UTMS 2010–17 October 29, 2010

Isomonodromic deformation and 4-dimensional Painlevé type equations

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Abstract

The Painlevé equations are classified into eight types, and all of them are obtained from degenerations of the sixth Painlevé equation. This is the case that the phase space has dimension two. When we consider the four-dimensional case, we have the four systems which are the sources like the sixth Painleé equation. In this paper, we write down the Hamiltonian equations corresponding to the all of four systems which are obtained from deformation theory of Fuchsian equations. These are the well-known Garnier system with two independent variables, a Fuji-Suzuki system, a Sasano system, and the last one which is new.

Keywords. integrable systems, Painlevé equations, isomonodromic deformation.

2010 Mathematical Subject Classification Numbers. 33E17, 34M55, 34M56.

1 Introduction

As special functions defined by nonlinear differential equations, the Painlevé transcendents which were found by Paul Painlevé and his coworkers ([19, 5]) have the next place to the elliptic functions. While the equations are studied in many view points, we have two important aspects among them for the sake of classification. What we ought to mention first is the theory of Okamoto's space of initial conditions; due to it we can characterize the Painlevé differential equations using theory of rational surfaces ([18]). The rational surfaces which appear here is similar to rational elliptic surfaces, but these are slightly different from the elliptic surfaces. This description corresponds to the fact that differential equations for the elliptic functions appear as autonomous limits of the Painlevé equations. Although there are six types of Painlevé equations according to the classical literature, we draw distinction among the third Painlevé equations into three types according to the theory of rational surfaces. Hence we have eight types of the Painlevé equations ([20]).

The other important view point is a deformation theory of linear equations. These kinds of research are started by R. Fuchs at the beginning of the twentieth century and he obtained the sixth Painlevé equation from the isomonodromic deformation of the second order Fuchsian equations with four essential regular singular points ([2]). The other seven Painlevé equations can be obtained from the degeneration of the sixth Painlevé equation and this procedure corresponds to the degeneration of the linear equations.

What we have mentioned so far are studies on the case of the two-dimensional phase space, while there appear many studies on higher dimensional case recently. In particular, many higher dimensional Painlevé type equations are obtained from similarity reductions of soliton equations. Among them Koike T. showed that the second and fourth Painlevé hierarchy which were found by Gordoa, Joshi, and Pickering can be regarded as some restrictions of degenerate Garnier systems ([6, 15]). However Noumi-Yamada systems, Sasano systems, and so on, still remain beyond the reach of the theory of the classical Painlevé equations and the Garnier systems.

It is to be expected that we can also construct a classification theory similarly for the four dimensional phase space's case. In particular we treat the second aspect, that is, the isomonodromic deformation theory of Fuchsian equations in this paper, motivated by the classification theory of four-dimensional Painlevé type equations. We will use a classification theory of Fuchsian equations, which is intensively studied recently. Oshima T. showed that any irreducible Fuchsian equation with four accessory parameters can be reduced to 13 types of equations by using Katz's two types of operations: the middle convolution and addition ([17]). Among these thirteen we have only four types which has the isomonodromic deformation. It is also shown that the isomonodromic deformation equation is invariant under the two operations of Katz's in the paper [7]. Therefore we can regard the four isomonodromic deformation equations as the "source" equations of all of four-dimensional Painlevé type equations; they are like the sixth Painlevé equation in the two dimensional case.

The aim of this paper is to give a Hamiltonian expression to each of these four differential systems. In the introduction, we will write down the results in advance.

The first system is the well-known Garnier system with two variables. Only this system has two-dimensional deformation, and is expressed as a partial differential system. This is obtained from a deformation of Fuchsian equations with the spectral type 11,11,11,11,11. There are two Hamiltonian functions corresponding to two time evolutions, and these are expressed by

$$\begin{aligned} t_i(t_i-1)H_i &= ti(t_i-1)H_{\mathrm{VI}}\begin{pmatrix}\theta^0,\theta^1,\theta^{t_1}+\theta^{t_2} & \\ \kappa_1,\kappa_2 & ;t_i;q_i,p_i\end{pmatrix} + (2q_ip_i+q_{i+1}p_{i+1}-\theta^1-\kappa_1)q_1q_2p_{i+1} \\ &\quad - \frac{q_1q_2}{t_i-t_{i+1}}(t_i(t_i-1)p_i^2+2t_i(t_{i+1}-1)p_1p_2+t_{i+1}(t_i-1)p_{i+1}^2) \\ &\quad - \theta^{t_{i+1}}\left(\frac{t_{i+1}(t_i-1)}{t_i-t_{i+1}}q_ip_{i+1}+\left((q_i-1)(q_i-t_i)-q_i\left(q_i+\frac{t_i(t_{i+1}-1)}{t_i-t_{i+1}}\right)\right)p_i\right) \\ &\quad - \frac{\theta^{t_i}t_i}{t_i-t_{i+1}}((t_1-1)p_1+(t_2-1)p_2), \qquad i\in\mathbb{Z}/2\mathbb{Z}. \end{aligned} \tag{1.1}$$

Here the function H_{VI} is the Hamiltonian function of the sixth Painlevé equation, and it is written as

$$\begin{aligned} t(t-1)H_{\mathrm{VI}}\begin{pmatrix}\alpha,\beta,\gamma & \\ \delta,\ \epsilon & ;t;q,p\end{pmatrix} &= q(q-1)(q-t)p^2 \\ &\quad +\{(\alpha+1)q(q-1)+(\delta-\epsilon-1)q(q-t)+\gamma(q-1)(q-t)\}p \\ &\quad +\epsilon(\beta+\epsilon)q-t(\gamma+\delta)\gamma-\alpha\gamma, \end{aligned} \tag{1.2}$$

where $\alpha+\beta+\gamma+\delta+\epsilon=0$.

The second system is obtained from a deformation of Fuchsian equations with the spectral type 21,21,111,111, and is found out to be a Fuji-Suzuki system which was obtained from a similarity reduction of a Drinfel'd-Sokolov hierarchy (see [3]). By using the Hamiltonian function of the sixth Painlevé equation, the Hamiltonian function is written as follows:

$$\begin{aligned}
&t(t-1)H\begin{bmatrix}21,21,111\\111\end{bmatrix}\begin{pmatrix}\theta^0_1,\theta^0_2,\theta^1,\theta^t & q_1,p_1\\ \kappa_1,\kappa_2,\kappa_3 & q_2,p_2\end{pmatrix}\\
=\;& t(t-1)H_{\mathrm{VI}}\begin{pmatrix}\theta^0_1-\theta^0_2,\theta^1+\kappa_3,\theta^t+\kappa_3 \\ \kappa_1-\kappa_3+\theta^0_2,\kappa_2+\theta^0_2\end{pmatrix};q_1,p_1\\
&+t(t-1)H_{\mathrm{VI}}\begin{pmatrix}\theta^0_1-\theta^0_2-\kappa_2,\theta^1+\theta^0_2+\kappa_2,\theta^t+\theta^0_2+\kappa_2 \\ \kappa_1,\quad \kappa_3\end{pmatrix};q_2,p_2\\
&+(q_1-t)(q_2-1)\{(p_1q_1-\kappa_2-\theta^0_2)p_2+(p_2q_2-\kappa_3)p_1\}.
\end{aligned}\tag{1.3}$$

The third one is obtained from a deformation of Fuchsian equations with the spectral type 31,22,22,1111, and is found out to be a Sasano system which was obtained from a higher dimensional generalization of Okamoto's space of initial conditions (see [21, 4]). The Hamiltonian function is written as follows:

$$\begin{aligned}
&t(t-1)H\begin{bmatrix}31,22,22\\1111\end{bmatrix}\begin{pmatrix}\theta^0,\ \theta^1,\ \theta^t & q_1,p_1\\ \kappa_1,\kappa_2,\kappa_3,\kappa_4 & q_2,p_2\end{pmatrix}\\
=\;& t(t-1)H_{\mathrm{VI}}\begin{pmatrix}-\theta^1-\theta^t-\kappa_1-\kappa_4,-\theta^0-\theta^t-\kappa_1-\kappa_4,-\theta^1-\kappa_2-\kappa_3 \\ \kappa_1-\kappa_2-\theta^1-\theta^t,\ \theta^1+\theta^t+\kappa_2+\kappa_4\end{pmatrix};q_1,p_1\\
&+t(t-1)H_{\mathrm{VI}}\begin{pmatrix}\theta^0,\theta^1-\theta^0-\kappa_1+\kappa_2-\kappa_3+\kappa_4,\theta^t \\ -\theta^1-\theta^t+\kappa_1-\kappa_2-\kappa_4,\ \kappa_3\end{pmatrix};q_2,p_2\\
&+2(q_1-t)p_1q_2((q_2-1)p_2-\kappa_3).
\end{aligned}\tag{1.4}$$

The last system is obtained from a deformation of Fuchsian equations with the spectral type 22,22,22,211, and this is thought to be a new system. The Hamiltonian function is written as follows:

$$\begin{aligned}
&t(t-1)H\begin{bmatrix}22,22,22\\211\end{bmatrix}\begin{pmatrix}\theta^0,\theta^1,\theta^t & q_1,p_1\\ \kappa_1,\kappa_2,\kappa_3 & q_2,p_2\end{pmatrix}\\
=\;& \frac{t(t-1)}{2}H_{\mathrm{VI}}\begin{pmatrix}2\theta^0,2\theta^1,2\theta^t \\ 2\kappa_1,\kappa_2+\kappa_3\end{pmatrix};q_1,p_1+(t-1)\left(p_1q_1/2+2p_2q_2\right)\\
&+(2q_1-1)(q_2^2p_2^2+(q_2p_2-\theta^0-\theta^1-\theta^t-\kappa_1-\kappa_2)^2)\\
&-2(q_1(q_1-1)-q_2)(q_1-t)p_2(q_2p_2-\theta^0-\theta^1-\theta^t-\kappa_1-\kappa_2)\\
&-\left(\frac{1}{2}(3q_1-t-1)p_1-\theta^1-\kappa_2-\kappa_3\right)p_1q_2\\
&-(2q_2p_2-\theta^0-\theta^1-\theta^t-\kappa_1-\kappa_2)\\
&\times\{(1-t)\theta^0+\theta^t+(\theta^1+\kappa_2+\kappa_3)(2q_1-t)-p_1\{3q_1^2-2(t+1)q_1-q_2+t\}\}.
\end{aligned}\tag{1.5}$$

The text is organized as follows: In the second section we give a brief exposition of some simple properties of the Schlesinger system. In the third section we will touch only a few aspects of a classification theory of Fuchsian equations. In the fourth section we will see that the space of accessory parameters of Fuchsian equations has a natural symplectic structure. The fifth and sixth sections are our main part. The fifth section contains a calculation on the Hamiltonian functions of the sixth Painlevé equation and the Garnier system. The sixth section deals with the four-dimensional case. Here we will give a procedure to obtain our four systems using tools which is stated so far. Concerning the sixth Painlevé equation and the Garnier system, the Hamiltonian functions are very well known, but our canonical coordinates are different from usual one and seem to be simpler than that.

It is to be expected that the degeneration scheme of these four systems gives all of four-dimensional Painlevé type equations. A study on the degeneration scheme will appear in the forthcoming paper ([12]).

Acknowledgments
The author express his sincere gratitude to Professor OSHIMA Toshio, Professor HARAOKA Yoshishige, Doctor KAWAKAMI Hiroshi, and Ms. NAKAMURA Akane for stimulating discussions and helpful advice. This work was partially supported by Grant-in-Aid no. 20740089, the Japan Society for the Promotion of Science.

2 Preliminary on Schlesinger system

Schlesinger system is derived from a deformation theory of Fuchsian equation in the form

$$\frac{d}{dx}Y = \left(\sum_{1=1}^{n} \frac{A_i}{x-u_i}\right) Y, \tag{2.1}$$

called Schlesinger normal form, and it is expressed as follows (see [22]; more general case is studied in the paper [9]):

$$\frac{\partial}{\partial u_j} A_i = \frac{[A_j, A_i]}{u_j - u_i}, \qquad (j \neq i), \tag{2.2}$$

$$\frac{\partial}{\partial u_i} A_i = -\sum_{j\neq i} \frac{[A_j, A_i]}{u_j - u_i}. \tag{2.3}$$

This is a system with unknown functions $A_1, \ldots, A_n \in M_m$, and the Garnier system is included in the Schlesinger system as the case that the size of matrices, m, is 2.

Each eigen value of each A_i is a conservative quantity. We understand this fact from the following argument: Conservation of $\mathrm{tr}A_i$ is obvious, and similarly $\mathrm{tr}(A_i^k)$ is invariant for $k \in \mathbb{Z}_{>0}$ because

$$\frac{\partial}{\partial u_j} A_i^k = \frac{[A_j, A_i^k]}{u_j - u_i}, \ (j \neq i), \qquad \frac{\partial}{\partial u_i} A_i^k = -\sum_{j\neq i} \frac{[A_j, A_i^k]}{u_j - u_i}.$$

Conservation of the eigen values is equivalent to that of $\mathrm{tr}(A_i^k)$ since these are power sums of the eigen values.

Furthermore $A_\infty = -\sum_{i=1}^{n} A_i$ is also a first integral.

In the second place we discuss independent variables. The number of independent variables is the number of singularities except infinity, that is, n but we can transform two of them into 0 and 1 respectively. By the change of independent variables $x = (u_2 - u_1)\xi + u_1$, the linear equation (2.1) becomes

$$\frac{1}{u_2 - u_1}\frac{d}{d\xi}Y = \frac{1}{u_2 - u_1}\left(\frac{A_1}{\xi} + \frac{A_2}{\xi - 1} + \sum_{i=3}^{n} \frac{A_i}{\xi - \frac{u_i - u_1}{u_2 - u_1}}\right)Y.$$

Correspondingly, put the variables as $(v_1, v_2, v_3, \ldots, v_n) = \left(u_1, u_2, \frac{u_3 - u_1}{u_2 - u_1}, \ldots \frac{u_n - u_1}{u_2 - u_1}\right)$. Then we get

$$\begin{aligned}
&v_1\frac{\partial}{\partial v_1}A_i = [A_i, A_\infty], \qquad \frac{\partial}{\partial v_2}A_i = 0,\\
&\frac{\partial}{\partial v_j}A_1 = \frac{[A_j, A_1]}{v_j}, \quad (j \neq 1, 2), \qquad \frac{\partial}{\partial v_j}A_2 = \frac{[A_j, A_2]}{v_j - 1}, \quad (j \neq 1, 2),\\
&\frac{\partial}{\partial v_j}A_i = \frac{[A_j, A_i]}{v_j - v_i}, \quad (j \neq i, \quad i, j \in \{3, \ldots, n\}),\\
&\frac{\partial}{\partial v_i}A_i = -\frac{[A_1, A_i]}{v_i} - \frac{[A_i, A_2]}{v_i - 1} - \sum_{j \neq 1,2,i} \frac{[A_j, A_i]}{v_j - v_i}, \quad (i \neq 1, 2).
\end{aligned}$$

Here A_i is constant with respect to the variable v_2, and is solved as a function in v_1 by using exponential and logarithmic functions since A_∞ is constant. Regarding them as trivial equations, the Schlesinger system has $n - 2$ independent variables $(v_3, \ldots, v_n)$.

In the last place we consider simultaneous similar transformations. Put $\widetilde{A_i} = W^{-1}A_iW$ and we get

$$\frac{\partial}{\partial u_j}\widetilde{A_i} = -\left[W^{-1}\frac{\partial W}{\partial u_j}, \widetilde{A_i}\right] + \frac{[\widetilde{A_j}, \widetilde{A_i}]}{u_j - u_i}.$$

In particular, when W is independent to u_j, the expression of the equations is left invariant. Because A_∞ is constant we can assume it is Jordan canonical form.

3 Classification theory of Fuchsian equations

Let's look at a geometrical picture of the Schlesinger system, in detail. The space of dependent variables is a quotient space of the direct product of spaces of matrices. Quotient is obtained from symmetries of simultaneous similar transformations. We have a mapping from this space to each eigen values of each matrices. When we fix the values of the mapping, we obtain the phase space of the dynamical system as the fiber of the mapping.

This fiber parameterizes the part of equations which is not determined by the eigen values, in another words, characteristic indices, and we call a coordinate of this fiber accessory parameters. It means that the dimension of the phase space, i.e., the fiber, equals to the number of accessory parameters.

When a degeneracy of eigen values occurs, the dimension of the fiber decreases. It is because symmetries G satisfying $G^{-1}A_iG = A_i$ increase. In order to classify the phase spaces, therefore, we need not only the number of singularities, $n+1$, and the size of matrices, m, but more detailed informations with respect to degeneracies of eigen values.

We now express these requisite informations by $n+1$-tuples of partitions of m, called spectral types. When a spectral type is given as

$$m_1^1 m_2^1 \ldots m_{l_1}^1, m_1^2 \ldots m_{l_2}^2, \ldots, m_1^n \ldots m_{l_n}^n, m_1^0 \ldots m_{l_0}^0 \qquad (\sum_{j=1}^{l_i} m_j^j = m \ \text{ for } \ 0 \leq \forall i \leq n),$$

it means that information about eigen values of A_i is given by the i-th partition; m_j^i $(1 \leq j \leq l_i$ same eigen values exist. However, the symmetry in non-diagonalizable case is smaller than that in diagonalizable case with same eigen values. General theory is constructed under general assumption including non-diagonalizable case, but for the simplicity, afterwards, we assume each A_i is diagonalizable in this article.

Let's begin an explanation of Katz's theory. We will consider two operations which are called Katz's operations. Though it is different from Katz's description, we give a description by using calculations of matrices, following Dettweiler and Reiter ([1]). The first one is called addition, and it is a transformation

$$A = (A_1, \ldots, A_n) \mapsto (A_1 + \alpha_1, \ldots, A_n + \alpha_n)$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$.

The second operation is called a middle convolution. Before going to a middle convolution, we will see a convolution first. A convolution is defined as a transformation $A \mapsto (G_1, \ldots, G_n)$ for $\lambda \in \mathbb{C}$, where

$$G_i = \begin{pmatrix} & & & O & & & \\ A_1 & A_2 & \cdots & \cdots & A_i + \lambda 1_m & \cdots & A_n \\ & & & O & & & \end{pmatrix} < i \quad \in M_{n \times m}(\mathbb{C}).$$

Then, we consider two invariant subspaces with respect to $G = (G_1, \ldots, G_n)$:

$$\mathcal{K} = \begin{pmatrix} \mathrm{Ker} A_1 \\ \vdots \\ \mathrm{Ker} A_n \end{pmatrix} \subset \mathbb{C}^{n \times m}, \quad \mathcal{L}_\lambda = \mathrm{Ker}(G_1 + \cdots + G_n),$$

and we express the action of $G = (G_1, \ldots, G_n)$ on the quotient space $\mathbb{C}^{n \times m}/\mathcal{K} + \mathcal{L}_\lambda$ as $\overline{G} = (\overline{G_1}, \ldots \overline{G_n})$. The transformation

$$A = (A_1, \ldots, A_n) \mapsto \overline{G} = (\overline{G_1}, \ldots \overline{G_n})$$

is called a middle convolution.

Here notice that a middle convolution changes the size of matrices in general. It is known that this transformation can be expressed by an integral transformation, namely, an Euler transformation of the solutions.

Under these preparation of terminology, we can state Katz's important theorem;

Theorem 1 (N. Katz[11]). *Any rigid, irreducible Fuchsian system is constructed by finite procedures of additions, and middle convolutions from a Fuchsian system of order one.*

Rigid equations mean equations without accessory parameters, in this context. We can solve any linear equation of order one, therefore this theorem proves the existence of integral representation of solutions of any irreducible and rigid Fuchsian equations, simultaneously.

Anyway, our concern is on equations with accessory parameters. In particular, we would like to draw a picture, or a table about 4-dimensional dynamical system as an extension of 2-dimensional phase space of the Painlevé equations.

We have also an important theorem for this problem again;

Theorem 2 (T. Oshima[17]). *Any irreducible Fuchsian system with 4 accessory parameters is constructed by finite procedures of additions and middle convolutions from a system of Fuchsian equations of the following 13 types:*

11,11,11,11,11

21,21,111,111 31,22,22,1111 22,22,22,211

211,1111,1111 221,221,11111 32,11111,11111 222,222,2211 33,2211,111111

44,2222,22211 44,332,11111111 55,3331,22222 66,444,2222211.

The first one corresponds to the 2-variable Garnier system; the number of independent variables is two because there are 5 singularities and configuration space has two-dimension. Furthermore we have three systems with one-dimensional deformation space. The other nine have no deformation, hence we only have to look at these 4 cases for study of 4-dimensional Schlesinger systems.

Of course, this story proceeds under the existence of the following theorem;

Theorem 3 (Haraoka-Filipuk[7]). *Schlesinger systems are invariant under the Katz's two operations.*

Incidentally, with the case of two accessory parameters, we know a similar theorem (Kostov[16]) and it can be reduced to the following four types:

11,11,11,11

111,111,111 22,1111,1111 33,222,111111.

There is only one system which has deformation, and this, namely the first one, corresponds to the sixth Painlevé equation. There is no contradiction with the fact that we can obtain all of two-dimensional Painlev'e equations as degenerations of the sixth Painlevé equation.

4 Space of accessory parameters

Now our aim of research is reduced to a study of deformation equations (Schlesinger systems) of types

$$11, 11, 11, 11, 11 \qquad 21, 21, 111, 111 \qquad 31, 22, 22, 1111 \qquad 22, 22, 22, 211.$$

Here we give general theory of structures of phase spaces, that is, spaces of accessory parameters.

The number of the accessory parameters is given by the following formula:

$$(n-1)m^2 - \sum_{i=0}^{n} \left(\sum_{j=1}^{l_i} (m_j^i)^2 \right) + 2. \tag{4.1}$$

We see, from this formula, that our four Fuchsian equations have four accessory parameters each.

We will take a coordinate for practical purposes. First of all, the residue matrix of a finite singular point has $m^2 - \sum_{j=1}^{l_i}(m_j^i)^2$-dimensional freedom when we fix the each eigen values. If m_j^i is 1 for any i, then it just equals to the number in which the number of eigen values is subtracted from the dimension of the space of matrices.

Secondly, we will consider a spontaneous similar transformation. We usually assume that the coefficient at infinity

$$A_\infty = A_0 = -\sum_{i=1}^{n} A_i \tag{4.2}$$

is a diagonal matrix when we study the Schlesinger systems. Since we can set $A_\infty = \mathrm{diag}(\theta_1^0, \dots, \theta_{l_0}^0)$, we have m^2 relations (4.2). Here notice that the relation of traces is a relation of eigen values, and it is not a restriction of accessory parameters. The relation of traces is called the Fuchs relation. Hence we obtained $m^2 - 1$ relations among accessory parameters.

Moreover it remains symmetries which is commutative with A_∞. We conclude that the dimension is

$$\sum_{i=1}^{n}\left(m^2 - \sum_{j=1}^{l_i}(m_j^i)^2\right) - (m^2-1) - (\sum_{j=1}^{l_0}(m_j^0)^2 - 1) = (n-1)m^2 - \sum_{i=0}^{n}\left(\sum_{j=1}^{l_i}(m_j^i)^2\right) + 2,$$

and it coincides with the formula.

Next, we look at a natural Poisson structure and a Hamiltonian function on this space of accessory parameters. It is well known that the space of n-tuple of matrices $M_m(\mathbb{C})^n$ has the Poisson structure which is called Kostant-Kirillov structure. This is introduced by defining a Poisson bracket for a pair of functions on $M_m(\mathbb{C})^n$ as follows:

$$\{(A_p)_{i,j}, (A_q)_{k,l}\} = \delta_{p,q}(\delta_{i,l}(A_p)_{k,j} - \delta_{k,j}(A_p)_{i,l}). \tag{4.3}$$

When we define the Poisson bracket above, the Schlesinger system can be rewritten in the form of

$$\frac{\partial}{\partial u_k} A_l = \{A_l, H_k\}, \tag{4.4}$$

by using the Hamiltonian function

$$H_k = \sum_{l \neq k} \frac{\mathrm{tr}(A_k A_l)}{u_k - u_l}. \tag{4.5}$$

It is easy to check out by a simple calculation.

It remains to set a canonical coordinate which is suitable for the dimension of the space of accessory parameters. Although it is not easy to find a natural canonical coordinate for this space, it is known that we can construct a higher dimensional space equipped with simple canonical coordinate (see Appendix of [8]).

We will decompose the each matrix A_i into a product of two matrices as

$$A_i = B^i \cdot C^i, \qquad B^i = (b_{kl}^i)_{k,l} \in M_{m,\mathrm{rank}A_i}(\mathbb{C}), \quad C^i = (c_{kl}^i)_{k,l} \in M_{\mathrm{rank}A_i,m}(\mathbb{C}).$$

If we set the Hamiltonian function as H_k above on the space of $2n$-tuples of matrices

$$(B^1, C^1, B^2, C^2, \ldots, B^n, C^n),$$

and we set b^i_{kl}, c^i_{kl} as canonical variables, then the function $A = (A_1, \ldots, A_n)$ which is constructed by b^i_{kl}, c^i_{kl} become the solution of the Schlesinger system. Here the symplectic form is expressed by

$$\omega = \sum_{i=1}^{n} \mathrm{tr}(dC^i \wedge dB^i) = \sum_{i=1}^{n} \sum_{k,l} dc^i_{kl} \wedge db^i_{lk}. \tag{4.6}$$

5 Schlesinger systems in the form of Hamiltonian systems

What we mentioned above gives us a Hamiltonian system on the space of $2n$-tuple of matrices

$$(B^1, C^1, \ldots, B^n, C^n)$$

with respect to a given spectral type. It remains to reduce this system into a system on the suitable dimensional space for each case.

Before we go ahead to the 4-dimensional Painlevé equations, we will touch simpler cases. These calculations are useful for our more complicated problems. Deformation theory of linear equations of the second order are well studied. As a classical result, the study of the isomonodromic deformation is initiated by R. Fuchs in 1906 (see [2]). He obtained the sixth Painlevé equation from a deformation of Fuchsian equations of the second order with four essential regular singular points. This would be the simplest example of isomonodromic deformations of Fuchsian equations.

This case is expressed by the spectral type 11,11,11,11. We will consider the deformation of Fuchsian equations of Schlesinger normal form, though R. Fuchs considered the deformation of single Fuchsian equations of second order. We can see the same problem in a famous paper of M.Jimbo and T. Miwa (see an appendix of [10]), but we will change the canonical variables from the Jimbo and Miwa's paper because of convenience of calculations for further problems. These canonical variable seems to be natural and simpler.

We will look at the table which consists of the eigen values of the residue matrices at each singularities, that is, characteristic indices. This is what we call Riemann scheme. In our case, we can normalize the Riemann scheme using appropriate additions as follows:

$$\begin{pmatrix} x=0 & x=1 & x=t & x=\infty \\ 0 & 0 & 0 & \kappa_1 \\ \theta^0 & \theta^1 & \theta^t & \kappa_2 \end{pmatrix}. \tag{5.1}$$

The characteristic indices satisfy the Fuchs relation:

$$\theta^0 + \theta^1 + \theta^t + \kappa_1 + \kappa_2 = 0. \tag{5.2}$$

To follow custom, we used a slightly different notation of eigen values from the previous section.

Here we consider a parameterization of Fuchsian equations of Schlesinger normal form,

$$\frac{d}{dx}Y = A(x)Y, \quad A(x) = \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}, \quad A_\infty = \mathrm{diag}(\kappa_1, \kappa_2),$$

with respect to this Riemann scheme. We also changed suffices of residue matrices from the previous section.

We change the dependent variable Y using a matrix P, which is independent on x, as $\widetilde{Y} = PY$. Then $\widetilde{Y}$ satisfies the equation

$$\frac{d}{dx}\widetilde{Y} = \widetilde{A}(x)\widetilde{Y}, \quad \widetilde{A}(x) = \frac{\widetilde{A}_0}{x} + \frac{\widetilde{A}_1}{x-1} + \frac{\widetilde{A}_t}{x-t}, \quad \widetilde{A}_\xi = PA_\xi P^{-1}, \quad \xi = 0, 1, t, \infty.$$

In particular, we can assume that $\widetilde{A}_\infty$ is lower triangular and $\widetilde{A}_0$ is upper triangular by the choice of P. Here, instead of A_ξ, decomposing $\widetilde{A}_\xi$ into a product of two matrices, put

$$\widetilde{A}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\theta^0, a^0), \quad \widetilde{A}_1 = \begin{pmatrix} b_1^1 \\ b_2^1 \end{pmatrix} (c_1^1, c_2^1), \quad \widetilde{A}_t = \begin{pmatrix} b_1^t \\ b_2^t \end{pmatrix} (c_1^t, c_2^t), \quad \widetilde{A}_\infty = \begin{pmatrix} \kappa_1 & 0 \\ a^\infty & \kappa_2 \end{pmatrix}.$$

When $b_1^1 b_1^t \neq 0$, we can assume that $b_1^1 = b_1^t = 1$. Then, in fact, it will be shown that the symplectic form is written as $\omega = dc_2^1 \wedge db_2^1 + dc_2^t \wedge db_2^t$. Now we prove this in more general settings:

Proposition 4. *Let* $(\widetilde{B}^i, \widetilde{C}^i) = (P^{-1}B^i, C^iP)$, $P \in GL_m$, $(\hat{C}^k, \hat{B}^k) = (B^kQ_k^{-1}, Q_kC^k)$, $Q_k \in GL_{\mathrm{rank}A_k}$.

(a) If $\mathrm{tr}(A_\infty(dP)P^{-1} \wedge (dP)P^{-1}) = 0$, *then* $\omega = \sum_{i=1}^n \mathrm{tr}(d\widetilde{C}^i \wedge d\widetilde{B}^i)$.

(b) If $d(\hat{C}^k\hat{B}^k) = 0$ *and* $\mathrm{tr}(\hat{C}_k\hat{B}_k(dQ_k)Q_k^{-1} \wedge (dQ_k)Q_k^{-1})$, *then* $\omega = \sum_{i\neq k} \mathrm{tr}(dC^i \wedge dB^i) + \mathrm{tr}(d\hat{C}^k \wedge d\hat{B}^k)$.

Proof. Here (a) is clear because $\mathrm{tr}(d\widetilde{C}^i \wedge d\widetilde{B}^i) = \mathrm{tr}(dC^i \wedge dB^i - B^iC^i(dP)P^{-1} \wedge (dP)P^{-1} - d(B^iC^i) \wedge (dP)P^{-1})$ and $\sum B^iC^i = A_\infty$ and $dA_\infty = 0$.

Similarly, (b) is trivial since $\mathrm{tr}(dC^k \wedge dB^k) = \mathrm{tr}(d\hat{C}^k \wedge d\hat{B}^k + \hat{C}^k\hat{B}^k(dQ_k)Q_k^{-1} \wedge (dQ_k)Q_k^{-1} + d(\hat{C}^k\hat{B}^k) \wedge (dQ_k)Q_k^{-1})$. □

As concerns the assumptions of the proposition, the assumption of (a) is satisfied when P is diagonal or when P is upper triangular and its diagonal elements are constants. The second half of the assumption of (b) is also satisfied when Q_k is diagonal or when Q_k is upper triangular and its diagonal elements are constants.

Now we have written down the equation in the term of the variables $a^0, a^\infty, b_2^1, b_2^t, c_1^1, c_1^t, c_2^1, c_2^t$, but we have several relations among these variables. We see that, from the traces of A_0, A_t,

$$\theta^1 = c_1^1 + b_2^1 c_2^1, \quad \theta^t = c_1^t + b_2^t c_2^t.$$

Furthermore, from the equation $A_0 + A_1 + A_t + A_\infty = 0$, we see

$$a^0 + c_2^1 + c_2^t = 0, \quad b_2^1 c_1^1 + b_2^t c_1^t + a^\infty = 0, \quad b_2^1 c_2^1 + b_2^t c_2^t + \kappa_2 = 0.$$

The last relation enable us to write the symplectic form as

$$\begin{aligned}\omega &= \frac{dc_2^1}{c_2^1}\wedge d(b_2^1c_2^1)+\frac{dc_2^t}{c_2^t}\wedge d(b_2^tc_2^t)\\ &= \left(\frac{dc_2^t}{c_2^t}-\frac{dc_2^1}{c_2^1}\right)\wedge d(b_2^tc_2^t)=d(c_2^t/c_2^1)\wedge d(b_2^tc_2^1).\end{aligned}$$

Here, by setting

$$p=\frac{1}{t}b_2^tc_2^1,\quad q=-t\frac{c_2^t}{c_2^1},$$

$\omega - dH\wedge dt$ can be written as

$$d(tp)\wedge d(q/t)-dH\wedge dt=dp\wedge dq-d(H+pq/t)\wedge dt,$$

where $H=\frac{1}{t}\mathrm{tr}(A_0A_t)+\frac{1}{t-1}\mathrm{tr}(A_1A_t)$. Define $H_{\mathrm{VI}}=\frac{1}{t}\mathrm{tr}(A_0A_t)+\frac{1}{t-1}\mathrm{tr}(A_1A_t)+pq/t$, and

$$\begin{aligned}t(t-1)H_{VI} &= (t-1)(\theta^0+p(q-t))(\theta^t+pq)\\ &\quad+(\theta^1+\kappa_2-p(q-t))(\theta^t t+\kappa_2 q-pq(q-t))+(t-1)pq,\end{aligned}$$

and fixing up the expression as

$$\begin{aligned}t(t-1)H_{\mathrm{VI}}\begin{pmatrix}\theta^0,\theta^1,\theta^t & \\ \kappa_1,\ \kappa_2 & ;t;q,p\end{pmatrix} &= q(q-1)(q-t)p^2\\ &\quad+\{(\theta^0+1)q(q-1)+(\kappa_1-\kappa_2-1)q(q-t)+\theta^t(q-1)(q-t)\}p\\ &\quad+\kappa_2(\theta^1+\kappa_2)q-t(\theta^t+\kappa_1)\theta^t-\theta^0\theta^t.\end{aligned}$$

This is the Hamiltonian function of the sixth Painlevé equation. Here the parameters θ^j and κ_j appear in a different way from the Jimbo-Miwa's Hamiltonian function because we used different canonical variables (cf. [10]). We can identify it with Jimbo-Miwa's expression through a Bäcklund transformation.

***Remark* 1.** When we put $\hat{A}_\xi=\begin{pmatrix}1&0\\0&c_2^1\end{pmatrix}\widetilde{A}_\xi\begin{pmatrix}1&0\\0&c_2^1\end{pmatrix}^{-1}$, we can write $\hat{A}_\xi$ simply in terms of p and q:

$$\hat{A}_0=\begin{pmatrix}1\\0\end{pmatrix}(\theta^0,-1+q/t),\ \hat{A}_1=\begin{pmatrix}1\\pq-\kappa_2\end{pmatrix}(\theta^1+\kappa_2-pq,1),\ \hat{A}_t=\begin{pmatrix}1\\tp\end{pmatrix}(\theta^t+pq,-q/t).\square$$

In the same manner we can see that the Garnier system is obtained from a deformation theory of second order Fuchsian equations with a general number of singularities. In the case we have the Riemann scheme

$$\begin{pmatrix}x=0 & x=1 & x=t_j & x=\infty\\ 0&0&0&\kappa_1\\ \theta^0&\theta^1&\theta^{t_j}&\kappa_2\end{pmatrix}_{j=1,\dots,L}, \tag{5.3}$$

and the Fuchs relation

$$\theta^0+\theta^1+\sum_{j=1}^{L}\theta^{t_j}+\kappa_1+\kappa_2=0. \tag{5.4}$$

We will consider a parameterization of Fuchsian equations in Schlesinger normal form

$$\frac{d}{dx}Y = A(x)Y, \quad A(x) = \frac{A_0}{x} + \frac{A_1}{x-1} + \sum_{j=1}^{L} \frac{A_{t_j}}{x-t_j}, \quad A_\infty = \mathrm{diag}(\kappa_1, \kappa_2)$$

with this Riemann scheme. Using a gauge transformation of Y, we change the equation into

$$\frac{d}{dx}\widetilde{Y} = \widetilde{A}(x)\widetilde{Y}, \quad \widetilde{A}(x) = \frac{\widetilde{A}_0}{x} + \frac{\widetilde{A}_1}{x-1} + \sum_{j=1}^{L} \frac{\widetilde{A}_{t_j}}{x-t_j}, \quad \widetilde{A}_\xi = PA_\xi P^{-1}, \quad \xi = 0, 1, t_j, \infty.$$

Set

$$\widetilde{A}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\theta^0, a^0), \ \widetilde{A}_1 = \begin{pmatrix} 1 \\ b_2^1 \end{pmatrix} (c_1^1, c_2^1), \ \widetilde{A}_{t_j} = \begin{pmatrix} 1 \\ b_2^{t_j} \end{pmatrix} (c_1^{t_j}, c_2^{t_j}), \ \widetilde{A}_\infty = \begin{pmatrix} \kappa_1 & 0 \\ a^\infty & \kappa_2 \end{pmatrix}.$$

and the relations can be written as

$$\theta^1 = c_1^1 + b_2^1 c_2^1, \qquad \theta^{t_j} = c_1^{t_j} + b_2^{t_j} c_2^{t_j},$$
$$a^0 + c_2^1 + \sum_{j=1}^{L} c_2^{t_j} = 0, \quad b_2^1 c_1^1 + \sum_{j=1}^{L} b_2^{t_j} c_1^{t_j} + a^\infty = 0, \quad b_2^1 c_2^1 + \sum_{j=1}^{L} b_2^{t_j} c_2^{t_j} + \kappa_2 = 0.$$

We will introduce the canonical variables

$$p_j = \frac{1}{t_j} b_2^{t_j} c_2^1, \qquad q_j = -t_j \frac{c_2^{t_j}}{c_2^1}.$$

Then the Hamiltonian functions are written as

$$H_i = \frac{1}{t_i}\mathrm{tr}(A_0 A_{t_i}) + \frac{1}{t_i - 1}\mathrm{tr}(A_1 A_{t_i}) + \sum_{l \neq i} \frac{1}{t_i - t_l}\mathrm{tr}(A_{t_l} A_{t_i}) + \frac{p_i q_i}{t_i}.$$

In terms of p_j, q_j, it is expressed as

$$\begin{aligned} t_i(t_i-1)H_i \;=\;& (t_i - 1)\left(\theta^0 - t_i p_i \left(1 - \sum_{l=1}^{L} \frac{q_l}{t_l}\right)\right)(\theta^{t_i} + p_i q_i) + (t_i - 1)p_i q_i \\ &+ \left(\theta^1 + \kappa_2 - \sum_{l=1}^{L} p_l q_l + t_i p_i\right)\left(t_i \theta^{t_i} + \left(\kappa_2 - \sum_{l=1}^{L} p_l q_l\right) q_i + t_i p_i q_i\right) \\ &+ \sum_{l \neq i} \frac{t_i(t_i-1)}{t_i - t_l}\left(\theta^{t_l} + p_l q_l - \frac{t_i}{t_l} p_i q_l\right)\left(\theta^{t_i} + p_i q_i - \frac{t_l}{t_i} p_l q_i\right). \end{aligned}$$

Furthermore, fix up the expression as

$$t_i(t_i - 1)H_i = \sum_{j,k=1}^{L} E_{ijk}(t,q) p_j p_k + \sum_{j=1}^{L} F_{ij}(t,q) p_j + \kappa_2(\theta^1 + \kappa_2) q_i + f(t)$$

and E_{ijk}, F_{ij} are expressed as

$$\begin{aligned}
E_{ijk} &= E_{ikj} = \begin{cases} q_iq_jq_k, & \text{if } i,j,k \text{ are distinct,} \\ q_iq_j\left(q_j - \frac{t_j(t_i-1)}{t_i-t_j}\right), & \text{if } j=k\neq i, \\ q_iq_j\left(q_i - \frac{t_i(t_j-1)}{t_j-t_i}\right), & \text{if } k=i\neq j, \\ q_i(q_i-1)(q_i-t_i) - \sum_{l\neq i}\frac{t_i(t_i-1)}{t_i-t_l}q_iq_l, & \text{if } i=j=k, \end{cases} \\
F_{ij} &= \begin{cases} -(\theta^1+\kappa_2)q_iq_j - \theta^{t_j}\frac{t_j(t_i-1)}{t_i-t_j}q_i - \theta^{t_i}\frac{t_i(t_j-1)}{t_j-t_i}q_j, & \text{if } i\neq j, \\ (\theta^0+1)q_i(q_i-1) + (\kappa_1-\kappa_2-1)q_i(q_i-t_i) + \theta^{t_i}(q_i-1)(q_i-t_i) & \\ \qquad + \sum_{l\neq i}\left\{\theta^{t_l}q_i\left(q_i - \frac{t_i(t_l-1)}{t_l-t_i}\right) - \theta^{t_i}\frac{t_i(t_i-1)}{t_i-t_l}q_l\right\}, & \text{if } i=j, \end{cases} \\
f(t) &= t_i(\theta^0+\theta^1+\kappa_2)\theta^{t_i} - \theta^0\theta^{t_i} + \sum_{l\neq i}\frac{t_i(t_i-1)}{t_i-t_l}\theta^{t_i}\theta^{t_l}.
\end{aligned}$$

***Remark* 2.** When we put $\hat{A}_\xi = \begin{pmatrix} 1 & 0 \\ 0 & c_2^1 \end{pmatrix} \widetilde{A}_\xi \begin{pmatrix} 1 & 0 \\ 0 & c_2^1 \end{pmatrix}^{-1}$, we can write $\hat{A}_\xi$ simply in terms of p_j and q_j:

$$\hat{A}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(\theta^0, -1 + \sum_{j=1}^{L} q_j/t_j), \quad \hat{A}_1 = \begin{pmatrix} 1 \\ \sum_{j=1}^{L} p_jq_j - \kappa_2 \end{pmatrix}(\theta^1 + \kappa_2 - \sum_{j=1}^{L} p_jq_j, 1),$$

$$\hat{A}_t = \begin{pmatrix} 1 \\ t_jp_j \end{pmatrix}(\theta^{t_j} + p_jq_j, -q_j/t_j). \qquad \square$$

This Hamiltonian function is same as the Kimura-Okamoto's polynomial Hamiltonian, although the parameters θ^j, κ_j appear differently, because of the different choice of canonical variables (cf. [14]). We can identify it with the Kimura-Okamoto's Hamiltonian through an appropriate Bäcklund transformation.

6 4-dimensional Painlevé type equations

Fuchsian equations with 4 accessory parameters are reduced to four types

$$11,11,11,11,11 \qquad 21,21,111,111 \qquad 31,22,22,1111 \qquad 22,22,22,211.$$

Therefore we will concentrate on obtaining the deformation in the form of a Hamiltonian system with respect to each four types.

6.1 11,11,11,11,11

This case corresponds to the Garnier system with $L = 2$ which is above mentioned. It has 2 independent variables. Using the Hamiltonian function of the sixth Painlevé equation, we can rewrite the Hamiltonian function as

$$
\begin{aligned}
t_i(t_i-1)H_i &= ti(t_i-1)H_{\mathrm{VI}}\left(\begin{matrix}\theta^0,\theta^1,\theta^{t_1}+\theta^{t_2} \\ \kappa_1,\kappa_2\end{matrix};t_i;q_i,p_i\right)+(2q_ip_i+q_{i+1}p_{i+1}-\theta^1-\kappa_1)q_1q_2p_{i+1} \\
&\quad -\frac{q_1q_2}{t_i-t_{i+1}}(t_i(t_i-1)p_i^2+2t_i(t_{i+1}-1)p_1p_2+t_{i+1}(t_i-1)p_{i+1}^2) \\
&\quad -\theta^{t_{i+1}}\left(\frac{t_{i+1}(t_i-1)}{t_i-t_{i+1}}q_ip_{i+1}+\left((q_i-1)(q_i-t_i)-q_i\left(q_i+\frac{t_i(t_{i+1}-1)}{t_i-t_{i+1}}\right)\right)p_i\right) \\
&\quad -\frac{\theta^{t_i}t_i}{t_i-t_{i+1}}((t_1-1)p_1+(t_2-1)p_2)+g(t), \qquad i\in\mathbb{Z}/2\mathbb{Z},
\end{aligned}
$$

Here $g(t)$ is written as $\theta^0\theta^{t_{i+1}}+\frac{t_i(t_i-1)}{t_i-t_{i+1}}\theta^{t_1}\theta^{t_2}$, but it does not change the Hamiltonian system because it contains no term of p_j and q_j.

As concerns this Garnier system with two variables, we already have degeneration scheme corresponding to the degeneration of the linear systems (see [13]).

6.2 21,21,111,111

We will consider a parameterization of Fuchsian equations in Schlesinger normal form

$$
\frac{d}{dx}Y=A(x)Y, \quad A(x)=\frac{A_0}{x}+\frac{A_1}{x-1}+\frac{A_t}{x-t}, \quad A_\infty=\mathrm{diag}(\kappa_1,\kappa_2,\kappa_3)
$$

with the Riemann scheme

$$
\begin{pmatrix} x=0 & x=1 & x=t & x=\infty \\ 0 & 0 & 0 & \kappa_1 \\ \theta_1^0 & 0 & 0 & \kappa_2 \\ \theta_2^0 & \theta^1 & \theta^t & \kappa_3 \end{pmatrix}. \tag{6.1}
$$

Here the Fuchs relation is

$$
\theta_1^0+\theta_2^0+\theta^1+\theta^t+\kappa_1+\kappa_2+\kappa_3=0. \tag{6.2}
$$

Using a gauge transformation of Y, we transform the equation into

$$
\frac{d}{dx}\widetilde{Y}=\widetilde{A}(x)\widetilde{Y}, \quad \widetilde{A}(x)=\frac{\widetilde{A}_0}{x}+\frac{\widetilde{A}_1}{x-1}+\frac{\widetilde{A}_t}{x-t}, \quad \widetilde{A}_\xi=PA_\xi P^{-1}, \quad \xi=0,1,t,\infty,
$$

where

$$
\widetilde{A}_0=\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}\begin{pmatrix}\theta_1^0 & a_1^0 & a_3^0\\ 0 & \theta_2^0 & a_2^0\end{pmatrix}, \quad \widetilde{A}_1=\begin{pmatrix}1\\b_2^1\\b_3^1\end{pmatrix}(c_1^1,c_2^1,c_3^1),
$$

$$
\widetilde{A}_t=\begin{pmatrix}1\\b_2^t\\b_3^t\end{pmatrix}(c_1^t,c_2^t,c_3^t), \quad \widetilde{A}_\infty=\begin{pmatrix}\kappa_1&0&0\\a_1^\infty&\kappa_2&0\\a_3^\infty&a_2^\infty&\kappa_3\end{pmatrix}.
$$

The relations can be written as

$$\begin{aligned}
&\theta^1 = c_1^1 + b_2^1 c_2^1 + b_3^1 c_3^1, \qquad \theta^t = c_1^t + b_2^t c_2^t + b_3^t c_3^t,\\
&a_1^0 + c_2^1 + c_2^t = 0, \quad a_2^0 + b_2^1 c_3^1 + b_2^t c_3^t = 0, \quad a_3^0 + c_3^1 + c_3^t = 0,\\
&a_1^\infty + b_2^1 c_1^1 + b_2^t c_1^t = 0, \quad a_2^\infty + b_3^1 c_2^1 + b_3^t c_2^t = 0, \quad a_3^\infty + b_3^1 c_1^1 + b_3^t c_1^t = 0,\\
&b_2^1 c_2^1 + b_2^t c_2^t + \theta_2^0 + \kappa_2 = 0, \quad b_3^1 c_3^1 + b_3^t c_3^t + \kappa_3 = 0.
\end{aligned}$$

Define canonical variables as

$$p_1 = \frac{1}{t} b_2^t c_2^1, \qquad q_1 = -t \frac{c_2^t}{c_2^1}, \qquad p_2 = \frac{1}{t} b_3^t c_3^1, \qquad q_2 = -t \frac{c_3^t}{c_3^1},$$

and the Hamiltonian function is written as

$$H \begin{bmatrix} 21, 21, 111 \\ 111 \end{bmatrix} = \frac{1}{t} \mathrm{tr}(A_0 A_t) + \frac{1}{t-1} \mathrm{tr}(A_1 A_t) + \frac{p_1 q_1 + p_2 q_2}{t}.$$

The traces are calculated as follows:

$$\begin{aligned}
\mathrm{tr}(A_0 A_t) &= c_1^t(\theta_1^0 + a_1^0 b_2^t + a_3^0 b_3^t) + c_2^t(\theta_2^0 b_2^t + a_2^0 b_3^t),\\
\mathrm{tr}(A_1 A_t) &= (c_1^1 + c_2^1 b_2^t + c_3^1 b_3^t)(c_1^t + c_2^t b_2^1 + c_3^t b_3^1),
\end{aligned}$$

and the Hamiltonian function is expressed as

$$\begin{aligned}
& t(t-1) H \begin{bmatrix} 21, 21, 111 \\ 111 \end{bmatrix} \begin{pmatrix} \theta_1^0, \theta_2^0, \theta^1, \theta^t & ; t; & q_1, p_1 \\ \kappa_1, \kappa_2, \kappa_3 & & q_2, p_2 \end{pmatrix}\\
= \;& t(t-1) H_{\mathrm{VI}} \begin{pmatrix} \theta_1^0 - \theta_2^0, \theta^1 + \kappa_3, \theta^t + \kappa_3 \\ \kappa_1 - \kappa_3 + \theta_2^0, \kappa_2 + \theta_2^0 \end{pmatrix} ; t; q_1, p_1 \Bigg)\\
& + t(t-1) H_{\mathrm{VI}} \begin{pmatrix} \theta_1^0 - \theta_2^0 - \kappa_2, \theta^1 + \theta_2^0 + \kappa_2, \theta^t + \theta_2^0 + \kappa_2 \\ \kappa_1, \quad \kappa_3 \end{pmatrix} ; t; q_2, p_2 \Bigg)\\
& + (q_1 - t)(q_2 - 1)\{(p_1 q_1 - \kappa_2 - \theta_2^0) p_2 + (p_2 q_2 - \kappa_3) p_1\} + g(t)
\end{aligned}$$

in the terms of p_j and q_j. Here

$$g(t) = (t(\theta^t + \theta_2^0 + \kappa_1) + \theta_1^0 - \theta_2^0)(\theta^t + \kappa_3) + (t(\theta^t + \theta_2^0 + \kappa_1 + \kappa_2) + \theta_1^0 - \theta_2^0 - \kappa_2)(\theta^t + \theta_2^0 + \kappa_2) - (t(\theta^t + \kappa_1) - \theta_1^0)\theta^t,$$

but it makes no difference on the Hamiltonian system.

Remark 3. When we put $\hat{A}_\xi = \mathrm{diag}(1, c_2^1, c_3^1) \widetilde{A}_\xi \mathrm{diag}(1, c_2^1, c_3^1)^{-1}$, we can write $\hat{A}_\xi$ simply in terms of p_j and q_j:

$$\begin{aligned}
\hat{A}_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_1^0 & -1 + q_1/t & -1 + q_2/t \\ 0 & \theta_2^0 & p_1(q_2 - q_1) + \kappa_2 + \theta_2^0 \end{pmatrix},\\
\hat{A}_1 &= \begin{pmatrix} 1 \\ p_1 q_1 - \kappa_2 - \theta_2^0 \\ p_2 q_2 - \kappa_3 \end{pmatrix} (\theta^1 + \theta_2^0 + \kappa_2 + \kappa_3 - p_1 q_1 - p_2 q_2, 1, 1),\\
\hat{A}_t &= \begin{pmatrix} 1 \\ t p_1 \\ t p_2 \end{pmatrix} (\theta^t + p_1 q_1 + p_2 q_2, -q_1/t, -q_2/t). \qquad \square
\end{aligned}$$

This Hamiltonian function coincide with that of Fuji-Suzuki system, that is, a higher order Painlevé system, which is obtained from a similarity reduction of Drinfel'd-Sokolov hierarchy of $A_5^{(1)}$ type (see [3]). It is known that the system has naturally an affine Weyl group symmetry of type $A_5^{(1)}$.

6.3 31,22,22,1111

We now proceed similarly to a parameterization of Fuchsian equations in Schlesinger normal form

$$\frac{d}{dx}Y = A(x)Y, \quad A(x) = \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}, \quad A_\infty = \mathrm{diag}(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$$

with the Riemann scheme

$$\begin{pmatrix} x=0 & x=1 & x=t & x=\infty \\ 0 & 0 & 0 & \kappa_1 \\ 0 & 0 & 0 & \kappa_2 \\ 0 & \theta^1 & \theta^t & \kappa_3 \\ \theta^0 & \theta^1 & \theta^t & \kappa_4 \end{pmatrix}. \tag{6.3}$$

Here the Fuchs relation is

$$\theta^0 + 2\theta^1 + 2\theta^t + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 0. \tag{6.4}$$

We consider a transformed equation

$$\frac{d}{dx}\widetilde{Y} = \widetilde{A}(x)\widetilde{Y}, \quad \widetilde{A}(x) = \frac{\widetilde{A}_0}{x} + \frac{\widetilde{A}_1}{x-1} + \frac{\widetilde{A}_t}{x-t}, \quad \widetilde{A}_\xi = PA_\xi P^{-1}, \quad \xi = 0, 1, t, \infty,$$

and set

$$\widetilde{A}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} (\theta^0, a_2^0, a_3^0, a_4^0), \quad \widetilde{A}_\infty = \begin{pmatrix} \kappa_1 & 0 & 0 & 0 \\ a_1^\infty & \kappa_2 & 0 & 0 \\ a_2^\infty & 0 & \kappa_3 & 0 \\ a_3^\infty & 0 & 0 & \kappa_4 \end{pmatrix}$$

$$\widetilde{A}_1 = \begin{pmatrix} 1_2 \\ B^{(1)} \end{pmatrix} (\theta^1 \cdot 1_2 - C^{(1)}B^{(1)},\, C^{(1)}), \quad \widetilde{A}_t = \begin{pmatrix} 1_2 \\ B^{(t)} \end{pmatrix} (\theta^t \cdot 1_2 - C^{(t)}B^{(t)},\, C^{(t)}),$$

$$B^{(\xi)} = \begin{pmatrix} b_{31}^\xi & b_{32}^\xi \\ b_{41}^\xi & b_{42}^\xi \end{pmatrix}, \quad C^{(\xi)} = \begin{pmatrix} c_{13}^\xi & c_{14}^\xi \\ c_{23}^\xi & c_{24}^\xi \end{pmatrix}, \quad \xi = 1, t.$$

Here we can write $\widetilde{A}_1$, $\widetilde{A}_t$ in the above form, because any vector in the image of each matrix is an eigen vector. The relations are as follows:

$$a_2^0 = b_{13}^1 c_{32}^1 + b_{14}^1 c_{42}^1 + b_{13}^t c_{32}^t + b_{14}^t c_{42}^t, \quad a_2^\infty = b_{31}^1 c_{23}^1 + b_{41}^1 c_{24}^1 + b_{31}^t c_{23}^t + b_{41}^t c_{24}^t,$$

$$(c_{13}^1 + c_{13}^t, c_{14}^1 + c_{14}^t) + (a_3^0, a_4^0) = 0, \quad c_{23}^1 + c_{23}^t = c_{24}^1 + c_{24}^t = 0,$$

$$\theta^1 B^{(1)} - B^{(1)}C^{(1)}B^{(1)} + \theta^t B^{(t)} - B^{(t)}C^{(t)}B^{(t)} + \begin{pmatrix} a_3^\infty & 0 \\ a_4^\infty & 0 \end{pmatrix} = 0,$$

$$\theta^0 + \theta^1 + \theta^t + \kappa_1 = c_{13}^1 b_{31}^1 + c_{14}^1 b_{41}^1 + c_{13}^t b_{31}^t + c_{14}^t b_{41}^t,$$

$$B^{(1)}C^{(1)} + B^{(t)}C^{(t)} + \begin{pmatrix} \kappa_3 & 0 \\ 0 & \kappa_4 \end{pmatrix} = 0.$$

What is left is to set canonical variables. We can calculate the symplectic form as follows:

$$\begin{aligned}
\omega &= \sum_{(i,j)=(1,3),(2,3),(1,4),(2,4)} dc^1_{ij}\wedge db^1_{ji} + dc^t_{ij}\wedge db^t_{ji} \\
&= d\left(\frac{c^t_{13}}{c^1_{13}}\right)\wedge d(b^t_{31}c^1_{13}) + d\left(\frac{c^t_{14}}{c^1_{14}}\right)\wedge d(b^t_{41}c^1_{14}) + d\left(\frac{c^1_{14}c^1_{23}}{c^1_{13}c^1_{24}}\right)\wedge\{(b^1_{32}-b^t_{32})\frac{c^1_{13}c^1_{24}}{c^1_{14}}\} \\
&= d\left(\frac{c^t_{14}}{c^1_{14}}\right)\wedge d(b^t_{31}c^1_{13}+b^t_{41}c^1_{14}) + d\left(\left(\frac{c^t_{13}}{c^1_{13}}-\frac{c^t_{14}}{c^1_{14}}\right)\frac{c^1_{13}c^1_{24}}{\det C^{(1)}}\right)\wedge d\left(\frac{b^t_{31}}{c^1_{24}}\det C^{(1)}\right) \\
&= d\left(\frac{c^t_{14}}{c^1_{14}}\right)\wedge d\left(c^1_{14}\left(b^t_{41}-b^t_{31}\frac{c^1_{23}}{c^1_{24}}\right)\right) + d\left(\frac{c^t_{13}c^1_{24}-c^t_{14}c^1_{23}}{\det C^{(1)}}\right)\wedge d\left(\frac{b^t_{31}}{c^1_{24}}\det C^{(1)}\right).
\end{aligned}$$

Define canonical variables as

$$p_1 = \frac{c^1_{14}}{t}\left(b^t_{41}-b^t_{31}\frac{c^1_{23}}{c^1_{24}}\right),\quad q_1 = -t\frac{c^t_{14}}{c^1_{14}},\quad p_2 = \frac{1}{t}\frac{b^t_{31}}{c^1_{24}}\det C^{(1)},\quad q_2 = -t\left(\frac{c^t_{13}c^1_{24}-c^t_{14}c^1_{23}}{\det C^{(1)}}\right),$$

and the Hamiltonian function is

$$H\begin{bmatrix}31,21,21\\1111\end{bmatrix} = \frac{1}{t}\mathrm{tr}(A_0A_t) + \frac{1}{t-1}\mathrm{tr}(A_1A_t) + \frac{p_1q_1+p_2q_2}{t}.$$

The traces are calculated as follows:

$$\begin{aligned}
\mathrm{tr}(A_0A_t) &= (\theta^0 + a^0_3b^t_{31} + a^0_4b^t_{41})(\theta^t - b^t_{31}c^t_{13} - b^t_{41}c^t_{14}) \\
&\quad -(a^0_2 + a^0_3b^t_{32} + a^0_4b^t_{42})(b^t_{31}c^t_{23} + b^t_{41}c^t_{24}) \\
\mathrm{tr}(A_1A_t) &= (\theta^1 - (b^1_{31}-b^t_{31})c^1_{13} - (b^1_{41}-b^t_{41})c^1_{14})(\theta^t - (b^t_{31}-b^1_{31})c^t_{13} - (b^t_{41}-b^1_{41})c^t_{14}) \\
&\quad +((b^1_{32}-b^t_{32})c^1_{13} + (b^1_{42}-b^t_{42})c^1_{14})((b^t_{31}-b^1_{31})c^t_{23} + (b^t_{41}-b^1_{41})c^t_{24}) \\
&\quad +((b^1_{31}-b^t_{31})c^1_{23} + (b^1_{41}-b^t_{41})c^1_{24})((b^t_{32}-b^1_{32})c^t_{13} + (b^t_{42}-b^1_{42})c^t_{14}) \\
&\quad +(\theta^t+\kappa_2)(\theta^1+\kappa_2),
\end{aligned}$$

and, in the terms of p_j, q_j, the Hamiltonian function is expressed as

$$\begin{aligned}
&t(t-1)H\begin{bmatrix}31,22,22\\1111\end{bmatrix}\left(\begin{matrix}\theta^0,\ \theta^1,\ \theta^t\\ \kappa_1,\kappa_2,\kappa_3,\kappa_4\end{matrix};t;\begin{matrix}q_1,p_1\\q_2,p_2\end{matrix}\right) \\
=\ & t(t-1)H_{\mathrm{VI}}\left(\begin{matrix}-\theta^1-\theta^t-\kappa_1-\kappa_4, -\theta^0-\theta^t-\kappa_1-\kappa_4, -\theta^1-\kappa_2-\kappa_3\\ \kappa_1-\kappa_2-\theta^1-\theta^t,\ \theta^1+\theta^t+\kappa_2+\kappa_4\end{matrix};t;q_1,p_1\right) \\
&+t(t-1)H_{\mathrm{VI}}\left(\begin{matrix}\theta^0,\theta^1-\theta^0-\kappa_1+\kappa_2-\kappa_3+\kappa_4,\theta^t\\ -\theta^1-\theta^t+\kappa_1-\kappa_2-\kappa_4,\ \kappa_3\end{matrix};t;q_2,p_2\right) \\
&+2(q_1-t)p_1q_2((q_2-1)p_2-\kappa_3) + g(t).
\end{aligned}$$

Here

$$g(t) = (\theta^1+\kappa_2+\kappa_3)((t-1)(2\theta^1+\theta^t+\kappa_1+\kappa_4)+t(2\theta^1+\kappa_2+\kappa_3)) - t\theta^t(2\theta^t-\kappa_3+2\kappa_4) + t\kappa_2(\theta^1+\kappa_2),$$

but it makes no difference on the Hamiltonian system.

This Hamiltonian function coincide with that of Sasano system, that is, a higher order Painlevé system, which is obtained from a generalization of space of initial conditions of the Painlevé equations (see [21]). Moreover it was also obtained by K. Fuji and T. Suzuki, from a similarity reduction of Drinfel'd-Sokolov hierarchy of $D^{(1)}_6$ type (cf. [4]). It is known that the system has naturally an affine Weyl group symmetry of type $D^{(1)}_6$.

6.4 22,22,22,211

We will consider a parameterization of Fuchsian equations in Schlesinger normal form

$$\frac{d}{dx}Y = A(x)Y, \quad A(x) = \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}, \quad A_\infty = \mathrm{diag}(\kappa_1, \kappa_1, \kappa_2, \kappa_3)$$

with the Riemann scheme

$$\left(\begin{array}{cccc} x=0 & x=1 & x=t & x=\infty \\ 0 & 0 & 0 & \kappa_1 \\ 0 & 0 & 0 & \kappa_1 \\ \theta^0 & \theta^1 & \theta^t & \kappa_2 \\ \theta^0 & \theta^1 & \theta^t & \kappa_3 \end{array}\right). \tag{6.5}$$

Here the Fuchs relation is as

$$2\theta^0 + 2\theta^1 + 2\theta^t + 2\kappa_1 + \kappa_2 + \kappa_3 = 0. \tag{6.6}$$

Consider a transformed equation

$$\frac{d}{dx}\widetilde{Y} = \widetilde{A}(x)\widetilde{Y}, \quad \widetilde{A}(x) = \frac{\widetilde{A}_0}{x} + \frac{\widetilde{A}_1}{x-1} + \frac{\widetilde{A}_t}{x-t}, \quad \widetilde{A}_\xi = PA_\xi P^{-1}, \quad \xi = 0, 1, t, \infty$$

and set

$$\widetilde{A}_0 = \begin{pmatrix} 1_2 \\ 0 \end{pmatrix} (\theta^0 \cdot 1_2, A^{(0)}), \quad \widetilde{A}_\infty = \begin{pmatrix} \kappa_1 \cdot 1_2 & O_2 \\ A^{(\infty)} & \mathrm{diag}(\kappa_2, \kappa_3) \end{pmatrix}$$

$$\widetilde{A}_1 = \begin{pmatrix} 1_2 \\ B^{(1)} \end{pmatrix} (\theta^1 \cdot 1_2 - C^{(1)}B^{(1)}, C^{(1)}), \quad \widetilde{A}_t = \begin{pmatrix} 1_2 \\ B^{(t)} \end{pmatrix} (\theta^t \cdot 1_2 - C^{(t)}B^{(t)}, C^{(t)}),$$

$$A^{(\infty)} = \begin{pmatrix} a^\infty_{31} & a^\infty_{32} \\ a^\infty_{41} & a^\infty_{42} \end{pmatrix}, \quad A^{(0)} = \begin{pmatrix} a^0_{13} & a^0_{14} \\ a^0_{23} & a^0_{24} \end{pmatrix},$$

$$B^{(\xi)} = \begin{pmatrix} b^\xi_{31} & b^\xi_{32} \\ b^\xi_{41} & b^\xi_{42} \end{pmatrix}, \quad C^{(\xi)} = \begin{pmatrix} c^\xi_{13} & c^\xi_{14} \\ c^\xi_{23} & c^\xi_{24} \end{pmatrix}, \quad \xi = 1, t.$$

The expression of $\widetilde{A}_1$ and $\widetilde{A}_t$ is established by the spectral type of equation. The relations are written as

$$A^{(0)} + C^{(1)} + C^{(t)} = 0, \qquad \theta^1 B^{(1)} - B^{(1)}C^{(1)}B^{(1)} + \theta^t B^{(t)} - B^{(t)}C^{(t)}B^{(t)} + A^{(\infty)} = 0$$
$$(\theta^0 + \theta^1 + \theta^t + \kappa_1) \cdot 1_2 = C^{(1)}B^{(1)} + C^{(t)}B^{(t)}, \qquad B^{(1)}C^{(1)} + B^{(t)}C^{(t)} + \begin{pmatrix} \kappa_2 & 0 \\ 0 & \kappa_3 \end{pmatrix} = 0.$$

We are left with the task of determining canonical variables. Calculating the symplectic form, we have

$$\omega = \mathrm{tr}(dC^{(1)} \wedge dB^{(1)} + dC^{(t)} \wedge dB^{(t)}) = \mathrm{tr}(d(C^{(1)-1}C^{(t)}) \wedge d(B^{(t)}C^{(1)})).$$

Here we can write the matrices in the form

$$C^{(1)-1}C^{(t)} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_1 \end{pmatrix}, \quad B^{(t)}C^{(1)} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_1 \end{pmatrix}$$

because of the relations. Hence we have

$$\begin{aligned}\omega &= 2dc_1 \wedge db_1 + dc_2 \wedge db_3 + dc_3 \wedge db_2 \\ &= 2dc_1 \wedge db_1 + d\left(\frac{b_2}{c_2}\right) \wedge d(c_2c_3).\end{aligned}$$

Define canonical variables as

$$p_1 = \frac{2}{t}b_1 = \frac{1}{t}\mathrm{tr}(B^{(t)}C^{(1)}), \quad q_1 = -tc_1 = \frac{t}{2}\mathrm{tr}(C^{(1)-1}C^{(t)}), \quad p_2 = \frac{1}{t^2}\frac{b_2}{c_2}, \quad q_2 = -t^2c_2c_3,$$

and the Hamiltonian function is

$$H\begin{bmatrix}22,22,22\\211\end{bmatrix} = \frac{1}{t}\mathrm{tr}(A_0A_t) + \frac{1}{t-1}\mathrm{tr}(A_1A_t) + \frac{p_1q_1 + 2p_2q_2}{t}.$$

The traces are expressed as

$$\begin{aligned}\mathrm{tr}(A_0A_t) &= \mathrm{tr}((\theta^0 \cdot 1_2 + A^{(0)}B^{(t)})(\theta^t \cdot 1_2 - C^{(t)}B^{(t)})), \\ \mathrm{tr}(A_1A_t) &= \mathrm{tr}((\theta^1 \cdot 1_2 - C^{(1)}(B^{(1)} - C^{(t)}))(\theta^t \cdot 1_2 - C^{(t)}(B^{(t)} - B^{(1)}))),\end{aligned}$$

and, in the terms of p_j, q_j, the Hamiltonian function can be written as

$$\begin{aligned}&t(t-1)H\begin{bmatrix}22,22,22\\211\end{bmatrix}\begin{pmatrix}\theta^0,\theta^1,\theta^t & q_1,p_1\\ \kappa_1,\kappa_2,\kappa_3 & q_2,p_2\end{pmatrix} \\ = \;&\frac{t(t-1)}{2}H_{\mathrm{VI}}\begin{pmatrix}2\theta^0,2\theta^1,2\theta^t\\2\kappa_1,\kappa_2+\kappa_3\end{pmatrix};q_1,p_1\Bigg) + (t-1)\left(p_1q_1/2 + 2p_2q_2\right) \\ &+(2q_1-1)(q_2^2p_2^2 + (q_2p_2 - \theta^0 - \theta^1 - \theta^t - \kappa_1 - \kappa_2)^2) \\ &-2(q_1(q_1-1) - q_2)(q_1 - t)p_2(q_2p_2 - \theta^0 - \theta^1 - \theta^t - \kappa_1 - \kappa_2) \\ &-\left(\frac{1}{2}(3q_1 - t - 1)p_1 - \theta^1 - \kappa_2 - \kappa_3\right)p_1q_2 \\ &-(2q_2p_2 - \theta^0 - \theta^1 - \theta^t - \kappa_1 - \kappa_2) \\ &\times\{(1-t)\theta^0 + \theta^t + (\theta^1 + \kappa_2 + \kappa_3)(2q_1 - t) - p_1\{3q_1^2 - 2(t+1)q_1 - q_2 + t\}\}.\end{aligned}$$

Remark 4. Let us mention a particular solution which is expressed by the terms of solutions of the sixth Painlevé equation. When $\theta^0 + \theta^1 + \theta^t + \kappa_1 + \kappa_2 = 0$, that is, $\kappa_2 = \kappa_3$, it is obvious that the Hamiltonian system has the particular solution. On this condition of parameters $q_2 = 0$ is a solution because $\frac{d}{dt}q_2 = 0$ on the condition $q_2 = 0$. Substituting these, we have

$$\frac{d}{dt}q_1 = \frac{1}{2}\frac{\partial H_{\mathrm{VI}}\begin{pmatrix}2\theta^0+1,2\theta^1,2\theta^t\\2\kappa_1-1,\kappa_2+\kappa_3\end{pmatrix}}{\partial p_1}, \qquad \frac{d}{dt}p_1 = -\frac{1}{2}\frac{\partial H_{\mathrm{VI}}\begin{pmatrix}2\theta^0+1,2\theta^1,2\theta^t\\2\kappa_1-1,\kappa_2+\kappa_3\end{pmatrix}}{\partial q_1}, \qquad \frac{d}{dt}q_2 = 0,$$

$$\begin{aligned}t(t-1)\frac{d}{dt}p_2 = \;&2q_1(q_1-1)(q_1-t)p_2^2 \\ &+2\{(1-t)(\theta^0+1) + \theta^t + (\theta^1 + \kappa_2 + \kappa_3)(2q_1 - t) - (3q_1^2 - 2(1+t)q_1 + t)p_1\}p_2 \\ &+\left(\frac{1}{2}(3q_1 - t - 1)p_1 - \theta^1 - \kappa_2 - \kappa_3\right)p_1.\end{aligned}$$

As a result, we can construct a particular solution, setting (q_1, p_1) is a solution of the sixth Painlevé equation, $q_2 = 0$, and p_2 is a solution of the Riccati equation, whose coefficients are expressed by the solution of the sixth Painlevé equation. □

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