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The Chas-Sullivan conjecture for a surface of infinite genus

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Abstract

Let $\Sigma_{\infty,1}$ be the inductive limit of compact oriented surfaces with one boundary component. We prove the center of the Goldman Lie algebra of the surface $\Sigma_{\infty,1}$ is spanned by the constant loop. A similar statement for a closed oriented surface was conjectured by Chas and Sullivan, and proved by Etingof. Our result is deduced from a computation of the center of the Lie algebra of oriented chord diagrams.

1 Introduction

Let S be a connected oriented surface and let $\hat{\pi} = \hat{\pi}(S) = [S^1, S]$ be the set of free homotopy classes of oriented loops on S. In 1986 Goldman [3] introduced a Lie algebra structure on the vector space $\mathbb{Q}\hat{\pi}$ spanned by the set $\hat{\pi}$. Nowadays this Lie algebra is called *the Goldman Lie algebra*, whose bracket is defined as follows. Let α, β be immersed loops on S such that their intersections consist of transverse double points. For each $p \in \alpha \cap \beta$, let $|\alpha_p \beta_p|$ be the free homotopy class of the loop first going the oriented loop α based at p, then going β based at p. Also let $\varepsilon(p; \alpha, \beta) \in \{\pm 1\}$ be the local intersection number of α and β at p, and set

$$[\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) |\alpha_p \beta_p| \in \mathbb{Q}\hat{\pi}.$$

He proved this descends to a Lie bracket on the vector space $\mathbb{Q}\hat{\pi}$. It is clear from the definition if α and β are freely homotopic to disjoint curves, then $[\alpha, \beta] = 0$. In the same paper, he proved a part of the opposite direction.

Theorem 1.0.1 (Goldman [3] Theorem 5.17)**.** *Let $\alpha, \beta \in \hat{\pi}$, where α is represented by a simple closed curve. Then $[\alpha, \beta] = 0$ in $\mathbb{Q}\hat{\pi}$ if and only if α and β are freely homotopic to disjoint curves.*

It is a fundamental problem to compute the center of a given Lie algebra. We denote the center of a Lie algebra $\mathfrak{g}$ by $Z(\mathfrak{g})$. If S is closed, then, from this theorem, $\hat{\pi} \cap Z(\mathbb{Q}\hat{\pi}) = \{1\}$. Here $1 \in \hat{\pi}$ is the constant loop. Chas and Sullivan conjectured the following, and Etingof proved it.

Theorem 1.0.2 (Etingof [1])**.** *If S is closed, the center $Z(\mathbb{Q}\hat{\pi})$ of the Lie algebra $\mathbb{Q}\hat{\pi}$ is spanned by the constant loop $1 \in \hat{\pi}$.*

His proof is based on symplectic geometry of the moduli space of flat $GL_N(\mathbb{C})$-bundles over the surface S. In this paper we study a variant of the Chas-Sullivan conjecture and give a supporting evidence for it. The variant, in the most general setting, is stated as follows.

Conjecture 1.0.3. *For any connected oriented surface S, the center $Z(\mathbb{Q}\hat{\pi})$ is spanned by the set $\hat{\pi} \cap Z(\mathbb{Q}\hat{\pi})$.*

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus g with one boundary component, ζ the simple loop going around the boundary in the opposite direction. Then, for $S = \Sigma_{g,1}$, we have $\hat{\pi} \cap Z(\mathbb{Q}\hat{\pi}) = \{\zeta^n; n \in \mathbb{Z}\}$ by Theorem 1.0.1. Hence Conjecture 1.0.3 for $S = \Sigma_{g,1}$ is given as follows.

Conjecture 1.0.4.

$$Z(\mathbb{Q}\hat{\pi}(\Sigma_{g,1})) = \bigoplus_{n\in\mathbb{Z}} \mathbb{Q}\zeta^n.$$

This conjecture is still open. We shall study a surface of infinite genus, instead. Gluing a compact connected oriented surface $\Sigma_{1,2}$ of genus 1 with 2 boundary components to the surface $\Sigma_{g,1}$ along the boundary, we obtain an embedding $i^g_{g+1} \colon \Sigma_{g,1} \hookrightarrow \Sigma_{g+1,1}$. We define a connected oriented surface $\Sigma_{\infty,1}$ as the inductive limit of these embeddings. Our main result supports Conjecture 1.0.3. The conjecture holds for the surface $S = \Sigma_{\infty,1}$:

Theorem 1.0.5.

$$Z(\mathbb{Q}\hat{\pi}(\Sigma_{\infty,1})) = \mathbb{Q}1.$$

Our method of proof differs from Etingof's proof of Theorem 1.0.2, and is based on our previous result [7] Theorem 1.2.1 which connects the Goldman Lie algebra $\mathbb{Q}\hat{\pi}(\Sigma_{g,1})$ to Kontsevich's "associative" formal symplectic geometry $\mathfrak{a}_g$. The notion of a *symplectic expansion* introduced by Massuyeau [11] plays a vital role there. Theorem 1.0.5 is deduced from a computation of the center of *the Lie algebra of oriented chord diagrams*, which is introduced in §3. This Lie algebra can be thought as the "limit" of the $\mathfrak{sp}$-invariants $(\mathfrak{a}_g)^{\mathfrak{sp}}$, $g \to \infty$, where $\mathfrak{sp} = \mathfrak{sp}_{2g}(\mathbb{Q})$, and its bracket is defined by a diagrammatic way. Along the proof we also prove a counterpart to Conjecture 1.0.4 in the formal symplectic geometry side, is true in a stable range (Theorem 3.2.8).

This paper is organized as follows. In §2, we recall symplectic expansions, Kontsevich's "associative" $\mathfrak{a}_g$, and our previous result. In §3, we give a description of the $\mathfrak{sp}$-invariants $(\mathfrak{a}_g)^{\mathfrak{sp}}$ by labeled chord diagrams. Looking at the bracket on $(\mathfrak{a}_g)^{\mathfrak{sp}}$, we arrive at the definition of the Lie algebra of oriented chord diagrams. We determine the center of this Lie algebra, and compute the center of the "associative" $\mathfrak{a}_g^-$, an extension of $\mathfrak{a}_g$, in a stable range. This gives a supporting evidence for Conjecture 1.0.4, since it enables us to approximate a given element of the center of $\mathbb{Q}\hat{\pi}(\Sigma_{g,1})$ by a polynomial in ζ (Corollary 3.2.9). In §4 we prove Theorem 1.0.5. A rough idea is as follows. Any element of $Z(\mathbb{Q}\hat{\pi}(\Sigma_{\infty,1}))$ lies in $Z(\mathbb{Q}\hat{\pi}(\Sigma_{g,1}))$ for some g. By the result in §3, this element is approximated by a polynomial in ζ. But we easily see the image of any positive power of ζ by the inclusion $i^g_\infty \colon \Sigma_{g,1} \to \Sigma_{\infty,1}$ does not lie in $Z(\mathbb{Q}\hat{\pi}(\Sigma_{\infty,1}))$, and conclude the element must be a multiple of the constant loop. In §5, we remark the bracket introduced in §3 naturally extends to the bracket on the space of *linear chord diagrams*.

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Contents

2 Symplectic expansion and formal symplecic geometry

In this section we fix an integer $g \geq 1$, and simply write $\Sigma = \Sigma_{g,1}$. Choose a basepoint $*$ on the boundary $\partial\Sigma$. The fundamental group $\pi := \pi_1(\Sigma, *)$ is a free group of rank 2g. The set $\hat{\pi} = \hat{\pi}(\Sigma)$ is exactly the set of conjugacy classes in the group π. We denote by $|\ |\colon \mathbb{Q}\pi \to \mathbb{Q}\hat{\pi}$ the natural projection.

2.1 Symplectic expansion

We begin by recalling the notion of a symplectic expansion introduced by Massuyeau [11]. Let $H := H_1(\Sigma; \mathbb{Q})$ be the first homology group of Σ. H is naturally isomorphic to $H_1(\pi; \mathbb{Q}) \cong \pi^{\rm abel} \otimes_{\mathbb{Z}} \mathbb{Q}$, the first homology group of π. Here $\pi^{\rm abel} = \pi/[\pi, \pi]$ is the abelianization of π. Under this identification, we write

$$[x] := (x \bmod [\pi, \pi]) \otimes_{\mathbb{Z}} 1 \in H, \quad \text{for } x \in \pi.$$

Let $\widehat{T}$ be the completed tensor algebra generated by H. Namely $\widehat{T} = \prod_{m=0}^{\infty} H^{\otimes m}$, where $H^{\otimes m}$ is the tensor space of degree m. This is a complete Hopf algebra over $\mathbb{Q}$ whose coproduct $\Delta\colon \widehat{T} \to \widehat{T}\hat{\otimes}\widehat{T}$ is given by $\Delta(X) = X\hat{\otimes}1 + 1\hat{\otimes}X$, $X \in H$. Here $\widehat{T}\hat{\otimes}\widehat{T}$ is the completed tensor product of the two $\widehat{T}$'s. The algebra $\widehat{T}$ has a decreasing filtration given by

$$\widehat{T}_p := \prod_{m\geq p} H^{\otimes m}, \quad \text{for } p \geq 1.$$

An element $u \in \widehat{T}$ is called group-like if $\Delta u = u\hat{\otimes}u$. As is known, the set of group-like elements is a subgroup of the multiplicative group of the algebra $\widehat{T}$. We regard ζ as a based loop with basepoint $*$. If we choose a symplectic generating system $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ of the fundamental group π, we have $\zeta = \prod_{i=1}^{g} \alpha_i\beta_i{\alpha_i}^{-1}{\beta_i}^{-1}$. Here, for γ_1 and $\gamma_2 \in \pi$, the product $\gamma_1\gamma_2 \in \pi$ is defined to be the based homotopy class of the loop traversing first γ_1 and then γ_2. The intersection form on the homology group H defines the symplectic form

$$\omega = \sum_{i=1}^{g} A_iB_i - B_iA_i \in H^{\otimes 2},$$

where $A_i = [\alpha_i]$ and $B_i = [\beta_i] \in H$. Here and throughout this paper we often omit $\otimes$ to express tensors. The exponential map $\exp\colon \widehat{T}_1 \to \widehat{T}$ is defined by $\exp(u) = \sum_{k=0}^{\infty}(1/k!)u^k \in \widehat{T}$ for $u \in \widehat{T}_1$.

Definition 2.1.1. (Massuyeau [11]) *A symplectic expansion θ of the fundamental group π of the surface Σ is a map $\theta\colon \pi \to \widehat{T}$ satisfying the conditions*

(1) $\theta(x) \equiv 1 + [x] \pmod{\widehat{T}_2}$ *for any $x \in \pi$,*

(2) $\theta(xy) = \theta(x)\theta(y)$ *for any x and $y \in \pi$,*

(3) $\theta(x)$ *is group-like for any $x \in \pi$,*

(4) $\theta(\zeta) = \exp(\omega)$.

Symplectic expansions do exist [11] Lemma 2.16, and they are infinitely many [7] Proposition 2.8.1. Several constructions of a symplectic expansion are known; *harmonic Magnus expansions* [6] via a transcendental method, a construction in [11] via the *LMO functor*; also there is an elementary method to associate a symplectic expansion with any (not necessary symplectic) free generators of π [10].

A map $\theta\colon \pi \to \widehat{T}$ satisfying the conditions (1) and (2) is called a *Magnus expansion* of the free group π [5].

2.2 Formal symplectic geometry

We recall the "associative" formal symplectic geometry $\mathfrak{a}_g$ introduced by Kontsevich [9]. Let $N\colon \widehat{T} \to \widehat{T}_1$ be a linear map defined by

$$N|_{H^{\otimes n}} = \sum_{k=0}^{n-1} \nu^k, \quad n \geq 1,$$

where ν is the cyclic permutation given by $X_1X_2\cdots X_n \mapsto X_2\cdots X_nX_1$ for $X_i \in H$, $n \geq 1$, and $N|_{H^{\otimes 0}} = 0$. By definition, a derivation on $\widehat{T}$ is a linear map $D\colon \widehat{T} \to \widehat{T}$ satisfying the Leibniz rule:

$$D(u_1u_2) = D(u_1)u_2 + u_1D(u_2),$$

for $u_1, u_2 \in \widehat{T}$. Since $\widehat{T}$ is freely generated by H as a complete algebra, any derivation on $\widehat{T}$ is uniquely determined by its values on H, and the space of derivations of $\widehat{T}$ is identified with $\mathrm{Hom}(H, \widehat{T})$. By the Poincaré duality, $\widehat{T_1} \cong H \otimes \widehat{T}$ is identified with $\mathrm{Hom}(H, \widehat{T})$:

$$\widehat{T_1} \cong H \otimes \widehat{T} \xrightarrow{\cong} \mathrm{Hom}(H, \widehat{T}), \quad X \otimes u \mapsto (Y \mapsto (Y \cdot X)u). \tag{2.2.1}$$

Here $(\ \cdot\ )$ is the intersection pairng on $H = H_1(\Sigma; \mathbb{Q})$.

Let $\mathfrak{a}_g^- = \mathrm{Der}_\omega(\widehat{T})$ be the space of derivations on $\widehat{T}$ killing the symplectic form ω. In view of (2.2.1) any derivation D is written as

$$D = \sum_{i=1}^{g} B_i \otimes D(A_i) - A_i \otimes D(B_i).$$

Since $-D(\omega) = \sum_{i=1}^{g} [B_i, D(A_i)] - [A_i, D(B_i)]$, we have $\mathfrak{a}_g^- = \mathrm{Ker}([\ ,\ ]\colon H \otimes \widehat{T} \to \hat{T})$. It is easy to see $\mathrm{Ker}([\ ,\ ]) = N(\widehat{T_1})$ (see [7] Lemma 2.6.2 (4)). Hence we can write

$$\mathfrak{a}_g^- = \mathrm{Ker}([\ ,\ ]\colon H \otimes \widehat{T} \to \widehat{T}) = N(\widehat{T_1}). \tag{2.2.2}$$

The Lie subalgebra $\mathfrak{a}_g := N(\widehat{T_2})$ is nothing but (the completion of) what Kontsevich [9] calls a_g. By a straightforward computation, the bracket on $\mathfrak{a}_g^-$ as derivations is given as follows.

Lemma 2.2.1. *We have*

$$\begin{aligned} &[N(X_1 \cdots X_n), N(Y_1 \cdots Y_m)] \\ = &-\sum_{s=1}^{n}\sum_{t=1}^{m} (X_s \cdot Y_t) N(X_{s+1} \cdots X_n X_1 \cdots X_{s-1} Y_{t+1} \cdots Y_m Y_1 \cdots Y_{t-1}) \end{aligned}$$

for $X_1, \dots, X_n, Y_1, \dots, Y_m \in H$.

We introduce a bilinear map $\mathcal{B}\colon H^{\otimes n} \times H^{\otimes m} \to N(H^{\otimes(n+m-2)})$ by

$$\mathcal{B}(X_1 \cdots X_n, Y_1 \cdots Y_m) := -(X_1 \cdot Y_1) N(X_2 \cdots X_n Y_2 \cdots Y_m) \tag{2.2.3}$$

for $X_s, Y_t \in H$. Then, Lemma 2.2.1 is written as

$$[Nu, Nv] = \sum_{s=0}^{n-1}\sum_{t=0}^{m-1} \mathcal{B}(\nu^s u, \nu^t v) = \mathcal{B}(Nu, Nv) \tag{2.2.4}$$

for $u \in H^{\otimes n}$ and $v \in H^{\otimes m}$.

2.3 "Completion" of the Goldman Lie algebra

The following result is proved in [7].

Theorem 2.3.1 ([7] Theorem 1.2.1)**.** *For any symplectic expansion θ, the map*

$$-N\theta\colon \mathbb{Q}\hat{\pi} \to N(\widehat{T_1}) = \mathfrak{a}_g^-, \quad \pi \ni x \mapsto -N\theta(x) \in N(\widehat{T_1})$$

is a well-defined Lie algebra homomorphism. The kernel is the subspace $\mathbb{Q}1$ spanned by the constant loop 1, and the image is dense in $N(\widehat{T_1}) = \mathfrak{a}_g^-$ with respect to the $\widehat{T_1}$-adic topology.

By this theorem, we may regard the formal symplectic geometry $\mathfrak{a}_g^-$ as a certain kind of completion of the Goldman Lie algebra $\mathbb{Q}\hat{\pi}$. We introduce a decreasing filtration of the Goldman Lie algebra $\mathbb{Q}\hat{\pi}$ defined by

$$\mathbb{Q}\hat{\pi}(p) := (N\theta)^{-1}N(\widehat{T}_p), \quad \text{for } p \geq 1.$$

Since $N(\widehat{T}_p)$ is a Lie subalgebra of $\mathfrak{a}_g^- = N(\widehat{T}_1)$, the subspace $\mathbb{Q}\hat{\pi}(p)$ is also a Lie subalgebra of $\mathbb{Q}\hat{\pi}$. Let $\theta'\colon \pi \to \widehat{T}$ be another Magnus expansion which is not necessarily symplectic. We denote by $[\widehat{T}, \widehat{T}]$ the derived ideal of $\widehat{T}$ as a Lie algebra, in other words, $[\widehat{T}, \widehat{T}]$ is the vector subspace generated by the set $\{uv - vu;\, u, v \in \widehat{T}\}$. Let $\varepsilon\colon \widehat{T} \to H^0 = \mathbb{Q}$ be the augmentation.

Lemma 2.3.2. *Fix $p \geq 1$. For $u \in \mathbb{Q}\pi$, the followings are equivalent.*

1. $|u| \in \mathbb{Q}\hat{\pi}(p)$, *namely,* $N\theta(u) \in N(\widehat{T}_p)$.
2. $\theta(u) - \varepsilon(u) \in \widehat{T}_p + [\widehat{T}, \widehat{T}]$.
3. $\theta'(u) - \varepsilon(u) \in \widehat{T}_p + [\widehat{T}, \widehat{T}]$.

In particular, the filtration $\{\mathbb{Q}\hat{\pi}(p)\}_{p=1}^{\infty}$ is independent of the choice of a Magnus expansion.

Proof. We have $N(X_1 \cdots X_n) - nX_1 \cdots X_n = \sum_{i=1}^n (X_i \cdots X_n X_1 \cdots X_{i-1} - X_1 \cdots X_n) = \sum_{i=2}^n [X_i \cdots X_n, X_1 \cdots X_{i-1}] \in [\widehat{T}, \widehat{T}]$ for $X_i \in H$. This means $Nu - nu \in [\widehat{T}, \widehat{T}]$ for any $u \in H^{\otimes n}$. If $u \in \mathrm{Ker} N \cap H^{\otimes n}$, then $u = -\frac{1}{n}(Nu - nu) \in [\widehat{T}, \widehat{T}]$. Clearly $N[\widehat{T}, \widehat{T}] = 0$. Hence we have

$$0 \to [\widehat{T}, \widehat{T}] \to \widehat{T}_1 \overset{N}{\to} \widehat{T}_1 \quad \text{(exact)}. \tag{2.3.1}$$

In particular, we have $(N|_{\widehat{T}_1})^{-1}(N(\widehat{T}_p)) = \widehat{T}_p + [\widehat{T}, \widehat{T}]$, which implies the conditions (1) and (2) are equivalent.

As was proved in [5] Theorem 1.3, there exists a filter-preserving algebra automorphism U of $\widehat{T}$ satisfying the equation $\theta' = U \circ \theta$. Then we have $U(\widehat{T}_p + [\widehat{T}, \widehat{T}]) = \widehat{T}_p + [\widehat{T}, \widehat{T}]$. Hence the conditions (2) and (3) are equivalent. This proves the lemma. □

Let $I\pi$ be the augmentation ideal of the group ring $\mathbb{Q}\pi$, i.e., the kernel of the augmentation $\varepsilon\colon \mathbb{Q}\pi \to \mathbb{Q}$. It is easy to show $\mathbb{Q}\hat{\pi}(p) = |\mathbb{Q}1 + (I\pi)^p|$, from which it also follows $\mathbb{Q}\hat{\pi}(p)$ is independent of a Magnus expansion. As a corollary of Theorem 2.3.1, we have

$$\bigcap_{p=1}^{\infty} \mathbb{Q}\hat{\pi}(p) = \mathrm{Ker} N\theta = \mathbb{Q}1, \quad \text{and} \tag{2.3.2}$$

$$Z(\mathbb{Q}\hat{\pi}) \subset (N\theta)^{-1} Z(\mathfrak{a}_g^-). \tag{2.3.3}$$

In view of this corollary (2.3.3), we are lead to consider the center $Z(\mathfrak{a}_g^-)$ of the Lie algebra $\mathfrak{a}_g^-$. The subspace $N(H^{\otimes 2})$ of $\mathfrak{a}_g^-$ is a Lie subalgebra naturally isomorphic to the Lie algebra of the symplectic group, $\mathfrak{sp} := \mathfrak{sp}_{2g}(\mathbb{Q})$. Hence $Z(\mathfrak{a}_g^-)$ is included in the $\mathfrak{sp}$-invariants $(\mathfrak{a}_g^-)^{\mathfrak{sp}} = (\mathfrak{a}_g)^{\mathfrak{sp}}$, i.e., the tensors anihilated by the action of $\mathfrak{sp}$. Here we use the fact $H^{\mathfrak{sp}} = 0$. The subspace $(\mathfrak{a}_g)^{\mathfrak{sp}}$ is a Lie subalgebra of $\mathfrak{a}_g$. Thus we obtain

$$Z(\mathfrak{a}_g^-) \subset Z((\mathfrak{a}_g)^{\mathfrak{sp}}). \tag{2.3.4}$$

3 Lie algebra of oriented chord diagrams

In this section, we describe the Lie algebra $(\mathfrak{a}_g)^{\mathfrak{sp}}$ in a stable range by introducing the Lie algebra of oriented chord diagrams. Following Morita [8] [12] [13] and Kontsevich [9], we make the symplectic form ω correspond to a labeled chord.

3.1 The $\mathfrak{sp}$-invariant tensors

Under the identification $\mathfrak{a}_g = N(\widehat{T}_2)$, we denote $(\mathfrak{a}_g)_{(n)} := \mathfrak{a}_g \cap H^{\otimes n} = N(H^{\otimes n}) \subset H^{\otimes n}$ for $n \geq 2$. We begin by recalling the $\mathfrak{sp}$-invariant tensors in the space $H^{\otimes n}$. It is a classical result of Weyl [14] ch. VI, §1, the space of $\mathfrak{sp}$-invariant tensors in $H^{\otimes n}$ is zero if n is odd, and generated by *linear chord diagrams* of $n/2$ chords if n is even. Let m be a positive integer. A *linear chord diagram* of m chords is a decomposition of the set of vertices $\{1, 2, \dots, 2m\}$ into m *unordered* pairs $\{\{i_1, j_1\}, \{i_2, j_2\}, \dots, \{i_m, j_m\}\}$. Further, a *labeled linear chord diagram* of m chords C is a set of m *ordered* pairs $\{(i_1, j_1), (i_2, j_2), \dots, (i_m, j_m)\}$ satisfying the condition $\{i_1, \dots, i_m, j_1, \dots, j_m\} = \{1, 2, \dots, 2m\}$. We denote by $\overline{C}$ the underlying linear chord diagram of C, $\overline{C} := \{\{i_1, j_1\}, \{i_2, j_2\}, \dots, \{i_m, j_m\}\}$. An $\mathfrak{sp}$-invariant tensor $a(C) \in H^{\otimes 2m}$ is defined by

$$a(C) := \begin{pmatrix} 1 & 2 & \cdots & 2m-1 & 2m \\ i_1 & j_1 & \cdots & i_m & j_m \end{pmatrix} (\omega^{\otimes m}) \in (H^{\otimes 2m})^{\mathfrak{sp}}.$$

Let C' be a labeled linear chord diagram obtained from C by a single label change. Namely, we have

$$C' = \{(i_1, j_1), \dots, (i_{k-1}, j_{k-1}), (j_k, i_k), (i_{k+1}, j_{k+1}), \dots, (i_m, j_m)\}$$

for a single k. Clearly we have $\overline{C'} = \overline{C}$ and $a(C') = -a(C)$. We denote by $\mathcal{LC}_m$ the $\mathbb{Q}$-linear space spanned by the labeled linear chord diagrams of m chords modulo the linear subspace generated by the set

$$\{C + C'; C' \text{ is obtained from } C \text{ by a single label change.}\},$$

and call it *the space of oriented linear chord diagrams of* m *chords.* We have a natural map

$$a\colon \mathcal{LC}_m \to (H^{\otimes 2m})^{\mathfrak{sp}}, \quad C \mapsto a(C).$$

Now we have

Lemma 3.1.1. *The map* $a\colon \mathcal{LC}_m \to (H^{\otimes 2m})^{\mathfrak{sp}}$ *is*

(1) *surjective for any* $m \geq 1$, *and*

(2) *an isomorphism if and only if* $m \leq g$.

The assertion (1) and the "if" part of (2) are Weyl's result stated above, while the "only if" part of (2) is due to Morita [13] p.797, Proposition 4.1. See also [12] p.361, Lemma 4.1.

Let $\nu \in \mathfrak{S}_{2m}$ be the cyclic permutation introduced in §2.2

$$\nu = \begin{pmatrix} 1 & 2 & 3 & \cdots & 2m \\ 2m & 1 & 2 & \cdots & 2m-1 \end{pmatrix}.$$

We denote by Z_{2m} the cyclic subgroup generated by ν in the group $\mathfrak{S}_{2m}$. For a labeled linear chord diagram $C = \{(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)\}$, we define

$$\nu^s(C) := \{(\nu^s(i_1), \nu^s(j_1)), (\nu^s(i_2), \nu^s(j_2)), \ldots, (\nu^s(i_m), \nu^s(j_m))\}, \quad s \in \mathbb{Z}.$$

Clearly we have $a(\nu^s C) = \nu^s a(C)$. This action descends to an action of Z_{2m} on the space $\mathcal{LC}_m$. We define the space $\mathcal{C}_m$ as the Z_{2m}-invariants in $\mathcal{LC}_m$

$$\mathcal{C}_m := (\mathcal{LC}_m)^{Z_{2m}},$$

and call it *the space of oriented chord diagrams of m chords.* If $m = 1$ and $C = \{(1, 2)\}$, then $\nu(C) = -C \in \mathcal{LC}_1$. Hence we have $\mathcal{C}_1 = 0$. We define

$$\mathcal{C} := \prod_{m=2}^{\infty} \mathcal{C}_m.$$

A *labeled chord diagram* of m chords $\mathcal{N}(C)$ is a collection of $2m$ labeled linear chord diagrams $C, \nu(C), \ldots, \nu^{2m-1}(C)$ for some C with m chords (to be more precise, we consider $\mathcal{N}(C)$ as an element of the $2m$-th symmetric product of the set of labeled linear chord diagrams of m-chords). We also denote $N(C) := \sum_{s=0}^{2m-1} \nu^s(C) \in \mathcal{C}_m$. We have $\mathcal{N}(\nu(C)) = \mathcal{N}(C)$ and

$$a(N(C)) = N(a(C)) \in (H^{\otimes 2m})^{Z_{2m}} = N(H^{\otimes 2m}) = (\mathfrak{a}_g)_{(2m)}.$$

Definition 3.1.2. *For a labeled linear chord diagram C of m chords, define the index of C as the cardinality of the set $\{\overline{\nu^s(C)}; 0 \leq s \leq 2m-1\}$. We also define the index of $\mathcal{N}(C)$ as the index of one of the diagrams in $\mathcal{N}(C)$. We say a diagram is of maximal index if its index is twice the number of chords.*

Clearly the index of $\mathcal{N}(C)$ is independent of the choice of a diagram.

Lemma 3.1.3. *Let C be a labeled linear chord diagram of m chords. Then $N(C) = 0 \in \mathcal{C}_m$ if and only if C is of odd index.*

Proof. We denote by $\overline{C}^\flat$ the labeled linear chord diagram on the underlying linear chord diagram $\overline{C}$ with *the standard label*, which means $i_k < j_k$ for any k. Clearly we have $N(C) = \pm N(\overline{C}^\flat)$. Let l be the index of C. Then, since $\nu(\overline{C}^\flat) = -\overline{\nu(C)}^\flat \in \mathcal{C}_m$, we have

$$N(\overline{C}^\flat) = \sum_{s=0}^{2m-1} \nu^s(\overline{C}^\flat) = \sum_{s=0}^{2m-1} (-1)^s \overline{\nu^s(C)}^\flat = \sum_{i=0}^{(2m/l)-1} (-1)^{li} \sum_{j=0}^{l-1} (-1)^j \overline{\nu^j(C)}^\flat$$

$$= \begin{cases} \frac{2m}{l} \sum_{j=0}^{l-1} (-1)^j \overline{\nu^j(C)}^\flat & \text{if } l \text{ is even,} \\ 0, & \text{if } l \text{ is odd.} \end{cases}$$

Here we remark $\overline{\nu^j(C)}^\flat$, $0 \leq j \leq l-1$, are linearly independent. This proves the lemma. □

The following is a corollary of Lemma 3.1.1.

Lemma 3.1.4. *The map $a \colon \mathcal{C}_m \to (\mathfrak{a}_g)^{\mathfrak{sp}}_{(2m)}$, $N(C) \mapsto a(N(C))$, is*

(1) *surjective for any $m \geq 1$, and*

(2) *an isomorphism if* $m \leq g$.

Proof. The assertion (1) immediately follows from Lemma 3.1.1 (1). The set of labeled linear chord diagrams with the standard label is a basis of the space $\mathcal{LC}_m$. Hence the assertion (2) follows from Lemma 3.1.1 (2) and the computation in the proof of Lemma 3.1.3. □

Hence the map $a\colon \mathcal{C} \to (\mathfrak{a}_g)^{\mathfrak{sp}}$ is an isomorphism in a stable range. So we compute the bracket on the Lie algebra $(\mathfrak{a}_g)^{\mathfrak{sp}}$ by means of the stable isomorphism a. Let C and C' be labeled chord diagrams given by

$$C = \{(i_1, j_1), (i_2, j_2), \dots, (i_m, j_m)\} \quad \text{and} \quad C' = \{(a_1, b_1), (a_2, b_2), \dots, (a_l, b_l)\}.$$

Then, by the formula (2.2.4), we have

$$[a(N(C)), a(N(C'))] = [N(a(C)), N(a(C))] = \sum_{s=0}^{2m-1} \sum_{t=0}^{2l-1} \mathcal{B}(a(\nu^s C), a(\nu^t C')).$$

In order to describe $\mathcal{B}(a(C), a(C'))$, we define an amalgamation of two labeled linear chord diagrams as follows. We may assume $1 \in \{i_1, j_1\} \cap \{a_1, b_1\}$ without loss of generality. A labeled chord diagram of $m + l - 1$ chords $C * C'$ is defined by

$$\{(x, y), (i_2-1, j_2-1), \dots, (i_m-1, j_m-1), (a_2+2m-2, b_2+2m-2), \dots, (a_l+2m-2, b_l+2m-2)\},$$

where

$$(x, y) := \begin{cases} (b_1 + 2m - 2, j_1 - 1), & \text{if } i_1 = a_1 = 1, \\ (j_1 - 1, a_1 + 2m - 2), & \text{if } i_1 = b_1 = 1, \\ (i_1 - 1, b_1 + 2m - 2), & \text{if } j_1 = a_1 = 1, \\ (a_1 + 2m - 2, i_1 - 1), & \text{if } j_1 = b_1 = 1. \end{cases}$$

We call it the *amalgamation* of the labeled linear chord diagrams C and C'. Then we have

$$\mathcal{B}(a(C), a(C')) = Na(C * C').$$

In fact, if we define a bilinear map $\mathcal{B}'\colon H^{\otimes 2} \times H^{\otimes 2} \to H^{\otimes 2}$ by $\mathcal{B}'(X_1X_2, Y_1Y_2) := -(X_1 \cdot Y_1)X_2Y_2$, $X_i, Y_j \in H$, then we have $\mathcal{B}'(\omega, \omega) = -\omega$. This means (x, y) should be $(b_1 + 2m - 2, j_1 - 1)$ in the case $i_1 = a_1 = 1$. Similar observations hold for the other three cases. Hence we obtain

Lemma 3.1.5.

$$[a(N(C), a(N(C'))] = \sum_{s=0}^{2m-1} \sum_{t=0}^{2l-1} a(N((\nu^s C) * (\nu^t C'))).$$

Here it should be remarked the right hand side in the above equality does *not* depend on the genus g. Since the map a is a stable isomorphism, the whole of the maps a induces a Lie algebra structure on the space $\mathcal{C}$. The bracket is given by

$$[N(C), N(C')] = \sum_{s=0}^{2m-1} \sum_{t=0}^{2l-1} N((\nu^s C) * (\nu^t C')). \tag{3.1.1}$$

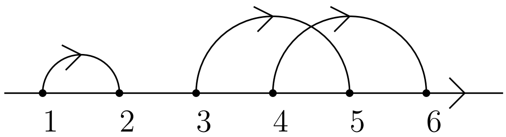
From Lemma 3.1.5 the map $a : \mathcal{C} \to {\mathfrak{a}_g}^{\mathfrak{sp}}$ is a Lie algebra homomorphism for each $g \geq 1$. In the next subsection, we will give a diagrammatic description of the Lie algebra $\mathcal{C}$, which will enable us to compute the center $Z((\mathfrak{a}_g)^{\mathfrak{sp}})$ in a stable range.

3.2 The center of the Lie algebra of oriented chord diagrams

In this subsection we give a diagrammatic description of a Lie algebra structure on $\mathcal{C} = \prod_m \mathcal{C}_m$ introduced by the formula (3.1.1) and compute its center.

We first recall the description of labeled linear chord diagrams by picture. Let $C = \{(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)\}$ be a labeled linear chord diagram of m chords. Fix a closed interval on the x-axis in the xy-plane and call it *the core* of the diagram. Put $2m$ distinct points on the interior of the core, and for each $1 \leq k \leq m$, draw an oriented simple path, called a *labeled chord*, in the upper half plane from the i_k-th point (with respect to the x-coordinate) to the j_k-th point. Hereafter we identify a labeled linear chord diagram with its picture. For example, the picture of $C = \{(1,2), (3,5), (4,6)\}$ is the following.

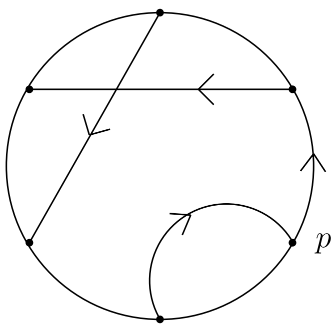
Figure 1: labeled linear chord diagram



We next recall a diagrammatic description of labeled chord diagrams in §3.1. In this subsection, a *labeled chord diagram* of m chords is a diagram in the xy-plane consisting of a circle, called *the core*, $2m$ vertices on the core, and oriented m simple paths, called *labeled chords*, connecting two vertices in the disk which bounds the core, such that the ends of the labeled chords exhaust the $2m$ vertices. We give an orientation to the core coming from that of the disk.

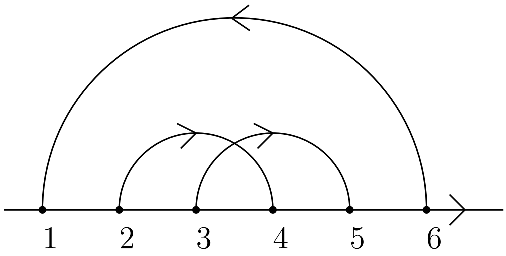
Given a labeled linear chord diagram C, we can produce a labeled chord diagram by connecting the two ends of the core by a simple path in the upper half plane avoiding the m chords. We call this operation *the closing of* C. For example, the closing of the diagram in Figure 1 is the following.

Figure 2



Conversely, given a vertex p of a labeled chord diagram D, we can produce a labeled linear chord diagram by cutting the core at a little short of p and embed the result into the xy-plane so that the cut core is included in the x-axis and the labeled chords are included in the upper half plane. We call this operation *the cut of* D *at* p, and denote the result by $C(D, p)$. For example, the cut of the diagram at p in Figure 2 is the following.

Figure 3: the cut $C(D, p)$



Let D be a labeled chord diagram of m chords, and p_0 a vertex of D. The collection of the cuts $C(D,p)$, where p runs through all the vertices of D, can be written as $\nu^k C(D,p_0)$, $0 \le k \le 2m-1$. This implies the two notions of labeled chord diagrams given in §3.1 and here are essentially the same. The sum $\sum_p C(D,p) \in \mathcal{LC}_m$ equals $N(C(D,p_0))$ hence is in $\mathcal{C}_m$. Let D' be a labeled chord diagram obtained from D by a single label change. Namely, D' is obtained from D by reversing the orientation of a single labeled chord. Then $\sum_p C(D,p) = -\sum_p C(D',p)$. Therefore the space $\mathcal{C}_m$ is also described as the $\mathbb{Q}$-linear space spanned by the labeled chord diagrams of m chords modulo the subspace generated by the set

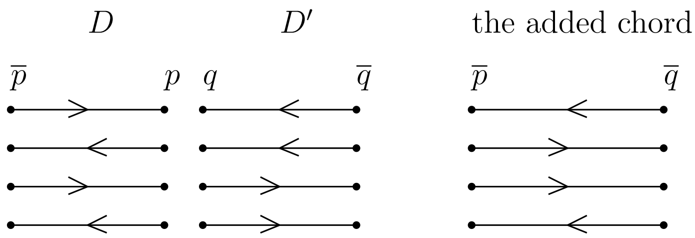
$$\{D + D';\ D' \text{ is obtained from } D \text{ by a single label change }\}.$$

We shall often regard a labeled chord diagram as an element of $\mathcal{C} := \prod_m \mathcal{C}_m$, if there is no confusion.

Let D and D' be labeled chord diagrams and let p and q be vertices of D and D', respectively. We shall produce a new labeled chord diagram $\mathcal{D}(D,p,D',q)$, which corresponds to an amalgamation in §3.1, by the following way. Let p_- and p_+ be the vertices of D adjacent to p, such that they are arranged as $p_- < p < p_+$ with respect to the cyclic ordering of vertices coming from the orientation of the core. Similarly, define q_- and q_+. Also, let $\overline{p}$ (resp. $\overline{q}$) be the vertex of D (resp. D') which is the other end of the edge though p (resp. q).

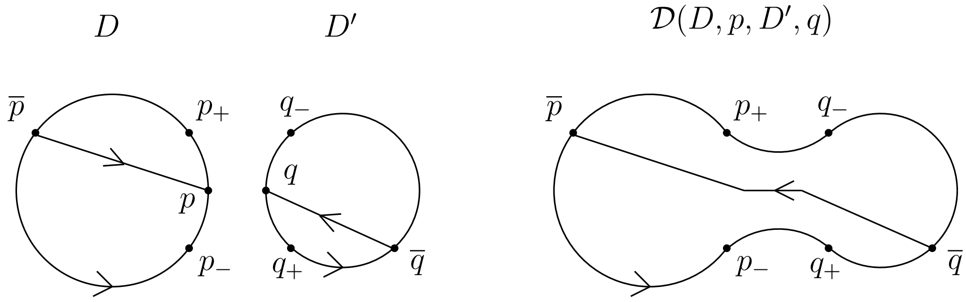
The first step is to place the cut $C(D',q)$ on the right of the cut $C(D,p_+)$, and regard the entirety as a labeled linear chord diagram. The second step is to remove the two chords through p or q, and add a labeled chord which connects $\overline{p}$ and $\overline{q}$ instead. The label of the added chord is determined by the following rule.

Figure 4: the label of the added chord



Finally, define $\mathcal{D}(D,p,D',q)$ to be the closing of the result of the second step. If D has m chords and D' has m' chords, then $\mathcal{D}(D,p,D',q)$ has $m+m'-1$ chords. We have $\mathcal{D}(D,p,D',q) = N(C(D,p) * C(D',q))$. A schematic picture of this operation is the following.

Figure 5: the new labeled chord diagram $\mathcal{D}(D,p,D',q)$



Definition 3.2.1. *Let D and D' be labeled chord diagrams of m and m' chords, respectively. Set*

$$[D, D'] := \sum_{(p,q)} \mathcal{D}(D, p, D', q) \in \mathcal{C}_{m+m'-1},$$

where the sum is taken over all pairs of the vertices of D and D'.

By construction, this formula is compatible with the formula (3.1.1). Hence it defines a well-defined Lie algebra structure on the space $\mathcal{C}$. But we continue a diagrammatic argument for its own interest. It is clear from the rule in Figure 4 that if D_1 is obtained from D by a single label change, then $[D, D'] = -[D_1, D']$. Therefore we can extend by linearlity the bracket in Definition 3.2.1 to a $\mathbb{Q}$-linear map $[\ ,\ ]\colon \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$.

Proposition 3.2.2. *The linear space $\mathcal{C} := \prod_m \mathcal{C}_m$ has a structure of Lie algebra with respect to the bracket defined above. Moreover, we have $[\mathcal{C}_m, \mathcal{C}_{m'}] \subset \mathcal{C}_{m+m'-1}$.*

We call this Lie algebra *the Lie algebra of oriented chord diagrams.*

Proof. The anti-symmetry of the bracket is clear from the rule in Figure 4. To prove the Jacobi identity, it suffices to show

$$[D, [D', D'']] = [D', [D, D'']] + [[D, D'], D''] \tag{3.2.1}$$

for any labeled chord diagrams D, D', and D''. Let p, q, and r be vertices of D, D', and D'', respectively. For simplicity, we denote $\mathcal{D}(D', q, D'', r) = D'''$. The contributions of p to the bracket $[D, D''']$ consists of the diagrams of the form $\mathcal{D}(D, p, D''', s)$, where s is a vertex of D'''. We consider the following four cases.

(1) s is the vertex corresponding to $\overline{q}$.

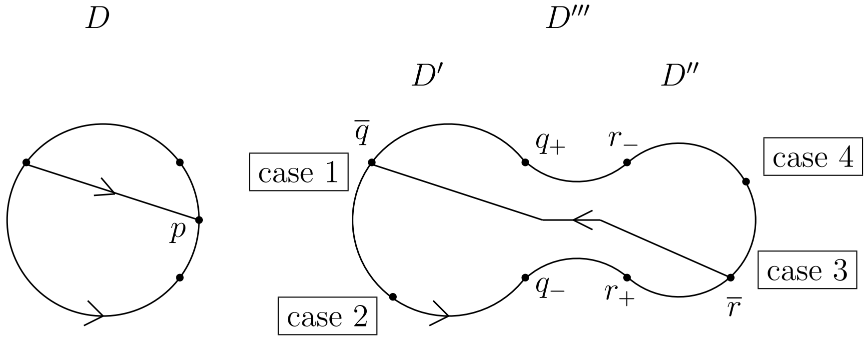
(2) s is a vertex corresponding to some vertex of D' other than $\overline{q}$.

(3) s is the vertex corresponding to $\overline{r}$.

(4) s is a vertex corresponding to some vertex of D'' other than $\overline{r}$.

See Figure 6 below. Consider the case (1). Then $\mathcal{D}(D, p, D''', s) = \mathcal{D}(D, p, D''', \overline{q})$. But this is also equal to $\mathcal{D}(D_0, q, D'', r)$, where $D_0 = \mathcal{D}(D, p, D', \overline{q})$. We can easily check the signs using Figure 4. Note that $\mathcal{D}(D_0, q, D'', r)$ will appear once when we compute the second term of the right hand side of (3.2.1). The same thing happens to each contribution of the case (2). By the same argument we see the cases (3) or (4) will appear once at the first term of the right hand side of (3.2.1).

Figure 6: the four cases



Now we consider the contributions $\mathcal{D}(D, p, D''', s)$ for all p, q, and r, and subtract them from the right hand side of (3.2.1). The remaining terms consist of two types. One comes from the first term, and is written as $\mathcal{D}(D', q, D_1, t)$, where $D_1 = \mathcal{D}(D, p, D'', r)$ and t is a vertex corresponding to some vertex of D. The other comes from the second term, and is written as $\mathcal{D}(D_2, u, D'', r)$, where $D_2 = \mathcal{D}(D, p, D', q)$ and u is a vertex corresponding to some vertex of D. By the same argument as before, we can see these two types cancel. This proves the Jacobi identity, hence completes the proof. □

An *isolated chord* in a labeled chord diagram is a labeled chord whose two ends are adjacent on the core.

Lemma 3.2.3. *Let D' be a labeled chord diagram having an isolated chord. Let q_0, q_1 be the ends of the isolated chord. Then*

$$\sum_p \mathcal{D}(D, p, D', q_0) + \sum_p \mathcal{D}(D, p, D', q_1) = 0$$

for any labeled chord diagram D.

Proof. We may assume q_1 is next to q_0 with respect to the orientation of the core. Let p_0 be a vertex of D and p_1 the vertex next to p_0. Then we have

$$\mathcal{D}(D, p_0, D', q_0) = -\mathcal{D}(D, p_1, D', q_1).$$

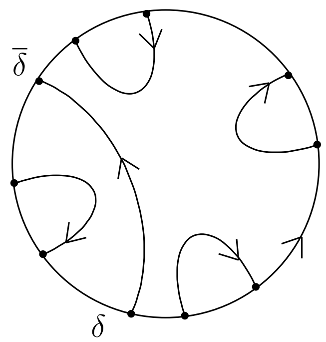
This proves the lemma. □

For $m \geq 1$, let $\Omega_m \in \mathcal{C}_m$ be the closing of the labeled linear chord diagram $I_m = \{(1,2), (3,4), \ldots, (2m-1, 2m)\}$. We say a vertex of Ω_m is *odd* (resp. *even*) if it corresponds to an odd (resp. even) numbered vertex in I_m. All the chords of Ω_m are isolated, hence by Lemma 3.2.3, $[D, \Omega_m] = 0$ for any labeled chord diagram D. Therefore, $\Omega_m \in Z(\mathcal{C})$.

For integers $a, b \geq 1$, define a labeled chord diagram $D(a, b)$ to be the closing of the labeled linear chord diagram

$$\{(1,2), (3,4), \ldots, (2a-1, 2a), (2a+1, 2a+2b+2), (2a+2, 2a+3), \ldots, (2a+2b, 2a+2b+1)\}.$$

Figure 7: $D(1, 3)$



Note that $D(a,b)$ is of maximal index, and $D(b,a) = -D(a,b) \in \mathcal{C}$. We denote by δ and $\bar{\delta}$, the vertices corresponding to $2a+1$ and $2a+2b+2$, respectively. See Figure 7. By Lemma 3.2.3, for any labeled chord diagram D, we have

$$[D, D(a,b)] = \sum_p \mathcal{D}(D, p, D(a,b), \delta) + \sum_p \mathcal{D}(D, p, D(a,b), \bar{\delta}). \tag{3.2.2}$$

We shall look into each term more detail. The diagram $\mathcal{D}(D, p, D(a,b), \delta)$ is obtained from D by inserting b isolated chords between p_- and p, and a isolated chords between p and p_+. Similarly the diagram $\mathcal{D}(D, p, D(a,b), \bar{\delta})$ is obtained from D by inserting a isolated chords between p_- and p, and b isolated chords between p and p_+, and reversing the orientation of the chord through p. Figure 9 is a picture of the results. Here, for simplicity we write a sequence of n isolated chords as Figure 8.

Figure 8: n isolated chords

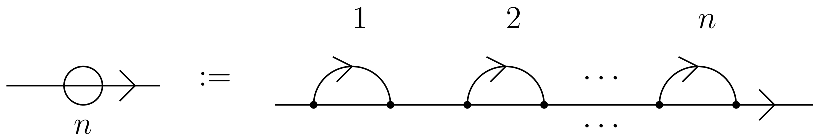
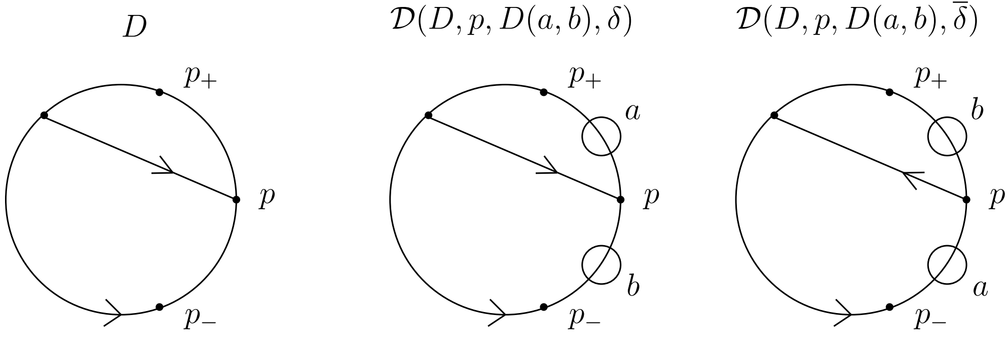


Figure 9: Inserting isolated chords



For a while, fix $m \geq 1$ and let $a = m$, $b = 2m+1$. The following two lemmas are the key in the sequel.

Lemma 3.2.4. *Let D_1 and D_2 be labeled chord diagrams of m chords, and let p_1 and p_2 be vertices of D_1 and D_2, respectively. Suppose $\mathcal{D}(D_1, p_1, D(a,b), \delta) = \pm\mathcal{D}(D_2, p_2, D(a,b), \delta)$ or $\mathcal{D}(D_1, p_1, D(a,b), \bar{\delta}) = \pm\mathcal{D}(D_2, p_2, D(a,b), \bar{\delta})$, in $\mathcal{C}$. Then the cuts $C(D_1, p_1)$ and $C(D_2, p_2)$ are equal in $\mathcal{LC}_m$ up to sign. In particular, $D_1 = \pm D_2 \in \mathcal{C}_m$.*

Proof. We only consider the case $\mathcal{D}(D_1, p_1, D(a,b), \delta) = \pm\mathcal{D}(D_2, p_2, D(a,b), \delta)$. We draw a picture of $C(D_i, p_i)$ as Figure 10. Here D'_i is the part of $C(D_i, p_i)$ between p_i and $\overline{p_i}$, D''_i the part on the right of $\overline{p_i}$, and the dotted line indicates the chords connecting the vertices in D'_i and D''_i. Then the diagrams $\mathcal{D}(D_i, p_i, D(a,b), \delta)$, $i = 1, 2$ look like Figure 11.

Figure 10: the cut $C(D_i, p_i)$

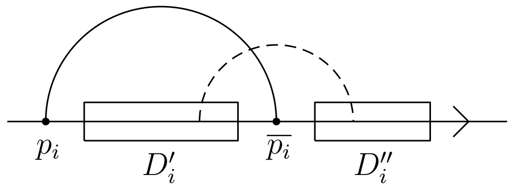
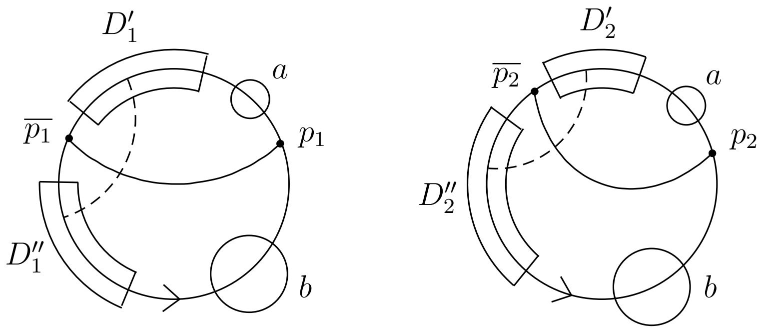


Figure 11: the two diagrams $\mathcal{D}(D_i, p_i, D(a,b), \delta)$

$\mathcal{D}(D_1, p_1, D(a,b), \delta)$ $\mathcal{D}(D_2, p_2, D(a,b), \delta)$



Observe that both the diagrams have a unique chord such that the number of vertices in the interior of the minor arc determined by the ends of the chord is $\geq 2m$. Namely, the chords $\{p_i, \overline{p_i}\}$, $i = 1, 2$. Moreover, the number of the vertices in the interior of the arc $\overline{p_i}p_i$ is $\geq 2(2m+1)$, and that of the arc $p_i\overline{p_i}$ is $\leq 4m$. These imply the diagrams $\mathcal{D}(D_i, p_i, D(a,b), \delta)$ are of maximal index, and by assumption these two diagrams must coincide when we forget the labels of chords, and the isomorphism between the two diagrams must maps p_1 to p_2 and $\overline{p_1}$ to $\overline{p_2}$. We conclude $C(D_1, p_1) = \pm C(D_2, p_2)$, and this proves the lemma. $\square$

Lemma 3.2.5. *Let D_1 and D_2 be labeled chord diagrams of m chords, and let p_1 and p_2 be vertices of D_1 and D_2, respectively. Suppose $\mathcal{D}(D_1, p_1, D(a,b), \delta) = \pm\mathcal{D}(D_2, p_2, D(a,b), \overline{\delta})$ in $\mathcal{C}$. Then one of the following two occurs: 1) $D_1 = D_2 = \Omega_m$ up to sign, and p_1 corresponds to an odd (resp. even) vertex and so does p_2 to an even (resp. odd) one, or 2) there exist $c, d \geq 1$ such that we have $D_1 = D_2 = D(c,d)$ up to sign, and p_1, p_2 correspond to $\delta, \overline{\delta}$, respectively.*

Proof. The picture of the diagram $\mathcal{D}(D_2, p_2, D(a,b), \overline{\delta})$ is obtained from the right diagram in Figure 11 by exchanging the role of a and b. By the same reason as before this diagram is of maximal index. If we forget the labels of chords, the diagrams $\mathcal{D}(D_1, p_1, D(a,b), \delta)$ and $\mathcal{D}(D_2, p_2, D(a,b), \overline{\delta})$ must be isomorphic by a unique map which maps p_1 to $\overline{p}_2$ and $\overline{p}_1$ to p_2. If D'_1 or D''_1 are the empty diagrams, the first conclusion follows. If D'_1 and D''_1 are both non-empty, the second conclusion follows. $\square$

As a corollary of the above two lemmas, we have:

Corollary 3.2.6. *Let D and D' be labeled chord diagrams of m chords, p and p' vertices of D and D', respectively, and let $d, d' \in \{\delta, \overline{\delta}\}$. Suppose $\mathcal{D}(D, p, D(a,b), d) = \pm\mathcal{D}(D', p', D(a,b), d')$ in $\mathcal{C}$. Then $D = \pm D' \in \mathcal{C}_m$.*

Now we are able to determine the center of $\mathcal{C}$.

Theorem 3.2.7.

$$Z(\mathcal{C}) = \prod_m \mathbb{Q}\Omega_m.$$

Proof. Since the Lie algebra $\mathcal{C}$ is graded, it suffices to show any homogeneous element of degree m which lies in the center $Z(\mathcal{C})$ is actually a multiple of Ω_m. Suppose $X \in \mathcal{C}_m \cap Z(\mathcal{C})$ and write X as

$$X = x\Omega_m + \sum_{(c,d)} x_{(c,d)} D(c,d) + \sum_i x_i D_i, \ x, x_{(c,d)}, x_i \in \mathbb{Q} \tag{3.2.3}$$

where the second term is a sum taken over $\{(c,d); 1 \leq c < d, m = c+d+1\}$, and the third term is a sum taken over labeled chord diagrams D_i not equal to $\pm\Omega_m$ and $\pm D(c,d)$. We may assume the index of any D_i is even, and $D_i \neq \pm D_j$ if $i \neq j$.

As in Lemmas 3.2.4 and 3.2.5, let $a = m$ and $b = 2m+1$. Then we have

$$0 = [X, D(a,b)] = \sum_{(c,d)} x_{(c,d)}[D(c,d), D(a,b)] + \sum_i x_i [D_i, D(a,b)].$$

We claim that the elements $[D(c,d), D(a,b)]$ and $[D_i, D(a,b)]$ are linearly independent in $\mathcal{C}$. Assuming this claim, we have $x_{(c,d)} = x_i = 0$ for all (c,d) and i. Thus $X = x\Omega_m$, and this will complete the proof.

Now we prove the claim. First we look at $[D(c,d), D(a,b)]$. For simplicity we denote $\mathcal{D}(D(c,d), \delta, D(a,b), \delta) = \mathcal{D}(\delta, \delta)$, etc. We have $\mathcal{D}(\delta,\delta) = D(b+c, a+d) = -D(a+d, b+c) = -\mathcal{D}(\overline{\delta}, \overline{\delta}) \in \mathcal{C}$, and similarly we have $\mathcal{D}(\delta, \overline{\delta}) = -\mathcal{D}(\overline{\delta}, \delta)$. Combining this with (3.2.2), we have

$$[D(c,d), D(a,b)] = \sum_{p \neq \delta, \overline{\delta}} \mathcal{D}(D(c,d), p, D(a,b), \delta) + \sum_{p \neq \delta, \overline{\delta}} \mathcal{D}(D(c,d), p, D(a,b), \overline{\delta}).$$

By Lemmas 3.2.4 and 3.2.5 and the fact that $D(c,d)$ is of maximal index, the $2(2m-2)$ diagrams appearing in this sum are distinct to each other, even if we forget the labels of chords. Therefore $[D(c,d), D(a,b)]$ is expressed as the sum of $2(2m-2)$ distinct labeled chord diagrams of maximal indices.

Next we look at $[D_i, D(a,b)]$. We denote by $\iota = \iota(D_i)$ the index of D_i. We have

$$[D_i, D(a,b)] = \sum_p \mathcal{D}(D_i, p, D(a,b), \delta) + \sum_p \mathcal{D}(D_i, p, D(a,b), \overline{\delta}).$$

Again by Lemmas 3.2.4 and 3.2.5 this sum equals $2m/\iota$ times the sum of $2\iota(D_i)$ distinct labeled chord diagrams of maximal indices.

Set $\Delta = \{D(c,d)\}_{(c,d)} \cup \{D_i\}_i$ and for each $D \in \Delta$, let $T_D \subset \mathcal{C}$ be the set of the diagrams appearing in $[D, D(a,b)]$ described as above. What we have observed is $[D, D(a,b)]$ is a non-zero multiple of $\sum_{\mathcal{D} \in T_D} \mathcal{D}$. Moreover, by Corollary 3.2.6, if $D, D' \in \Delta$, $D \neq D'$, then $T_D \cap (\pm T_{D'}) = \emptyset$. This shows $[D, D(a,b)]$, $D \in \Delta$ are linearly independent and proves the claim. ∎

The following theorem could be a supporting evidence for Conjecture 1.0.4.

Theorem 3.2.8. *Denote* $m(g) := \left[\frac{g-1}{4}\right] + 1$ *for* $g \geq 1$. *Then we have*

$$Z(\mathfrak{a}_g^-) + N(\widehat{T}_{2m(g)}) = \bigoplus_{m=2}^{\infty} \mathbb{Q}N(\omega^m) + N(\widehat{T}_{2m(g)}) \subset N(\widehat{T}_1) \subset \mathfrak{a}_g^-.$$

Proof. Let $u \in Z(\mathfrak{a}_g^-)$ be a homogeneous element of degree $< 2m(g)$. From (2.3.4), we have $u \in Z((\mathfrak{a}_g)^{\mathfrak{sp}})$. By Lemma 3.1.4 (2), there uniquely exists $X \in \mathcal{C}_m$, where $m < m(g)$, such that $a(X) = u$. As in (3.2.3), we write X as a linear combination of labeled chord diagrams of m chords. We want to show $X = x\Omega_m$. By Lemma 3.1.4 (1) and $u \in Z((\mathfrak{a}_g)^{\mathfrak{sp}})$, we have $a([X, D]) = 0$ for any labeled chord diagram. We observe that if D is of m chords, the number of chords of the diagrams appearing in $[D, D(a, b)]$ is $m + a + b = 4m + 1 \leq g$ since $m < m(g)$. Hence by the claim in the proof of Theorem 3.2.7, we see the elements $a([D, D(a, b)])$, $D \in \Delta$ are linearly independent in $(\mathfrak{a}_g)^{\mathfrak{sp}}$.

Therefore tracing the proof of Theorem 3.2.7 with applications of the map a to the equations in $\mathcal{C}$, we obtain $X = x\Omega_m$ hence u is a multiple of $a(\Omega_m) = N(\omega^m)$. The other inclusion is clear since the map a is surjective. □

As a corollary, we obtain

Corollary 3.2.9. *For any* $u \in Z(\mathbb{Q}\hat{\pi}(\Sigma_{g,1}))$, *there exists a polynomial* $f(\zeta) \in \mathbb{Q}[\zeta] \subset \mathbb{Q}\pi$ *such that*

$$u \equiv |f(\zeta)| \pmod{\mathbb{Q}\hat{\pi}(2m(g))}.$$

Proof. We have $N\theta(u) \in Z(\mathfrak{a}_g^-)$ by (2.3.3). From Theorem 3.2.8 there exists a polynomial $h(\omega) \in \mathbb{Q}[\omega]$ such that $N\theta(u) \equiv Nh(\omega) \pmod{N(\widehat{T}_{2m(g)})}$. Since θ is symplectic, we have $\theta(\zeta^n) = \sum_{k=0}^{\infty}(1/k!)n^k\omega^k$. From Vandermonde's determinant

$$\det\left(\frac{1}{k!}j^k\right)_{1\leq j,k\leq 2m(g)-1} = \left(\prod_{k=1}^{2m(g)-1} k!\right)^{-1} \prod_{j_i<j_2}(j_2 - j_1) \neq 0,$$

there exists a polynomial $f(\zeta) \in \mathbb{Q}[\zeta]$ such that $\theta(f(\zeta)) \equiv h(\omega) \pmod{\widehat{T}_{2m(g)}}$. Hence we have $N\theta(u) \equiv N\theta(f(\zeta)) \pmod{N(\widehat{T}_{2m(g)})}$, and so $u \equiv |f(\zeta)| \pmod{\mathbb{Q}\hat{\pi}(2m(g))}$, as was to be shown. □

4 Surface of infinite genus

In this section we prove Theorem 1.0.5.

4.1 Inductive system of surfaces

As in Introduction, we consider the embedding

$$i_{g+1}^g : \Sigma_{g,1} \to \Sigma_{g+1.1}$$

given by gluing the surface $\Sigma_{1,2}$ to the surface $\Sigma_{g,1}$ along the boundary. These embeddings constitute an inductive system of oriented surfaces $\{\Sigma_{g,1}, i_g^h\}_{h\leq g}$. Here $i_g^h : \Sigma_{h,1} \to \Sigma_{g,1}$

is the composite of the embeddings $i^h_{h+1}, i^{h+1}_{h+2}, \cdots, i^{g-1}_g$. Choose a basepoint $*_g$ on the boundary $\partial\Sigma_{g,1}$. For the rest of the paper, we often write simply

$$\pi^{(g)} = \pi_1(\Sigma_{g,1}, *_g), \quad \hat{\pi}^{(g)} = \hat{\pi}(\Sigma_{g,1}), \quad H^{(g)} = H_1(\Sigma_{g,1}; \mathbb{Q}) \quad \text{and} \quad \widehat{T}^{(g)} = \prod_{m=1}^{\infty} (H^{(g)})^{\otimes m}.$$

Lemma 4.1.1. *The inclusion map i^h_g induces an injective map of homotopy sets $i^h_g : \hat{\pi}^{(h)} \to \hat{\pi}^{(g)}$. In particular, the map*

$$i^h_g : \mathbb{Q}\hat{\pi}^{(h)} \to \mathbb{Q}\hat{\pi}^{(g)}$$

on the Goldman Lie algebras is an injective homomorphism of Lie algebras.

Proof. Choose a simple path $\ell : [0,1] \to \Sigma_{g,1} \setminus \overset{\circ}{\Sigma}_{h,1}$ connecting the basepoint $*_g$ to $*_h$. Here we denote by $\overset{\circ}{\Sigma}_{h,1}$ the interior of the surface $\Sigma_{h,1}$. The map $i^g_h : \pi^{(h)} \to \pi^{(g)}$ given by $x \mapsto \ell x \ell^{-1}$ is an injective homomorphism which induces the map $i^h_g : \hat{\pi}^{(h)} \to \hat{\pi}^{(g)}$. There exists a group homomorphism $r^g_h : \pi^{(g)} \to \pi^{(h)}$ satisfying $r^g_h \circ i^h_g = 1_{\pi^{(h)}}$. In fact, if $\{x_1, \ldots, x_{2h}\} \subset \pi^{(h)}$ is a free generating system of $\pi^{(h)}$, we may choose $x_i \in \pi^{(g)}$ for $2h+1 \leq i \leq 2g$ such that $\{\ell x_1 \ell^{-1}, \ldots, \ell x_{2h} \ell^{-1}, x_{2h+1}, \ldots, x_{2g}\}$ is a free generating system of $\pi^{(g)}$. If we define r^g_h by $r^g_h(\ell x_i \ell^{-1}) = x_i$ for $1 \leq i \leq 2h$, and $r^g_h(x_j) = 1$ for $2h+1 \leq j \leq 2g$, then we have $r^g_h \circ i^h_g = 1_{\pi^{(h)}}$.

Let x and y be elements in $\pi^{(h)}$. Suppose $i^h_g(x)$ is conjugate to $i^h_g(y)$. Then there exists an element $z \in \pi^{(g)}$ such that $i^h_g(y) = z i^h_g(x) z^{-1}$. Applying the homomorphism r^g_h, we obtain $y = r^g_h(z) x r^g_h(z)^{-1}$. Hence x is conjugate to y. This proves the first half of the lemma.

From the first half, the map $i^h_g : \mathbb{Q}\hat{\pi}^{(h)} \to \mathbb{Q}\hat{\pi}^{(g)}$ is injective. It is a homomorphism of Lie algebras by the definition of the Goldman bracket. □

Recall from §2.3 a decreasing filtration $\mathbb{Q}\hat{\pi}(p)$.

Lemma 4.1.2. *For any $p \geq 1$ and $h \leq g$, we have*

$$\mathbb{Q}\hat{\pi}^{(h)}(p) = (i^h_g)^{-1}(\mathbb{Q}\hat{\pi}^{(g)}(p))$$

Proof. Choose a Magnus expansion $\theta' : \pi^{(h)} \to \widehat{T}^{(h)}$ and extend it to a Magnus expansion $\theta'' : \pi^{(g)} \to \widehat{T}^{(g)}$. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}\pi^{(h)} & \xrightarrow{\theta'} & \widehat{T}^{(h)} \\ {\scriptstyle i^h_g}\downarrow & & {\scriptstyle i^h_g}\downarrow \\ \mathbb{Q}\pi^{(g)} & \xrightarrow{\theta''} & \widehat{T}^{(g)}. \end{array}$$

Here the right i^h_g is induced by the inclusion homomorphism $i^h_{g*} : H^{(h)} = H_1(\Sigma_{h,1}; \mathbb{Q}) \to H^{(g)} = H_1(\Sigma_{g,1}; \mathbb{Q})$. Using the map induced by r^g_h, we obtain $(i^h_g)^{-1}(\widehat{T}^{(g)}_p) = \widehat{T}^{(h)}_p$ and $(i^h_g)^{-1}([\widehat{T}^{(g)}, \widehat{T}^{(g)}]) = [\widehat{T}^{(h)}, \widehat{T}^{(h)}]$. Hence, for $u \in \mathbb{Q}\hat{\pi}^{(h)}$, the condition $\theta''(i^h_g(u)) - \varepsilon(u) \in \widehat{T}^{(g)}_p + [\widehat{T}^{(g)}, \widehat{T}^{(g)}]$ is equivalent to $\theta'(u) - \varepsilon(u) \in \widehat{T}^{(h)}_p + [\widehat{T}^{(h)}, \widehat{T}^{(h)}]$. From Lemma 2.3.2, these conditions are equivalent to $i^h_g|u| \in \mathbb{Q}\hat{\pi}^{(g)}(p)$ and $|u| \in \mathbb{Q}\hat{\pi}^{(h)}(p)$, respectively. This proves the lemma. □

We denote by $\Sigma_{\infty,1}$ the inductive limit of the system $\{\Sigma_{g,1}, i^h_g\}_{h\leq g}$

$$\Sigma_{\infty,1} := \varinjlim_{g\to\infty} \Sigma_{g,1}.$$

This is an oriented connected paracompact surface. We regard $\Sigma_{g,1}$ as a subsurface of $\Sigma_{\infty,1}$ and denote the inclusion map by $i^g_\infty : \Sigma_{g,1} \to \Sigma_{\infty,1}$. For any compact subset $K \subset \Sigma_{\infty,1}$, there exists a sufficiently large g such that $K \subset \Sigma_{g,1}$. In particular, the Goldman Lie algebra $\mathbb{Q}\hat{\pi}(\Sigma_{\infty,1})$ is exactly the inductive limit of the Lie algebras $\mathbb{Q}\hat{\pi}(\Sigma_{g,1})$'s

$$\mathbb{Q}\hat{\pi}(\Sigma_{\infty,1}) = \varinjlim_{g\to\infty} \mathbb{Q}\hat{\pi}(\Sigma_{g,1}). \tag{4.1.1}$$

From Lemma 4.1.1, the inclusion homomorphism

$$i^g_\infty : \mathbb{Q}\hat{\pi}(\Sigma_{g,1}) \to \mathbb{Q}\hat{\pi}(\Sigma_{\infty,1}) \tag{4.1.2}$$

is injective.

4.2 Proof of Theorem 1.0.5

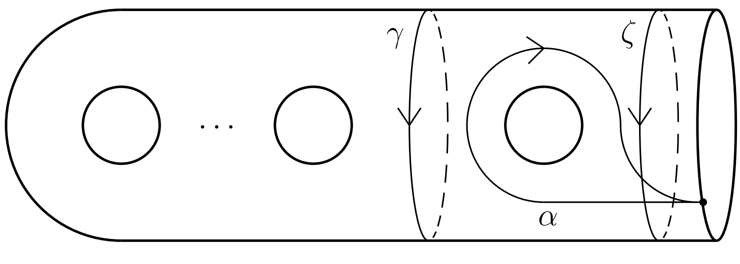
In this subsection we prove Theorem 1.0.5. It is clear $\mathbb{Q}1 \subset Z(\mathbb{Q}\hat{\pi}(\Sigma_{\infty,1}))$. We assume there exists an element $u \in Z(\mathbb{Q}\hat{\pi}(\Sigma_{\infty,1})) \setminus \mathbb{Q}1$, and deduce a contradiction. By (4.1.1), we have $u \in \mathbb{Q}\hat{\pi}(\Sigma_{g_0,1})$ for some $g_0 \geq 1$. From (2.3.2) and the assumption $u \notin \mathbb{Q}1$, there exists some $p \geq 1$ such that $u \notin \mathbb{Q}\hat{\pi}^{(g_0)}(p)$. We choose the minimum p satisfying this property. By Lemma 4.1.2, we have

$$i^{g_0}_g(u) \notin \mathbb{Q}\hat{\pi}^{(g)}(p) \tag{4.2.1}$$

for any $g \geq g_0$.

Choose $q \geq p$. There exists some $g_1 \geq g_0$ such that $2m(g) \geq q$ for any $g \geq g_1$. Denote $h := g_1$ and $g := h + 1$. Choose a non-null homologous based loop $\alpha \in \pi^{(g)} = \pi_1(\Sigma_{g,1}, *_g)$ inside the subsurface $\Sigma_{1,2} \subset \Sigma_{g,1}$. We denote the boundary loops of $\Sigma_{h,1}$ and $\Sigma_{g,1}$ by γ and ζ, respectively. The loops γ and α are disjoint. See Figure 12.

Figure 12: $\Sigma_{h,1}$ and $\Sigma_{g,1}$



From (4.1.2) and Lemma 4.1.1, the homomorphisms

$$\mathbb{Q}\hat{\pi}(\Sigma_{h,1}) \xrightarrow{i^h_g} \mathbb{Q}\hat{\pi}(\Sigma_{g,1}) \xrightarrow{i^g_\infty} \mathbb{Q}\hat{\pi}(\Sigma_{\infty,1})$$

are injective. Hence we may regard $u \in Z(\mathbb{Q}\hat{\pi}(\Sigma_{h,1})) \cap Z(\mathbb{Q}\hat{\pi}(\Sigma_{g,1}))$. By Corollay 3.2.9 we have polynomials $f_h(\gamma) \in \mathbb{Q}[\gamma]$ and $f_h(\zeta) \in \mathbb{Q}[\zeta]$ such that

$$u \equiv |f_h(\gamma)| \pmod{\mathbb{Q}\hat{\pi}^{(h)}(q)}, \quad \text{and} \quad u \equiv |f_g(\zeta)| \pmod{\mathbb{Q}\hat{\pi}^{(g)}(q)}.$$

By Lemma 4.1.2 we have

$$|f_h(\gamma)| \equiv u \equiv |f_g(\zeta)| \pmod{\mathbb{Q}\hat{\pi}^{(g)}(q)},$$

Choose a symplectic expansion $\theta : \pi^{(g)} \to \widehat{T}^{(g)}$. For the rest of the proof, we drop the suffix $^{(g)}$. If $v \in \mathbb{Q}\hat{\pi}(q)$, then $N\theta(v) \in N(\widehat{T}_q)$ and so $(N\theta(v))\theta(\alpha) = (N\theta(v))\theta(\alpha - 1) \in \widehat{T}_{q+1-2} = \widehat{T}_{q-1}$, since $\theta(\alpha - 1) \in \widehat{T}_1$. Hence we have $(N\theta(f_h(\gamma)))\theta(\alpha) \equiv (N\theta(f_g(\zeta)))\theta(\alpha)$ (mod $\widehat{T}_{q-1}$). Moreover we have $(N\theta(f_h(\gamma)))\theta(\alpha) = 0$ by [7] Theorem 1.2.2, since the free loop γ and the based loop α are disjoint. Thus we obtain

$$(N\theta(f_g(\zeta)))\theta(\alpha) \in \widehat{T}_{q-1}. \tag{4.2.2}$$

On the other hand, we have $|f_g(\zeta)| \notin \mathbb{Q}\hat{\pi}(p)$ because $u \notin \mathbb{Q}\hat{\pi}(p)$. $\theta(f_g(\zeta))$ is a power series in the symplectic form ω. Hence, since p is the minimum, p is odd ≥ 5, and $\theta_{p-1}(f_g(\zeta)) = c\omega^{(p-1)/2}$ for some non-zero constant $c \in \mathbb{Q}$. Here $\theta_{p-1} : \pi \to H^{\otimes(p-1)}$ is the degree $(p-1)$ part of the expansion θ. Then we have

$$\begin{aligned}(N\theta(f_g(\zeta)))\theta(\alpha) &\equiv cN(\omega^{(p-1)/2})([\alpha]) \\ &\equiv ((p-1)/2)c(-[\alpha]\omega^{(p-3)/2} + \omega^{(p-3)/2}[\alpha]) \not\equiv 0 \pmod{\widehat{T}_{p-1}}.\end{aligned}$$

Namely we have $(N\theta(f_g(\zeta)))\theta(\alpha) \notin \widehat{T}_{p-1} \supset \widehat{T}_{q-1}$. This contradicts (4.2.2). Hence we obtain $Z(\mathbb{Q}\hat{\pi}(\Sigma_{\infty,1})) \subset \mathbb{Q}1$. This completes the proof of Theorem 1.0.5.

5 Appendix: The Lie algebra of linear chord diagrams

The Lie bracket on the space $\mathcal{C}$ of oriented chord diagrams is extended to a bracket on the space of linear chord diagrams

$$[\ ,\ ] : \mathcal{LC}_m \otimes \mathcal{LC}_{m'} \to \mathcal{LC}_{m+m'-1},$$

which makes the direct sum

$$\mathcal{LC} := \bigoplus_{m=1}^{\infty} \mathcal{LC}_m$$

a Lie algebra. We define the bracket by using the stable isomorphism $a : \mathcal{LC} \to \mathrm{Der}(T)^{\mathfrak{sp}}$ in Lemma 3.1.1 and the Lie algebra structure on $\mathrm{Der}(T)$, the derivation algebra of T. Here $T := \bigoplus_{m=0}^{\infty} H^{\otimes m}$ is the tensor algebra of H, the rational symplectic vector space of genus $g \geq 1$. As before, we identify the dual $H^* = \mathrm{Hom}(H, \mathbb{Q})$ with H by the Poincaré duality $H \cong H^*$, $X \mapsto (Y \mapsto Y \cdot X)$. Then the restriction map to the subspace H identifies the space $\mathrm{Der}(T)$ with the space $\mathrm{Hom}(H, T) = H^* \otimes T = H \otimes T = \bigoplus_{m=1}^{\infty} H^{\otimes m}$.

It should be remarked the set of linear chord diagrams of m chords with the standard label is a basis of the space $\mathcal{LC}_m$. Here $C = \{(i_1, j_1), \dots, (i_m, j_m)\}$ is a linear chord diagram of m chords with the standard label, if and only if $\{i_1 \dots, i_m, j_1, \dots, j_m\} = \{1, 2, \dots, 2m\}$ and $i_k < j_k$ for any k (see the proofs of Lemmas 3.1.3 and 3.1.4). For the rest of this appendix, we regard $\mathcal{LC}$ as the vector space spanned by the (unlabeled) linear chord diagrams. Thus we identify the labeled linear chord diagram C with the fixed-point free involution $\sigma(C) := (i_1, j_1) \cdots (i_m, j_m) \in \mathfrak{S}_{2m}$. The invariant tensor $a(C) \in (H^{\otimes 2m})^{\mathfrak{sp}}$ is defined as in §3.1 and the map $a : \mathcal{LC}_m \to (H^{\otimes 2m})^{\mathfrak{sp}}$ is a stable isomorphism (Lemma

3.1.1). This stable isomorphism induces a Lie algebra structure on the space $\mathcal{LC}$ such that $a : \mathcal{LC} \to \mathrm{Der}(T)^{\mathfrak{sp}}$ is a Lie algebra homomorphism.

In order to describe the bracket on $\mathcal{LC}$, we introduce new amalgamations of two linear chord diagrams. Let C and C' be linear chord diagrams of m and l chords, respectively. They are regarded as involutions $\sigma = \sigma(C) \in \mathfrak{S}_{2m}$ and $\sigma' = \sigma(C') \in \mathfrak{S}_{2l}$. For $2 \le t \le 2l$, we define the t-th amalgamation $C *_t C'$ as an involution $\sigma'' = \sigma(C *_t C') \in \mathfrak{S}_{2m+2l-2}$ by

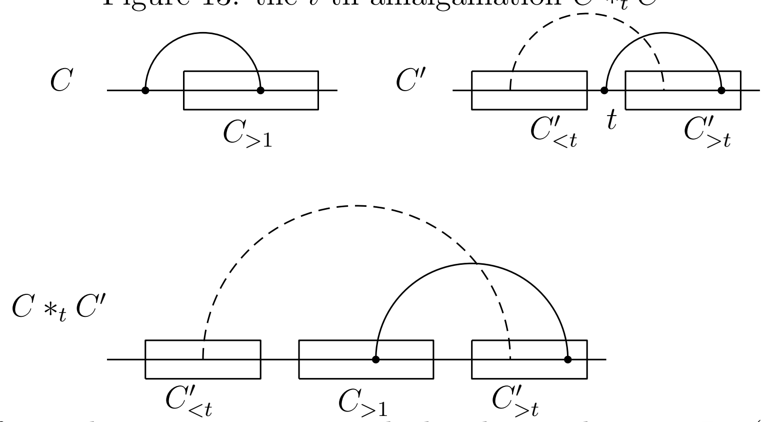
$$\begin{aligned}
&\sigma''(\sigma(1)+t-2) := f_{m,t}(\sigma'(t))\\
&\sigma''(f_{m,t}(\sigma'(t))) := \sigma(1)+t-2\\
&\sigma''(k) := \begin{cases} f_{m,t}(\sigma'(k)), & \text{if } k \le t-1 \text{ and } k \ne \sigma'(t),\\ \sigma(k-t+2)+t-2, & \text{if } t \le k \le t+2m-2 \text{ and } k \ne \sigma(1)+t-2,\\ f_{m,t}(\sigma'(k-2m+2)), & \text{if } t+2m-1 \le k \text{ and } k-2m+2 \ne \sigma'(t). \end{cases}
\end{aligned}$$

Here $f_{m,t} : \{1, \dots, t-1, t+1, \dots, 2l\} \to \{1, 2, \dots, 2m+2l-2\}$ is defined by

$$f_{m,t}(k) := \begin{cases} k, & \text{if } k \le t-1,\\ k+2m-2, & \text{if } k \ge t+1. \end{cases}$$

In other words, we delete the t-th vertex from C' and the first vertex from C, insert the deleted C into the t-th hole of the deleted C', and connect the vertices $\sigma(C)(1)$ and $\sigma(C')(t)$. The resulting linear chord diagram with the standard label is exactly the t-th amalgamation $C *_t C' \in \mathcal{LC}_{m+l-1}$. See Figure 13. Interchanging the role of C and C', we can define the s-th amalgamation $C' *_s C$ for $2 \le s \le 2m$.

Figure 13: the t-th amalgamation $C *_t C'$



By a straightforward computation we see the bracket on the space $\mathrm{Der}(T) = \bigoplus_{m=1}^{\infty} H^{\otimes m}$ is given by

$$\begin{aligned}
[X_1 \cdots X_p, Y_1 \cdots Y_q] &= \sum_{t=2}^{q} (Y_t \cdot X_1) Y_1 Y_2 \cdots Y_{t-1} X_2 \cdots X_p Y_{t+1} \cdots Y_q\\
&\quad - \sum_{s=1}^{p} (X_s \cdot Y_1) X_1 X_2 \cdots X_{s-1} Y_2 \cdots Y_q X_{s+1} \cdots X_p
\end{aligned}$$

for X_s, $Y_t \in H$. Hence, by a similar argument to §3.1, we have

$$[C, C'] = -\sum_{t=2}^{2l} C *_t C' + \sum_{s=2}^{2m} C' *_s C. \tag{5.0.3}$$

It is easy to compute the center and the homology of the Lie algebra $\mathcal{LC}$. We denote $E_0 := -\frac{1}{2}\{\{1,2\}\} \in \mathcal{LC}_1$. Then we have $(-2E_0) *_t C = C *_2 (-2E_0) = C$ for any t. Hence $\mathcal{LC}_m$ is just the eigenspace of the operator $\mathrm{ad}(E_0)$ corresponding to the eigenvalue $m-1(\geq 0)$. This observation implies the center of $\mathcal{LC}$ vanishes

$$Z(\mathcal{LC}) = 0. \tag{5.0.4}$$

Using the Lie derivative $\mathcal{L}_{E_0}$, we can prove the standard chain complex $C_*(\mathcal{LC})$ is quasi-isomorphic to the E_0-invariant subcomplex $C_*(\mathcal{LC})^{E_0} = C_*(\mathcal{LC}_1)$. Thus we obtain

$$H_*(\mathcal{LC}) = \begin{cases} \mathbb{Q}, & \text{if } * = 0, 1, \\ 0, & \text{otherwise.} \end{cases} \tag{5.0.5}$$

We denote by $W_1 := \mathbb{Q}[x]\frac{d}{dx}$ the Lie algebra of polynomial vector fields in one variable x. The subalgebras $L_0 := x\mathbb{Q}[x]\frac{d}{dx}$ and $L_1 := x^2\mathbb{Q}[x]\frac{d}{dx}$ play important roles in Gel'fand-Fuks theory (cf., e.g., [2]). The formula (5.0.3) implies immediately the surjection

$$\kappa : \mathcal{LC} \to L_0$$

assigning $-2x^m\frac{d}{dx}$ to each linear chord diagram of m chords is a Lie algebra homomorphism. The vector field $\kappa(E_0) = x\frac{d}{dx}$ is just the Euler operator.

By analogy with the Lie subalgebra L_1, we consider the Lie algebra $\mathcal{LC}^1 := \bigoplus_{m=2}^{\infty} \mathcal{LC}_m$. The homology group $H_*(\mathcal{LC}^1)$ is decomposed into the eigenspaces of the action of E_0. We denote by $H_*(\mathcal{LC}^1)_{(k)}$ the eigenspace corresponding to the eigenvalue $k \geq 1$. The first homology group $H_1(\mathcal{LC}^1)_{(k)}$ does not vanish for any integer $k \geq 1$, and its dimension diverges when k goes to the infinity. The proof will appear elsewhere. The generating function of the Euler characteristics $\sum_{k=1}^{\infty} \chi(H_*(\mathcal{LC}^1)_{(k)})x^k$ can be computed as

$$-3x - 12x^2 - 61x^3 - 570x^4 - 6600x^5 - 91910x^6 - 1460655x^7 - 26064990x^8 - \cdots .$$

This is completely different from the homology of the Lie subalgebra L_1 given by Goncharova [4].

References

[1] P. Etingof, Casimirs of the Goldman Lie algebra of a closed surface, Intern. Math. Res. Notices **2006**, 1-5 (2006)

[2] D. B. Fuks, Cohomology of infinite-dimensional Lie algebras, Consultants Bureau, New York (1986)

[3] W. M. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface groups representations, Invent. Math. **85**, 263-302 (1986)

[4] L. V. Goncharova, Cohomology of Lie algebra of formal vector fields on the line, Functional Anal. Appl. **7** (2), 6-14 (1973)

[5] N. Kawazumi, Cohomological aspects of Magnus expansions, preprint, math.GT/0505497 (2005)

[6] N. Kawazumi, Harmonic Magnus expansion on the universal family of Riemann surfaces, preprint, math.GT/0603158 (2006)

[7] N. Kawazumi and Y. Kuno, The logarithms of Dehn twists, preprint, arXiv:1008.5017 (2010)

[8] N. Kawazumi and S. Morita, The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes, Math. Res. Lett. **3**, 629–641 (1996)

[9] M. Kontsevich, Formal (non)-commutative symplectic geometry, in: "The Gel'fand Mathematical Seminars, 1990-1992", Birkhäuser, Boston, 173-187 (1993)

[10] Y. Kuno, A combinatorial construction of symplectic expansions, to appear in: Proc. Amer. Math. Soc., available at arXiv:1009.2219

[11] G. Massuyeau, Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant, preprint, arXiv:0809.4629 (2008)

[12] S. Morita, Structure of the mapping class groups of surfaces: a survey and a prospect, Geom. Topol. Monogr. **2**, 349–406 (1999)

[13] S. Morita, Generators for the tautological algebra of the moduli space of curves, Topology **42**, 787–819 (2003)

[14] H. Weyl, The classical groups, Princeton University Press, 2nd. ed. Princeton (1953)

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The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

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