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by

Shigeo Kusuoka and Takenobu Nakashima

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

A remark on credit risk models and copula

Shigeo KUSUOKA *and Takenobu NAKASHIMA †

Abstract

In the present paper, the authors study the relationship between a family of copula functions parameterized by finite dimensional parameters and dynamical default time models, and that there are some constraint conditions for a family of copula functions, if it describes a dynamical default time model.

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1 Introduction

Björk-Christernsen [3] considered the relationship between a family of forwardrate curves parameterized by finite factors and a dynamical interest rate model free of arbitrage, and showed that there are some constraint conditions for a family of forwardrate curves which comes from a dynamical interest rate model free of arbitrage.

Recently, copula models are widely used to describe a family of default time. But it is not clear that such copula models are consistent with dynamical credit risk models. In the present paper, we study the relationship between a family of copula functions parameterized by finite dimensional parameters and dynamical default time models. Although we consider rather restricted dynamical default time models, we show that there are some constraint conditions for a family of copula functions.

The setup in this paper is the following. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $W(t) = (W^k(t))_{k=1,\ldots,d}$, $t \geqq 0$, be a d-dimensinal standard Wiener process, and $\mathcal{G}_t = \sigma\{W(s),\ s \in [0,t]\} \vee \mathcal{N}$, where $\mathcal{N} = \{B \in \mathcal{F};\ P(B) = 0 \text{ or } 1\}$. Let $N \geqq 2$, $\tau_i : \Omega \to [0,\infty)$, $i = 1, \ldots, N$, be random variables, and let $\mathcal{F}_t = \mathcal{G}_t \vee \sigma\{\tau_i \wedge t, i = 1, \ldots, N\}$. Let $\xi_i : [0,\infty) \times \Omega \to [0,\infty)$, $i = 1, \ldots, N$, be $\mathcal{G}$-progressively measurable processes.

First, we assume the following conditions.

(SC) $\displaystyle (\prod_{i\in I} 1_{\{\tau_i>t\}})P(\tau_i > t_i, i \in I|\mathcal{F}_t) = (\prod_{i\in I} 1_{\{\tau_i>t\}})E[\exp(-\sum_{i\in I}\int_t^{t_i} \xi_i(s)ds)|\mathcal{G}_t]\ a.s.$

for any $I \subset \{1, \ldots, N\}$ and $t, t_i \in [0,\infty), i \in I$ with $t \leqq \min_{i\in I} t_i$.

(PO) For any $t \geqq 0$,

$$P(\bigcap_{i=1}^{N}\{\tau_i > t\})|\mathcal{G}_t] > 0\ a.s.$$

*Graduate School of Mathematical Sciences, The University of Tokyo, research supported by the 21st century COE project, Graduate School of Mathematical Sciences, The University of Tokyo

†Nomura Securities Company, Graduate School of Mathematical Sciences, The University of Tokyo

We also assume the following technical assumptions.

(A-1) For any $T > 0$,

$$\sum_{i=1}^{N} \int_0^T E[\xi_i(t)^4]dt < \infty.$$

(A-2) For any $i = 1, \ldots, N$,

$$\int_0^\infty \xi_i(t) = \infty \quad a.s. \text{ and } \int_a^b \xi_i(t) > 0 \quad a.s. \text{ for any } a, b > 0 \text{ with } b > a.$$

(A-3) $\displaystyle \sum_{i=1}^{N} \int_0^\infty (1+t)^2 E[\xi_i(t)^2 \exp(-2\int_0^t \xi_i(s)ds)]dt < \infty.$

Let $\theta : [0,\infty)\times\Omega \to \mathbf{R}^M$ be a $\mathcal{G}$-Ito process, i.e., θ is $\mathcal{G}$-progressively measurable, $\theta(t,\omega)$ is continuous in t for all $\omega \in \Omega$, and there are $\mathbf{R}^M$-valued $\mathcal{G}$-progressively measurable processes η_k, $k = 1, \ldots, d$, and b satisfying

$$P(\sum_{k=1}^{d} \int_0^T |\eta_k(t)|^2 dt + \int_0^T |b(t)|dt < \infty) = 1, \text{ for any } T > 0,$$

and

$$\theta(t) = \theta(0) + \sum_{k=1}^{d} \int_0^t \eta_k(s)dW^k(s) + \int_0^t b(s)ds. \tag{1}$$

Let Θ be an open subset in $\mathbf{R}^M$ and $K \in C([0,1]^N \times \Theta; [0,1])$. We assume the following, moreover.

(A-4) $P(\theta(t) \in \bar{\Theta}$ for all $t \geqq 0) = 1$, where $\bar{\Theta}$ is the closure of Θ in $\mathbf{R}^M$.

(A-5) the support of probability law of $\theta(t,\omega)$ under $e^{-t}dt \otimes P(d\omega)$ contains a non-empty open set in Θ, i.e., there is a non-empty open set U_0 in Θ such that for any $\theta_0 \in U_0$ and $\varepsilon > 0$

$$\int_0^\infty P(|\theta(t) - \theta_0| < \varepsilon)\, e^{-t}dt > 0.$$

(CP) $K(\cdot,\theta) : [0,1]^N \to [0,1]$ is a copula function for any $\theta \in \Theta$, and

$$\prod_{i=1}^{N} 1_{\{\tau_i > t\}} 1_\Theta(\theta(t)) P(\tau_i > t_i, i = 1, \ldots, N|\mathcal{F}_t)$$

$$= \prod_{i=1}^{N} 1_{\{\tau_i > t\}} 1_\Theta(\theta(t)) K(P(\tau_1 > t_1|\mathcal{F}_t), \ldots, (P(\tau_N > t_N|\mathcal{F}_t), \theta(t)) \ a.s.$$

for any $t, t_1, \ldots, t_N > 0$ with $t < \min_{i=1,\ldots,N} t_i$.

We call a family $((\Omega,\mathcal{F},P), (W_t^k)_{k=1,\ldots,d}, (\tau_i)_{i=1,\ldots,N}, (\xi_i(t))_{i=1,\ldots,N}, \theta(t), \Theta, K)$ satisfying the above assumptions a dynamical default time copula model, and we call K the associated family of copula functions to this model.

Definition 1 *Let Θ be an open subset in $\mathbf{R}^M$. We say that $K \in C([0,1]^N \times \Theta; [0,1])$ is an admissible family of copula functions, if there is a dynamical default time copula model and K is the associated family of copula functions to the model.*

The purpose of the present paper is to show that there are some analytic constraint conditions for an admissible family of copula functions. For example we will prove the following.

Let $N, M \geqq 1$, Θ be a non-vois open subset in $\mathbf{R}^M$. Let $\mathcal{C}_{(N)}(\Theta)$ denote the subset of $C([0,1]^N \times \Theta; [0,1])$ consisting of elements K such that $K(\cdot,\theta) : [0,1]^N \to [0,1]$ is a copula function for any $\theta \in \Theta$, and $K|_{(0,1)^N \times \Theta}$ is a C^∞ function.

Let D_n be an increasing sequence of compact subsets in Θ such that $\bigcup_n^\infty D_n = \Theta$. Then we can regard $\mathcal{C}_{(N)}(\Theta)$ as a Polish space with a metric function dis given by

$$dis(K_1, K_2)$$

$$= \sum_{n=1}^{\infty} 2^{-n} \wedge \sup\{|K_1(x,\theta) - K_2(x,\theta)|;\ x \in [0,1]^N,\ \theta \in D_n\}$$

$$+ \sum_{n=1}^{\infty} 2^{-n} \wedge (\sum_{\alpha_1,\ldots,\alpha_{N+M}=0}^{n} \sup\{|\frac{\partial^{\alpha_1+\cdots\alpha_{N+M}}(K_1-K_2)}{\partial x_1^{\alpha_1} \partial x_N^{\alpha_N} \partial \theta_1^{\alpha_{N+1}} \partial \theta_M^{\alpha_{N+M}}}(x,\theta);\ x \in [1/4n.1-1/4n]^N,\ \theta \in D_n\}).$$

Our main result is the following.

Theorem 2 *Let $N \geqq 3$, $M \geqq 1$, and Θ be a no-void open subset in $\mathbf{R}^M$. Then the subset of $\mathcal{C}_{(N)}(\Theta)$ whose elements are admissible families of copula functions is a set of the first category in Baire's sense.*

We also show that a family of Gumbel copula functions of 3 variables is not admissible by relying on numerical computation in Section 7.

2 Preliminary results

Let $\xi : [0,\infty) \times \Omega \to [0,\infty)$ be a $\mathcal{G}$-progressively measurable satisfying the following three conditions.

(B-1) $\displaystyle\int_0^T E[\xi(t)^4] dt < \infty$ for any $T > 0$.

(B-2) $\int_0^\infty \xi(t) dt = \infty$ *a.s.*, and $\int_a^b \xi(t) dt > 0$ *a.s.* for any $b > a \geqq 0$.

(B-3) $\displaystyle E[\int_0^\infty (1+t)^2 \xi(t)^2 \exp(-2\int_0^t \xi(r)dr) dt] < \infty.$

For each $s \geqq 0$, let $\{M(t,s); t \geqq 0\}$ is a continuous martingale given by

$$M(t,s) = E[\exp(-\int_0^s \xi(r)dr)|\mathcal{G}_t], \qquad t \geqq 0.$$

Proposition 3 *There is $f : [0,\infty) \times [0,\infty) \times \Omega \to (0,\infty)$ satisfying the following.*
(1) *For any $t, s \geqq 0$,*

$$f(t,s) = E[\exp(-\int_{t\wedge s}^s \xi(r)dr)|\mathcal{G}_t] = \exp(\int_0^{t\wedge s} \xi(r)dr) M(t,s) \qquad a.s.$$

(2) *For any $\omega \in \Omega$ $f(\cdot, * : \omega) : [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous.*
(3) *If $s_2 > s_1 > t$, then $f(t,s_1) > f(t,s_2) > 0$, $\omega \in \Omega$.*
(4) *For any $t \geqq 0$*

$$f(t;t) = 1, \quad \lim_{s\uparrow\infty} f(t;s) = 0.$$

Proof. Note that for $0 \leqq s_1 \leqq s_2$

$$E[\sup_{t\in[0,\infty)} |M(t,s_1) - M(t,s_2)|^4] \leqq 4E[|M(s_1,s_1) - M(s_2,s_2)|^4]$$

$$\leqq 4E[|\int_{s_1}^{s_2} \xi(r)dr|^4] \leqq 4(s_2 - s_1)^3(\int_{s_1}^{s_2} E[\xi(r)^4]dr).$$

So by Kolmogorov's continuity theorem and the assumption (B-1), we see that there is a $\tilde{M} : [0,\infty) \times [0,\infty) \times \Omega \to [0,\infty)$ such that $\tilde{M}(\cdot,*,\omega) \to [0,\infty)$ is continuous and $P(\tilde{M}(t,s) = M(t,s)) = 1$, $t, s \geqq 0$. Let

$$\tilde{f}(t,s) = \exp(\int_0^{t\wedge s} \xi(r)dr)\tilde{M}(t,s) \qquad t, s \geqq 0.$$

Then $\tilde{f}(t,s)$ is continuous in (t,s). Let $0 \leqq s_1 < s_2$. Then

$$\tilde{f}(t,s_1) - \tilde{f}(t,s_2) = \exp(\int_0^t \xi(r)dr)(\tilde{M}(t,s_1) - \tilde{M}(t,s_2)), \qquad t \in [0,s_1].$$

By the assumption (B-2), we have

$$\tilde{M}(s_1,s_1) - \tilde{M}(s_1,s_2) = \exp(-\int_0^{s_1} \xi(r)dr)E[1 - \exp(-\int_{s_1}^{s_2} \xi(r)dr))|\mathcal{G}_{s_1}] > 0 \ a.s.$$

Let $\tau_0 = \inf\{t \geqq 0;\ \tilde{M}(t,s_1) - \tilde{M}(t,s_2) = 0\} \wedge s_1$. Then we see that

$$\tilde{M}(\tau,s_1) - \tilde{M}(\tau,s_2) = E[\tilde{M}(s_1,s_1) - \tilde{M}(s_1,s_2)|\mathcal{G}_\tau] > 0, \qquad a.s.$$

So we see that

$$\inf_{t\in[0,s_1]}(\tilde{M}(t,s_1) - \tilde{M}(t,s_2)) > 0 \ a.s.$$

So we see that

$$\inf_{t\in[0,s_1]}(\tilde{f}(t,s_1) - \tilde{f}(t,s_2)) > 0 \ a.s.$$

for any $s_1, s_2 \in \mathbf{Q}$ with $s_2 > s_1 \geqq 0$. So there is an $\Omega_1 \in \mathcal{F}$ with $P(\Omega_1) = 1$ such that $\tilde{f}(t,s_1,\omega) > \tilde{f}(t,s_2,\omega)$ for any $\omega \in \Omega_1$, $s_1, s_2 \in \mathbf{Q}$ with $s_2 > s_1 > 0$ and $t \in [0,s_1)$. Since $\tilde{f}(t,s)$ is continuous in (t,s), we see that $\tilde{f}(t,s)$ is non-increasing in s. So we see that $\tilde{f}(t,s_1,\omega) > \tilde{f}(t,s_2,\omega)$ for any $\omega \in \Omega_1$, $t, s_1, s_2 \in [0,\infty)$ with $s_2 > s_1 > t$. Similarly we can show that there is an $\Omega_2 \in \mathcal{F}$ with $P(\Omega_2) = 1$ such that $\tilde{f}(t,s,\omega) > 0$ for any $\omega \in \Omega_2$, $t, s \in [0,\infty)$.

We see that

$$E[\varliminf_{s\to\infty} \tilde{f}(t,s)] \leqq \varliminf_{s\to\infty} E[\tilde{f}(t,s)] = \varliminf_{s\to\infty} E[\exp(-\int_t^s \xi(r)dr] = 0.$$

Since $\lim_{s\to\infty} \tilde{f}(t,s,\omega)$ exists for $\omega \in \Omega_1$, we see that $\lim_{s\to\infty} \tilde{f}(t,s,\omega) = 0$ a.s. Also, it is easy to see $\tilde{f}(t,t) = 1$ *a.s.* Therefore we can take a good version f of $\tilde{f}$ satisfying the assertion. ∎

Proposition 4 *There exist* $\hat{\sigma}_k : [0,\infty] \times [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1,\dots,d$, *satisfying the following.*
(1) $\hat{\sigma}_k(t,\cdot,\omega) : [0,\infty] \to \mathbf{R}$, $k = 1,\dots,d$, *is continuous for any* $t \in [0,\infty)$ *and* $\omega \in \Omega$. *Moreover,* $\hat{\sigma}_k(t,s,\omega) = 0$, $t \geqq s$, *and* $\lim_{s\to\infty} \hat{\sigma}_k(t,s,\omega) = 0$ *for any* $t \in [0,\infty)$ *and* $\omega \in \Omega$.
(2) $\hat{\sigma}_k(\cdot,s) : [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1,\dots,N$, *is* $\mathcal{G}$*-progressively measurable for any* $s \geqq 0$ *and*

$$M(t,s) = M(0,s) + \sum_{k=1}^{d} \int_0^t \hat{\sigma}_k(r,s)dW^k(r), \quad t \geqq 0, \ a.s.$$

for any $s > 0$.

Proof. For each $s \geqq 0$, let $N(t;s), t \in [0,\infty)$ be a continuous martingale given by

$$N(t;s) = E[\xi(s)\exp(-\int_0^s \xi(r)dr)|\mathcal{G}_t].$$

By Ito's representation theorem, we see that for any $s \geqq 0$ there exist $\mathcal{G}$-progressively measurable processes $c_k(\cdot,s) : [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1,\dots,d$, such that

$$N(t;s) = N(0;s) + \sum_{k=1}^{d} \int_0^t c_k(r,s)dW^k(r), \quad t \geqq 0.$$

Since the map from $[0,\infty)$ to $L^2(\Omega,\mathcal{F},P)$ coresponding s to $N_t(s)$ is measurable, we may assume that $c_k : [0,\infty) \times [0,\infty) \times \Omega \to \mathbf{R}$ is measurable. Note that

$$N(0;s)^2 + \sum_{k=1}^{d} \int_0^\infty E[c_k(r,s)^2]dr$$

$$= \lim_{t\to\infty} E[N(t;s)^2] \leqq E[\xi(s)^2 \exp(-2\int_0^s \xi(r)dr)].$$

Therefore by the assumption (B-3), we see that

$$E[\int_{[0,\infty)\times[0,\infty)} (1+s)^2 c_k(r,s)^2 drds] < \infty, \qquad k = 1,\dots,d.$$

Let us define $\tilde{\sigma}_k : [0,\infty) \times [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1,\dots,N$, by

$$\tilde{\sigma}_k(t;s) = \begin{cases} -\int_0^s c_k(t,u)du, & \text{if } \int_{[0,\infty)}(1+u)^2 c_k(t,u)^2 ds < \infty, \\ 0, & \text{otherwise }. \end{cases}$$

Then we see that $\tilde{\sigma}_k(\cdot,s) : [0,\infty) \times \Omega \to \mathbf{R}$, is $\mathcal{G}$-progressively measurable for any $s \geqq 0$ and $\tilde{\sigma}_k(t,\cdot) : [0,\infty) \to \mathbf{R}$ is continuous. Also, by stochastic Fubini's theorem, we have

$$-\int_0^s N_0(u)du + \sum_{k=1}^{d} \int_0^t \tilde{\sigma}_k(r,s)dW^k(r)$$

$$= -\int_0^s N_0(u)du - \sum_{k=1}^{d} \int_0^t (\int_0^s c_k(r,u)du)dW^k(r)$$

$$= -\int_0^s N_t(u)du = E[\exp(-\int_0^s \xi_i(u)du) - 1|\mathcal{G}_t] = M(t;s) - 1 \qquad a.s.$$

So we see that

$$M(t;s) = M(0,s) + \sum_{k=1}^d \int_0^t \tilde{\sigma}_k(r,s)dW^k(r).$$

Note that for $0 < s_1 < s_2 < \infty$, we have

$$|\tilde{\sigma}_k(t,s_1) - \tilde{\sigma}_k(t,s_2)|^2 \leqq (\int_{s_1}^{s_2} |c_k(t,u)|du)^2$$

$$\leqq (\int_{s_1}^{s_2} (1+u)^{-2}du)(\int_{s_1}^{s_2} (1+u)^2 c_k(t,u)^2 du) \leqq \int_{s_1}^{\infty} (1+u)^2 c_k(t,u)^2 du.$$

So we have

$$E[\int_{[0,\infty)} dt(\sup_{s_1,s_2>s} |\tilde{\sigma}_k(t,s_1) - \tilde{\sigma}_k(t,s_2)|^2)]$$

$$\leqq E[\int_{[0,\infty)\times[s,\infty)} (1+u)^2 c_k(r,u)^2 drdu] \to 0, \qquad s \to \infty.$$

Therefore we see that

$$\sup_{s_1,s_2>s} |\tilde{\sigma}_k(t,s_1) - \tilde{\sigma}_k(t,s_2)|^2 \to 0, \qquad s \to \infty \qquad dt \otimes P(d\omega) - a.e.(t,\omega).$$

This implies that $\tilde{\sigma}_k(t,s)$ converges as $s \to \infty$ for $dt \otimes P(d\omega) - a.e.(t,\omega)$.

Also, we see by (B-2) that

$$E[\int_0^\infty (\underline{\lim}_{s\to\infty} \tilde{\sigma}_k(t,s)^2)dt] \leqq \underline{\lim}_{s\to\infty} E[\int_0^\infty \tilde{\sigma}_k(t,s)^2 dt] \leqq \underline{\lim}_{s\to\infty} E[\exp(-2\int_0^s \xi_i(u)du)] = 0.$$

Thus we see that $\tilde{\sigma}_k(t,s) \to 0$, $s \to \infty$ for $dt \otimes P(d\omega) - a.e.(t,\omega)$.

Let $\hat{\sigma}_k$, $k = 1, \ldots, d$, be given by

$$\hat{\sigma}_k(t;s) = \begin{cases} \tilde{\sigma}_k(t;s), & \text{if } \tilde{\sigma}_k(t,s) \to 0, \text{ as } s \to \infty, \\ 0, & \text{otherwise ,} \end{cases}$$

Then we have our assertion. ∎

By Ito's formula, we have

$$f(t;s) = f(0,s) + \int_0^{t\wedge s} \xi(r)f(r;s)dr + \int_0^t \exp(\int_0^{r\wedge s} \xi(u)du)\hat{\sigma}_k(r,s)dW^k(r), \quad t \geqq 0,$$

for any $s \geqq 0$. So we have the following as a corollary to Proposition 4.

Corollary 5 *There exist $\tilde{\sigma}_k : [0,\infty] \times [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1, \ldots, d$, such that*
(1) $\tilde{\sigma}_k(t,\cdot,\omega) : [0,\infty] \to \mathbf{R}$, $k = 1, \ldots, d$, is continuous for any $t \in [0,\infty)$ and $\omega \in \Omega$. Moreover, $\tilde{\sigma}_k(t,s,\omega) = 0$, $t \geqq s$, and $\lim_{s\to\infty} \tilde{\sigma}_k(t,s,\omega) = 0$ for any $t \in [0,\infty)$ and $\omega \in \Omega$.
(2) $\tilde{\sigma}_{i,k}(\cdot,s) : [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1, \ldots, N$, is $\mathcal{G}$-progressively measurable for any $s \geqq 0$ and

$$f(t,s) = f(0,s) + \int_0^{t\wedge s} \xi(r)f(r;s)dr + \sum_{k=1}^d \int_0^t \tilde{\sigma}_k(r,s)dW^k(r), \quad t \geqq 0, \ a.s.$$

for any $s > 0$.

By Proposition 3, we have the following immediately.

Proposition 6 *Let* $T : [0,\infty) \times (0,1] \times \Omega \to [0,\infty]$ *be given by*

$$T(t,x) = \inf\{s \geqq t;\ f(t,s) < x\}, \qquad x \in (0,1].$$

Then $T(t,\cdot,\omega) : (0,1] \to [0,\infty)$ *is continuous and strictly decreasing and* $\lim_{x \downarrow 0} T(t,x,\omega) = \infty$ *for any* $t \geqq 0$ *and* $\omega \in \Omega$.

Now let $X : [0,\infty) \times [0,\infty) \times (0,1] \times \Omega \to (0,1]$ be given by

$$X(t,s;x) = f(t \vee s, T(s,x)), \qquad t,s \geqq 0,\ x \in (0,1].$$

Then we see that $\lim_{x \to 0} X(t,s,x,\omega) = 0$. So by defining $X(t,s,0) = 0$, we can define $X : [0,\infty) \times [0,\infty) \times [0,1] \times \Omega \to [0,1]$ such that $X(\cdot,*,**,\omega) : [0,\infty) \times [0,\infty) \times [0,1] \to [0,1]$ is continuous for any $\omega \in \Omega$, $X(t,s,\cdot,\omega) : [0,1] \to [0,1]$ is continuous and non-decreasing, and

$$f(t \vee s, r) = X(t,s,f(s,r)), \qquad r \geqq s \geqq 0,\ t \geqq 0.$$

Then we see that $X(t,t,x) = x$, and for $t \geqq s \geqq r \geqq 0$,

$$X(t,s,X(s,r,x)) = f(t,T(s,f(s,T(r,x)))) = f(t,T(r,x) \vee s) = f(t,T(r,x)) = X(t,r,x).$$

Let $Y(t,s) = \inf\{x \in [0,1]; X(t,s,x) = 1\}$. Then we see that $T(s,x) \geqq t$ iff $x \geqq Y(t,s)$, and that $Y(t,s)$ is $\mathcal{G}_s$-measurable.

3 A remark on support

Let $(\Omega,\mathcal{F},P)$ be a probability measure, Θ be a non-empty open set in $\mathbf{R}^M$, and $\mathcal{M}_0$ be a Polish space. Also, let $\xi : [0,\infty) \times \Omega \to [0,\infty)$, $\theta : [0,\infty) \times \Omega \to \bar{\Theta}$, and $Y : [0,\infty) \times \Omega \to \mathcal{M}_0$ be measurable processes. Remind that $\bar{\Theta}$ is the closure of Θ in $\mathbf{R}^M$. We assume that $\theta(\cdot,\omega) \to \bar{\Theta}$ is continuous for all $\omega \in \Omega$ and that $P(\int_a^b \xi(t)dt > 0) = 1$ for any $a,b \geqq 0$ with $a < b$.

Let $\tilde{\Omega} = [0,\infty) \times \Omega$. Let ν_0 be a probability measure on $[0,\infty)$, given by $\nu(dt) = e^{-t}dt$, and ν be a probability measure on $(\tilde{\Omega},\mathcal{B}([0,\infty)) \times \mathcal{F})$ given by $\nu = \nu_0 \otimes P$. Then ξ, (resp. θ, Y) can be regarded as a $[0,\infty)$ (resp. Θ, $\mathcal{M}_0$)-valued random variable defined in a probability space $(\tilde{\Omega},\mathcal{B}([0,\infty)) \times \mathcal{F},\nu)$.

Let μ be a probability law of (ξ,Y,θ) and μ_θ be a probability law of θ unde ν. Then μ and μ_θ be probability measures on $[0,\infty) \times \mathcal{M}_0 \times \bar{\Theta}$ and $\bar{\Theta}$ respectively. Let Γ and Γ_θ be the support of probability measures μ and μ_θ respectively. Then Γ and Γ_θ are closed subsets of $[0,\infty) \times \mathcal{M}_0 \times \bar{\Theta}$ and $\bar{\Theta}$ respectively. Let $\pi : [0,\infty) \times \mathcal{M}_0 \times \bar{\Theta} \to \bar{\Theta}$ ne a natural projection and let $\Gamma_0 = \pi((0,\infty) \times \mathcal{M}_0 \times \Theta)$.

Then we have the following.

Proposition 7 *The closure of* Γ_0 *contains* $\Gamma_\theta \cap \Theta$.

Proof. Let $\Phi : \tilde{\Omega} \to [0,\infty) \times \mathcal{M}_0 \times \bar{\Theta}$ be given by $\Phi(t,\omega) = (\xi(t,\omega), Y(t,\omega), \theta(t,\omega))$. Let $A = \Phi^{-1}(\Gamma)$. Then we have

$$1 = \nu(A) = \int_\Omega \nu_0(A_\omega)P(d\omega),$$

where $A_\omega = \{t \in [0,\infty);\ (t,\omega) \in A\}$. Let

$$B = \{\omega \in \Omega;\ \nu_0(A_\omega) = 1,\ \int_r^{r'} \xi(t,\omega)dt > 0 \text{ for any } r.r' \in \mathbf{Q} \text{ with } r < r' \ \}.$$

Then we see that $P(B) = 1$. Let $A' = A \cap ([0,\infty) \times B)$. Then we see that $\nu(A') = 1$. Let $\theta_0 \in \Gamma_\theta \cap \Theta$. Then for any $n \geqq 1$,

$$\nu(\{(t,\omega) \in A';\ |\theta(t,\omega) - \theta_0| < \frac{1}{2n}\}) > 0.$$

Therefore there is a $(t_n, \omega_n) \in A'$ such that $|\theta(t_n,\omega_n) - \theta_0| < 1/2n$. For any $m \geqq 1$, we see that $\int_{t_n}^{t_n+1/m} \xi(t,\omega_n)dt > 0$, and so there is a $s_{n,m} \in (t_n, t_n + 1/m) \cap A_{\omega_n}$ such that $\xi(s_{n,m},\omega_n) > 0$. Since $\theta(t,\omega_n)$ is continuous in t, we see that there is a $m(n) \geqq 1$ such that $|\theta(s_{n,m(n)},\omega_n) - \theta(t_n,\omega_n)| < 1/2n$. Now let $\xi_n = \xi(s_{n,m(n)},\omega_n)$, $\theta_n = \theta(s_{n,m(n)},\omega_n)$, and $y_n = Y(s_{n,m(n)},\omega_n)$. Then we see that $(\xi_n, y_n, \theta_n) \in \Gamma$, $\xi_n > 0$, and $|\theta_n - \theta_0| < 1/n$. Since Θ is open, $\theta_n \in \Theta$ for sufficiently large n. So we have our assertion. ∎

4 Fundamental Relations

Let $(\Omega, \mathcal{F}, P, (W_t^k)_{k=1,\dots,d}, (\tau_i)_{i=1,\dots,N}, (\xi_i(t))_{i=1,\dots,N}, \theta(t), \Theta, K)$ be a dynamical default time copula model as in Introduction. We also assume that $K|_{(0,1)^N \times \Theta}$ is C^2. We think about conditions which K must satisfy.

By Proposition 3, we see that there are $f_i : [0,\infty) \times [0,\infty) \times \Omega \to (0,\infty)$, $i = 1,\dots,N$, such that

$$f_i(t,s) = E[\exp(-\int_{t\wedge s}^s \xi_i(r)dr)|\mathcal{G}_t] \quad a.s. \qquad t,s \geqq 0,$$

$f_i(\cdot,* : \omega) : [0,\infty) \times [0,\omega)$, are continuous for any $\omega \in \Omega$, $f_i(t,s_1,\omega) > f_i(t,s_2,\omega) > 0$ for $s_2 > s_1 > t$, $\omega \in \Omega$, and

$$f(t;t,\omega) = 1, \quad \lim_{s\uparrow\infty} f(t;s,\omega) = 0, \qquad t \geqq 0,\ \omega \in \Omega.$$

Also by Corollary 5, we see that there are $\tilde{\sigma}_{i,k} : [0,\infty] \times [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1,\dots,d$, $i = 1,\dots,N$, satisfying the following.
(1) $\tilde{\sigma}_{i,k}(t,\cdot,\omega) : [0,\infty] \to \mathbf{R}$, $k = 1,\dots,d$, is continuous for any $t \in [0,\infty)$ and $\omega \in \Omega$.
(2) $\tilde{\sigma}_{i,k}(t,s,\omega) = 0$, $t \geqq s$, and $\lim_{s\to\infty} \tilde{\sigma}_{i,k}(t,s,\omega) = 0$ for any $t \in [0,\infty)$ and $\omega \in \Omega$.
(3) $\sigma_{i,k}(\cdot,s) : [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1,\dots,N$, is $\mathcal{G}$-progressively measurable for any $s \geqq 0$.
(4) For any $s > 0$

$$f_i(t,s) = f_i(0,s) + \int_0^{t\wedge s} \xi_i(r)f_i(r,s)dr + \sum_{k=1}^d \int_0^t \tilde{\sigma}_{i,k}(r,s)dW^k(r).$$

Let $T_i : [0,\infty) \times (0,1) \times \Omega \to (0,\infty)$, $i = 1,\ldots,N$, be given by

$$T_i(t,x) = \inf\{s \geqq t, f_i(t,s) \leqq x\}, \qquad x \in [0,1).$$

Then by Proposition 6 we see that $T_i(t,\cdot,\omega) : (0,1) \to (0,\infty)$ is continuous and strictly decreasing, $\lim_{x\downarrow 0} T_i(t,x,\omega) = \infty$ and $\lim_{x\uparrow 1} T_i(t,x,\omega) = 0$ for any $t \geqq 0$ and $\omega \in \Omega$. Let $\sigma_{i,k} : [0,\infty) \times (0,1) \times \Omega \to \mathbf{R}$, $i = 1,\ldots,N$, $k = 1,\ldots,d$, be given by

$$\sigma_{i,k}(t,x) = \tilde{\sigma}_{i,k}(t, T_i(t,x)) \qquad t \geqq 0,\ x \in (0,1).$$

Then we see that

$$\lim_{x\downarrow 0} \sigma_{i,k}(t,x) = 0, \qquad \lim_{x\uparrow 1} \sigma_{i,k}(t,x) = 0.$$

So we can extend this $\sigma_{i,k}$ as a function $\sigma_{i,k} : [0,\infty)\times[0,1]\times\Omega \to \mathbf{R}$, for which $\sigma_{i,k}(t,\cdot,\omega) : [0,1] \to \mathbf{R}$ is continuous for any $t \geqq 0$, $\omega \in \Omega$, and $\sigma_{i,k}(t,0) = \sigma_{i,k}(t,1) = 0$.

Let ν be a probability measure on $(0,\infty) \times \Omega$ given by $\nu(dt,d\omega) = e^{-t}dtP(d\omega)$. By the assumption (SC), we see that

$$1_{\{\tau_i > t\}} f_i(t;s) = 1_{\{\tau_i > t\}} P(\tau_i > s|\mathcal{F}_t) \quad a.s. \quad s \geqq t,\ i = 1,\ldots,N,$$

and

$$\prod_{i=1}^N 1_{\{\tau_i > t\}} \exp(-\sum_{i=1}^N \int_0^t \xi_i(r)dr) P(\tau_1 > s_1,\ldots,\tau_N > s_n|\mathcal{F}_t)$$

$$= \prod_{i=1}^N 1_{\{\tau_i > t\}} E[\exp(-\sum_{i=1}^N \int_0^{s_i} \xi_i(r)dr)|\mathcal{G}_t]\ a.s.$$

for $t \in [0, \min_{i=1,\ldots,N} s_i]$. So by the assumption (CP) we have

$$\prod_{i=1}^N 1_{\{\tau_i > t\}} 1_\Theta(\theta(t)) \exp(-\sum_{i=1}^N \int_0^t \xi_i(r)dr) K(f_1(t;s_1),\ldots,f_N(t;s_N),\theta(t))$$

$$= \prod_{i=1}^N 1_{\{\tau_i > t\}} 1_\Theta(\theta(t)) E[\exp(-\sum_{i=1}^N \int_0^{s_i} \xi_i(r)dr)|\mathcal{G}_t]\ a.s.$$

for $t \in [0, \min_{i=1,\ldots,N} s_i]$. Therefore by the assuption (PO), we have

$$1_\Theta(\theta(t)) \exp(-\sum_{i=1}^N \int_0^t \xi_i(r)dr) K(f_1(t;s_1),\ldots,f_N(t;s_N),\theta(t))$$

$$= 1_\Theta(\theta(t)) E[\exp(-\sum_{i=1}^N \int_0^{s_i} \xi_i(r)dr)|\mathcal{G}_t]\ a.s.$$

for $t \in [0, \min_{i=1,\ldots,N} s_i]$.

Now let us take a non-empty open set U in $\mathbf{R}^M$ such that $\bar{U} \subset \Theta$ and fix it for a while. For $T \geqq 0$, let $\tau_T^U : \Omega \to [0,\infty)$ be given by

$$\tau_T^U = \inf\{t \geqq T;\ \theta(t) \notin \bar{U}\} \wedge (T+1).$$

Then by the assumption (CP), we see that for any $s_1, \ldots, s_N \geqq T$

$$1_U(\theta(T)) \exp(-\sum_{i=1}^N \int_0^{t\wedge\tilde{\tau}} \xi_i(r)dr) K(f_1(t\wedge\tilde{\tau}, s_1), \ldots, f_N(t\wedge\tilde{\tau}; s_N), \theta(t\wedge\tilde{\tau})),\ t \in [T, T+1],$$

is a $\{\mathcal{G}_t\}_{t\in[T,T+1]}$-maringale, where $\tilde{\tau} = \tau_T^U \wedge \min_{i=1,\ldots,N} s_i$.

Note that $\theta(t)$ is an Ito process satisfying Equation (1). Therefore, applying Ito's formula and comparing finite total variation process, we have for any $s_1, \ldots, s_N \geqq T$

$$\begin{aligned}
1_U(\theta(T)) 1_{[t,T+1]}(\tau_T^U \wedge \min_{i=1,\ldots,N} s_i)\{-(\sum_{i=1}^N \xi_i(t)) K(f_1(t; s_1), \ldots, f_N(t; s_N), \theta(t)) \\
+ \sum_{i=1}^N \xi_i(t) f_i(t; s_i) \frac{\partial K}{\partial x_i}(f_1(t; s_1), \ldots, f_N(t; s_N), \theta(t)) \\
+ \sum_{j=1}^M b^j(t) \frac{\partial K}{\partial \theta^j}(f_1(t; s_1), \ldots, f_N(t; s_N), \theta(t)) \\
+ \frac{1}{2} \sum_{i,i'=1}^N \sum_{k=1}^d \tilde{\sigma}_{i,k}(t; s_i) \tilde{\sigma}_{i',k}(t; s_{i'}) \frac{\partial^2 K}{\partial x_i \partial x_{i'}}(f_1(t; s_1), \ldots, f_N(t; s_N), \theta(t)) \\
+ \frac{1}{2} \sum_{j,j'=1}^M \sum_{k=1}^d \eta_k^j(t) \eta_k^{j'}(t) \frac{\partial^2 K}{\partial \theta_j \theta_{j'}}(f_1(t; s_1), \ldots, f_N(t; s_N), \theta(t)) \\
+ \sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^d \tilde{\sigma}_{i,k}(t; s_i) \eta_k^j(t) \frac{\partial^2 K}{\partial x_i \theta_j}(f_1(t; s_1), \ldots, f_N(t; s_N), \theta(t))\} = 0
\end{aligned} \tag{2}$$

for $\nu - a.e.(t, \omega) \in (T, T+1) \times \Omega$.

Note that the left hand side of Equation (2) is right continuous in $s_1, \ldots, s_N$. So we see that there is an $B_T^U \in \mathcal{B}((T, T+1)) \times \mathcal{F}$ such that $\nu(((T, T+1) \times \Omega) \setminus B_T^U) = 0$ and Equation (2) holds for all $(t, \omega) \in B_T^U$ and $s_1, \ldots, s_N \in [T, \infty)$.

Also, substituting $s_i = T_i(t, x_i)$, $i = 1, \ldots, N$, to Equation (2), we see that for all $(t, \omega) \in B_T^U$

$$\begin{aligned}
1_U(\theta(T)) 1_{[t,T+1]}(\tau_T^U)\{-(\sum_{i=1}^N \xi_i(t)) K(x_1, \ldots, x_N, \theta(t)) \\
+ \sum_{i=1}^N \xi_i(t) x_i \frac{\partial K}{\partial x_i}(x_1, \ldots, x_N, \theta(t)) + \sum_{j=1}^M b^j(t) \frac{\partial K}{\partial \theta^j}(x_1, \ldots, x_N, \theta(t)) \\
+ \frac{1}{2} \sum_{i,i'=1}^N \sum_{k=1}^d \sigma_{i,k}(t; x_i) \sigma_{i',k}(t; x_{i'}) \frac{\partial^2 K}{\partial x_i \partial x_{i'}}(x_1, \ldots, x_N, \theta(t)) \\
+ \frac{1}{2} \sum_{j,j'=1}^M \sum_{k=1}^d \eta_k^j(t) \eta_k^{j'}(t) \frac{\partial^2 K}{\partial \theta_j \theta_{j'}}(x_1, \ldots, x_N, \theta(t))
\end{aligned}$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{d}\sigma_{i,k}(t;x_i)\eta^j_k(t)\frac{\partial^2 K}{\partial x_i\theta_j}(x_1,\ldots,x_N,\theta(t))\}=0 \qquad (3)$$

for any $x_1,\ldots,x_N\in(0,1)$.

Let $J_2=\{(j,j')\in\{0,1,\ldots,M\}\times\{1,\ldots,M\};\ j\leqq j'\}$. We define linear operators $S^{(2)}_{ii'}$, , $i,i'=1,\ldots,N$, $i<i'$, $S^{(1)}_{ij}$, $i=1,\ldots,N$, $j=0,1,\ldots,M$, and $S^{(0)}_{jj'}$, $(j,j')\in J_2$, from $C^2((0,1)^N\times\Theta)$ to $C((0,1)^N\times\Theta)$ by

$$(S^{(2)}_{ii'}F)(x,\theta)=\frac{\partial^2 F}{\partial x_i\partial x_{i'}}(x,\theta), \qquad 1\leqq i<i'\leqq N,$$

$$(S^{(1)}_{i0}F)(x,\theta)=\frac{\partial^2 F}{(\partial x_i)^2}(x,\theta), \qquad i=1,\ldots,N,$$

$$(S^{(1)}_{ij}F)(x,\theta)=\frac{\partial^2 F}{\partial x_i\partial\theta_j}(x,\theta), \qquad i=1,\ldots,N,\ j=1,\ldots,M,$$

$$(S^{(0)}_{jj'}F)(x,\theta)=\frac{\partial^2 F}{\partial\theta_j\partial\theta_{j'}}(x,\theta), \qquad 1\leqq j\leqq j'\leqq N,$$

$$(S^{(0)}_{0j'}F)(x,\theta)=\frac{\partial F}{\partial\theta_{j'}}(x,\theta), \qquad 1\leqq j'\leqq N,$$

for any $F\in C^2((0,1)^N\times\Theta)$.

Also, let us define $a^{(2)}_{ii'}:[0,\infty)\times[0,1]\times[0,1]\times\Omega\to\mathbf{R}$, $i,i'=1,\ldots,N$, $i<i'$, $a^{(1)}_{ij}:[0,\infty)\times[0,1]\times\Omega\to\mathbf{R}$, $i=1,\ldots,N$, $j=0,1,\ldots,M$, and $a^{(0)}_{jj'}:[0,\infty)\times[0,1]\times\Omega\to\mathbf{R}$, $(j,j')\in J_2$, by the following.

$$a^{(2)}_{ii'}(t,x_i,x_{i'})=\sum_{k=1}^{d}\sigma_{i,k}(t;x_i)\sigma_{i',k}(t,x_{i'}), \qquad 1\leqq i<i'\leqq N,$$

$$a^1_{i0}(t,x_i)=\frac{1}{2}\sum_{k=1}^{d}\sigma_{i,k}(t;x_i)^2 \qquad i=1,\ldots,N,$$

$$a^{(1)}_{ij}(t,x_i)=\sum_{k=1}^{d}\hat\sigma_{i,k}(t;x_i)\eta_{j,k}(t), \qquad i=1,\ldots,N,\ j=1,\ldots,M,$$

$$a^{(0)}_{jj'}(t)=\sum_{k=1}^{d}\eta_{j,k}(t)\eta_{j',k}(t), \qquad j,j'=1,\ldots,M,\ \text{with } j<j',$$

$$a^{(0)}_{jj}(t)=\frac{1}{2}\sum_{k=1}^{d}\eta_{j,k}(t)^2, \qquad j=1,\ldots,M,$$

and

$$a^{(0)}_{0j'}(t)=b_{j'}(t), \qquad j'=1,\ldots,M.$$

Then we have for all $(t,\omega)\in B^U_T$

$$1_U(\theta(T))1_{[t,T+1]}(\tau^U_T)\{\sum_{i=1}^{N}\xi_i(t)(x_i\frac{\partial K}{\partial x_i}(x,\theta(t))-K(x,\theta(t)))$$

$$+\sum_{1\leqq i<i'\leqq N} a_{ii'}^{(2)}(t,x_i,x_{i'})(S_{ii'}^{(2)}K)(x,\theta(t))+\sum_{i=1}^{N}\sum_{j=0}^{d} a_{ij}^{(1)}(t,x_i)(S_{ij}^{(1)}K)(x,\theta(t))$$

$$+\sum_{(j,j')\in J_2} a_{jj'}^{(0)}(t)(S_{jj'}^{(0)}K)(x,\theta(t))\}=0, \qquad x_1,\ldots,x_N\in(0,1). \tag{4}$$

Now let U_n, $n=1,2,\ldots$, be non-empty open sets in $\mathbf{R}^M$ such that the closure of U_n is contained in Θ for each n, and $\bigcup_{n=1}^{\infty} U_n=\Theta$. Since $\theta(t)$ is continuous in t, we see that

$$\{(t,\omega)\in(0,\infty)\times\Omega;\ \theta(t,\omega)\in\Theta\}$$

$$=\bigcup_{T\in\mathbf{Q}_{\geqq 0}}\bigcup_{n=1}^{\infty}\{(t,\omega)\in(T,T+1)\times\Omega;\ \theta(T,\omega)\in U_n,\ t\leqq\tau_T^{U_n}(\omega)\}.$$

So let

$$B_0=\bigcup_{T\in\mathbf{Q}_{\geqq 0}}\bigcup_{n=1}^{\infty}(B_T^{U_n}\cap\{(t,\omega)\in(T,T+1)\times\Omega;\ \theta(T,\omega)\in U_n,\ t\leqq\tau_T^{U_n}(\omega)\})$$

and $B_1=B_0\cup\{(t,\omega)\in(0,\infty)\times\Omega;\ \theta(t,\omega)\notin\Theta\}$. Then we see that $\nu(B_1)=1$. Also, we see that for all $(t,\omega)\in B_1$

$$1_\Theta(\theta(t))\{\sum_{i=1}^{N}\xi_i(t)(x_i\frac{\partial K}{\partial x_i}(x,\theta(t))-K(x,\theta(t)))$$

$$+\sum_{1\leqq i<i'\leqq N} a_{ii'}^{(2)}(t,x_i,x_{i'})(S_{ii'}^{(2)}K)(x,\theta(t))+\sum_{i=1}^{N}\sum_{j=0}^{d} a_{ij}^{(1)}(t,x_i)(S_{ij}^{(1)}K)(x,\theta(t))$$

$$+\sum_{(j,j')\in J_2} a_{jj'}^{(0)}(t)(S_{jj'}^{(0)}K)(x,\theta(t))\}=0, \qquad x_1,\ldots,x_N\in(0,1). \tag{5}$$

Let $\mathcal{C}_2$ be the set of continuous functions $a:[0,1]\times[0,1]\to\mathbf{R}$ with $a(0,x)=a(1,x)$ $=a(1,x)=a(x,1)=0$, $x\in[0,1]$, and $\mathcal{C}_1$ be the set of continuous functions $\tilde{a}:[0,1]\to\mathbf{R}$ with $a(0)=a(1)=0$. Then we see that $a_{ii'}^{(2)}(t,\cdot,*)\in\mathcal{C}^2$, $1\leqq i<i'\leqq N$, $a_{ij}^{(1)}\in\mathcal{C}^1$, $i=1,\ldots,N$, $j=0,1,\ldots,M$, for $\nu-a.e.\ (t,\omega)$.

Also, let $\mathcal{M}=(\mathcal{C}_2)^{N(N-1)/2}\times(\mathcal{C}_1)^{N(1+M)}\times\mathbf{R}^{J_2}$. Then $\mathcal{M}$ is a Poilish space. Let $Y(t,\omega)$ $=((a_{ii'}^{(2)}(t,\cdot,*,\omega))_{1\leqq i<i'\leqq N},(a_{ij}^{(1)}(t,\cdot,\omega))_{i=1,\ldots,N,j=0,1,\ldots,M},(a_{jj'}^{(0)}(t,\omega))_{(j,j')\in J_2})$. Then we see that $Y(t,\omega)\in\mathcal{M}$ for $\nu-a.e.\ (t,\omega)$. Therefore under the probability measure ν on $[0,\infty)\times\Omega$, $((\xi_i(t,\omega))_{i=1,\ldots,N},Y(t,\omega),\theta(t,\omega))$ is a $[0,\infty)^N\times\mathcal{M}\times\bar{\Theta}$-valued ranndom variable. Let Γ be the support of the probability law of this random variable.

Then we have the following.

Proposition 8 *If* $((\xi_i)_{i=1,\ldots,N},(\tilde{a}_{ii'}^{(2)})_{1\leqq i<i'\leqq N},(\tilde{a}_{ij}^{(1)})_{i=1,\ldots,N,j=0,1,\ldots,M},(\tilde{a}_{jj'}^{(0)})_{jj'})_{(j,j')\in J_2},\theta)$ *belongs to* Γ, *and* $\theta\in\Theta$,*then*

$$\sum_{i=1}^{N}\xi_i(x_i\frac{\partial K}{\partial x_i}(x,\theta)-K(x,\theta))$$

$$+\sum_{1\leqq i<i'\leqq N}\tilde{a}^{(2)}_{ii'}(x_i,x_{i'})(S^{(2)}_{ii'}K)(x,\theta)+\sum_{i=1}^N\sum_{j=0}^d\tilde{a}^{(1)}_{ij}(x_i)(S^{(1)}_{ij}K)(x,\theta)$$

$$+\sum_{(j,j')\in J_2}\tilde{a}^{(0)}_{jj'}(S^{(0)}_{jj'}K)(x,\theta)=0,\ \textit{for all } x=(x_1,\ldots,x_N)\in(0,1)^N$$

and

$$\sum_{1\leqq i<i'\leqq N}\tilde{a}^{(2)}_{ii'}(x_i,x_{i'})z_iz_{i'}+\sum_{i=1}^N\sum_{j=0}^d\tilde{a}^{(1)}_{i0}(x_i)z_i^2$$

$$+\sum_{i=1}^N\sum_{j=1}^d\tilde{a}^{(1)}_{ij}(x_i)z_iy_j+\sum_{(j,j')\in J_2}\tilde{a}^{(0)}_{jj'}y_jy_{j'}\geqq 0$$

for all $x=(x_1,\ldots,x_N)\in(0,1)^N$ *and* $z_1,\ldots,z_N,y_1,\ldots,y_M\in\mathbf{R}$.

Let Γ_θ be the support of $\theta(t,\omega)$ under ν. Let $\pi:[0,\infty)^N\times\mathcal{M}\times\bar{\Theta}\to\bar{\Theta}$ be the natural projection, and let $\Gamma_0=\pi(\Gamma\cap((0,\infty)\times[0,\infty)^{N-1}\times\mathcal{M}\times\Theta))$. Then Proposition 7 implies that the closure of Γ_0 contains $\Gamma_\theta\cap\Theta$.

Then we have the following from the previous Proposition.

Lemma 9 *Let* $N\geqq 2$, $M\geqq 1$, Θ *be an open set in* $\mathbf{R}^M$, *and* $K\in C([0,1]^N\times\Theta;[0,1])$. *Assume that* K *is an admissible family of copula functions and that* $K|_{(0,1)^N\times\Theta}$ *is* C^2. *Then there is a subset* A *of* Θ *such that the closure of* A *contains non-void open set in* Θ, *and for any* $\theta\in A$, *there are* $\xi_1>0$, $\xi_i\geqq 0$, $i=2,\ldots,N$, $\tilde{a}^{(2)}_{ii'}\in\mathcal{C}_1$, $1\leqq i<i'\leqq N$, $\tilde{a}^{(1)}_{ij}\in\mathcal{C}_1$, $i=1,\ldots,N$, $j=0,1,\ldots,M$, *and* $\tilde{a}^{(0)}_{jj'}\in\mathbf{R}$, $(j,j')\in J_2$, *such that*

$$\sum_{i=1}^N\xi_i(x_i\frac{\partial K}{\partial x_i}(x,\theta)-K(x,\theta))$$

$$+\sum_{1\leqq i<i'\leqq N}\tilde{a}^{(2)}_{ii'}(x_i,x_{i'})(S^{(2)}_{ii'}K)(x,\theta)+\sum_{i=1}^N\sum_{j=0}^d\tilde{a}^{(1)}_{ij}(x_i)(S^{(1)}_{ij}K)(x,\theta)$$

$$+\sum_{(j,j')\in J_2}\tilde{a}^{(0)}_{jj'}(S^{(0)}_{jj'}K)(x,\theta)=0,\ \textit{for all } x=(x_1,\ldots,x_N)\in(0,1)^N$$

and

$$\sum_{1\leqq i<i'\leqq N}\tilde{a}^{(2)}_{ii'}(x_i,x_{i'})z_iz_{i'}+\sum_{i=1}^N\sum_{j=0}^d\tilde{a}^{(1)}_{i0}(x_i)z_i^2$$

$$+\sum_{i=1}^N\sum_{j=1}^d\tilde{a}^{(1)}_{ij}(x_i)z_iy_j+\sum_{(j,j')\in J_2}\tilde{a}^{(0)}_{jj'}y_jy_{j'}\geqq 0$$

for all $x=(x_1,\ldots,x_N)\in(0,1)^N$ *and* $z_1,\ldots,z_N,y_1,\ldots,y_M\in\mathbf{R}$.

5 Verification

Let $N, M \geqq 1$, and Θ be an open set in $\mathbf{R}^M$. Let $n \geqq 1$, and $\vec{z} = (z_{ik})_{i=1,\dots,N,k=1,\dots,n} \in (0,1)^{nN}$.
For $\vec{k} = (k_1, \dots, k_N) \in \{1, \dots, n\}^N$, and $\vec{z} \in (0,1)^{nN}$, let $Z_i(\vec{z};\vec{k}) = z_{ik_i}$, $i = 1, \dots, N$, and $\vec{Z}(\vec{z};\vec{k}) = (z_{1k_1}, \dots, z_{Nk_N}) \in (0,1)^N$.

Let $K \in C([0,1]^N \times \Theta; [0,1])$ be an admissible family of copula functions, and assume that $K|_{(0,1)^N \times \Theta}$ is C^2. Now let A be a subset in Θ as in Lemma 9 Then for any $\theta \in A$, there are ξ_i, $i = 1, \dots, N$, $\tilde{a}^{(2)}_{ii'}$, $1 \leqq i < i' \leqq N$, $\tilde{a}^{(1)}_{ij}$, $i = 1, \dots, N, j = 0, 1, \dots, M$, and $\tilde{a}^{(0)}_{jj'}$, $(j,j') \in J_2$, be as in Lemma 9. Then we have

$$\sum_{i=1}^{N} \xi_i(Z_i(\vec{z};\vec{k}))\frac{\partial K}{\partial x_i}(\vec{Z}(\vec{z};\vec{k}),\theta) - K(\vec{Z}(\vec{z};\vec{k}),\theta))$$

$$+ \sum_{1 \leqq i < i' \leqq N} \tilde{a}^{(2)}_{ii'}(Z_i(\vec{z};\vec{k}), Z_{i'}(\vec{z};\vec{k}))(S^{(2)}_{ii'}K)(\vec{Z}(\vec{z};\vec{k}),\theta) + \sum_{i=1}^{N}\sum_{j=0}^{d} \tilde{a}^{(1)}_{ij}(Z_i(\vec{z};\vec{k}))(S^{(1)}_{ij}K)(\vec{Z}(\vec{z};\vec{k}),\theta)$$

$$+ \sum_{(j,j') \in J_2} \tilde{a}^{(0)}_{jj'}(S^{(0)}_{jj'}K)(\vec{Z}(\vec{z};\vec{k}),\theta) = 0.$$

That is

$$\sum_{i=1}^{N} \xi_i(Z_i(\vec{z};\vec{k})\frac{\partial K}{\partial x_i}(\vec{Z}(\vec{z};\vec{k}),\theta) - K(\vec{Z}(\vec{z};\vec{k}),\theta))$$

$$+ \sum_{p,q=1}^{n} \sum_{1 \leqq i < i' \leqq N} \tilde{a}^{(2)}_{ii'}(z_{ip}, z_{i'q})\delta_{p,k_i}\delta_{q,k_{i'}}(S^{(2)}_{ii'}K)(\vec{Z}(\vec{z};\vec{k}),\theta)$$

$$+ \sum_{p=1}^{n}\sum_{i=1}^{N}\sum_{j=0}^{d} \tilde{a}^{(1)}_{ij}(z_{ip})\delta_{p,k_i}(S^{(1)}_{ij}K)(\vec{Z}(\vec{z};\vec{k}),\theta)$$

$$+ \sum_{(j,j') \in J_2} \tilde{a}^{(0)}_{jj'}(S^{(0)}_{jj'}K)(\vec{Z}(\vec{z};\vec{k}),\theta) = 0, \qquad \vec{k} \in \{1, \dots, n\}^N. \tag{6}$$

Let

$$C^{(2)}_n = \{(i,i') \in \{1,2,\dots,N\}^2;\ i < i'\} \times \{1,2,\dots,n\}^2$$

and

$$C^{(1)}_n = \{1,2,\dots,N\} \times \{0,1,\dots,M\} \times \{1,2,\dots,n\}.$$

For any $G \in C^2((0,1)^N \times \Theta)$, $n \geqq 1$, and $\vec{k} \in \{1, \dots, n\}^N$, we define continuous functions defined in $(0,1)^{nN} \times \Theta$, $(M^{(n)I}_i G)(\cdot,\vec{k})$, $i = 1, \dots, N$, $(M^{(n)(2)}_{ii'pq}G)(\cdot,\vec{k})$, $(i,i',p,q) \in C^{(2)}_n$, $(M^{(n)(1)}_{ijp}G)(\cdot,\vec{k})$, $(i,j,p) \in C^{(1)}_n$, $(M^{(n)(0)}_{jj'}G)(\cdot,\vec{k})$, $(j,j') \in J_2$, by

$$(M^{(n)I}_i G)(\vec{z},\theta,\vec{k}) = Z_i(\vec{z},\vec{k})\frac{\partial G}{\partial x_i}(\vec{Z}(\vec{z};\vec{k}),\theta) - G(\vec{Z}(\vec{z};\vec{k}),\theta) \quad i = 1, \dots, N,$$

$$(M^{(n)(2)}_{ii'pq}G)(\vec{z},\theta,\vec{k}) = \delta_{p,k_i}\delta_{q,k_{i'}}(S^{(2)}_{ii'}G)(\vec{Z}(\vec{z};\vec{k}),\theta) \quad (i,i',p,q) \in C^{(2)}_n,$$

$$(M^{(n)(1)}_{ijp}G)(\vec{z},\theta,\vec{k}) = \delta_{p,k_i}(S^{(1)}_{ij}G)(\vec{Z}(\vec{z};\vec{k}),\theta), \quad (i,j,p) \in C^{(1)}_n,$$

$$(M^{(n)(0)}_{jj'}G)(\vec{z},\theta,\vec{k}) = (S^{(0)}_{jj'}G)(\vec{Z}(\vec{z};\vec{k}),\theta), \quad (j,j') \in J_2,$$

for any $\vec{z} \in (0,1)^{nN}$ and $\theta \in \Theta$.

Let $C_{n0} = C^{(2)}_n \cup C^{(1)}_n \cup J_2$, and $C_n = \{1,\dots,N\} \cup C_{n0}$. Note that the cardinal $\#(C_{n0})$ of C_{n0} is equal to $n^2N(N-1)/2 + nN(M+1) + M(M+3)/2$, and $\#(C_n) = N + \#(C_{n0})$.

For any $G \in C^2((0,1)^N \times \Theta)$, $n \geqq 1$, and $\gamma \in C_n$ we define a continuous function $(\vec{M}^{(n)}G)_\gamma : (0,1)^{nN} \times \Theta \to \mathbf{R}^{\{1,\dots,n\}^N}$ by

$$(\vec{M}^{(n)}G)_\gamma(\vec{z},\theta)$$

$$= \begin{cases} ((M^{(n)(2)}_{ii'pq}G)(\vec{z},\theta,\vec{k}))_{\vec{k}\in\{1,\dots,n\}^N} & \text{if } \gamma = (i,i',p,q) \in C^{(2)}_n, \\ (M^{(n)(1)}_{ijp}G)(\vec{z},\theta,\vec{k}))_{\vec{k}\in\{1,\dots,n\}^N} & \text{if } \gamma = (i,j,p) \in C^{(1)}_n, \\ (M^{(n)(0)}_{jj'}G)(\vec{z},\theta,\vec{k}))_{\vec{k}\in\{1,\dots,n\}^N} & \text{if } \gamma = (j,j') \in J_2, \\ ((M^{(n)I}_{i}G))(\vec{z},\theta,\vec{k}))_{\vec{k}\in\{1,\dots,n\}^N} & \text{if } \gamma = i \in \{1,\dots,N\}. \end{cases}$$

For any $G \in C^2((0,1)^N \times \Theta)$, $n \geqq 1$, $\vec{z} \in (0,1)^{nN}$ and $\theta \in \Theta$, let $V_n(G,\vec{z},\theta)$ (resp. $V_{n0}(G,\vec{z},\theta)$) be the vector subspace of $\mathbf{R}^{\{1,\dots,n\}^N}$ spaned by $\{(\vec{M}^{(n)}G)_\gamma(\vec{z},\theta);\ \gamma \in C_n\}$ (resp. $\{(\vec{M}^{(n)}G)_\gamma(\vec{z},\theta);\ \gamma \in C_{n0}\}$). Also, let $N_{(n)}(G,\vec{z},\theta)$ be a vector space in $\mathbf{R}^N$ given by

$$N_{(n)}(G,\vec{z},\theta) = \{(v_1,\dots,v_N) \in \mathbf{R}^N;\ \sum_{i=1}^n v_i(\vec{M}^{(n)}G)_i(\vec{z},\theta) \in V_{n0}(G,\vec{z},\theta)\}.$$

Then we have $N_{(n)}(K,\vec{z},\theta) \cap [0,\infty)^N \neq \{0\}$, for any $\theta \in A$.

Therefore we have the following.

Lemma 10 *Let $N \geqq 2$, $M \geqq 1$, and Θ be an open subset of $\mathbf{R}^M$ Let $K \in C([0,1]^N \times \Theta; [0,1])$. Assume that K is an admissible family of copula functions, and that $K|_{(0,1)^N\times\Theta}$ is C^2. Then there is a subset A of Θ such that the closure of A contains non-void open set in Θ, and for any $\theta \in A$ and $\vec{z} \in (0,1)^{nN}$, $N_{(n)}(K,\vec{z},\theta) \cap [0,\infty)^N \neq \{0\}$.*

As a corollary we have the following.

Corollary 11 *Let $N \geqq 2$, $M \geqq 1$, and Θ be an open subset of $\mathbf{R}^M$. Let $K \in C([0,1]^N \times \Theta; [0,1])$. Assume that K is an admissible family of copula functions, and that $K|_{(0,1)^N\times\Theta}$ is C^2. Then for any $n \geqq 1$ and $\vec{z} \in (0,1)^{nN}$, there is a non-void open subset U of Θ such that*

$$\dim V_n(K,\vec{z},\theta) \leqq \#(C_n) - 1, \qquad \theta \in U.$$

Proof. Since $\dim V_n(K,\vec{z},\theta) \leqq n^N$, the assertion is obvious in the case that $n^N \leqq \#(C_n) - 1$. So assume that $n^N \geqq \#(C_n)$.

Let A be a subset in Θ as in Lemma 10. It is easy to see that for any $\theta \in A$

$$\dim V_n(K,\vec{z},\theta) = \dim V_{n0}(K,\vec{z},\theta) + N - \dim N_{(n)}(G,\vec{z},\theta) \leqq \#(C_n) - 1.$$

Let H be the set of injections from C_n to $\{1,\ldots,n\}^N$ and let

$$\varphi(\theta) = \sum_{h\in H} \det((\vec{M}^{(n)}G)_{\gamma_1}(\vec{z},\theta,h(\gamma_2)))_{\gamma_1,\gamma_2\in C_n})^2, \quad \theta\in\Theta.$$

Then we see $\varphi(\theta)=0$ for $\theta\in A$. Since $\varphi:\theta\to\mathbf{R}$ is continuous, we see that $\varphi(\theta)=0$ for $\theta\in\bar{A}$. So we see that

$$\dim V_n(K,\vec{z},\theta) \leqq \#(C_n)-1, \qquad \theta\in\bar{A}.$$

This implies our assertion. ∎

6 Proof of Theorem 2

Now let $N\geqq 2$, $M\geqq 1$ and $n\geqq 1$. We say that $h: C_n^{(2)}\cup C_n^{(1)}\to\{1,\ldots,n\}^N$ is a matching map, if h is injective and satisfying the following.

$$h((i,i',p,q))_i = p, \quad h((i,i',p,q))_{i'} = q \quad \text{ for any } (i,i',p,q)\in C_n^{(2)},$$

and

$$h((i,j,p))_i = p \quad \text{ for any } (i,j,p)\in C_n^{(1)}.$$

Proposition 12 *Let $N\geqq 3$. Assume that there is a matching map $h_0: C_n^{(2)}\cup C_n^{(1)}\to\{1,\ldots,n\}^N$, and that $\#(C_n)\leqq n^N$. Let $0<c_{i1}<c_{i2}<\ldots<c_{in}<1$, $i=1,\ldots,N$, $\vec{c}=(c_{ik})_{i=1,\ldots,N,k=1,\ldots,n}\in(0,1)^{nN}$, and $\theta_0\in\mathbf{R}^M$. Then there is a $K\in\mathcal{C}_{(N)}(\mathbf{R}^M)$ such that $\dim V_n(K,\vec{c},\theta_0)=\#(C_n)$.*

Proof. From the assumption, there is an injective map $h: C_n\to\{1,\ldots,n\}^N$ such that the restriction of h to $C_n^{(2)}\cup C_n^{(1)}$ is equal to h_0. Note that $\vec{Z}(\vec{c};\vec{k})$, $\vec{k}\in\{1,\ldots,n\}^N$, are distinct points. Let

$$\varepsilon_0 = \min\{|\vec{Z}(\vec{c};\vec{k})-\vec{Z}(\vec{c};\vec{k}')|;\ \vec{k},\vec{k}'\in\{1,\ldots,n\}^N, \vec{k}\neq\vec{k}'\},$$

$$\varepsilon_1 = \min\{c_{i1};\ i=1,\ldots,N\}, \wedge\min\{1-c_{in};\ i=1,\ldots,N\},$$

and $\varepsilon=\varepsilon_0\wedge\varepsilon_1$. Let $\varphi_0\in C_0^\infty(\mathbf{R}^N)$ and $\varphi_1\in C_0^\infty(\mathbf{R}^M)$ such that $\varphi_0(x)=1$, $|x|<\varepsilon/3$, $\varphi_0(x)=0$, $|x|>2\varepsilon/3$, and $\varphi_1(\theta)=1$, $|\theta|<1$.

Let $F:\mathbf{R}^N\times\mathbf{R}^M\times C_n\to\mathbf{R}$ be given by the following.

$$F(x,\theta,i) = -\varphi_0(x-\vec{Z}(\vec{c};h(i))), \qquad i\in I,$$

$$F(x,\theta,(0,j)) = \varphi_0(x-\vec{Z}(\vec{c};h((0,j)))(\theta_j-\theta_{0j})\varphi_1(\theta-\theta_0), \qquad j=1,\ldots,M,$$

$$F(x,\theta,(j,j)) = \frac{1}{2}\varphi_0(x-\vec{Z}(\vec{c};h((j,j)))(\theta_j-\theta_{0j})^2\varphi_1(\theta-\theta_0), \qquad j=1,\ldots,M,$$

$$F(x,\theta,(j,j')) = \varphi_0(x-\vec{Z}(\vec{c};h((j,j')))(\theta_j-\theta_{0j})(\theta_{j'}-\theta_{0j'})\varphi_1(\theta-\theta_0), \qquad 1\leqq j<j'\leqq M,$$

$$F(x,\theta,(i,i',p,q))$$
$$= \varphi_0(x-\vec{Z}(\vec{c};h((i,i',p,q)))(x_i-\vec{Z}(\vec{c};h((i,i',p,q))_i)(x_{i'}-\vec{Z}(\vec{c};h((i,i',p,q))_{i'}),$$

$(i, i', p, q) \in C_n^{(2)}$,

$$F(x, \theta, (i, 0, p)) = \frac{1}{2}\varphi_0(x - \vec{Z}(\vec{c}; h((i, 0, p)))(x_i - \vec{Z}(\vec{c}; h((i, 0, p))_i)^2, \qquad (i, 0, p) \in C_n^{(1)},$$

$$F(x, \theta, (i, j, p))$$
$$= \varphi_0(x - \vec{Z}(\vec{c}; h((i, j, p)))(x_i - \vec{Z}(\vec{c}; h((i, j, p, q))_i)(\theta_j - \theta_{0j})\varphi_1(\theta - \theta_0), \quad (i, j, p) \in C_n^{(1)}, j \geqq 1.$$

Then we see that $(\vec{M}^{(n)}F(\cdot, u))_\gamma(\vec{c}, \theta_0)_{h(\alpha)} = \delta_{\gamma\alpha}$, $\gamma, \alpha \in C_n$, and that $(\vec{M}^{(n)}F(\cdot, u))_\gamma(\vec{c}, \theta_0)_{\vec{k}} = 0$, $\gamma \in C_n$, $\vec{k} \in \{1, \ldots, n\}^N$, with $\vec{k} \neq h(\gamma)$. Now let $F_0 \in C_0^\infty(\mathbf{R}^N \times \mathbf{R}^M)$ be given by

$$F_0(\cdot, *) = \sum_{\gamma \in C_n} F(\cdot, *, \gamma).$$

Then we see that derivatives of F_0 of any order are bounded functions defined in $\mathbf{R}^N \times \mathbf{R}^M$,

$$\det((\vec{M}^{(n)}F_0)_\gamma(\vec{c}, \theta_0, h(\alpha))_{\gamma, \alpha \in C_n}) = 1,$$

and $F_0(x_1, \ldots, x_N, \theta) = 0$, if $x_i < \varepsilon/3$ or $x_i > 1 - (\varepsilon/3)$ for some $i = 1, \ldots, N$.

Let
$$f_0(x, \theta) = \frac{\partial^N F_0}{\partial x_1 \ldots \partial x_N}(x, \theta)$$

Then we have
$$F_0(x, \theta) = \int_0^{x_1} \cdots \int_0^{x_N} f_0(y_1, \ldots, y_N, \theta) dy_1 \ldots dy_N.$$

Let $c = \sup\{|f_0(x, \theta)|;\ (x, \theta) \in \mathbf{R}^N \times \mathbf{R}^M\} < \infty$, and let

$$G_s(x, \theta) = \int_0^{x_1} \cdots \int_0^{x_N} (1 + s f_0(y_1, \ldots, y_N, \theta)) dy_1 \ldots dy_N$$
$$= x_1 \cdots x_N + s F_0(x_1, \ldots, x_N.\theta)$$

for $s \in \mathbf{R}$, $x \in [0, 1]^N$, $\theta \in \mathbf{R}^M$. Then

$$p(s) = \det((\vec{M}^{(n)}G_s)_\gamma(\vec{c}, \theta_0)_{h(\alpha)})_{\gamma, \alpha \in C_n})$$

is a polynomial in s and
$$\lim_{s \to \infty} s^{-\#(C_n)} p(s) = 1.$$

Therefore there is a $\tilde{s}$ with $0 < \tilde{s} < 1/(2c + 1)$ such that

$$\det((\vec{M}^{(n)}G_{\tilde{s}})_\gamma(\vec{c}, \theta_0)_{h(\alpha)})_{\gamma, \alpha \in C_n}) \neq 0.$$

Note that $1 + \tilde{s}h(y_1, \ldots, y_N, \theta) > 1/(2c+1)$. So it is easy to see that $G_{\tilde{s}}(\cdots, \theta)$ is a copula function for all $\theta \in \mathbf{R}^M$.

This shows our assertion. ∎

Proposition 13 *Let $N = 3$, $M \geqq 1$ and $n \geqq 1$. Suppose that $n \geqq M + 3$, and $n \equiv 1$ or 5 mod 6. Then there is a matching map $h : C_n^{(2)} \cup C_n^{(1)} \to \{1, \ldots, n\}^N$.*

Proof. First we prove the following.

Claim. Let $p, q, r = 1, \dots, n$. If $p \neq q$ and $r \equiv 2q - p \bmod n$, then $r \neq p, q$, $p \not\equiv 2r - q \bmod n$, and $q \not\equiv 2p - r \bmod n$.

Actually if $r = p$, we have $2p \equiv 2q \bmod n$, which implies $p = q$. If $r = q$, we have $q \equiv p \bmod n$. which implies $p = q$. If $p \equiv 2r - q \bmod n$, then $3p \equiv 3q \bmod n$, which implies $p = q$. If $q \equiv 2p - r \bmod n$, then $3q \equiv 3p \bmod n$, which implies $p = q$. Therefore we have our Claim.

Let us define $h : C_n^{(2)} \cup C_n^{(1)} \to \{1, \dots, n\}^N$ by the following.

$h|_{C_n^{(2)}}$ is given by the following. For $p, q = 1, \dots, n$, with $p \neq q$

$$h(1,2,p,q) = (p,q,r), \quad h(1,3,q,p) = (q,r,p) \quad h(2,3,p,q) = (r,p,q),$$

where $r = 1, \dots, n$, with $r \equiv 2q - p \bmod n$. For $p = 1, \dots, n$,

$$h(1,2,p,p) = (p,p,r), \quad h(1,3,p,p) = (p,r,p) \quad h(2,3,p,p) = (r,p,p),$$

where $r = 1, \dots, n$, with $r \equiv p - 1 \bmod n$.

$h|_{C_n^{(1)}}$ is given by the following. For $p = 1, \dots, n$, and $j = 0, 1, \dots, M$,

$$h(1,j,p) = (p,r,r), \quad h(2,j,p) = (r,p,r) \quad h(3,j,p) = (r,r,p),$$

where $r = 1, \dots, n$, with $r \equiv p + j + 2 \bmod n$.

By the above Claim, we can easily check h is a matching map. ∎

Proposition 14 *Let $N \geqq 4$, $M \geqq 1$ and $n \geqq 1$. Suppose that $n \geqq M + 2$. Then there is a matching map $h : C_n^{(2)} \cup C_n^{(1)} \to \{1, \dots, n\}^N$.*

Proof. First take a map $R : \{i, \dots, n\}^2 \to \{1, \dots, n\}$ such that $R(p,q) \neq p, q$ and $R(p,p) \equiv p + 1 \bmod n$. Since $n \geqq 3$, we can take such a map. Now let us define $h : C_n^{(2)} \cup C_n^{(1)} \to \{1, \dots, n\}^N$ by the following.

$h(i, i', p, q) = (k_1, \dots, k_N)$, $(i, i', p, q) \in C_n^{(2)}$, where $k_i = p$, $k_{i'} = q$, $k_r = R(p,q)$, $r \neq i, i'$. $h(i,j,p) = (k_1, \dots, k_N)$, $(i,j,p) \in C_n^{(1)}$, where $k_i = p$, $k_r = 1, \dots, n$, with $k_r \equiv p + j + 1 \bmod n$.

Since $p + 2 \not\equiv p \bmod n$, we can easily check h is a matching map. ∎

Now let us prove Theorem 2.

Let $N \geqq 3$, $M \geqq 1$, and Θ is a non-empty open set in $\mathbf{R}$. By Propositions 13 and 14, there are $n \geqq 1$ and an injective map $h : C_n \to \{1, \dots, n\}^N$ for which $h|_{C_n^{(2)} \cup C_n^{(1)}}$ is a matching map. Fix such an n and let us take $\vec{c} \in \mathbf{R}^{Nn}$ such that $0 < c_{i1} < c_{i2} < \dots < c_{in} < 1$, $i = 1, \dots, N$, $i = 1, \dots, N$. Let $D(\theta)$, $\theta \in \Theta$, be a set given by

$$D(\theta) = \{K \in \mathcal{C}_{(N)}(\Theta);\ \dim V_n(K, \vec{c}, \theta_0) = \#(C_n)\}.$$

Then by Propositions 12, we see that $D(\theta) \neq$ for all $\theta \in \Theta$. Let H be the set of injections from C_n to $\{1, \dots, n\}^N$. Let $\varphi : \mathcal{C}_{(N)}(\Theta) \to \mathbf{R}$ be given by

$$\varphi(K) = \sum_{h \in H} \det((\vec{M}^{(n)} K)_{\gamma_1}(\vec{z}, \theta, h(\gamma_2)))_{\gamma_1, \gamma_2 \in C_n})^2.$$

Then we see that $\varphi : \mathcal{C}_{(N)}(\Theta) \to \mathbf{R}$ is continuous and $D(\theta) = \{K \in \mathcal{C}_{(N)}(\Theta);\ \varphi(K) > 0\}$. So we see that $D(\theta)$ is an open subset $\mathcal{C}_{(N)}(\Theta)$.

Let $G \in D(\theta)$. For any $K \in \mathcal{C}_{(N)}(\Theta)$ and $s \in [0,1]$, $(1-s)K + sG_0 \in \mathcal{C}_{(N)}(\Theta)$. Also, $\varphi((1-s)K + sG)$ is a polynomial in s, and so is not equal to 0 except finite s's. Therefore there is a $\{s_\ell\}_{\ell=1}^{\infty} \subset [0,1]$ such that $s_\ell \downarrow 0$, $\ell \to \infty$, and $(1-s_\ell)K + s_\ell)G) \in D(\theta)$, $\ell \geqq 1$. This observation shows that $D(\theta)$ is dense in $\mathcal{C}_{(N)}(\Theta)$ for all $\theta \in \Theta$.

Now let $\{\theta_m\}_{m=1}^{\infty}$ be a dense set in Θ, and let

$$D = \bigcap_{m=1}^{\infty} D(\theta_m).$$

Then by Corollary 11, we see that any element of D is not admissible family of copula functions. This proves Theorem 2.

7 Remarks

Let $N \geqq 3$, $M \geqq 1$, and Θ be an open set in $\mathbf{R}^M$. Let $K \in \mathcal{C}_{(N)}(\Theta)$. Assume that $n \geqq N$.

Let

$$A^{(2)}$$
$$= \{(1,2,p,q);\ p,q = 1,\dots,n\} \cup \{(1,i,p,q);\ i' = 3,\dots,N,\ p = 1,\dots,n,\ q = 2,\dots,n\}$$
$$\cup \{(i,i',p,q);\ 2 \leqq i < i' \leqq N,\ p,q = 2,\dots,n\} \subset C_n^{(2)}.$$

Also, let $\vec{k}_{ii'pq} \in \mathbf{R}^{\{1,\dots,n\}^N}$, $(i,i',p,q) \in C_n^{(2)}$, be given by

$$\vec{k}_{ii'pq} = (1,\dots,1,\underset{i}{p},1,\dots,1,\underset{i'}{q},1,\dots,1).$$

Then we have the following.

Proposition 15 *Let $\theta_0 \in \Theta$, and assume that*

$$\frac{\partial^2 K}{\partial x_i \partial x_{i'}}(x,\theta_0) > 0, \qquad x \in (0,1)^N,\ 1 \leqq i < i' \leqq N.$$

Then for any $n \geqq N$, and $\vec{z} \in (0,1)^{3n}$,

$$\dim V_{n0}(K,\vec{c},\theta_0) \geqq \#(A^{(2)}) = \frac{N(N-1)}{2}n^2 - nN(N-2) + \frac{(N-1)(N-2)}{2}.$$

Proof. Remind that

$$(M_{ii'pq}^{(n)(2)}K)(\vec{z},\theta_0,\vec{k}) = \delta_{p,k_i}\delta_{q,k_{i'}}\frac{\partial^2 K}{\partial x_i \partial x_{i'}}(z_{1k_1},\dots,z_{Nk_N},\theta_0), \quad p,q = 1,\dots,n.$$

So for $(i,i',p,q),(j,j',r,\ell) \in A^{(2)}$, we see that $(M_{ii'pq}^{(n)(2)}K)(\vec{z},\theta_0,\vec{k}_{jj'r\ell}) = 0$ if $i > j$, or if $i = j$ and $i' > j$, and that $(M_{ii'pq}^{(n)(2)}K)(\vec{z},\theta_0,\vec{k}_{ii'r\ell}) = \delta_{p,r}\delta_{q,\ell}c_{ii'pq}$, for some positive numbers $c_{ii'pq}$. So we see that $\{(\vec{M}^{(n)}K)_\gamma(\vec{z},\theta_0);\ \gamma \in A^{(2)}\}$ is linearly independent. So we have our assertion. ∎

From now on we think of a special case. We assume that K is a family of Archimedian copula functions, i.e., there are smooth functions $\varphi : (0,1)\times\Theta \to (0,\infty)$ and $\rho : (0,\infty)\times\Theta \to (0,1)$ such that

$$K(x_1,\ldots,x_N,\theta) = \rho(\sum_{k=1}^{N}\varphi(x_k,\theta),\theta), \qquad x_1,\ldots,x_N \in (0,1),\ \theta\in\Theta.$$

Then $\rho(\cdot,\theta)$ must be the inverse function of $\varphi(\cdot,\theta)$.

Then we have the following.

Proposition 16 *Let*

$$m_0 = \frac{N(N-1)}{2}n^2 + N(M+3-N)n - \frac{(N-1)(2M+4-N)}{2} + \frac{M(M+3)}{2}.$$

Then we have the following.
(1) $\dim V_{n0}(K,\vec{z},\theta) \leqq m_0$. *and* $\dim V_n(K,\vec{z},\theta) \leqq m_0+1$ *for any* $\vec{z}\in(0,1)^{Nn}$, *and* $\theta\in\Theta$.
(2) *Assume that* Θ *is connected and that* $\varphi : (0,1)\times\Theta\to(0,\infty)$ *is real analytic. If there exists a* $\theta_0\in\Theta$ *such that* $\dim V_n(K,\vec{z},\theta_0) = m_0+1$, *then* K *is not an admissible family of copula functions.*

Since the proof is rather long, we will give it in the next section.

Now let us think of a family of Gumbel copula functions. Let $N=3$, $M=1$, and $\Theta=(0,1)$. Let $K\in\mathcal{C}_{(3)}((0,1))$ be given by

$$K(x_1,x_2,x_3,\theta) = \exp(-(\sum_{i=1}^{3}(-\log x_i)^\theta)^{1/\theta}), \qquad x_1,x_2,x_3\in(0,1),\ \theta\in(0,1).$$

Then letting $\varphi(x,\theta) = (-\log x)^\theta$, $\rho(y,\theta) = \exp(-y^\theta)$, we see that

$$K(x_1,x_2,x_3,\theta) = \rho(-\sum_{i=1}^{3}\varphi(x_i,\theta)), \qquad x_1,x_2,x_3\in(0,1),\ \theta\in(0,1).$$

Let $n=5$. Then we have $m_0=89$. So by Proposition 16 we see that if there exist $\theta_0\in(0,1)$ and $\vec{z}=(z_{ip})_{i=1,2,3,p=1,\ldots,5}\in(0,1)^{15}$ such that $\dim V_n(K,\vec{z},\theta_0)=90$, we see that K is not admissible family of copula functions.

By using numerical computation, we check that $\dim V_n(K,\vec{z},\theta_0)=90$ for $(z_{i1},\ldots,z_{i5})$ $=(0.55,0.65,0.75,0.85,0.95)$, $i=1,2,3$, and $\theta_0=0.4$ or 0.6. Actuary, we compute the dimension of the vector subspace in $\mathbf{R}^{\{1,2,3,4,5\}^3}$ spanned by $e^{(2)}_{ii'pq}(\vec{z},\theta_0)$, $(i,i',p,q)\in A^{(2)}$, $e^{(1)}_{ijp}(\vec{z},\theta_0)$, $(i,j,p)\in A^{(1)}$, $e^{(0)}_{jj'}(\vec{z},\theta_0)$, $(j,j')\in J_2$, and $e_0(\vec{z},\theta_0)$ given in the next section by applying Householder transformation for the associated matrix, and we are convinced that it is 90. As we show in the next section, the dimension of this vector subspace is the same as $\dim V_n(K,\vec{z},\theta_0)$.

8 Proof of Proposition 16

For $\vec{z} \in (0,1)^{3n}$, $\theta \in \Theta$ and $\vec{k} \in \{1, \ldots, n\}^3$, let

$$\Phi(\vec{z}, \theta, \vec{k}) = \sum_{i=1}^{3} \varphi(z_{ik_i}, \theta),$$

$$e^{(2)}_{ii'pq}(\vec{z}, \theta, \vec{k}) = \delta_{pk_i}\delta_{qk_{i'}}\frac{\partial^2 \rho}{\partial y^2}(\Phi(\vec{z}, \theta, \vec{k}), \theta), \quad (i, i', p, q) \in C^{(2)}_n,$$

$$e^{(1)}_{i0p}(\vec{z}, \theta, \vec{k}) = \delta_{pk_i}\frac{\partial \rho}{\partial y}(\Phi(\vec{z}, \theta, \vec{k}), \theta), \quad (i, 0, p) \in C^{(1)}_n,$$

$$e^{(1)}_{ijp}(\vec{z}, \theta, \vec{k}) = \delta_{pk_i}\frac{\partial^2 \rho}{\partial \theta_j \partial y}(\Phi(\vec{z}, \theta, \vec{k}), \theta) \quad (i, j, p) \in C^{(1)}_n,\ j \geqq 1,$$

$$e^{(0)}_{jj'}(\vec{z}, \theta, \vec{k}) = \frac{\partial^2 \rho}{\partial \theta_j \partial \theta_{j'}}(\Phi(\vec{z}, \theta, \vec{k}), \theta), \quad (j, j') \in J_2,\ j \geqq 1,$$

$$e^{(0)}_{0j}(\vec{z}, \theta, \vec{k}) = \frac{\partial \rho}{\partial \theta_j}(\Phi(\vec{z}, \theta, \vec{k}), \theta). \quad (0, j) \in J_2,$$

and

$$e_0(\vec{z}, \theta, \vec{k}) = -\rho(\Phi(\vec{z}, \theta, \vec{k}), \theta).$$

Let $e^{(2)}_{ii'pq}(\vec{z}, \theta)$, $(i, i', p, q) \in C^{(2)}_n$, $e^{(1)}_{ijp}(\vec{z}, \theta)$, $(i, j, p) \in C^{(1)}_n$, $e^{(0)}_{jj'}(\vec{z}, \theta)$, $(j, j') \in J_2$, and $e_0(\vec{z}, \theta)$ be elements of $\mathbf{R}^{\{1,\ldots,n\}^3}$ given by $e^{(2)}_{ii'pq}(\vec{z}, \theta) = (e^{(2)}_{ii'pq}(\vec{z}, \theta, \vec{k}))_{\vec{k} \in \{1,\ldots,n\}^3}$ etc.

Let $A^{(1)}$ be the subset of $C^{(1)}_n$ given by

$$A^{(1)}$$

$$\begin{aligned} &= \{(1, j, p);\ j = 0, 1, \ldots, M,\ p = 1, \ldots, n\} \\ &\quad \cup \{(i, j, p);\ i = 2, \ldots, N,\ j = 0, 1, \ldots, M,\ p = 2, \ldots, n\}, \end{aligned}$$

and $A^{(2)}$ be the subset of $C^{(2)}_n$ given in the previous section.

Let $U_0(\vec{z}, \theta)$ be a vector subspace in $\mathbf{R}^{\{1,\ldots,n\}^N}$ spaned by $\{e^{(2)}_{ii'pq}(\vec{z}, \theta);\ (i, i', p, q) \in A^{(2)}\}$, $\{e^{(1)}_{ijp}(\vec{z}, \theta);\ (i, j, p) \in A^{(1)}\}$ and $\{e^{(0)}_{jj'}(\vec{z}, \theta);\ (j, j') \in J_2\}$. Since $\#(A^{(2)}) + \#(A^{(1)}) + \#(J_2)$ $= m_0$, we see that $\dim U_0(\vec{z}, \theta) \leqq m_0$.

First, we prove the following.

Proposition 17 (1) $e^{(2)}_{ii'pq}(\vec{z}, \theta) \in U_0(\vec{z}, \theta)$ *for all* $(i, i', p, q) \in C^{(2)}_n$.
(2) $e^{(1)}_{ijp}(\vec{z}, \theta) \in U_0(\vec{z}, \theta)$ *for all* $(i, j, p) \in C^{(1)}_n$.

Proof. Let

$$\tilde{e}^{(2)}_{1p}(\vec{z}, \theta) = \sum_{q=1}^{n} e^{(2)}_{12pq}(\vec{z}, \theta) \in U_0(\vec{z}, \theta), \quad p = 1, \ldots, n,$$

and

$$\tilde{e}^{(2)}_{ip}(\vec{z}, \theta) = \sum_{q=1}^{n} e^{(2)}_{1iqp}(\vec{z}, \theta) \in U_0(\vec{z}, \theta), \quad i = 2, \ldots, N-1,\ p = 2, \ldots, n.$$

Then we see that

$$e^{(2)}_{1ip1}(\vec{z},\theta) = \hat{e}^{(2)}_{1p}(\vec{z},\theta) - \sum_{q=2}^{n} e^{(2)}_{1ipq}(\vec{z},\theta) \in U_0(\vec{z},\theta), \quad i = 3,\dots,N,\ p = 1,\dots,n.$$

So we see that $e^{(2)}_{1ipq}(\vec{z},\theta) \in U_0(\vec{z},\theta)$, $i = 2,\dots,N$, $p,q = 1,\dots,n$.

Also, we see taht

$$e^{(2)}_{ii'1q}(\vec{z},\theta) = \hat{e}^{(2)}_{i'q}(\vec{z},\theta) - \sum_{p=2}^{n} e^{(2)}_{ii'pq}(\vec{z},\theta) \in U_0(\vec{z},\theta), \quad i = 2,\dots,N,\ q = 2,\dots,n,$$

and so

$$e^{(2)}_{ii'11}(\vec{z},\theta) = \hat{e}^{(2)}_{i1}(\vec{z},\theta) - \sum_{q=2}^{n} e^{(2)}_{ii'1q}(\vec{z},\theta) \in U_0(\vec{z},\theta), \quad i = 2,\dots,N.$$

These show that the assetion (1).

Let

$$\tilde{e}^{(1)}_{j}(\vec{z},\theta) = \sum_{p=1}^{n} e^{(1)}_{1jp}(\vec{z},\theta) \in U_0(\vec{z},\theta), \quad j = 0,\dots,M.$$

Then we see that

$$e^{(1)}_{ij1}(\vec{z},\theta) = \hat{e}^{(1)}_{i}(\vec{z},\theta) - \sum_{p=2}^{n} e^{(1)}_{ijp}(\vec{z},\theta) \in U_0(\vec{z},\theta), \quad j = 0,\dots,M.$$

This proves the assertion (2). ∎

Now note that

$$(M^{(n)(2)}_{ii'pq}K)(\vec{z},\theta,\vec{k}) = \frac{\partial\varphi}{\partial x}(z_{ip},\theta)\frac{\partial\varphi}{\partial x}(z_{i'q},\theta)e^{(2)}_{ii'pq}(\vec{z},\theta,\vec{k}), \quad (i,i',p,q) \in C^{(2)}_n,$$

$$(M^{(n)(1)}_{i0p}K)(\vec{z},\theta,\vec{k}) = \frac{\partial^2\varphi}{\partial x^2}(z_{ip},\theta)e^{(1)}_{i0p}(\vec{z},\theta,\vec{k}) + (\frac{\partial\varphi}{\partial x}(z_{ip},\theta))^2\tilde{e}^{(2)}_{ip}(\vec{z},\theta,\vec{k})), \quad (i,0,p) \in C^{(1)}_n,$$

$$(M^{(n)(1)}_{ijp}K)(\vec{z},\theta,\vec{k}) = \frac{\partial\varphi}{\partial x}(z_{ip},\theta)e^{(1)}_{ijp}(\vec{z},\theta,\vec{k})$$

$$+\frac{\partial\varphi}{\partial x}(z_{ip},\theta)(\sum_{q=1}^{n}\frac{\partial\varphi}{\partial\theta_j}(z_{iq},\theta)e^{(2)}_{iq}(\vec{z},\theta,\vec{k})))+\frac{\partial^2\varphi}{\partial\theta_j\partial x}(z_{ip},\theta)e^{(1)}_{i0p}(\vec{z},\theta,\vec{k}), \quad (i,j,p) \in C^{(1)}_n,\ j \geqq 1,$$

$$(M^{(n)(0)}_{0j}K)(\vec{z},\theta,\vec{k}) = \sum_{i=1}^{N}\sum_{p=1}^{n}\frac{\partial\varphi}{\partial\theta}(z_{ip},\theta)e^{(1)}_{i0p}(\vec{z},\theta,\vec{k}) + e^{(0)}_{0j}(\vec{z},\theta,\vec{k}), \quad (0,j) \in J_2,$$

$$(M^{(n)(0)}_{jj'}K)(\vec{z},\theta,\vec{k})$$

$$= \sum_{1\leqq i<i'\leqq N}\sum_{p,q=1}^{n}(\frac{\partial\varphi}{\partial\theta_j}(z_{ip},\theta)\frac{\partial\varphi}{\partial\theta_{j'}}(z_{i'q},\theta) + \frac{\partial\varphi}{\partial\theta_j}(z_{ip},\theta)\frac{\partial\varphi}{\partial\theta_{j'}}(z_{i'q},\theta))e^{(2)}_{ii'pq}(\vec{z},\theta,\vec{k})$$

$$+\sum_{i=1}^{N}\sum_{p=1}^{n}\frac{\partial\varphi_j}{\partial\theta_j}(z_{ip},\theta)\frac{\partial\varphi_{j'}}{\partial\theta_{j'}}(z_{ip},\theta)\tilde{e}^{(2)}_{ip}(\vec{z},\theta,\vec{k})$$

$$+\sum_{i=1}^{N}\sum_{p=1}^{n}\frac{\partial\varphi}{\partial\theta_j}(z_{ip},\theta)e^{(1)}_{ij'p}(\vec{z},\theta,\vec{k})+\frac{\partial\varphi}{\partial\theta_{j'}}(z_{ip},\theta)e^{(1)}_{ijp}(\vec{z},\theta,\vec{k})$$

$$+\sum_{i=1}^{N}\sum_{p=1}^{n}\frac{\partial^2\varphi}{\partial\theta_j\partial\theta_{j'}}(z_{ip},\theta)e^{(2)}_{i0p}(\vec{z},\theta,\vec{k})+e_{jj'}(\vec{z},\theta,\vec{k}).$$

Therefore from the assumption, we see that

$$V_{n0}(K,\vec{z},\theta)\subset U_0(\vec{z},\theta).$$

So we have the first assertion of Proposition 16 (1).

We remark that if

$$\frac{\partial\varphi}{\partial x}(z_{ip},\theta)>0,\qquad \frac{\partial^2\varphi}{(\partial x)^2}(z_{ip},\theta)\neq 0,$$

for any $i=1,\ldots,N$ and $p=1,\ldots,n$, then

$$U_0(\vec{z},\theta)=V_{n0}(K,\vec{z},\theta).$$

Now note that

$$(M^{(n)I}_i K)(\vec{z},\theta,\vec{k})=e_0(\vec{z},\theta,\vec{k})+\sum_{p=1}^{n}z_{ip}\frac{\partial\varphi}{\partial x}(z_{ip},\theta)e^{(1)}_{i0p}(\vec{z},\theta,\vec{k}).$$

So we have $(\vec{M}^{(n)}K)_i(\vec{z},\theta)-e_0(\vec{z},\theta)\in V_{n0}(K,\vec{z},\theta)$, $i\in I$ This implies that the second assertion of Proposition 16 (1).

Now let us prove the assertion (2) of Proposition 16. Suppose that K is an admissible family of copula functions. Then by Lemma 9, we see that there is a subset A of Θ such that the closure of A contains a non-void open subset of Θ and for any $\theta\in A$ there are $\xi_i\geqq 0$, $i=1,\ldots,N$, such that $\sum_{i\in I}\xi_i>0$ and $\sum_{i\in I}\xi_i(\vec{M}^{(n)}K)_i(\vec{z},\theta)\in V_{n0}(K,\vec{z},\theta)$. Then we see that $e_0(\vec{z},\theta)\in V_{n0}(K,\vec{z},\theta)$. This implies that $V_n(K,\vec{z},\theta)\subset U_0(\vec{z},\theta)$.

Then by the assertion (1), we see that $\dim V_n(K,\vec{z},\theta)\leqq m_0$, $\theta\in A$. Let H_1 (resp. H_2) be the set of injective maps from $\{1,\ldots,m_0\}$ to C_n (resp. $\{1,\ldots,n\}^N$.) Now let

$$f(\theta)=\sum_{h_1\in H_1}\sum_{h_2\in H_2}\det(((\vec{M}^{(n)}K)_{h_1(r)})(\vec{z},\theta,h_2(\ell)))_{r,\ell=1,\ldots,m_0})^2,\ \theta\in\Theta.$$

Then we see that $f(\theta)=0$, $\theta\in A$. From the assumption, we see that $f:\Theta\to\mathbf{R}$ is real analytic. So we see that $f(\theta)=0$, $\theta\in\bar{A}\cap\Theta$. Since $\bar{A}\cap\Theta$ contains a non-void open set and Θ is connected, we see that $f=0$ on Θ. In particular, $f(\theta_0)=0$. But this implies that $\dim V_n(K,\vec{z},\theta_0)\leqq m_0-1$. This contradicts to the assumption. Therefore K is not admissible.

This completes the proof of Proposition 16.

9 Examples of dynamical default time copula models

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $W(t) = (W^k(t))_{k=1,\dots,d}$, $t \geqq 0$, be a d-dimensinal standard Wiener process. Let $N \geqq 2$, and $Z_1, \dots, Z_N$ be a independent identically distributed random variables whose distributions are uniform distribution on $(0,1)$. We assume that $\sigma\{Z_1, \dots, Z_N\}$ and $\sigma\{W(t), t \geqq 0\}$ are independent. Let $M \geqq 1$. Let $\sigma_k : \mathbf{R}^M \to \mathbf{R}^M$, $k = 0, 1, \dots, d$, be Lipschitz continuous functions and $h_i : \mathbf{R}^M \to (0, \infty)$, $i = 1, \dots, N$, be continuous functions.

Let Y be the unique solution to the following stochastic differential equation on $\mathbf{R}^M$.

$$dY(t, y) = \sum_{k=1}^{d} \sigma_k(t, Y(t, y))dW^k(t) + \sigma_0(t, Y(t, y))dt,$$

$$Y(0, y) = y \in \mathbf{R}^M.$$

Let $y_0 \in \mathbf{R}^M$. We also assume that

$$P(\int_0^\infty h_i(Y(t, y_0))dt = \infty) = 1, \qquad i = 1, \dots, N,$$

and the support of the distribution of $Y(t, y_0)$ under $e^{-t} \otimes P(d\omega)$ contains non-empty open set.

Now let us define random times $\tau_1, \dots, \tau_N$ by

$$\tau_i = \inf\{t > 0;\ \exp(-\int_0^t h_i(Y(s, y_0))ds) < Z_i\}, \qquad i = 1, \dots, N.$$

Then we see that

$$(\prod_{i \in I} 1_{\{\tau_i > t\}})P(\tau_i > t_i,\ i \in I | \mathcal{F}_t)$$

$$= (\prod_{i \in I} 1_{\{\tau_i > t\}})E[\exp(-\sum_{i \in I} \int_t^{t_i} h_i(Y(s, y_0)ds)|Y(t, y_0)]$$

for $t, t_1, \dots .t_N \geqq 0$ with $t < \min\{t_i;\ i \in I\}$ (c.f. [1],[2],[4]). Let

$$H(s_1, \dots, s_N, y) = E[\exp(-\sum_{i=1}^{N} \int_0^{s_i} h_i(Y(r, y)dr)] \qquad s_1, \dots, s_N \geqq 0,\ y \in \mathbf{R}^M,$$

and

$$H_i(s, y) = E[\exp(-\int_0^s h_i(Y(r, y)dr)]. \qquad i = 1, \dots, N,\ s \geqq 0,\ y \in \mathbf{R}^M.$$

Then $H_i(\cdot, y) : [0, \infty) \to (0, 1]$, $i = 1, \dots, N$, is strictly decreasing surjective function. So the inverse functions $H_i^{-1}(\cdot, y) : (0, 1] \to [0, \infty)$, $i = 1, \dots, N$, exist. Let $K : [0, 1]^N \times \mathbf{R}^M \to [0, 1]$ be given by

$$K(x_1, \dots, x_N, y) = \begin{cases} H(H_1^{-1}(\cdot, y)(x_1), \dots, H_N^{-1}(\cdot, y), y), & \text{if } x_1, \dots, x_N \in (0, 1], \\ 0, & \text{if one of } x_1, \dots, x_N = 0. \end{cases}$$

Then we have

$$(\prod_{i=1}^{N} 1_{\{\tau_i > t\}}) P(\tau_i > t_i,\ i = 1, \dots, N | \mathcal{F}_t)$$

$$= (\prod_{i=1}^{N} 1_{\{\tau_i > t\}}) K(P(\tau_1 > t_1 | \mathcal{F}_t), \dots, P(\tau_N > t_N, Y(t, y_0))\ a.s.$$

for any $t \geqq 0$, and $t_1, \dots, t_N \in [t, \infty)$.

So we see that K is an admissible family of copula functions.

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Shigeo KUSUOKA
Graduate School of Mathematical Sciences,
The University of Tokyo,
Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, Japan

Takenobu NAKASHIMA
Graduate School of Mathematical Sciences,
The University of Tokyo,
Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, Japan
Nomura Securities Company

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012