

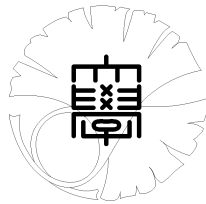
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by

M.ABE, S.HAMANO and J.NOBUCHI



UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

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Makoto Abe, Sachiko Hamano and Junjiro Noguchi

Abstract

After the solution of Cousin II problem by K. Oka III in 1939, he thought an *extra-zero problem* in 1945 (his posthumous paper) asking if it is possible to solve an arbitrarily given Cousin II problem adding some extra-zeros whose support is disjoint from the given one. By the secondly named author, some special case was affirmatively confirmed in dimension two and a counter-example in dimension three or more was given. The purpose of the present paper is to give a complete solution of this problem with examples and to discuss some new questions.

1 Introduction.

After the solution of Cousin II problem by K. Oka [8] III he thought the following *extra-zero problem* in 1945 (his posthumous paper [9], No. 2, p. 31 Problem 2; see §2):

Oka's Extra-Zero Problem. “Let X be a domain of holomorphy and let D be an effective divisor on X . Find an effective divisor E on X such that their supports have no intersection,

$$(\text{Supp } D) \cap (\text{Supp } E) = \emptyset,$$

and Cousin II problem for $D + E$ is solvable on X .”

Let $L(D)$ denote the line bundle determined by D , let $N(D) = L(D)|_{(\text{Supp } D)} \rightarrow \text{Supp } D$ be the normal bundle of D over the support $\text{Supp } D$ of D , and $\mathbf{1}_X$ denote the trivial line bundle over X . Then Cousin II problem is equivalent to ask if $L(D) \cong \mathbf{1}_X$. *Oka Principle* ([8], III) says that $L(D) \cong \mathbf{1}_X$ if and only if the the first Chern class $c_1(L(D)) = 0$ in the cohomology group $H^2(X, \mathbf{Z})$. Since the problem is trivial for D such that $L(D) \cong \mathbf{1}_X$, Oka's extra-zero problem makes sense for D with $c_1(L(D)) \neq 0$.

In [4] a counter-example was constructed in $\dim X \geq 3$, and if $\dim X = 2$, some partial affirmative answer was shown.

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The purpose of this paper is to give a complete answer to Oka's extra-zero problem with examples and some new questions based on this problem, on which we would like to put equal emphasis as well (cf. §§4 and 5). The following is our main theorem.

Theorem 1.1. *Let D be an effective Cartier divisor on a Stein space X . Then Oka's extra-zero problem is solvable if and only if $c_1(N(D)) = 0$ in $H^2(\text{Supp } D, \mathbf{Z})$.*

In particular, if $\dim X = 2$, Oka's extra-zero problem is always solvable.

The last statement is due to $H^2(\text{Supp } D, \mathbf{Z}) = 0$, since $\dim \text{Supp } D = 1$.

N.B. K. Oka [9] almost proved Theorem 1.1 (see Theorem 2.1). Referring to Oka's Theorem 2.1, one may say that Theorem 1.1 is an infinitesimalization of the topological condition from a neighborhood of D to D itself. This is not difficult now by many well-established results.

The divisor E in Oka's extra-zero problem is called an *extra-zero* of D . By definition $L(E) = L(-D)$. Thus the problem is equivalent to find a holomorphic section $\sigma \in \Gamma(X, L(-D))$ such that

$$(1.2) \quad \text{Supp}(\sigma) \cap \text{Supp } D = \emptyset.$$

Here we consider only σ whose zero set is nowhere dense in X and hence defines a divisor (σ) on X . From this viewpoint it is interesting to see

Proposition 1.3. *Let the notation be as in Theorem 1.1. Then Oka's extra-zero problem is solvable if and only if there exists a section $\tau \in \Gamma(X, L(D))$ with nowhere dense zero set and*

$$(1.4) \quad \text{Supp}(\tau) \cap \text{Supp } D = \emptyset.$$

N.B. For τ in (1.4) we required that the zero set of τ is nowhere dense in X . This is, however, not a restriction. For if τ vanishes constantly on an irreducible component X' of X , then we take a section $\tau' \in \Gamma(X, L(D))$ such that $\tau'|_{X'} \not\equiv 0$ and $\tau' \equiv 0$ on every irreducible component of X other than X' . Then $\{\tau + \tau' = 0\} \subset \{\tau = 0\}$ as sets and $\{\tau + \tau'\}|_{X'} \not\equiv 0$. In this way we may modify τ so that its zero set is nowhere dense in X . This is the same for σ in (1.2).

Acknowledgment. After the counter-example constructed by [4] which is a reducible divisor, Professor T. Ueda asked if there is an irreducible counter-example; his question forms a part of the motivation of the present paper. Professor S. Takayama gave an interesting example of §4. Professor T. Tsuboi kindly answered to the thirdly named author a number of questions on the triangulation of complex analytic subsets. The authors are very grateful to all of them.

2 Oka's notes.

Here we summarize in short the contents of the posthumous paper [9]. We should first notice that it is dated 28 February 1945 before Oka's Coherence Theorem ([8] VII).

Roughly speaking, he developed the following study.

- (i) He wished to reformulate Cousin II problem by relaxing the conclusion so that it is solvable on every domain of holomorphy.
- (ii) He recalled the Oka Principle for Cousin II problem on a domain of holomorphy, and reduced the essential key-part of the problem to the following:

Let $\bar{\Omega} \Subset \mathbf{C}^n$ be a bounded closed domain with a holomorphically convex neighborhood. Let D be divisor on a neighborhood $\bar{\Omega}$. Then the Cousin II problem for D is solvable in a neighborhood of $\bar{\Omega}$ iff $c_1(L(D)) = 0$ in a neighborhood of $\bar{\Omega}$.¹

- (iii) He then posed the Extra-Zero Problem as Problem 2. Let Ω and D (effective) be as in the above item. Then he asks to find an effective divisor E in a neighborhood of $\bar{\Omega}$ such that $\text{Supp } D \cap \text{Supp } E = \emptyset$ and Cousin II problem for $D + E$ is solvable in a neighborhood of $\bar{\Omega}$.
- (iv) He proved a result as Theorem 8 which is stated as follows:

Theorem. The extra-zero problem is solvable for D in a neighborhood of $\bar{\Omega}$ if and only if there is a neighborhood V of $\bar{\Omega}$ with $c_1(L(D)|_V) = 0$.

- (v) After confirming the above topological obstruction for the extra-zero problem, he proved that there always exists an effective divisor F in a neighborhood of $\bar{\Omega}$ such that Cousin II problem for $D + F$ is solvable. Furthermore he proved that there are at most $n + 1$ holomorphic functions $f_j, 1 \leq j \leq n + 1$, in a neighborhood of $\bar{\Omega}$ such that in a neighborhood W of every point of $D \cap \bar{\Omega}$ one of zeros of f_j is exactly $D \cap W$.

Taking account of the above items (ii) and (iv), we may assume that he obtained or at least recognized the following statement.

Theorem 2.1. (Oka [9]) *Let $\Omega \subset \mathbf{C}^n$ be a domain of holomorphy, and let D be an effective divisor on Ω . Then the extra-zero problem for D is solvable if and only if there is a neighborhood V of D satisfying $c_1(L(D)|_V) = 0$ in V .*

¹Here his term is "balayable" used in Oka [8] III; the meaning is that the given Cousin II distribution is continuously deformable to a zero-free continuous Cousin II distribution. The Cousin II problem on a domain X of holomorphy is solvable iff D is "balayable" on X .

K. Oka wrote that it strongly attracts his interest from a number of viewpoints to decide if this Extra-Zero Problem is always solvable or there is a counter-example, and the problem would have a wide influence in future.²

It is now necessary to know what is the most general form of his statement (Theorem 2.1), and it is Theorem 1.1.

3 Proofs.

(a) Proof of Theorem 1.1. Suppose first that Oka's extra-zero problem is solvable. Let E be an extra-zero of D , and let $\sigma \in \Gamma(X, L(E))$ with $(\sigma) = E$. Set

$$Y = \text{Supp } D.$$

Then the restriction $\sigma|_U$ to $U = X \setminus \text{Supp } E$ has no zero over the neighborhood U of Y . Therefore $L(-D)|_U = L(E)|_U \cong \mathbf{1}_U$, and then $N(D) \cong \mathbf{1}_Y$, so that $c_1(N(D)) = 0$.

Conversely, assume that $c_1(N(D)) = 0$. Note that $c_1(N(D)) \in H^2(Y, \mathbf{Z})$ is a restriction of $c_1(L(D)) \in H^2(X, \mathbf{Z})$. By Mihalache [5] there is a Stein neighborhood V of Y for which there is a strong deformation retract $V \rightarrow Y$. Therefore we have

$$H^2(V, \mathbf{Z}) \cong H^2(Y, \mathbf{Z}).$$

It follows that $c_1(L(D))|_V = 0$.³ Since V is a Stein space, we have that $L(D)|_V = \mathbf{1}_V$, and hence $L(-D)|_V = \mathbf{1}_V$. Thus there is a section $\sigma \in \Gamma(V, L(-D))$ without zero on V . By the Fundamental Theorem of Oka-Cartan (Oka [8] I-II, VII-VIII; Grauert-Remmert [3]) the restriction $\sigma|_Y$ extends to a holomorphic section $\tilde{\sigma} \in \Gamma(X, L(-D))$ with nowhere dense zero set. Thus the divisor $(\tilde{\sigma})$ gives rise to an extra-zero of D .

(b) Proof of Proposition 1.3. We keep the notation used in the above (a). Suppose that Oka's extra-zero problem is solvable. Then the above $\sigma \in \Gamma(X, L(E))$ has no zero on Y . Therefore, $N(D) = L(D)|_Y = L(-E)|_Y \cong \mathbf{1}_Y$. By the Fundamental Theorem of Oka-Cartan $\sigma^{-1}|_Y$ holomorphically extends to a section $\tau \in \Gamma(X, L(D))$ with nowhere dense zero set. By definition $\text{Supp } (\tau) \cap Y = \emptyset$.

Suppose the existence of $\tau \in \Gamma(X, L(D))$ with nowhere dense zero set such that $\text{Supp } (\tau) \cap Y = \emptyset$. Then the same argument implies the existence of $\sigma \in \Gamma(X, L(-D))$ with nowhere dense zero set such that $\text{Supp } (\sigma) \cap Y = \emptyset$, and hence (σ) is an extra-zero of D .

²He did not give an explicit problem here.

³The existence of such a neighborhood V for a chosen element $c_1(L(D))$ is sufficient for our argument here. For that purpose, it suffices to know the two facts: (1) There is a system of neighborhoods of Y which admit deformation retracts to Y (Whitney-Bruhat, Comment. Math. Helv. 1959); (2) There is a system of neighborhoods of Y which are Stein (Siu, Invent. Math. 1976).

4 Examples.

(a) The first solvable non-trivial example for Oka's extra-zero problem was given by [4] Theorem 1. Using a similar idea we give another example. Let $X = (\mathbf{C}^*)^2$ with $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. Then the torus $T = S^1 \times S^1 \subset X$ gives the generator of $H_2(X, \mathbf{Z}) \cong H^2(X, \mathbf{Z})$. K. Stein [10], §4 computed the divisor D on X corresponding to T , i.e., $c_1(L(D)) = T$. Let $(z, w) \in X$ be the natural coordinates. Then the analytic hypersurface given by

$$(4.1) \quad D^+ : \quad w = z^i = e^{i \log z}$$

has the first Chern class T . Stein [10] also obtained an analytic function $F^+(z, w)$ that defines D :

$$(4.2) \quad F^+(z, w) = \exp\left(\frac{(\log z)^2}{4\pi} + \frac{\log z}{1-i}\right) \prod_{\nu=0}^{\infty} \left(1 - \frac{w}{e^{i \log z + 2\nu\pi}}\right) \times \prod_{\mu=1}^{\infty} \left(1 - \frac{1}{we^{-i \log z + 2\mu\pi}}\right),$$

where we take a branch $\log 1 = 0$. Then $\langle c_1(L(D^+)), T \rangle = 1$, and so Cousin II problem for D^+ is not solvable. Let \mathcal{L}_z denote the analytic continuation as the variable z runs over the unit circle in the anti-clockwise direction. Then $\mathcal{L}_z \log z = \log z + 2\pi i$, and

$$\mathcal{L}_z F^+(z, w) = w F^+(z, w), \quad \mathcal{L}_w F^+(z, w) = F^+(z, w).$$

Set

$$D^- : \quad w = z^{-i} = e^{-i \log z},$$

$$F^-(z, w) = F^+\left(\frac{1}{z}, w\right).$$

Then $L(D^+ + D^-) \cong \mathbf{1}_X$, however $D^+ \cap D^- \neq \emptyset$. We have by (4.2) that

$$(4.3) \quad F^-(z, w) = \exp\left(\frac{(\log z)^2}{4\pi} - \frac{\log z}{1-i}\right) \prod_{\nu=0}^{\infty} \left(1 - \frac{w}{e^{-i \log z + 2\nu\pi}}\right) \times \prod_{\mu=1}^{\infty} \left(1 - \frac{1}{we^{i \log z + 2\mu\pi}}\right),$$

$$\mathcal{L}_z F^-(z, w) = \frac{1}{w} F^-(z, w), \quad \mathcal{L}_w F^-(z, w) = F^-(z, w).$$

By Theorem 1.1 there is an extra-zero E of D^+ , but it is unknown what is E . Therefore it is very interesting to ask

Question. 4.4. *Find an analytic expression of E .*

On the other hand we may give an example for Proposition 1.3. Let $\lambda \in \mathbf{C}$ such that the real part $\Re \lambda \notin 2\pi\mathbf{Z}$ and set

$$D_\lambda^+ : \quad w = e^\lambda z^i.$$

Then $D_\lambda^+ \cap D^+ = \emptyset$, $L(D_\lambda^+) = L(D^+)$, and D_λ^+ is the zero of the analytic function

$$F_\lambda^+(z, w) = F^+(z, e^{-\lambda}w).$$

Set $\Omega = \{\xi \in \mathbf{C}; |\Re \xi| < \pi, |\Im \xi| < \pi\}$. Then, it is interesting to observe that the holomorphic mapping

$$(4.5) \quad \Phi : \zeta \in \mathbf{C} \rightarrow (e^\zeta, e^{i\zeta}) \in (\mathbf{C}^*)^2 = X$$

is into-biholomorphic; this describes precisely why D^+ is “balayable” in a neighborhood of D^+ (see §2 (ii) and its footnote).

(b) Examples for Theorem 1.1 with $c_1(N(D)) \neq 0$.

(1) (Reducible divisor) A counter example in $\dim X \geq 3$ is given in [4] in a domain of \mathbf{C}^n ($n \geq 3$). Using a similar idea, we give another counter example of a divisor on $(\mathbf{C}^*)^3$ for which Oka’s extra-zero problem has no solution.

Now we let $X = (\mathbf{C}^*)^2 \times \mathbf{C}^* = (\mathbf{C}^*)^3$ with projection $p : X \rightarrow (\mathbf{C}^*)^2$. Let $D^+ \subset (\mathbf{C}^*)^2$ be as in the above (a), and set

$$(4.6) \quad \begin{aligned} D_1 &= D^+ \times \mathbf{C}^*, & D_2 &= (\mathbf{C}^*)^2 \times \{1\}, \\ D &= D_1 + D_2. \end{aligned}$$

Since $L(D_2) \cong \mathbf{1}_X$, $L(D) \cong L(D_1) \cong p^*L(D^+)$. Therefore $N(D)|_{D_2} \cong L(D^+) \not\cong \mathbf{1}_{D_2}$ with $D_2 \cong (\mathbf{C}^*)^2$, so that $N(D) \not\cong \mathbf{1}_D$. One sees that D has no extra-zero on X .

(2) (Irreducible divisor) The above example of D is reducible, and we like to have an irreducible analytic hypersurface that has no extra-zero. We are going to modify the example of (1).

Let $\mathbf{Z}[i] = \mathbf{Z} + i\mathbf{Z}$ be the lattice of Gaussian integers and put

$$\mathbf{C} \rightarrow \mathbf{C}/\mathbf{Z}[i] \cong \mathbf{C}^* \xrightarrow{\lambda_0} \mathbf{C}^*/\mathbf{Z} = E \cong \mathbf{C}/\mathbf{Z}[i].$$

Then E is an elliptic curve with complex multiplication $a \in E \rightarrow ia \in E$. Set

$$\iota : (a, b) \in E^2 \rightarrow (ia, b) \in E^2$$

and let $\Delta \subset E^2$ be the diagonal divisor. Set

$$\begin{aligned} D_1 &= \iota^* \Delta, \\ \lambda_1 &= \lambda_0 \times \lambda_0 : E^2 \rightarrow E^2, \\ \hat{D}_1 &= \lambda^* D_1 \subset (\mathbf{C}^*)^2. \end{aligned}$$

Note that the example of Stein [10], §4 ($w = z^i$ in $(\mathbf{C}^*)^2$) is a connected component \hat{D}'_1 of \hat{D}_1 . It follows that the Chern class

$$c_1(L(\hat{D}_1)) \neq 0 \quad \text{in } H^2((\mathbf{C}^*)^2, \mathbf{Z}).$$

(This is equivalent to the non-solvability of Cousin II for \hat{D}_1 , or to the non-triviality of the line bundle $L(\hat{D}_1)$ over $(\mathbf{C}^*)^2$.) In fact, letting $T = S^1 \times S^1 \in H_2((\mathbf{C}^*)^2, \mathbf{Z})$ denote the generator, we get

$$(4.7) \quad \langle c_1(L(\hat{D}_1)), T \rangle = \langle c_1(L(\hat{D}'_1)), T \rangle = 1.$$

Now we set

$$\begin{aligned} \lambda_2 : X &= (\mathbf{C}^*)^3 \times \mathbf{C}^* \rightarrow E^2 \times E \quad (\text{the quotient map}), \\ D_2 &= D_1 \times E + E^2 \times \{0\}, \\ \hat{D}_2 &= \lambda_2^* D_2. \end{aligned}$$

Then $L(\lambda_2^*(E^2 \times \{0\}))$ is trivial on X and so $L(\hat{D}_2) = L(\lambda_2^*(D_1 \times E))$, which is the pull-back of $L(\hat{D}_1)$ over $(\mathbf{C}^*)^2$ by the projection $X \rightarrow (\mathbf{C}^*)^2$. Therefore, $L(\hat{D}_2) \not\cong \mathbf{1}_X$, where $\mathbf{1}_X$ denotes the trivial line bundle over X .

Furthermore, we see that the normal bundle $N(\hat{D}_2) = L(\hat{D}_2)|_{\hat{D}_2} \rightarrow \hat{D}_2$ is non-trivial. For $N(\hat{D}_2)|_{\{1\} \times (\mathbf{C}^*)^2} \cong L(\hat{D}_1)$. Therefore we obtain

Lemma 4.8. *Let the notation be as above. Then $L(\hat{D}_2) \not\cong \mathbf{1}_X$ and $N(\hat{D}_2) \not\cong \mathbf{1}_{\hat{D}_2}$.*

N.B. This means that Cousin II problem for \hat{D}_2 on X is not solvable and there is no extra zero for \hat{D}_2 .

We would like to deform \hat{D}_2 to a smooth irreducible divisor, but this is not trivial. Thus we are going to deform D_2 on E^3 , but D_2 is not ample. To make it ample, we add the divisor $\{1\} \times E^2$ to D_2 with setting

$$D_3 = D_2 + \{1\} \times E^2,$$

which is then ample, and we put $\hat{D}_3 = \lambda_2^* D_3$ on X . Since $\lambda_2^* L(\{1\} \times E^2) = L(\lambda_2^{-1}\{1\} \times (\mathbf{C}^*)^2) \cong \mathbf{1}_X$,

$$L(\hat{D}_3) \cong L(\hat{D}_2).$$

Thus Lemma 4.8 holds for \hat{D}_3 , too:

Lemma 4.9. *Let the notation be as above. We have that $L(\hat{D}_3) \not\cong \mathbf{1}_X$ and $N(\hat{D}_3) \not\cong \mathbf{1}_{\hat{D}_3}$.*

It is well known that $L(3D_3)$ is very ample. We take a smooth irreducible hyperplane section D_4 by a holomorphic section of $L(3D_3)$, and set

$$\hat{D}_4 = \lambda_2^* D_4.$$

Proposition 4.10. (Example) *Let the notation be as above. Then \hat{D}_4 is a smooth irreducible divisor on X such that $L(\hat{D}_4) \not\cong \mathbf{1}_X$ and $N(\hat{D}_4) \not\cong \mathbf{1}_{\hat{D}_4}$; equivalently,*

$$\begin{aligned} c_1(L(\hat{D}_4)) &\neq 0 \quad \text{in } H^2(X, \mathbf{Z}), \\ c_1(N(\hat{D}_4)) &\neq 0 \quad \text{in } H^2(\hat{D}_4, \mathbf{Z}). \end{aligned}$$

Proof It is clear due to the construction that \hat{D}_4 is smooth and irreducible (or connected). Now we look at the 2-cycle T in (4.7). We regard $T = S^1 \times S^1 \times \{1\} \in H_2(X, \mathbf{Z})$. Then this cycle T comes from a 2-cycle of E^3 , which is again denoted by the same $T \in H_2(E^3, \mathbf{Z})$. Then it follows that

$$(4.11) \quad \langle c_1(L(D_4)), T \rangle = 3,$$

so that $c_1(L(\hat{D}_4)) \neq 0$.

It remains to show that $c_1(N(\hat{D}_4)) \neq 0$. By Lefschetz' hyperplane-section theorem the natural morphism

$$H_2(D_4, \mathbf{Z}) \rightarrow H_2(E^3, \mathbf{Z}) \rightarrow 0$$

is surjective, and then there is a 2-cycle $T' \in H_2(D_4, \mathbf{Z})$ which is mapped to T . Then T' can be lifted to a 2-cycle in $H_2(\hat{D}_4, \mathbf{Z})$, denoted by the same T' . We see by (4.11) that

$$\langle c_1(N(\hat{D}_4)), T' \rangle = \pm 3.$$

Thus $c_1(N(\hat{D}_4)) \neq 0$; this finishes the proof. *q.e.d.*

(3) (Takayama's irreducible example) Let $z_j = x_j + iy_j, 1 \leq j \leq n$ be the natural complex coordinates of \mathbf{C}^n with the standard basis $e_j, 1 \leq j \leq n$. Then $e_j, ie_j, 1 \leq j \leq n$ form real basis of \mathbf{C}^n and we define a lattice $\Gamma \subset \mathbf{C}^n$ defined by

$$\Gamma = \langle e_1, \dots, e_n, ie_1, \dots, ie_n \rangle.$$

We set $A = \mathbf{C}^n / \Gamma$ and a sequence of covering maps,

$$(4.12) \quad \mathbf{C}^n \xrightarrow{\rho} (\mathbf{C}^*)^n \xrightarrow{\pi} A,$$

where ρ is the quotient map by $\langle ie_1, e_2, \dots, e_n \rangle$ and π is that by $\langle e_1, ie_2, \dots, ie_n \rangle$. We set $X = (\mathbf{C}^*)^n$.

Let L be the line bundle whose Chern class is represented by

$$\omega = di \sum_{j=1}^n dz_j \wedge d\bar{z}_j + i \sum_{j \neq k} dz_j \wedge d\bar{z}_k, \quad d \in \mathbf{Z}.$$

Then L is ample for $d \geq 2$, and very ample if $d \geq 4$.

Claim 4.13. $\pi^*\omega \neq 0$ in $H^2((X, \mathbf{Z}))$; in particular, the pairing, $\omega \cdot (ie_1 \wedge e_j) \neq 0$, $j \geq 2$ where $ie_1 \wedge e_j \in H_2(A, \mathbf{Z})$.

Proof. We consider the two pull-back morphisms

$$\pi^* : H^q(A, \mathbf{Z}) \rightarrow H^q(X, \mathbf{Z}), \quad q \geq 1.$$

Then $\pi^*dx_1 = 0$, and $\pi^*dy_k = 0$, $k \geq 2$; on the other hand, $\pi^*dy_1 \neq 0$, and $\pi^*dx_k = 0$, $k \geq 2$. It follows that

$$idz_j \wedge dz_j = 2dx_j \wedge dy_j = 0 \pmod{dx_1, dy_k, k \geq 2}.$$

Therefore we have

$$\pi^*i(dz_1 \wedge d\bar{z}_j + dz_j \wedge d\bar{z}_1) = \pi^*(idy_1 \wedge d_j + dx_j \wedge (-idy_1)) = -2\pi^*(dy_1 \wedge dx_j)$$

for $j \geq 2$. □

Now we assume $n \geq 3$ and $d \geq 4$. Then L is very ample.

Proposition 4.14. (Example) *We take a smooth irreducible divisor $D \in |L|$ and set $\tilde{D} = \pi^{-1}D \subset X$. Then the divisor \tilde{D} is smooth irreducible and has no extra-zero on X .*

Proof Since $H_1(D, \mathbf{Z}) \cong H_1(A, \mathbf{Z})$ (Lefschetz' Theorem), \tilde{D} is connected. Again by Lefschetz' Theorem the natural morphism is surjective:

$$H_2(D, \mathbf{Z}) \rightarrow H_2(A, \mathbf{Z}) \rightarrow 0.$$

There is an element $\xi \in H_2(D, \mathbf{Z})$ which is mapped to $ie_1 \wedge e_j$ ($j \geq 2$).

Let $\iota : D \hookrightarrow A$ be the inclusion map and let $\tilde{\iota} : \tilde{D} \rightarrow X$ be the lifting. It follows from (4.12) that there is an element $\tilde{\xi} \in H_2(\tilde{D}, \mathbf{Z})$ with $\tilde{\iota}_*\tilde{\xi} = \xi$. Note that $c_1(L(\tilde{D})) = \pi^*\omega$. We have that

$$c_1(L(\tilde{D})) \cdot \tilde{\xi} = \omega \cdot (ie_1 \wedge e_j) \neq 0.$$

Therefore we see that $c_1(L(\tilde{D})) \neq 0$ and that $c_1(N(\tilde{D})) \neq 0$; equivalently, the smooth irreducible divisor \tilde{D} has no extra-zero on X . *q.e.d.*

5 Intersections of analytic cycles.

We would like to consider what is the intersection theory of analytic cycles on Stein manifolds.

(a) The prototype of intersection theory is Bezout's Theorem such that for two cycles A_1, A_2 on a variety

$$\deg(A_1 \cdot A_2) = \deg A_1 \cdot \deg A_2.$$

Cornalba-Shiffman [2] however gave a counter example of analytic curves C_1 and C_2 of \mathbf{C}^2 such that the "orders" of C_j are zero, but the "order" of $C_1 \cdot C_2$ is infinite.

From the viewpoint of Oka's extra-zero problem, however, we should have by Theorem 1.1

$$C_1 \cdot C_2 = 0.$$

More in general, because of Theorem 1.1 there is no global intersections of "divisors" and "curves" on Stein manifolds (nor of divisors and analytic cycles on \mathbf{C}^n ($n \geq 2$)). In fact, let D be a divisor on a Stein manifold X and let C be an analytic curve in X . Then the intersection

$$D \cdot C = \langle c_1(L(D)), C \rangle = \langle c_1(L(D)|_C), C \rangle = \langle c_1(\mathbf{1}_C), C \rangle = 0.$$

This suggests that some topological structure must be involved in the possible intersection theory on a Stein manifold. Therefore it is interesting to propose

Question. 5.1. *What is the global intersection theory of analytic cycles on Stein manifolds?*

We may consider at least three kind of intersections on a Stein manifold X :

- (i) Intersections of zeros (divisor) of holomorphic functions X :
- (ii) Intersections of hypersurfaces of X with trivial normal bundles:
- (iii) Intersections of hypersurfaces of X .

N.B. It is noticed that the normal bundle of the zeros (divisor) of a holomorphic function is trivial.

Question. 5.2. *What are the differences of these intersections? In particular, characterize the complete intersections of hypersurfaces with trivial normal bundles on a Stein manifold.*

N.B. If X is affine algebraic, there are intersections in algebraic category and there is a difference even in a simplest case as follows. Let $X \subset \mathbf{C}^2$ be an affine elliptic curve with a point at infinity, and let $a \in X$ be a point, which is an algebraic divisor. There is no regular rational function on X with exact zero a , but there exists such a holomorphic function on X .

Question. 5.3. *Let X be Stein and algebraic. Let D be an effective algebraic divisor on X with $c_1(L(D)) = 0$ (resp. $c_1(N(D)) = 0$). Does there exist a holomorphic function $f \in \mathcal{O}(X)$ with zero divisor D (resp. locally in a neighborhood of $\text{Supp } D$) such that the order of f at the infinity is at most one.*

(b) We set $X = (\mathbf{C}^*)^3$ and would like to discuss the global intersections of analytic cycles on X . As observed in (a), there is no intersection between analytic curves and divisors on X . therefore we may restrict ourselves to deal with the intersections of divisors on X . The first homology group of X is

$$H_1(X, \mathbf{Z}) \cong \mathbf{Z}^3,$$

which is generated by $e_1 = S^1 \times \{1\}^2$, $e_2 = \{1\} \times S^1 \times \{1\}$ and $e_3 = \{1\}^2 \times S^1$. Then their products generate the higher homology groups and in particular,

$$H^2(X, \mathbf{Z}) \cong H_2(X, \mathbf{Z}) \cong \mathbf{Z}^3.$$

(c) *Stein's example from the viewpoint of the value distribution theory.* Let $f : \zeta \in \mathbf{C} \rightarrow (e^\zeta, e^{i\zeta}) \in (\mathbf{C}^*)^2 = X$ be the example (4.1) due to Stein in §4. Then f is algebraically non-degenerate; that is, there is no proper algebraic subset $Y \subset X$ with $f(\mathbf{C}) \subset Y$. In fact, let $P(z, w) (\neq 0)$ be any non-zero polynomial in $(z, w) \in X$. We write

$$P(z, w) = \sum_{j,k} c_{j,k} z^j w^k.$$

Suppose that $f(\mathbf{C}) \subset \{P = 0\}$. Then

$$\sum_{j,k} c_{j,k} e^{(j+ik)\zeta} \equiv 0.$$

This is absurd, since $e^{(j+ik)\zeta}$ are linearly independent over \mathbf{C} .

According to the main result of Noguchi-Winkelmann-Yamanoi [6], [7], and Corvaja-Noguchi [1], the intersection set $f(\mathbf{C}) \cap D$ is infinite for an arbitrary algebraic divisor D on X . For an extra-zero E of $D^+ = f(\mathbf{C})$ we have

$$f(\mathbf{C}) \cap E = \emptyset.$$

Problem 5.4. *Let $g : \mathbf{C} \rightarrow X$ be an analytically non-degenerate entire curve. Then, is $g(\mathbf{C}) \cap A \neq \emptyset$ for an arbitrary analytic divisor A of X ? Moreover, is $g(\mathbf{C}) \cap A$ an infinite set?*

Here it is natural to generalize X to a semi-abelian variety.

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⁴It is now very difficult to find a complete correct record of Oka's publications without errors. The most commonly referred reference of K. Oka's works is "*Kiyoshi Oka Collected Papers, translated by R. Narasimhan, commentaries by H. Cartan, edited by R. Remmert, Springer-Verlag, (1984)*", where there are some mistakes in the records and moreover all records of the received dates were deleted. Therefore the authors think that it is meaningful and useful to provide a complete list of his publications with the received dates in one place.

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Division of Mathematical and Information Sciences
 Faculty of Integrated Arts and Sciences
 Hiroshima University
 1-7-1 Kagamiyama
 Higashi-Hiroshima, 739-8521 Japan
 e-mail: abem@hiroshima-u.ac.jp

Department of Mathematics
 Faculty of Human Development and Culture
 Fukushima University
 Kanayagawa, Fukushima, 960-1296 Japan

e-mail: hamano@educ.fukushima-u.ac.jp

Graduate School of Mathematical Sciences

The University of Tokyo

Komaba, Meguro, Tokyo, 153-8914 Japan

e-mail: noguchi@ms.u-tokyo.ac.jp

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ADDRESS:

Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012