\Omega \subset R^d$ be a bounded domain with smooth boundary $\partial\Omega$ and, $\overline{\Omega} = \Omega \cup \partial\Omega$, $\nu = \nu(x) = (\nu_1, ..., \nu_d)$ denote the unit outward normal vector to $\partial\Omega$ at x, and $\Gamma \subset \partial\Omega$ be a subboundary. We consider

$$Av(x) = \operatorname{div}(a(x)\nabla v(x)) + b(x)v(x), \quad x \in \Omega,$$

where a and b are sufficiently smooth and $a > 0$ on $\overline{\Omega}$. We set

$$\frac{\partial v}{\partial \nu_A}(x) = a(x) \sum_{i=1}^{d} \frac{\partial v}{\partial x_j} \nu_j, \qquad x \in \partial\Omega.$$

We consider

Inverse heat conduction problem. Determine

$$f(x,t) \equiv \frac{\partial u}{\partial \nu_A}(x,t), \qquad x \in \partial\Omega \setminus \Gamma,\ 0 < t < T$$

and

$$u(x,0), \qquad x \in \Omega$$

from

$$\partial_t u(x,t) = Au(x,t), \qquad x \in \Omega,\ 0 < t < T,$$
$$\frac{\partial u}{\partial \nu_A}(x,t) = g(x,t), \qquad x \in \Gamma,\ 0 < t < T$$

and

$$u(x,t) = h(x,t), \qquad x \in \Gamma,\ 0 < t < T.$$

Similarly to (2.7), we can assume that $g(x,t) = 0$, $x \in \Gamma$, $0 < t < T$. We consider the set of the eigenvalues of $-A$ with the boundary condition $\frac{\partial v}{\partial \nu_A} = 0$ on $\partial\Omega$ and number them according to the multiplicities:

$$\lambda_1 \le \lambda_2 \le .... \tag{1}$$

That is, if for an eigenvalue λ_i, there exist ℓ linearly independent eigenfunctions $v_1, ..., v_\ell$: $-Av_j = \lambda_i v_j$ in Ω and $\frac{\partial v_j}{\partial \nu_A} = 0$ on $\partial\Omega$ for $j = 1, .., \ell$, then λ_i appear ℓ-times in the sequence (1).

By Itô [13] and $g = 0$ on Γ, there exists the fundamental solution $G(t, x, y)$, $x, y \in \Omega, t > 0$ such that we can represent $u = u(x, t)$ by

$$u(x,t) = \sum_{n=1}^{\infty} B_n(x) e^{-\lambda_n t} + \int_0^t \int_{\partial\Omega\setminus\Gamma} G(t-s, x, y) f(y, s) dS_y ds, \quad x \in \overline{\Omega},\, 0 < t < T \tag{2}$$

with some functions $B_n(x)$, provided that u satisfies some conditions on the smoothness. In order to numerically reconstruct $f(x, t) = \frac{\partial u}{\partial \nu_A}(x, t)$, $x \in \partial\Omega\setminus\Gamma$, $0 < t < T$, we introduce a suitable system $\{\psi_k\}_{1\le k\le N_1}$ of linearly independent functions on $(\partial\Omega \setminus \Gamma) \times (0, T)$ and we approximate f by a finite sum $\sum_{k=1}^{N_1} a_k\psi_k$ and replace $\sum_{n=1}^{\infty} B_n(x)e^{-\lambda_n t}$ by $\sum_{n=1}^{N_2} B_n(x)e^{-\lambda_n t}$ with suitable natural number N_2. Then, choosing $x_\ell \in \Gamma$, $\ell = 1, ..., L_1$, $t_m \in (0, T)$, $m = 1, ..., M_1$, on the basis of (2), we search an approximation $f(x, t) \sim \sum_{k=1}^{N_1} a_k\psi_k(x, t)$ satisfying

$$h(x_\ell, t_m) = \sum_{k=1}^{N_1} a_k G_k(x_\ell, t_m) + \sum_{n=1}^{N_2} B_n(x_\ell) e^{-\lambda_n t_m}, \quad 1 \le \ell \le L_1,\, 1 \le m \le M_1. \tag{3}$$

Here we set

$$G_k(x, t) = \int_0^t \int_{\partial\Omega\setminus\Gamma} G(t-s, x, y)\psi_k(y, s) dS_y ds, \quad 1 \le k \le N_1. \tag{4}$$

Considering (3) as a linear system and solving with respect to $a_1, ..., a_{N_1}$, $B_n(x_\ell)$, $\ell = 1, ..., L_1$, $n = 1, ..., N_2$, we can obtain an approximation $\sum_{k=1}^{N_1} a_k\psi_k$ for $f = \frac{\partial u}{\partial \nu_A}$ on $(\partial\Omega \times \Gamma) \times (0, T)$. In order to calculate the coefficient matrix of the linear system (3), we have to obtain N_2 eigenvalues $\lambda_1, ...\lambda_{N_2}$ and $G_k(x_\ell, t_m)$. For λ_k, we can use a suitable numerical method for finding eigenvalues. Moreover, under suitable conditions on ψ_k, since we can verify (e.g., [13]) that G_k satisfies

$$\partial_t G_k = AG_k(x, t), \qquad x \in \Omega,\, 0 < t < T,$$

$$\frac{\partial G_k}{\partial \nu_A}(x, t) = \begin{cases} \psi_k(x, t), & x \in \partial\Omega \setminus \Gamma,\, 0 < t < T, \\ 0, & x \in \Gamma,\, 0 < t < T, \end{cases}$$

$$G_k(x, 0) = 0, \qquad x \in \Omega,$$

we can calculate $G_k(x_\ell, t_m)$ by numerically solving this initial value - boundary value problem. Due to the ill-posedness of the inverse heat conduction problem, we have to choose appropriate N_1, N_2 and apply a suitable regularization in stably solving the linear system (3), but here we will not discuss details. We note that in the one dimensional case in Section 2, $\partial\Omega \setminus \Gamma$ consists of one point $x = 0$, and that we choose

$$\psi_k(x,t) = \psi_k(t) = \begin{cases} 1, & 0 < t \le t_k, \\ 0, & t > t_k. \end{cases}$$

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School of Mathematical Sciences, Fudan University, Shanghai 200433, CHINA

E-mail address: ybwang@fudan.edu.cn

School of Mathematical Sciences, Fudan University, Shanghai 200433, CHINA

E-mail address: jcheng@fudan.edu.cn

Energy & Environment Research Lab., Advanced Technology Research Laboratories, Technical Development Bureau, Nippon Steel Corporation, 20-1 Shintomi Futtsu Chiba, 293-8511, Japan

E-mail address: nakagawa.junichi@nsc.co.jp

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo, 153-8914 Japan

E-mail address: myama@ms.u-tokyo.ac.jp

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012