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Global uniqueness in determining the potential for the two dimensional Schrödinger equation from cauchy data measured on disjoint subsets of the boundary

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GLOBAL UNIQUENESS IN DETERMINING THE POTENTIAL FOR THE TWO DIMENSIONAL SCHRÖDINGER EQUATION FROM CAUCHY DATA MEASURED ON DISJOINT SUBSETS OF THE BOUNDARY

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Abstract. We discuss the inverse boundary value problem of determining the potential in the two dimensional stationary Schrödinger equation from the pair of all Dirichlet data supported on an open subset Γ_+ and all the corresponding Neumann data measured on an open subset Γ_-. We prove global uniqueness, under some conditions, for the case that Γ_+ and Γ_- are disjoint. We construct appropriate complex geometrical optics solutions using Carleman estimates with a singular weight to prove the main result.

1. Introduction

We consider the problem of determining a complex-valued potential q for the Shrödinger equation $\Delta + q$ in a bounded two dimensional domain from the following boundary data. Let $\partial\Omega = \overline{\Gamma_- \cup \Gamma_+ \cup \Gamma_0}$ where $\Gamma_- \cap \Gamma_+ = \Gamma_0 \cap \Gamma_\pm = \emptyset$. The input is located on Γ_+ and the output is measured on Γ_-. It is well-known that this problem is closely related to Calderón's problem in the situation when the voltage potential is applied on the surface Γ_+ and the current is measured on the surface Γ_-.

The unique determination of the potential q in the two dimensional case initially was proved in the case of full data, $\Gamma_+ = \Gamma_- = \partial\Omega$ under some restrictions on the potential q, [8], [9], [10]. Recently A. Bukhgeim [3] removed these restrictions for the case of full Cauchy data.

For the case of partial data, in [6] the authors showed that the potential q can be uniquely determined if $\Gamma_+ = \Gamma_- = \tilde\Gamma$ and $\tilde\Gamma$ is an arbitrary fixed open set on $\partial\Omega$.

The main result of this manuscript is the unique identifiability of the potential q under some geometric conditions on the sets $\Gamma_\pm$. To the best of our knowledge this is the first unique determination result for Calderón's problem when the voltage is applied and the current is measured on disjoint surfaces.

In a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ we consider the Shrödinger equation

$$\Delta u + qu = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0\cup\Gamma_-} = 0.$$

Let Γ_+, Γ_- and $\Gamma_0 \subset \partial\Omega$ be non-empty open subsets of the boundary such that $\partial\Omega = \overline{\Gamma_+ \cup \Gamma_- \cup \Gamma_0}$, $\overline{\Gamma}_+ \cap \overline{\Gamma_-} = \emptyset$, $\Gamma_\pm = \cup_{j=1}^2 \Gamma_{\pm,j}$, where sets $\Gamma_{\pm,j}$ are open in $\partial\Omega$.

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We introduce the following set of Cauchy data:

$$\mathcal{C}_q = \left\{ \left(u|_{\Gamma_+}, \frac{\partial u}{\partial \nu}\Big|_{\Gamma_-} \right) \mid (\Delta + q)u = 0 \,\text{in}\, \Omega,\ u|_{\Gamma_0\cup\Gamma_-} = 0,\ u \in H^1(\Omega) \right\}. \tag{1.1}$$

We need the following geometric assumption on the position of the sets $\Gamma_{\pm,j}$ on $\partial\Omega$.

Assumption A. *If we are starting the clockwise movement from some point of the set $\Gamma_{\pm,j}$ before arriving to another component $\Gamma_{\pm,k}$ we have to pass through a component $\Gamma_{\mp,\ell}$.*

Our main result gives global uniqueness by measuring the Cauchy data with input on Γ_+ and output on Γ_-.

Theorem 1.1. *Let $q_j \in C^{2+\alpha}(\overline{\Omega})$, $j = 1, 2$ for some $\alpha > 0$ and let q_j be complex-valued. Then if*

$$\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$$

we have

$$q_1 \equiv q_2.$$

Now we apply the above result to Calderón's problem. A bounded and positive function $\gamma(x)$ models the electrical conductivity of Ω. Then a potential $u \in H^1(\Omega)$ satisfies the Dirichlet problem

$$\begin{aligned} \operatorname{div}(\gamma\nabla u) &= 0 \text{ in } \Omega, \\ u\big|_{\Gamma_0\cup\Gamma_-} &= f, \end{aligned} \tag{1.2}$$

where $f \in H^{\frac{1}{2}}(\partial\Omega)$ is a given boundary voltage potential. The Cauchy data is defined by

$$\mathcal{A}_\gamma = \{(u|_{\Gamma_+}, \frac{\partial u}{\partial \nu}\Big|_{\Gamma_-})|\operatorname{div}(\gamma\nabla u) = 0 \text{ in } \Omega, \quad u\big|_{\Gamma_0\cup\Gamma_-} = 0, \quad u|_{\Gamma_+} = f\}. \tag{1.3}$$

Theorem 1.2. *Let $\gamma_j \in C^{4+\alpha}(\overline{\Omega})$, $j = 1, 2$ for some $\alpha > 0$ and let γ_j be positive functions. Assume $\mathcal{A}_{\gamma_1} = \mathcal{A}_{\gamma_2}$. Then $\gamma_1 \equiv \gamma_2$.*

Uniqueness for C^2 conductivities for the case when $\Gamma_0 = \Gamma_\pm = \partial\Omega$ was proved in [8]. The regularity condition was relaxed in [2] and [1]. In particular in [1] uniqueness was shown for arbitrary L^∞ conductivities. For the case of partial data when $\Gamma_+ = \Gamma_-$ is an arbitrary open subset of the boundary global uniqueness was shown in [6] for $C^{3+\epsilon}(\overline{\Omega})$ conductivities.

A brief outline of the paper is as follows.In section 2 we give some preliminary results and estimates needed in the construction of the appropriate complex geometrical optics solutions. In section 3 we construct these solutions. In section 4 we prove the main result.

2. **Preliminary results**

Throughout the paper we use the following notations.

Notations. $i = \sqrt{-1}$, $x_1, x_2, \xi_1, \xi_2 \in \mathbb{R}^1$, $z = x_1 + ix_2$, $\zeta = \xi_1 + i\xi_2$, $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$. $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$. The tangential derivative on the boundary is given by $\partial_{\vec{\tau}} = \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}$, with $\nu = (\nu_1, \nu_2)$ the unit outer normal to $\partial\Omega$, $B(\widehat{x}, \delta) = \{x \in \mathbb{R}^2 | |x - \widehat{x}| < \delta\}$, $f : \mathbb{R}^2 \to \mathbb{R}^1$,

f'' is the Hessian matrix with entries $\frac{\partial^2 f}{\partial x_k \partial x_j}$, $\mathcal{L}(X,Y)$ denotes the Banach space of all bounded linear operators from a Banach space X to another Banach space Y.

Let $\Phi(z) = \varphi(x_1,x_2) + i\psi(x_1,x_2) \in C^2(\overline{\Omega})$ be a holomorphic function in Ω with real-valued φ and ψ:

$$\partial_{\overline{z}}\Phi(z) = 0 \quad \text{in } \Omega. \tag{2.1}$$

Denote by $\mathcal{H}$ the set of critical points of the function Φ

$$\mathcal{H} = \{z \in \overline{\Omega} | \partial_z \Phi(z) = 0\}.$$

Assume that Φ has no critical points on $\overline{\Gamma_+} \cup \overline{\Gamma_-}$, and that all the critical points are nondegenerate:

$$\mathcal{H} \cap \partial\Omega = \{\emptyset\}, \quad \partial_z^2 \Phi(z) \neq 0, \quad \forall z \in \mathcal{H}. \tag{2.2}$$

Then we know that Φ has only a finite number of critical points which

$$\mathcal{H} = \{\widetilde{x}_1, ..., \widetilde{x}_\ell\}.$$

Assume that Φ satisfies

$$\Gamma_0 = \{x \in \partial\Omega | (\nu, \nabla\varphi) = 0\}, \quad \Gamma_- = \{x \in \partial\Omega | (\nu, \nabla\varphi) < 0\}. \tag{2.3}$$

Consider the boundary value problem

$$\begin{cases} L(x,D)u = \Delta u + qu = f \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

For this problem we have the following Carleman estimate with boundary terms.

Proposition 2.1. *Suppose that Φ satisfies (2.1), (2.2), (2.3) $u \in H_0^1(\Omega)$ and $q \in L^\infty(\Omega)$. Then there exist $\tau_0 = \tau_0(L,\Phi)$ and $C_1 = C_1(L,\Phi)$ independent of u and τ such that for all $|\tau| > \tau_0$*

$$\begin{aligned} &|\tau| \|ue^{\tau\varphi}\|^2_{L^2(\Omega)} + \|ue^{\tau\varphi}\|^2_{H^1(\Omega)} + \|\frac{\partial u}{\partial \nu} e^{\tau\varphi}\|^2_{L^2(\Gamma_0 \cup \Gamma_-)} + \tau^2 \| |\frac{\partial \Phi}{\partial z}| ue^{\tau\varphi}\|^2_{L^2(\Omega)} \\ &\leq C_1 (\|(L(x,D)u)e^{\tau\varphi}\|^2_{L^2(\Omega)} + |\tau| \int_{\Gamma_+} |\frac{\partial u}{\partial \nu}|^2 e^{2\tau\varphi} d\sigma). \end{aligned} \tag{2.4}$$

Let us introduce the operators:

$$\partial_{\overline{z}}^{-1} g = \frac{1}{2\pi i} \int_\Omega \frac{g(\zeta,\overline{\zeta})}{\zeta - z} d\zeta \wedge d\overline{\zeta} = -\frac{1}{\pi} \int_\Omega \frac{g(\zeta,\overline{\zeta})}{\zeta - z} d\xi_2 d\xi_1,$$

$$\partial_z^{-1} g = -\frac{1}{2\pi i} \overline{\int_\Omega \frac{\overline{g}(\zeta,\overline{\zeta})}{\zeta - z} d\zeta \wedge d\overline{\zeta}} = -\frac{1}{\pi} \int_\Omega \frac{g(\zeta,\overline{\zeta})}{\overline{\zeta} - \overline{z}} d\xi_2 d\xi_1 = \overline{\partial_{\overline{z}}^{-1} \overline{g}}.$$

Then we have (e.g., p.47 and p.56 in [11]):

Proposition 2.2. A) *Let $m \geq 0$ be an integer number and $\alpha \in (0,1)$. Then $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(C^{m+\alpha}(\overline{\Omega}), C^{m+\alpha+1}(\overline{\Omega}))$.*
B) *Let $1 \leq p \leq 2$ and $1 < \gamma < \frac{2p}{2-p}$. Then $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(L^p(\Omega), L^\gamma(\Omega))$.*

We define two other operators:

$$R_{\Phi,\tau}g = e^{\tau(\overline{\Phi}-\Phi)}\partial_{\overline{z}}^{-1}(ge^{\tau(\Phi-\overline{\Phi})}), \quad \widetilde{R}_{\Phi,\tau}g = e^{\tau(\overline{\Phi}-\Phi)}\partial_{z}^{-1}(ge^{\tau(\Phi-\overline{\Phi})}).$$

In [7] we prove the following

Proposition 2.3. *Let $g \in C^\alpha(\overline{\Omega})$ for some positive α. The function $R_{\Phi,\tau}g$ is a solution to*

$$\partial_{\overline{z}}R_{\Phi,\tau}g - \tau(\overline{\partial_z\Phi})R_{\Phi,\tau}g = g \quad in\ \Omega. \tag{2.5}$$

The function $\widetilde{R}_{\Phi,\tau}g$ solves

$$\partial_z\widetilde{R}_{\Phi,\tau}g + \tau(\partial_z\Phi)\widetilde{R}_{\Phi,\tau}g = g \quad in\ \Omega. \tag{2.6}$$

Using the stationary phase argument we show

Proposition 2.4. *Let $g \in L^1(\Omega)$ and the function Φ satisfy (2.1), (2.2). Then*

$$lim_{|\tau|\to+\infty}\int_\Omega ge^{\tau(\Phi(z)-\overline{\Phi(z)})}dx = 0.$$

Denote

$$\mathcal{O}_\epsilon = \{x \in \Omega | dist(x, \partial\Omega) \le \epsilon\}.$$

We have

Proposition 2.5. *Let $\alpha > 0$, $g \in C^{2+\alpha}(\Omega), g|_{\mathcal{O}_\epsilon} = 0$ and $g|_{\mathcal{H}} = 0$. Then*

$$\left\|R_{\Phi,\tau}g + \frac{g}{\tau\overline{\partial_z\Phi}}\right\|_{L^2(\Omega)} + \left\|\widetilde{R}_{\Phi,\tau}g - \frac{g}{\tau\partial_z\Phi}\right\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad as\ |\tau| \to +\infty. \tag{2.7}$$

Consider the following problem

$$L(x, D)u = fe^{\tau\varphi} \quad \text{in } \Omega, \quad u|_{\Gamma_0\cup\Gamma_-} = ge^{\tau\varphi}. \tag{2.8}$$

We have

Proposition 2.6. *(see [7]) Let $q \in L^\infty(\Omega)$. There exists $\tau_0 > 0$ such that for all $\tau > \tau_0$ there exists a solution to the boundary value problem (2.8) such that*

$$\frac{1}{\sqrt{|\tau|}}\|\nabla ue^{-\tau\varphi}\|_{L^2(\Omega)} + \sqrt{|\tau|}\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \le C_2(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2},\tau}(\Gamma_0)}). \tag{2.9}$$

Let ϵ be a sufficiently small positive number. If $suppf \subset G_\epsilon = \{x \in \Omega | dist(x, \mathcal{H}) > \epsilon\}$ and $g = 0$ then there exists $\tau_0 > 0$ such that for all $\tau > \tau_0$ there exists a solution to the boundary value problem (2.8) such that

$$\|\nabla ue^{-\tau\varphi}\|_{L^2(\Omega)} + |\tau|\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \le C_3(\epsilon)\|f\|_{L^2(\Omega)}. \tag{2.10}$$

We have

Proposition 2.7. *Let $q \in L^\infty(\Omega)$ and let $supp\, g \subset \Gamma_-$ and $g/\sqrt{|\partial_\nu\varphi|} \in L^2(\Gamma_-)$. Then there exists $\tau_0 > 0$ such that for all $\tau > \tau_0$ there exist a solution to (2.8) such that*

$$\sqrt{|\tau|}\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \le C_4\|g/\sqrt{|\partial_\nu\varphi|}\|_{L^2(\Gamma_-)}.$$

3. **Complex geometrical optics solutions**

In this section, we construct complex geometrical optics solutions for the Schrödinger equation $\Delta + q_j$ with q_j satisfying the conditions of Theorem 1.1. Consider

$$L_1(x, D)u = \Delta u + q_1 u = 0 \quad \text{in } \Omega. \tag{3.1}$$

We will construct solutions to (3.1) of the form

(3.2)
$$u_1(x) = e^{\tau\Phi(z)}(a(z)+a_0(z)/\tau)+e^{\tau\overline{\Phi(z)}}\overline{(a(z)+a_1(z)/\tau)}+e^{\tau\varphi}u_-+e^{\tau\varphi}u_{11}+e^{\tau\varphi}u_{12}, \quad u_1|_{\Gamma_0\cup\Gamma_-} = 0.$$

Thanks to assumption A, the set Γ_0 consists of four arcs : $\Gamma_0 = \Gamma_{0,1} \cup \Gamma_{0,2} \cup \Gamma_{0,3} \cup \Gamma_{0,4}$, the set Γ_- consists of two arcs $\Gamma_- = \Gamma_{-,1} \cup \Gamma_{-,2}$ and the set Γ_+ also consists of two arcs $\Gamma_+ = \Gamma_{+,1} \cup \Gamma_{+,2}$. Denote the endpoints of the arc $\Gamma_{0,j}$ as $\hat{x}_{j,\pm}$.

Proposition 3.1. *Let $\tilde{x} \in \Omega$ be an arbitrary point. There exists a smooth holomorphic function a in Ω such that*

$$a(\tilde{x}) \neq 0, \quad \mathrm{Re}\, a|_{\Gamma_0} = 0, \quad \nabla^k a(\hat{x}_{j,\pm}) = 0 \quad \forall k \in \{1, \dots, 100\}, \quad \forall j \in \{0, \dots, 4\}.$$

Proof. Consider the following linear operator

$$\mathcal{R}(v) = (w(\tilde{x}), w(\hat{x}_{j,\pm}), \dots, \partial_z^{100} w(\hat{x}_{j,\pm})),$$

where

$$\partial_{\bar{z}} w = 0 \quad \text{in } \Omega, \quad \mathrm{Re}\, w = v \quad \text{on } \partial\Omega, \quad \mathit{supp}\, v \subset \Gamma_+.$$

Clearly image of the operator $\mathcal{R}$ is closed. Let $b(x)$ be a smooth holomorphic function in Ω such that $b(\tilde{x}) = 1$ and $\mathrm{Re}\, b|_{\Gamma_0\cup\Gamma_-} = 0$. By Proposition 5.1 there exists a sequence of holomorphic functions $\{w_k\}_{k=1}^\infty \subset C^{100+\alpha}(\bar{\Omega})$ such that

$$w_k \to 0 + i\mathrm{Im}\, b \quad \text{in } C^{100+\alpha}(\Gamma_0 \cup \Gamma_-) \quad \text{and} \quad w_k(\tilde{x}) \to 0.$$

Using the classical results on solvability of the Cauchy-Riemann equations we construct the sequence of holomorphic functions $\tilde{w}_k$ such that

$$\tilde{w}_k \to 0 \quad \text{in } C^{100+\alpha}(\bar{\Omega}), \quad \mathrm{Re}\, \tilde{w}_k = \mathrm{Re}\, w_k \quad \text{on } \Gamma_0 \cup \Gamma_-.$$

Consider the sequence $v_k = b + (\tilde{w}_k - w_k)$. We have $\mathcal{R}(v_k) \to (1, 0, \dots, 0)$. The proof of proposition finished. □

Without the loss of generality, using some conformal mapping if necessary, we may assume that Γ_- and Γ_+ belong to the line $\{x_2 = 0\}$ and domain Ω itself is located below the line $x_2 = 0$.

We construct the holomorphic function Φ with domain Ω_Φ, such that $\Omega \subset \Omega_\Phi$, satisfies (2.1), (2.2) and

$$\mathrm{Im}\, \Phi|_{\Gamma_0} = 0, \quad \frac{\partial \mathrm{Re}\, \Phi}{\partial \nu}|_{\Gamma_-} < 0, \quad \frac{\partial \mathrm{Re}\, \Phi}{\partial \nu}|_{\Gamma_+} > 0. \tag{3.3}$$

The domain Ω_Φ has the following properties:

$$\Omega \subset \Omega_\Phi, \quad \Gamma_0 \subset \partial\Omega_\Phi, \quad (\Gamma_+ \cup \Gamma_-) \cap \partial\Omega_\Phi = \emptyset, \quad \partial\Omega_\Phi \in C^{10}. \tag{3.4}$$

Therefore, thanks to Assumption A, the set $\partial\Omega_\Phi \setminus \partial\Omega$ consists of four disconnected curves which we denote as $\Gamma_{\Phi,1}, \Gamma_{\Phi,2}, \Gamma_{\Phi,3}, \Gamma_{\Phi,4}$. Counting the clockwise assume that $\Gamma_{\Phi,1}$ located between $\Gamma_{0,1}$ and $\Gamma_{0,2}$, $\Gamma_{\Phi,2}$ located between $\Gamma_{\Phi,2}$ located between $\Gamma_{0,2}$ and $\Gamma_{0,3}$, $\Gamma_{\Phi,3}$ located between $\Gamma_{0,3}$ and $\Gamma_{0,4}$, $\Gamma_{\Phi,4}$ located between $\Gamma_{0,4}$ and $\Gamma_{0,1}$. Assume in addition that each component $\Gamma_{\Phi,k}$ can be parameterize by the function $\tilde\gamma_k \in C^{12}[\hat x_{k,+}, \hat x_{k+1,-}]$, where $\hat x_{k,+}, \hat x_{k+1,-}$ are the endpoints of the arcs $\Gamma_{0,k}$ and $\hat x_{5,-} = \hat x_{1,-}$. Let us start the construction of the function Φ. Consider functions γ_j with domain $\mathbb{R}^1$ such that γ_j is positive on $(\hat x_{j,+}, \hat x_{j+1,-})$, otherwise γ_j is zero,

$$\frac{d^k\gamma_j}{dt^k}(\hat x_{j,+}) = \frac{d^k\gamma_j}{dt^k}(\hat x_{j+1,-}) = 0 \quad \forall k \in \{0,\dots,10\}, \quad \frac{d^{11}\gamma_j}{dt^{11}}(\hat x_{j,+}) \ne 0, \quad \frac{d^{11}\gamma_j}{dt^{11}}(\hat x_{j+1,-}) \ne 0,$$

there exists some small positive $\hat\epsilon$ such that

(3.5)

$$\gamma_j(x_1) = (x_1 - \hat x_{j,+})^{11} \quad \forall x_1 \in (\hat x_{j,+}, \hat x_{j,+} + \hat\epsilon), \quad \gamma_j(x_1) = (\hat x_{j+1,-} - x_1)^{11} \quad \forall x_1 \in (\hat x_{j+1,-} - \hat\epsilon, \hat x_{j+1,-}).$$

We introduce the domain Ω_δ for any small positive δ as follows. From below it is bounded by the boundary of $\partial\Omega$ form above by segments $\Gamma_{0,k}$ and the graphs of functions $\delta\gamma_j$.

By ν_δ we denote the outward unit normal derivative to $\partial\Omega_\delta$ and by $\vec\tau_\delta$ we denote the clockwise unit tangent derivative to $\partial\Omega_\delta$. We set

$$\Gamma_{\delta,k} = \{(x_1, \delta\gamma_k(x_1)) | x_1 \in [x_{k,+}, x_{k+1,-}]\}.$$

Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ be rational positive numbers

$$\mathcal{C}_k = \frac{m_k}{n_k} \quad m_k, n_k \in \mathbb{Z} \tag{3.6}$$

and $\tilde\psi$ be harmonic function in Ω, continuous on $\bar\Omega$ such that

$$\begin{cases} \tilde\psi = \mathcal{C}_1 \quad \text{on } \Gamma_{0,1}, \quad \tilde\psi = \mathcal{C}_3 \quad \text{on } \Gamma_{0,3};\\ \tilde\psi = -\mathcal{C}_2 \quad \text{on } \Gamma_{0,2}, \quad \tilde\psi = -\mathcal{C}_4 \quad \text{on } \Gamma_{0,4};\\ \partial_{\vec\tau}\tilde\psi < 0 \quad \text{on } (\hat x_{1,+}, \hat x_{2,-}) \cup (\hat x_{3,+}, \hat x_{4,-});\\ \partial_{\vec\tau}\tilde\psi > 0 \quad \text{on } (\hat x_{2,+}, \hat x_{3,-}) \cup (\hat x_{4,+}, \hat x_{1,-});\\ \tilde\psi \in C^5(\partial\Omega), \quad \tilde\psi \in C^\infty(\partial\Omega \setminus \cup_{k=1}^4 \Gamma_{0,k}). \end{cases} \tag{3.7}$$

Moreover we assume that

$$\begin{array}{l} \lim_{x_1 \to \hat x_{k,+}+0} \partial_{x_1}\tilde\psi(x_1,0)/(\hat x_{k,+} - x_1)^6 < 0 \;\; k = 1,3,\\ \lim_{x_1 \to \hat x_{k,+}+0} \partial_{x_1}\tilde\psi(x_1,0)/(\hat x_{k,+} - x_1)^6 > 0 \;\; k = 2,4,\\ \lim_{x_1 \to \hat x_{k,-}+0} \partial_{x_1}\tilde\psi(x_1,0)/(\hat x_{k,-} - x_1)^6 < 0 \;\; k = 2,4,\\ \lim_{x_1 \to \hat x_{k,-}-0} \partial_{x_1}\tilde\psi(x_1,0)/(\hat x_{k,-} - x_1)^6 > 0 \;\; k = 1,3. \end{array}$$

Let function ψ_δ be the harmonic function in Ω_δ such that for any $j \in \{1,\dots,4\}$

$$\psi_\delta = \tilde\psi \quad \text{on} \cup_{k=1}^4 \Gamma_{0,k}, \quad \tilde\psi_\delta(x_1, \delta\gamma_j(x_1)) = \tilde\psi(x_1,0) \quad \text{on } [\hat x_{j,+}, \hat x_{j+1,-}]. \tag{3.8}$$

For all sufficiently small δ, counting clockwise, between $\Gamma_{0,1}$ and $\Gamma_{0,2}$ function ψ_δ is monotone decreasing, also it is monotone decreasing on the arc between $\Gamma_{0,3}$ and $\Gamma_{0,4}$. This function is monotone increasing on the arcs between $\Gamma_{0,2}$ and $\Gamma_{0,3}$ and also on ark between $\Gamma_{0,4}$ and $\Gamma_{0,1}$. Once the function ψ_δ is constructed, using the Cauchy-Riemann equation, we construct the function φ_δ such that the function $\varphi_\delta + i\psi_\delta$ is holomorphic. The following inequalities are true for all sufficiently small positive δ

$$\frac{\partial \varphi_\delta}{\partial \nu_\delta}|_{\Gamma_{\delta,1}\cup\Gamma_{\delta,3}} < 0, \quad \frac{\partial \varphi_\delta}{\partial \nu_\delta}|_{\Gamma_{\delta,2}\cup\Gamma_{\delta,4}} > 0, \tag{3.9}$$

$$\lim{}_{x_1\to \hat x_{k,+}+0}\frac{\partial \varphi_\delta}{\partial \nu_\delta}(x_1,\delta\gamma_k(x_1))/(\hat x_{k,+}-x_1)^6 > 0 \; k=1,3, \tag{3.10}$$
$$\lim{}_{x_1\to \hat x_{k,+}-0}\frac{\partial \varphi_\delta}{\partial \nu_\delta}(x_1,\delta\gamma_{k-1}(x_1))/(\hat x_{k,-}-x_1)^6 > 0 \; k=2,4.$$

$$\lim{}_{x_1\to \hat x_{k,+}+0}\frac{\partial \varphi_\delta}{\partial \nu_\delta}(x_1,\delta\gamma_{k+1}(x_1))/(\hat x_{k,+}-x_1)^6 < 0 \; k=2,4, \tag{3.11}$$
$$\lim{}_{x_1\to \hat x_{k,+}-0}\frac{\partial \varphi_\delta}{\partial \nu_\delta}(x_1,\delta\gamma_{5-k}(x_1))/(\hat x_{k,-}-x_1)^6 < 0 \; k=1,3.$$

At the endpoints of $\Gamma_{0,2}$ and $\Gamma_{0,4}$ function ψ_δ reach its minimum and at endpoints of $\Gamma_{0,1}$ and $\Gamma_{0,3}$ function ψ_δ reach its maximum. By (3.8) we have

$$(\varphi_\delta,\psi_\delta)\to(\tilde\varphi,\tilde\psi) \quad \text{in } C^2(\bar\Omega) \quad \text{as } \delta\to+0. \tag{3.12}$$

Here $\tilde\varphi$ is a harmonic function in Ω such that $\partial_{\bar z}(\tilde\varphi + i\tilde\psi)\equiv 0$.

By (3.9)-(3.11) for all sufficiently small positive δ the holomorphic function $\varphi_\delta + i\psi_\delta$ satisfies (2.3).

Consider the domain $\mathcal{G}_- = \{(x_1,x_2)|\hat x_{2,+}\le x_1\le \hat x_{3,-}, -\delta\gamma_2(x_1)\le x_2\le 0\}\cup\{(x_1,x_2)|\hat x_{4,+}\le x_1\le \hat x_{1,-}, -\delta\gamma_4(x_1)\le x_2\le 0\}$. We claim that there exist a positive constant C_δ such that

$$\varphi_\delta(x)-\varphi_\delta(x_1,-x_2)\ge C_\delta\ell(x) \quad \forall x\in\mathcal{G}_-, \; \frac{\partial\varphi_\delta}{\partial x_2}(x)\le -C_\delta\ell_1(x) \quad \forall x\in\mathcal{G}_-, \tag{3.13}$$

where $\ell(x) = \min_{y\in\{\hat x_{2,+},\hat x_{4,+}\,\hat x_{3,-},\hat x_{1,-}\}}|x_1-y|^7|x_2|, \ell_1(x) = \min_{y\in\{\hat x_{2,+},\hat x_{4,+}\,\hat x_{3,-},\hat x_{1,-}\}}|x_1-y|^6$. Indeed, suppose that the second inequality in (3.13) fails for all small positive δ. By (3.9), (3.12) this is possible only for a sequence of the points x_δ such that it converges to the set $\mathcal{D}_- = \{x_{1,+},x_{2,-},x_{3,+},x_{4,-}\}$. Taking if it is necessary a subsequence we may assume that x_δ converges to the single point of the set $\mathcal{D}_-$. Let it be the point $\hat x_{2,+}$. By the Cauchy-Riemann equations $\frac{\partial\varphi_\delta}{\partial\nu_\delta} = -\frac{\partial\psi_\delta}{\partial\vec\tau_\delta}$ for any point of $\partial\Omega_\delta$. So by (3.10) there exists a positive constants $\hat C$ and ϵ independent of δ such that

$$\frac{\partial\varphi_\delta}{\partial\nu_\delta}\le -\hat C(x_1-\hat x_{1,+})^6 \quad \text{on } \{x|x\in\Gamma_{\delta,2}, \text{dist}(\hat x_{2,+},x)<\epsilon\}.$$

Taking into account that by (3.5) $\vec\nu_\delta = (8(x_1-\hat x_{2,+})^7,1)/(1+64(x_1-\hat x_{2,+})^{14})^{\frac12}$ we obtain

$$\frac{\partial\varphi_\delta}{\partial x_2}(x)\le -\frac{\hat C}{2}(x_1-\hat x_{2,+})^6 \quad \forall x\in\{(x_1,x_2)|x_1\in[\hat x_{2,+},\hat x_{2,+}+\epsilon], x_2=\delta\gamma_2(x_1)\}.$$

Using (3.5), (3.12) and the Taylor's formula for any $x \in \{(x_1, x_2)|x_1 \in [\hat x_{2,+}, \hat x_{2,+}+\epsilon], -\delta\gamma_2(x_1) \le x_2 \le \delta\gamma_2(x_1)\}$ we have

$$\frac{\partial\varphi_\delta}{\partial x_2}(x_1,x_2) = \frac{\partial\varphi_\delta}{\partial x_2}(x_1,\delta\gamma_2(x_1)) + \frac{\partial^2\varphi_\delta}{\partial x_2^2}(x_1,\zeta)(x_2-\delta\gamma_2(x_1)) \le -\frac{\hat C}{2}(x_1-\hat x_{2,+})^6 + 2C_5\gamma_2(x_1) =$$

$$-\frac{\hat C}{4}(x_1-\hat x_{2,+})^6 + 2C(x_1-\hat x_{2,+})^{11} \le -\frac{C_6}{4}(x_1-\hat x_{2,+})^6. \tag{3.14}$$

So we finish the proof of second inequality in (3.13)

Let $x \in \mathcal{G}_-$. Using (3.14) we have

$$\varphi_\delta(x) - \varphi_\delta(x_1,-x_2) \le \varphi_\delta(x_1,0) - \varphi_\delta(x) =$$

$$\int_{x_2}^0 \partial_\xi\varphi_\delta(x_1,\xi)d\xi \le -\frac{C_6}{2}\int_{x_2}^0 (x_1-\hat x_{2,+})^6 d\xi = -\frac{C_6}{2}(x_1-\hat x_{2,+})^6 x_2. \tag{3.15}$$

The proof of (3.13) is finished.

Consider the domain $\mathcal{G}_+ = \{(x_1,x_2)|\hat x_{2,+} \le x_1 \le \hat x_{3,-}, -\delta\gamma_2(x_1) \le x_2 \le 0\} \cup \{(x_1,x_2)|\hat x_{4,+} \le x_1 \le \hat x_{1,-}, -\delta\gamma_4(x_1) \le x_2 \le 0\}$. similarly one can prove that all sufficiently small positive δ there exist a positive constant $\tilde C_\delta$ such that for any x from $\mathcal{G}_+$

$$\varphi_\delta(x) - \varphi_\delta(x_1,-x_2) \le -\tilde C_\delta \tilde\ell(x) \quad \text{and} \quad \frac{\partial\varphi_\delta}{\partial x_2}(x) \ge \tilde C_\delta \tilde\ell_1(x), \tag{3.16}$$

where $\tilde\ell(x) = \min_{y\in\{\hat x_{2,+},\hat x_{4,+},\hat x_{1,-},\hat x_{3,-}\}} |x_1-y|^7|x_2|, \tilde\ell_1(x) = \min_{y\in\{\hat x_{2,+},\hat x_{4,+},\hat x_{1,-},\hat x_{3,-}\}} |x_1-y|^6$. At this point we fix the parameter δ such that inequalities (3.13), (3.16) hold true. The holomorphic function $\varphi_\delta + i\psi_\delta$ satisfies (2.3), all internal critical points if they exist are nondegenerate. This function might have some critical points in the set $\{\hat x_{j,\pm}, j = 1,\dots,4\}$. Let tangential derivative of ψ_δ will not be equal to zero on some open set $\tilde\Gamma$. By Corollary 5.1 there exists a harmonic function $\hat\varphi + i\hat\psi$ such that $\operatorname{Im}\hat\psi = 0$ on $\partial\Omega_\delta$ and $\frac{\partial\hat\varphi}{\partial\vec\tau}|_{\hat x_{j,\pm}}$ not equal to zero for all j. Then the function $\varphi_\delta + \epsilon\hat\varphi + i(\psi_\delta + \hat\psi)$ does not have critical points on the set $\{\hat x_{j,\pm}, j = 1,\dots,4\}$ for all small positive ϵ. In fact this function can not have more then one internal critical point. Indeed it is known (see e.g. [12]) that if $\hat x$ is the internal critical point of the harmonic function ψ the set $\{x \in \partial\Omega|\psi(x) = \psi(\hat x)\}$ consists of at least four points. Moreover the set $\{x|\psi(x) = \psi(\hat x)\}$ consists of two continuous curves intersecting at $\hat x$. These curves divide domain Ω in four domains $\Omega = \cup_{k=1}^4\Omega_k$. If there exists another internal critical point $\hat x_1$ it should belong to some domain Ω_k. But in this case the condition that there exist four different points x_j from $\partial\Omega_k$ such that $\psi(\hat x_1) = \psi(x_j)$ obviously fails. Construction of of the weight function Φ is complete. If an internal critical point of Φ exists we denote it as $\tilde x$.

The amplitude function $a(z)$ is not identically zero on $\overline\Omega$ and has the following properties:

$$a \in C^6(\overline\Omega_\Phi), \quad \partial_{\overline z} a \equiv 0, \ \operatorname{Re} a|_{\Gamma_0} = 0, \ |a(x)| \le C_7|x-\hat x_{j,\pm}|^{100} \quad \forall j \in \{1,\dots,4\}. \tag{3.17}$$

Such a function can be constructed in the following way: Using the C^4 conformal mapping Π we map the domain Ω_Φ into bounded domain $\mathcal{O}$ with $\partial\mathcal{O} \in C^\infty$. Applying the Proposition 3.1 we construct the holomorphic function $\mathcal{A} \in C^{120}(\overline{\mathcal{O}})$ such that $\operatorname{Re}\mathcal{A}|_{\Pi(\Gamma_0)} = 0$ and $\partial_z\mathcal{A}(\hat x_{j,\pm}) = 0$ for any k less or equal 100. Then we set $a(x) = \mathcal{A}\circ\Pi$.

Let the polynomials $M_1(z)$ and $M_3(\bar z)$ satisfy

$$\partial_z^j(\partial_{\bar z}^{-1}(aq_1)-M_1)(x)=0,\quad x\in\mathcal{H}\cup\{\hat x_{k,\pm},k=1,\dots,4\},\ j=0,1,2, \tag{3.18}$$

$$\partial_{\bar z}^j(\partial_z^{-1}(\overline{a}q_1)-M_3)(x)=0,\quad x\in\mathcal{H}\cup\{\hat x_{k,\pm},k=1,\dots,4\},\ j=0,1,2, \tag{3.19}$$

and

$$\partial_z^k M_1(\hat x_{j,\pm})=\partial_{\bar z}^k M_3(\hat x_{j,\pm})=0\quad\forall k\in\{3,\dots,100\}\ \text{and}\ \forall j\in\{1,\dots,4\}. \tag{3.20}$$

By (3.18)-(3.20) and (3.17) we have

$$|\partial_{\bar z}^{-1}(aq_1)-M_1(z)|\le C_8|x-\hat x_{k,\pm}|^{100},\ \ |\partial_z^{-1}(\overline{a}q_1)-M_3(\overline{z})|\le C_9|x-\hat x_{k,\pm}|^{100}\quad\forall k\in\{1,\dots,4\}. \tag{3.21}$$

Finally $a_0,a_1\in C^6(\bar\Omega_\Phi)$ are the holomorphic functions such that

$$(a_0+\overline{a_1})|_{\Gamma_0}=\frac{(\partial_{\bar z}^{-1}(aq_1)-M_1)}{4\partial_z\Phi}+\frac{(\partial_z^{-1}(\overline{a}q_1)-M_3)}{4\overline{\partial_z\Phi}},$$

and there exists a positive constant C such that

$$|a_k(x)|\le C_9|x-\hat x_{j,\pm}|^3\ \ \forall j\in\{1,\dots,4\},\ \forall k\in\{0,1\}. \tag{3.22}$$

We introduce the function $u_-(\tau,\cdot)$ by formula

$$e^{\tau\varphi}u_-(\tau,x)=-\chi_\tau(e^{\tau\overline{\Psi}}\overline{\hat a}+e^{\tau\tilde\Psi}\hat a)+w_\tau(x)e^{\tau\varphi}, \tag{3.23}$$

where $\Psi(z)$ is the holomorphic function

$$\Psi(z)=\varphi(x_1,-x_2)-i\psi(x_1,-x_2)\quad x\in\mathcal{G}_-\cup\mathcal{G}_+. \tag{3.24}$$

In order to construct w_τ we introduce the following functions

$$\hat a(x_1,x_2)=\mathrm{Re}\,a(x_1,-x_2)-i\mathrm{Im}\,a(x_1,-x_2)\quad x\in\mathcal{G}_-\cup\mathcal{G}_+, \tag{3.25}$$

and

$$\hat a_k(x_1,x_2)=\mathrm{Re}\,a_k(x_1,-x_2)-i\mathrm{Im}\,a_k(x_1,-x_2)\quad x\in\mathcal{G}_-\cup\mathcal{G}_+,\ k\in\{0,1\}. \tag{3.26}$$

The function χ_τ is constructed in the following way. Let $\mu\in C_0^\infty(-2,2)$ and $\mu|_{[-1,1]}=1$. We set

$$\chi_\tau(x)=\begin{cases}(1-\mu((x_1-\hat x_{2,+})\tau^{\frac{1}{80}})-\mu((x_1-\hat x_{3,-})\tau^{\frac{1}{80}}))\mu(x_2\tau^{\frac17})\\ \text{for}\quad x\in\mathcal{V}_1=\{(x_1,x_2)|\hat x_{2,+}\le x_1\le\hat x_{3,-},-\delta\gamma_2(x_1)\le x_2\le 0\},\\ (1-\mu((x_1-\hat x_{4,+})\tau^{\frac{1}{80}})-\mu((x_1-\hat x_{1,-})\tau^{\frac{1}{80}}))\mu(x_2\tau^{\frac17})\\ \text{for}\quad x\in\mathcal{V}_2=\{(x_1,x_2)|\hat x_{4,+}\le x_1\le\hat x_{1,-},-\delta\gamma_4(x_1)\le x_2\le 0\},\\ 0\quad\text{for}\quad x\notin\mathcal{V}_1\cup\mathcal{V}_2.\end{cases} \tag{3.27}$$

For all sufficiently large τ

$$\mathrm{supp}\,\chi_\tau\cap\Omega\subset\mathcal{G}_-. \tag{3.28}$$

Let function w_τ be solution to the following boundary value problem:

$$\begin{aligned}\Delta(w_\tau e^{\tau\varphi})+q_1(w_\tau e^{\tau\varphi})=r_\tau&=\chi_\tau q_1(e^{\tau\overline{\Psi}}\overline{(\hat a+\hat a_1/\tau)}+e^{\tau\Psi}(\hat a+\hat a_0/\tau))\\ &+[\chi_\tau,\Delta](e^{\tau\overline{\Psi}}\overline{(\hat a+\hat a_1/\tau)}+e^{\tau\Psi}(\hat a+\hat a_0/\tau))\quad\text{in }\Omega,\end{aligned} \tag{3.29}$$

$$(w_\tau e^{\tau\varphi})|_{\Gamma_0\cup\Gamma_-} = 0. \tag{3.30}$$

Denote $g_\tau = [\chi_\tau, \Delta](e^{\tau\overline{\Psi}}\overline{(\hat a + \hat a_1/\tau)} + e^{\tau\Psi}(\hat a + \hat a_0/\tau))$. We claim that

$$\|g_\tau e^{-\tau\varphi}\|_{L^2(\Omega)} = O(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty. \tag{3.31}$$

Indeed the operator $[\chi_\tau, \Delta]$ is the first order operator :$[\chi_\tau, \Delta] = 2(\nabla\chi_\tau, \nabla) + \Delta\chi_\tau$ where

$$\|\nabla\chi_\tau\|_{L^\infty(\Omega)} = O(\tau^{\frac{1}{10}}), \quad \|\Delta\chi_\tau\|_{L^\infty(\Omega)} = O(\tau^{\frac{1}{5}}) \text{ as } |\tau| \to +\infty. \tag{3.32}$$

By (3.27) there exists τ_0 such that for all $\tau \ge \tau_0$ we have

$$\text{supp}\,\Delta\chi_\tau,\ \text{supp}\nabla\chi_\tau \subset \mathcal{I}_1(\tau)\cup\mathcal{I}_2(\tau),$$

where

$$\mathcal{I}_1(\tau) = \{(x_1,x_2)|\frac{1}{\tau^{\frac{1}{7}}} \le x_2 \le \frac{2}{\tau^{\frac{1}{7}}},\ x_1 \in [\hat x_{2,+} + \frac{2}{\tau^{\frac{1}{80}}}, \hat x_{3,-} - \frac{2}{\tau^{\frac{1}{80}}}] \cup [\hat x_{4,+} + \frac{2}{\tau^{\frac{1}{80}}}, \hat x_{1,-} - \frac{2}{\tau^{\frac{1}{80}}}]\},$$

$$\mathcal{I}_2(\tau) = \{(x_1,x_2)|0 \le x_2 \le \frac{2}{\tau^{\frac{1}{7}}},\ x_1 \in [\hat x_{2,+} + \frac{1}{\tau^{\frac{1}{80}}}, \hat x_{2,+} + \frac{2}{\tau^{\frac{1}{80}}}] \cup [\hat x_{3,-} - \frac{2}{\tau^{\frac{1}{80}}}, \hat x_{3,-} - \frac{1}{\tau^{\frac{1}{80}}}]$$
$$\cup[\hat x_{4,+} + \frac{1}{\tau^{\frac{1}{80}}}, \hat x_{4,+} + \frac{2}{\tau^{\frac{1}{80}}}] \cup [\hat x_{1,-} - \frac{2}{\tau^{\frac{1}{80}}}, \hat x_{1,-} - \frac{1}{\tau^{\frac{1}{80}}}]\}.$$

Observe that

$$\mathcal{I}_1(\tau)\cup\mathcal{I}_2(\tau) \subset \mathcal{G}_+. \tag{3.33}$$

Applying (3.17), (3.12) (3.32), (3.33) we have

$$\begin{aligned}
&\|e^{-\tau\varphi}[\chi_\tau, \Delta](e^{\tau\overline{\Psi}}\overline{(\hat a + \hat a_1/\tau)} + e^{\tau\Psi}(\hat a + \hat a_0/\tau))\|_{L^\infty(\mathcal{I}_1)} \\
&\le \|e^{-\tau\varphi}\Delta\chi_\tau(e^{\tau\overline{\Psi}}\overline{(\hat a + \hat a_1/\tau)} + e^{\tau\Psi}(\hat a + \hat a_0/\tau))\|_{L^\infty(\mathcal{I}_1)} \\
&+2\|e^{-\tau\varphi}(e^{\tau\overline{\Psi}}(\nabla\chi_\tau, \nabla)\overline{(\hat a + \hat a_1/\tau)} + e^{\tau\Psi}(\nabla\chi_\tau, \nabla)(\hat a + \hat a_0/\tau))\|_{L^\infty(\mathcal{I}_1)} \\
&+2\|\tau e^{-\tau\varphi}(e^{\tau\overline{\Psi}}(\nabla\chi_\tau, \nabla\overline{\Psi})\overline{(\hat a + \hat a_1/\tau)} + e^{\tau\Psi}(\nabla\chi_\tau, \nabla\Psi)(\hat a + \hat a_0/\tau))\|_{L^\infty(\mathcal{I}_1)} \\
&\le |\tau|^3 \sup_{x\in\mathcal{I}_1(\tau)} e^{-\tau\varphi+\tau\mathrm{Re}\Psi} \le |\tau|^3 \sup_{x\in\mathcal{I}_1(\tau)} e^{-\tau\tilde C_\delta \ell(x)} \le |\tau|^3 e^{-\tau\tilde C_\delta \tau^{\frac{-7}{80}}\tau^{\frac{-1}{7}}} = O(\frac{1}{\tau^2}) \quad \text{as } \tau \to +\infty.
\end{aligned} \tag{3.34}$$

Using (3.17), (3.22), (3.32) we have

$$\begin{aligned}
&\|e^{-\tau\varphi}[\chi_\tau, \Delta](e^{\tau\overline{\Psi}}\overline{(\hat a + \hat a_1/\tau)} + e^{\tau\Psi}(\hat a + \hat a_0/\tau))\|_{L^\infty(\mathcal{I}_2)} \\
&\le \|e^{-\tau\varphi}\Delta\chi_\tau(e^{\tau\overline{\Psi}}\overline{(\hat a + \hat a_1/\tau)} + e^{\tau\Psi}(\hat a + \hat a_0/\tau))\|_{L^\infty(\mathcal{I}_2)} \\
&+2\|e^{-\tau\varphi}(e^{\tau\overline{\Psi}}(\nabla\chi_\tau, \nabla)\overline{(\hat a + \hat a_1/\tau)} + e^{\tau\Psi}(\nabla\chi_\tau, \nabla)(\hat a + \hat a_0/\tau))\|_{L^\infty(\mathcal{I}_2)} \\
&+2\|\tau e^{-\tau\varphi}(e^{\tau\overline{\Psi}}(\nabla\chi_\tau, \nabla\overline{\Psi})\overline{(\hat a + \hat a_1/\tau)} + e^{\tau\Psi}(\nabla\chi_\tau, \nabla\Psi)(\hat a + \hat a_0/\tau))\|_{L^\infty(\mathcal{I}_2)} \\
&\le \|\Delta\chi_\tau(\overline{(\hat a + \hat a_1/\tau)} + (\hat a + \hat a_0/\tau))\|_{L^\infty(\mathcal{I}_2)} \\
&+2\|(\nabla\chi_\tau, \nabla)\overline{(\hat a + \hat a_1/\tau)} + (\nabla\chi_\tau, \nabla)(\hat a + \hat a_0/\tau)\|_{L^\infty(\mathcal{I}_2)} \\
&+2\|\tau((\nabla\chi_\tau, \nabla\overline{\Psi})\overline{(\hat a + \hat a_1/\tau)} + (\nabla\chi_\tau, \nabla\Psi)(\hat a + \hat a_0/\tau))\|_{L^\infty(\mathcal{I}_2)} = O(\frac{1}{\tau^2}) \quad \text{as } \tau \to +\infty.
\end{aligned} \tag{3.35}$$

The inequality (3.34) and (3.35) implies (3.31) immediately. By (3.13) and (3.24)

$$\|e^{-\tau\varphi}\chi_\tau q_1(e^{\tau\overline{\Psi}}\overline{(\hat a+\hat a_1/\tau)}+e^{\tau\Psi}(\hat a+\hat a_0/\tau))\|_{L^2(\Omega)}=o(1)\quad \text{as } \tau\to+\infty. \tag{3.36}$$

Using (3.31), (3.36) and the fact that supp $\chi_\tau\cap\mathcal{H}=\emptyset$ we can apply the Proposition 2.6 to obtain the solution to the boundary value problem (3.29), (3.30) such that

$$\|w_\tau\|_{L^2(\Omega)}=o(\frac{1}{\tau})\quad \text{as}\quad \tau\to+\infty. \tag{3.37}$$

The function u_{11} is given by

$$u_{11}=-\frac14 e^{i\tau\psi}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1)-M_1))-\frac14 e^{-i\tau\psi}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1)-M_3)) \tag{3.38}$$
$$-\frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1)-M_1)}{4\partial_z\Phi}-\frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a}q_1)-M_3)}{4\overline{\partial_z\Phi}},$$

where functions $e_1,e_2\in C^\infty(\Omega)$ are constructed so that

$$e_1+e_2\equiv 1 \text{ on } \overline{\Omega}, e_2 \text{ vanishes in some neighborhood of } \mathcal{H} \text{ and } e_1 \text{ vanishes in a neighborhood of } \partial\Omega. \tag{3.39}$$

Let u_{12} be solution to the inhomogeneous problem

$$\Delta(u_{12}e^{\tau\varphi})+q_1u_{12}e^{\tau\varphi}=-q_1u_{11}e^{\tau\varphi}+h_1e^{\tau\varphi}\quad \text{in } \Omega, \tag{3.40}$$

$$u_{12}=d_{1,\tau}+d_{2,\tau}+d_{3,\tau}\quad \text{on } \Gamma_0\cup\Gamma_-, \tag{3.41}$$

where

$$h_1=e^{\tau i\psi}\Delta\left(\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1)-M_1)}{4\tau\partial_z\Phi}\right)+e^{-\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a}q_1)-M_3)}{4\tau\overline{\partial_z\Phi}}\right) -a_0q_1e^{i\tau\psi}/\tau-\overline{a_1}q_1e^{-i\tau\psi}/\tau, \tag{3.42}$$

and $d_{1,\tau}=(\frac{e^{i\tau\psi}}{4}\tilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1)-M_1))+\frac{e^{-i\tau\psi}}{4}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1)-M_3)))$, $d_{2,\tau}=\chi_{\Gamma_-}(1-\chi_\tau)\text{Re}\,\{e^{i\tau\psi}a\}$, $d_{3,\tau}=\frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1)-M_1)}{4\partial_z\Phi}+\frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a}q_1)-M_3)}{4\overline{\partial_z\Phi}}-\frac{a_0e^{\tau i\psi}+\overline{a_1}e^{-\tau i\psi}}{\tau}$.

By (3.17), (3.22) there exists a constant C_{10} independent of τ such that

$$\left\|d_{3,\tau}/\sqrt{|\frac{\partial\varphi}{\partial\nu}|}\right\|_{L^2(\Gamma_-)}\le\frac{C_{10}}{|\tau|}. \tag{3.43}$$

So applying the Proposition 2.7 we obtain solution for the initial value problem $L_1(x,D)(e^{\tau\varphi}u_{12,I})=0, u_{12,I}|_{\Gamma_0}=0, u_{12,I}|_{\Gamma_-}=d_{3,\tau}$ which satisfies the estimate

$$\|u_{12,I}\|_{L^2(\Omega)}=o(\frac{1}{\tau})\quad \text{as}\quad \tau\to+\infty. \tag{3.44}$$

Since

$$\|q_1u_{11}+h_1\|_{L^2(\Omega)}\le C_{11}(\delta)/|\tau|^{1-\delta}\quad \forall\delta\in(0,1)$$

and by the stationary phase argument $\|d_{1,\tau}\|_{L^2(\Gamma_0\cup\Gamma_-)} = O(\frac{1}{\tau^2})$ there exists a solution to the initial value problem $L_1(x,D)(e^{\tau\varphi}u_{12,II}) = 0, u_{12,II}|_{\Gamma_0\cup\Gamma_-} = d_{1,\tau}$ which satisfies the estimate

$$\|u_{12,II}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad \tau \to +\infty. \tag{3.45}$$

Finally, by (3.17) $\|d_{1,\tau}\|_{L^2(\Gamma_0\cup\Gamma_-)} = O(\frac{1}{\tau^2})$. So applying the Proposition 2.6 we obtain solution to the initial value problem $L_1(x,D)(e^{\tau\varphi}u_{12,III}) = 0, u_{12,III}|_{\Gamma_0\cup\Gamma_-} = d_{2,\tau}$ which satisfies the estimate

$$\|u_{12,III}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad \tau \to +\infty. \tag{3.46}$$

Setting $u_{12} = u_{12,III} + u_{12,II} + u_{12,I}$ we obtain solution to (3.40), (3.41) satisfying

$$\|u_{12}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty. \tag{3.47}$$

Now consider the sequence of τ_j such that

$$\tau_j = 2\pi j n_1 n_2 n_3 n_4. \tag{3.48}$$

For each τ_j from this sequence our solution u_1 satisfies the zero Dirichlet boundary condition on $\Gamma_0 \cup \Gamma_-$.

Consider now the Schrödinger equation

$$L_2(x,D)v = \Delta v + q_2 v = 0 \quad \text{in } \Omega. \tag{3.49}$$

We will construct solutions to (3.49) of the form

$$v(x) = e^{-\tau\Phi}(a + b_0/\tau) + e^{-\tau\overline{\Phi}}\overline{(a + b_1/\tau)} + e^{-\tau\varphi}v_{11} + e^{-\tau\varphi}v_{12}, \quad v|_{\Gamma_0} = 0. \tag{3.50}$$

The construction of v repeats the corresponding steps of the construction of u_1. The only difference is that instead of q_1 and τ, we use q_2 and $-\tau$ respectively. Let polynomials $M_2(z), M_4(\bar z)$ be such that

$$\partial_z^j(\partial_{\bar z}^{-1}(aq_1) - M_2)(x) = 0, \quad x \in \mathcal{H} \cup \{\hat x_{k,\pm}, k = 1,\dots,4\},\ j = 0,1,2, \tag{3.51}$$

$$\partial_{\bar z}^j(\partial_z^{-1}(\overline{a}q_1) - M_4)(x) = 0, \quad x \in \mathcal{H} \cup \{\hat x_{k,\pm}, k = 1,\dots,4\},\ j = 0,1,2, \tag{3.52}$$

and

$$\partial_z^k M_2(\hat x_{j,\pm}) = \partial_{\bar z}^k M_4(\hat x_{j,\pm}) = 0 \quad \forall k \in \{3,\dots,100\} \text{ and } \forall j \in \{1,\dots,4\}. \tag{3.53}$$

Finally b_0, b_1 are holomorphic functions such that

$$(b_0 + \overline{b}_1)|_{\Gamma_0} = -\frac{(\partial_{\bar z}^{-1}(aq_2) - M_2)}{4\partial_z\Phi} - \frac{(\partial_z^{-1}(\overline{a}q_2) - M_4)}{4\overline{\partial_z\Phi}}.$$

and there exists a positive constant C_{12} such that

$$|b_k(x)| \le C_{12}|x - \hat x_{j,\pm}|^3 \ \forall j \in \{1,\dots,4\},\ \forall k \in \{0,1\}. \tag{3.54}$$

Let

$$\hat b_j(x_1,x_2) = \mathrm{Re}\, b_j(x_1,-x_2) - i\mathrm{Im}\, b_j(x_1,-x_2) \quad \forall x \in \mathcal{G}_+,\ j \in \{0,2\}. \tag{3.55}$$

We set

$$e^{-\tau\varphi}v_+(\tau,x) = -\tilde\chi_\tau(e^{-\tau\overline{\Psi}}\overline{(a+\hat b_0/\tau)} + e^{-\tau\Psi}(a+\hat b_1/\tau)) + \tilde w_\tau(x)e^{-\tau\varphi}. \tag{3.56}$$

The function $\tilde\chi_\tau$ is constructed in the following way. Then we set

$$\tilde\chi_\tau(x) = \begin{cases} (1-\mu((x_1-\hat x_{1,+})\tau^{\frac{1}{80}}) - \mu((x_1-\hat x_{2,-})\tau^{\frac{1}{80}}))\mu(x_2\tau^{\frac17}) \\ \text{for} \quad x\in\mathcal{V}_3 = \{(x_1,x_2)|\hat x_{1,+}\le x_1\le \hat x_{2,-}, -\delta\gamma_1(x_1)\le x_2\le 0\}, \\ (1-\mu((x_1-\hat x_{3,+})\tau^{\frac{1}{80}}) - \mu((x_1-\hat x_{4,-})\tau^{\frac{1}{80}}))\mu(x_2\tau^{\frac17}) \\ \text{for} \quad x\in\mathcal{V}_4 = \{(x_1,x_2)|\hat x_{3,+}\le x_1\le \hat x_{4,-}, -\delta\gamma_3(x_1)\le x_2\le 0\}, \\ 0 \quad \text{for} \quad x\notin \mathcal{V}_3\cup\mathcal{V}_4. \end{cases} \tag{3.57}$$

Let function $\tilde w_\tau$ be solution to the following boundary value problem:

$$\begin{aligned} \Delta(\tilde w_\tau e^{-\tau\varphi}) + q_2(\tilde w_\tau e^{-\tau\varphi}) &= \tilde\chi_\tau q_2(e^{-\tau\overline{\Psi}}\overline{(a+\hat b_0/\tau)} + e^{-\tau\Psi}(a+\hat b_1/\tau)) \\ &+[\tilde\chi_\tau,\Delta](e^{-\tau\overline{\Psi}}\overline{(a+\hat b_0/\tau)} + e^{-\tau\Psi}(a+\hat b_1/\tau)) \quad \text{in } \Omega, \end{aligned} \tag{3.58}$$

$$\begin{aligned} (\tilde w_\tau e^{-\tau\varphi})|_{\Gamma_0\cup\Gamma_-} &= -e^{-\tau\Phi}(a+\hat b_0/\tau) + e^{-\tau\overline{\Phi}}\ \overline{(a+\hat b_1/\tau)} \\ &+\tilde\chi_\tau(e^{-\tau\overline{\Psi}}\overline{(a+\hat b_0/\tau)} + e^{-\tau\Psi}(a+\hat b_1/\tau)). \end{aligned} \tag{3.59}$$

Denote $\tilde g_\tau = [\tilde\chi_\tau,\Delta](e^{-\tau\overline{\Psi}}\overline{(a+\hat b_0/\tau)} + e^{-\tau\Psi}(a+\hat b_1/\tau))$. We claim that

$$\|\tilde g_\tau e^{\tau\varphi}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty. \tag{3.60}$$

Indeed the operator $[\tilde\chi_\tau,\Delta]$ is the first order operator :$[\tilde\chi_\tau,\Delta] = 2(\nabla\tilde\chi_\tau,\nabla)+\Delta\tilde\chi_\tau$ where

$$\|\nabla\tilde\chi_\tau\|_{L^\infty(\Omega)} = O(\tau^{\frac{1}{10}}), \quad \|\Delta\tilde\chi_\tau\|_{L^\infty(\Omega)} = O(\tau^{\frac15}) \text{ as } |\tau|\to+\infty. \tag{3.61}$$

By (3.57) we have

$$\text{supp}\,\nabla\tilde\chi_\tau,\ \text{supp}\,\Delta\tilde\chi_\tau \subset \tilde{\mathcal{I}}_1(\tau)\cup\tilde{\mathcal{I}}_2(\tau),$$

where

$$\tilde{\mathcal{I}}_1(\tau) = \{(x_1,x_2)|\frac{1}{\tau^{\frac17}}\le x_2\le\frac{2}{\tau^{\frac17}},\ x_1\in[\hat x_{1,+}+\frac{2}{\tau^{\frac{1}{80}}}, \hat x_{2,-}-\frac{2}{\tau^{\frac{1}{80}}}]\cup[\hat x_{3,+}+\frac{2}{\tau^{\frac{1}{80}}}, \hat x_{4,-}-\frac{2}{\tau^{\frac{1}{80}}}]\},$$

$$\begin{aligned} \tilde{\mathcal{I}}_2(\tau) = \{(x_1,x_2)|0\le x_2\le\frac{2}{\tau^{\frac17}},\ x_1\in[\hat x_{1,+}+\frac{1}{\tau^{\frac{1}{80}}}, \hat x_{1,+}+\frac{2}{\tau^{\frac{1}{80}}}]\cup[\hat x_{2,-}-\frac{2}{\tau^{\frac{1}{80}}}, \hat x_{2,-}-\frac{1}{\tau^{\frac{1}{80}}}] \\ \cup[\hat x_{3,+}+\frac{1}{\tau^{\frac{1}{80}}}, \hat x_{3,+}+\frac{2}{\tau^{\frac{1}{80}}}]\cup[\hat x_{4,-}-\frac{2}{\tau^{\frac{1}{80}}}, \hat x_{4,-}-\frac{1}{\tau^{\frac{1}{80}}}]\}. \end{aligned}$$

Observe that

$$\tilde{\mathcal{I}}_1(\tau)\cup\tilde{\mathcal{I}}_2(\tau)\subset\Gamma_-. \tag{3.62}$$

Applying (3.17), (3.32), (3.62) we have

$$\begin{aligned}(3.63)\qquad &\|e^{\tau\varphi}[\tilde\chi_\tau,\Delta](e^{-\tau\overline{\Psi}}\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\hat a+\hat b_1/\tau))\|_{L^\infty(\tilde{\mathcal I}_1)}\\
&\le \|e^{\tau\varphi}\Delta\tilde\chi_\tau(e^{-\tau\overline{\Psi}}\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\hat a+\hat b_1/\tau))\|_{L^\infty(\tilde{\mathcal I}_1)}\\
&+2\|e^{\tau\varphi}(e^{-\tau\overline{\Psi}}(\nabla\tilde\chi_\tau,\nabla)\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\nabla\tilde\chi_\tau,\nabla)(\hat a+\hat b_1/\tau))\|_{L^\infty(\tilde{\mathcal I}_1)}\\
&+2\|\tau e^{\tau\varphi}(e^{-\tau\overline{\Psi}}(\nabla\tilde\chi_\tau,\nabla\overline{\Psi})\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\nabla\tilde\chi_\tau,\nabla\Psi)(\hat a+\hat b_1/\tau))\|_{L^\infty(\tilde{\mathcal I}_1)}\\
&\le |\tau|^3\sup_{x\in\tilde{\mathcal I}_1(\tau)} e^{\tau\varphi-\tau\mathrm{Re}\Psi}\le |\tau|^3\sup_{x\in\tilde{\mathcal I}_1(\tau)} e^{-\tau\tilde C_\delta\tilde\ell(x)}\le |\tau|^3 e^{-\tau\tau^{\frac{-7}{80}}\tilde C_\delta\tau^{\frac{-1}{7}}}=O(\frac{1}{\tau^2})\quad\text{as } |\tau|\to+\infty.\end{aligned}$$

Using (3.17), (3.22), (3.32) we have

$$\begin{aligned}(3.64)\qquad &\|e^{\tau\varphi}[\tilde\chi_\tau,\Delta](e^{\tau\overline{\Psi}}\overline{(\hat a+\hat b_0/\tau)}+e^{\tau\Psi}(\hat a+\hat b_1/\tau)\|_{L^\infty(\tilde{\mathcal I}_2)}\\
&\le \|e^{\tau\varphi}\Delta\tilde\chi_\tau(e^{-\tau\overline{\Psi}}\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\hat a+\hat b_1/\tau))\|_{L^\infty(\tilde{\mathcal I}_2)}\\
&+2\|e^{\tau\varphi}(e^{-\tau\overline{\Psi}}(\nabla\tilde\chi_\tau,\nabla)\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\nabla\tilde\chi_\tau,\nabla)(\hat a+\hat b_1/\tau))\|_{L^\infty(\tilde{\mathcal I}_2)}\\
&+2\|\tau e^{\tau\varphi}(e^{-\tau\overline{\Psi}}(\nabla\tilde\chi_\tau,\nabla\overline{\Psi})\overline{(\hat a+\hat b_0/\tau)}+e^{-\tau\Psi}(\nabla\tilde\chi_\tau,\nabla\Psi)(\hat a+\hat b_1/\tau))\|_{L^\infty(\tilde{\mathcal I}_2)}\\
&\le \|\Delta\tilde\chi_\tau(\overline{(\hat a+\hat b_0/\tau)}+(\hat a+\hat b_1/\tau))\|_{L^\infty(\tilde{\mathcal I}_2)}\\
&+2\|(\nabla\tilde\chi_\tau,\nabla)\overline{(\hat a+\hat b_0/\tau)}+(\nabla\tilde\chi_\tau,\nabla)(\hat a+\hat b_1/\tau)\|_{L^\infty(\tilde{\mathcal I}_2)}\\
&+2\|\tau((\nabla\tilde\chi_\tau,\nabla\overline{\Psi})\overline{(\hat a+\hat b_0/\tau)}+(\nabla\tilde\chi_\tau,\nabla\Psi)(\hat a+\hat b_1/\tau))\|_{L^\infty(\tilde{\mathcal I}_2)}=O(\frac{1}{\tau^2})\quad\text{as } |\tau|\to+\infty.\end{aligned}$$

The inequality (3.63) and (3.64) implies (3.60) immediately. Using (3.31), (3.36) and the fact that $\mathrm{supp}\,\chi_\tau\cap\mathcal H=\emptyset$ we can apply the Proposition 2.6 to obtain the solution to the boundary value problem (3.29),(3.30) such that

$$(3.65)\qquad \|w_\tau\|_{L^2(\Omega)}=o(\frac{1}{\tau})\quad\text{as}\quad |\tau|\to+\infty.$$

The function v_{11} is given by

$$\begin{aligned}(3.66)\qquad v_{11}=&-\frac14 e^{-i\tau\psi}\widetilde R_{\Phi,-\tau}(e_1(\partial_{\overline z}^{-1}(q_2a)-M_2))-\frac14 e^{i\tau\psi}R_{\Phi,\tau}(e_1(\partial_z^{-1}(q_2\overline a)-M_4))\\
&+\frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline z}^{-1}(aq_2)-M_2)}{4\partial_z\Phi}+\frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline aq_2)-M_4)}{4\overline{\partial_z\Phi}}.\end{aligned}$$

Denote

$$\begin{aligned}h_2=&e^{-\tau i\psi}\Delta\left(\frac{e_2(\partial_{\overline z}^{-1}(aq_2)-M_2)}{4\tau\partial_z\Phi}\right)+e^{\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline aq_2)-M_4)}{4\tau\overline{\partial_z\Phi}}\right)\\
&-\frac{b_0}{\tau}q_2e^{-i\tau\psi}-\frac{\overline{b_1}}{\tau}q_2e^{i\tau\psi}.\end{aligned}$$

The function v_{12} is a solution to the problem:

$$(3.67)\qquad \Delta(v_{12}e^{-\tau\varphi})+q_2v_{12}e^{-\tau\varphi}=-q_2v_{11}e^{-\tau\varphi}-h_2e^{-\tau\varphi}\quad\text{in }\Omega,$$

$$v_{12}|_{\Gamma_0\cup\Gamma_+} = \tilde d_{1,\tau} + \tilde d_{2,\tau} + \tilde d_{3,\tau}, \tag{3.68}$$

where $\tilde d_{1,\tau} = \frac{e^{i\tau\psi}}{4}\tilde R_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_2) - M_2)) + \frac{e^{-i\tau\psi}}{4}R_{\Phi,\tau}(e_1(\partial_z^{-1}(\overline{a}q_2) - M_4))$, $\tilde d_{2,\tau} = \chi_{\Gamma_+}(1 - \chi_\tau)\text{Re}\,\{e^{-\tau i\psi}a\}$, $\tilde d_{3,\tau} = \frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_2)-M_2)}{4\partial_z\Phi} + \frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a}q_2)-M_4)}{4\partial_z\Phi} - \frac{b_0e^{-\tau i\psi}+\overline{b_1}e^{\tau i\psi}}{\tau}$.

By (3.17), (3.22) there exists a constant C independent of τ such that

$$\left\|\tilde d_{3,\tau}/\sqrt{\frac{\partial\varphi}{\partial\nu}}\right\|_{L^2(\Gamma_+)} \le \frac{C_{13}}{|\tau|}. \tag{3.69}$$

Applying the Proposition 2.7 we obtain the solution to the initial value problem $L_2(x,D)(e^{-\tau\varphi}v_{12,I}) = 0, v_{12,I}|_{\Gamma_0} = 0, v_{12,I}|_{\Gamma_+} = \tilde d_{3,\tau}$ which satisfies the estimate

$$\|v_{12,I}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad \tau\to+\infty. \tag{3.70}$$

Since

$$\|q_2v_{11}+h_2\|_{L^2(\Omega)} \le C_{14}(\delta)/|\tau|^{1-\delta} \quad \forall\delta\in(0,1)$$

and by the stationary phase argument $\|\tilde d_{1,\tau}\|_{L^2(\Gamma_0\cup\Gamma_+)} = O(\frac{1}{\tau^2})$ there exists a solution to the initial value problem $L_2(x,D)(e^{-\tau\varphi}v_{12,II}) = 0, v_{12,II}|_{\Gamma_0\cup\Gamma_+} = \tilde d_{1,\tau}$ which satisfies the estimate

$$\|v_{12,II}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad \tau\to+\infty. \tag{3.71}$$

Finally, by (3.17) $\|\tilde d_{1,\tau}\|_{L^2(\Gamma_0\cup\Gamma_+)} = O(\frac{1}{\tau^2})$. So applying the Proposition 2.6 we obtain solution to the initial value problem $L_2(x,D)(e^{-\tau\varphi}v_{12,III}) = 0, v_{12,III}|_{\Gamma_0\cup\Gamma_+} = \tilde d_{2,\tau}$ which satisfies the estimate

$$\|v_{12,III}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \quad \tau\to+\infty. \tag{3.72}$$

Setting $v_{12} = v_{12,III} + v_{12,II} + v_{12,I}$ we obtain solution to (3.40), (3.41) satisfying

$$\|v_{12}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } \tau\to+\infty. \tag{3.73}$$

For each τ_j defined by (4.19) this sequence our solution v satisfies the zero Dirichlet boundary condition on $\Gamma_0\cup\Gamma_-$.

4. **Proof of the theorem.**

Proposition 4.1. *Let function Ψ determined in (3.24) and holomorphic function Φ constructed in Section 3 has an internal critical point $\tilde x$. Then for any potentials $q_1, q_2 \in C^{2+\alpha}(\overline{\Omega})$, $\alpha > 0$ with the same Cauchy data and for any holomorphic function a satisfying (3.17) and $M_1(z), M_2(z), M_3(\overline{z}), M_4(\overline{z})$ as in Section 3, then*

$$2\frac{\pi(q|a|^2)(\widetilde x)Re\,e^{2i\tau_j Im\Phi(\widetilde x)}}{|(det\,Im\Phi'')(\widetilde x)|^{\frac12}} + \int_\Omega q(a(a_0+b_0)+\overline{a}(\overline{a}_1+\overline{b}_1))dx \tag{4.1}$$

$$+\ \frac14\int_\Omega\left(qa\frac{\partial_{\overline{z}}^{-1}(aq_2)-M_2}{\partial_z\Phi}+q\overline{a}\frac{\partial_z^{-1}(q_2\overline{a})-M_4}{\overline{\partial_z\Phi}}\right)dx$$
$$-\ \frac14\int_\Omega\left(qa\frac{\partial_{\overline{z}}^{-1}(aq_1)-M_1}{\partial_z\Phi}+q\overline{a}\frac{\partial_z^{-1}(\overline{a}q_1)-M_3}{\overline{\partial_z\Phi}}\right)dx$$
$$+\ \int_{\Gamma_-}q|a|^2Re\left\{\frac{1}{\partial_{x_2}(\overline{\Psi}-\Phi)}\right\}d\sigma-\int_{\Gamma_+}q|a|^2Re\left\{\frac{1}{\partial_{x_2}(\overline{\Psi}-\Phi)}\right\}d\sigma=o(1)\quad \textit{as } \tau_j\to+\infty$$

where $q=q_1-q_2$ *and the sequence* τ_j *given by (4.19).*

Proof. Let u_1 be a solution to (3.1) and satisfy (3.2), and u_2 be a solution to the following equation

$$\Delta u_2+q_2u_2=0\quad\text{in }\Omega,\quad u_2|_{\partial\Omega}=u_1|_{\partial\Omega}.$$

Since the Cauchy data are equal, we have

$$\nabla u_2=\nabla u_1\quad\text{on }\Gamma_-.$$

Denoting $u=u_1-u_2$, we obtain

$$\Delta u+q_2u=-qu_1\quad\text{in }\Omega,\quad u|_{\partial\Omega}=\frac{\partial u}{\partial\nu}|_{\Gamma_-}=0. \tag{4.2}$$

Let v satisfy (3.49) and (3.50). We multiply (4.2) by v, integrate over Ω and we use $v|_{\Gamma_0}=0$ and $\frac{\partial u}{\partial\nu}=0$ on $\widetilde{\Gamma}$ to obtain $\int_\Omega qu_1vdx=0$. By (3.2), (3.50) and (3.47), (3.73), we have

$$\begin{aligned}0=\int_\Omega qu_1vdx=\int_\Omega q(a^2+\overline{a}^2+|a|^2e^{\tau_j(\Phi-\overline{\Phi})}+|a|^2e^{\tau_j(\overline{\Phi}-\Phi)}\\ +\frac{1}{\tau_j}(a(a_0+b_0)+\overline{a}(\overline{a}_1+\overline{b}_1))+u_{11}e^{\tau_j\varphi}(ae^{-\tau_j\Phi}+\overline{a}e^{-\tau_j\overline{\Phi}})\\ +(ae^{\tau_j\Phi}+\overline{a}e^{\tau_j\overline{\Phi}})v_{11}e^{-\tau_j\varphi})dx\\ +\int_\Omega q(e^{-\tau_j\Phi}a+e^{-\tau_j\overline{\Phi}}\overline{a})u_-e^{\tau_j\varphi}dx\\ +\int_\Omega q(e^{\tau_j\Phi}a+e^{\tau_j\overline{\Phi}}\overline{a})v_+e^{-\tau_j\varphi}dx+o\left(\frac{1}{\tau_j}\right),\quad \tau_j>0.\end{aligned}\tag{4.3}$$

The first and second terms in the asymptotic expansion of (4.3) are independent of τ_j, so that

$$\int_\Omega q(a^2+\overline{a}^2)dx=0. \tag{4.4}$$

Let the functions functions e_1,e_2 be defined in (3.39). We have

$$\int_\Omega q(|a|^2e^{\tau_j(\Phi-\overline{\Phi})}+|a|^2e^{\tau_j(\overline{\Phi}-\Phi)})dx=\int_\Omega e_1q(|a|^2e^{\tau_j(\Phi-\overline{\Phi})}+|a|^2e^{\tau_j(\overline{\Phi}-\Phi)})dx$$
$$+\ \int_\Omega e_2q(|a|^2e^{\tau_j(\Phi-\overline{\Phi})}+|a|^2e^{\tau_j(\overline{\Phi}-\Phi)})dx.$$

By the Cauchy-Riemann equations, we see that $\mathrm{sgn}(\mathrm{Im}\,\Phi''(\widetilde{x}_k)) = 0$, where $\mathrm{sgn}\, A$ denotes the signature of the matrix A, that is, the number of positive eigenvalues of A minus the number of negative eigenvalues (e.g., [5], p.210). Moreover we note that

$$\det \mathrm{Im}\,\Phi''(z) = -(\partial_{x_1}\partial_{x_2}\varphi)^2 - (\partial_{x_1}^2\varphi)^2 \neq 0.$$

To see this, suppose that $\det \mathrm{Im}\,\Phi''(z) = 0$. Then $\partial_{x_1}\partial_{x_2}\varphi(\mathrm{Re}\, z, \mathrm{Im}\, z) = \partial_{x_1}^2\varphi(\mathrm{Re}\, z, \mathrm{Im}\, z) = 0$ and the Cauchy-Riemann equations imply that all second order partial derivatives of functions φ, ψ at the point z are zero. This fact contradicts the assumption that critical points of the function Φ are nondegenerate.

Observe that if Φ has the critical point in Ω it can not have any critical point on Γ_0. Then by (2.2) $\tilde{x}$ is the only critical point of this function on $\bar{\Omega}$. Using stationary phase (see p.215 in [5]), we obtain

$$\int_\Omega e_1 q(|a|^2 e^{\tau_j(\Phi-\overline{\Phi})} + |a|^2 e^{\tau_j(\overline{\Phi}-\Phi)})dx = 2\frac{\pi q|a|^2(\widetilde{x})\mathrm{Re}\, e^{2\tau_j i \mathrm{Im}\,\Phi(\widetilde{x})}}{\tau_j|(\det \mathrm{Im}\,\Phi'')(\widetilde{x})|^{\frac{1}{2}}} + o\left(\frac{1}{\tau_j}\right). \tag{4.5}$$

Integrating by parts we have

$$\begin{aligned}
&\int_\Omega e_2 q(|a|^2 e^{\tau_j(\Phi-\overline{\Phi})} + |a|^2 e^{\tau_j(\overline{\Phi}-\Phi)})dx \\
= &\int_\Omega e_2 q|a|^2 \left(\frac{(\nabla\psi, \nabla e^{\tau_j(\Phi-\overline{\Phi})})}{2i\tau_j|\nabla\psi|^2} - \frac{(\nabla\psi, \nabla e^{\tau_j(\overline{\Phi}-\Phi)})}{2i\tau_j|\nabla\psi|^2}\right) dx \\
= &-\int_\Omega \mathrm{div}\left(\frac{e_2 q|a|^2\nabla\psi}{2i\tau_j|\nabla\psi|^2}\right)(e^{\tau_j(\Phi-\overline{\Phi})} - e^{\tau_j(\overline{\Phi}-\Phi)})dx \\
+ &\int_{\partial\Omega} \frac{q|a|^2}{2i\tau_j|\nabla\psi|^2}\frac{\partial\psi}{\partial\nu}(e^{\tau_j(\Phi-\overline{\Phi})} - e^{\tau_j(\overline{\Phi}-\Phi)})d\sigma \\
= &-\int_{\mathrm{supp}\, e_2} \mathrm{div}\left(\frac{e_2 q|a|^2\nabla\psi}{2i\tau_j|\nabla\psi|^2}\right)(e^{\tau_j(\Phi-\overline{\Phi})} - e^{\tau_j(\overline{\Phi}-\Phi)})dx \\
+ &\int_{\Gamma_-\cup\Gamma_+} \frac{q|a|^2}{2i\tau_j|\nabla\psi|^2}\frac{\partial\psi}{\partial\nu}(e^{2\tau_j i\psi} - e^{-2\tau_j i\psi})d\sigma + o(\frac{1}{\tau_j}) \quad \text{as } \tau_j \to +\infty.
\end{aligned}$$

In the last equality, we used that $e^{\tau_j(\Phi-\overline{\Phi})} - e^{\tau_j(\overline{\Phi}-\Phi)} = 0$ on Γ_0 which follows since by (2.3) $\mathrm{Im}\,\Phi = 0$ on Γ_0, and (3.17) in order to show that $\mathrm{div}\left(\frac{e_2 q|a|^2\nabla\psi}{2i\tau_j|\nabla\psi|^2}\right)$ and $\frac{q|a|^2}{2i\tau_j|\nabla\psi|^2}$ are bounded functions. Applying Proposition 2.4 we obtain

$$\int_\Omega e_2 q(|a|^2 e^{\tau_j(\Phi-\overline{\Phi})} + |a|^2 e^{\tau_j(\overline{\Phi}-\Phi)})dx = o(\frac{1}{\tau_j}) \quad \text{as } \tau_j \to +\infty.$$

Since the function ψ is strictly monotone on $\Gamma_-\cup\Gamma_+$ we have

$$\int_{\Gamma_-\cup\Gamma_+} \frac{q|a|^2}{2i\tau_j|\nabla\psi|^2}\frac{\partial\psi}{\partial\nu}(e^{2\tau_j i\psi} - e^{-2\tau_j i\psi})d\sigma = o(\frac{1}{\tau_j}) \quad \text{as } \tau_j \to +\infty.$$

Therefore

$$\int_\Omega q(|a|^2 e^{\tau_j(\Phi-\overline{\Phi})} + |a|^2 e^{\tau_j(\overline{\Phi}-\Phi)})dx = o\left(\frac{1}{\tau_j}\right). \tag{4.6}$$

Next we claim that

(4.7)
$$\int_\Omega q(e^{-\tau_j\Phi}a+e^{-\tau_j\overline{\Phi}}\overline{a})u_-e^{\tau_j\varphi}dx=\int_{\Gamma_-}\frac{q|a|^2}{\tau_j}\mathrm{Re}\left\{\frac{1}{\partial_{x_2}(\overline{\Psi}-\Phi)}\right\}d\sigma+o(\frac{1}{\tau_j})\quad\text{as }\tau_j\to+\infty,$$

and

$$\int_\Omega q(e^{\tau_j\Phi}a+e^{\tau_j\overline{\Phi}}\overline{a})v_+e^{-\tau_j\varphi}dx=\int_{\Gamma_+}\frac{q|a|^2}{\tau_j}\mathrm{Re}\left\{\frac{1}{\partial_{x_2}(\overline{\Psi}-\Phi)}\right\}d\sigma+o(\frac{1}{\tau_j})\quad\text{as }\tau_j\to+\infty. \tag{4.8}$$

Indeed, by (3.56) and (3.24)

$$\mathcal{K}=\int_\Omega q(e^{-\tau_j\Phi}a+e^{-\tau_j\overline{\Phi}}\overline{a})u_-e^{\tau_j\varphi}dx=\int_\Omega q(e^{-\tau_j\Phi}a+e^{-\tau_j\overline{\Phi}}\overline{a})\chi_{\tau_j}(e^{\tau_j\overline{\Psi}}\overline{(a+\hat a_0/\tau_j)}$$
$$+e^{\tau_j\Psi}(a+\hat a_1/\tau_j))dx=\int_\Omega q\chi_{\tau_j}(a\overline{(a+\hat a_0/\tau_j)}e^{\tau_j(\overline{\Psi}-\Phi)}+\bar a\overline{(a+\hat a_0/\tau_j)}e^{\tau_j(\overline{\Psi}-\bar\Phi)}$$
$$+a(a+\hat a_1/\tau_j)e^{\tau_j(\Psi-\Phi)}+\bar a(a+\hat a_1/\tau_j)e^{\tau_j(\Psi-\bar\Phi)})dx=$$
$$\int_{\partial\Omega}q\chi_{\tau_j}(a\overline{(a+\hat a_0/\tau_j)}\frac{\nu_2}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\bar a\overline{(a+\hat a_0/\tau_j)}e^{2\tau_ji\psi}\frac12\frac{(\nu_1+i\nu_2)}{\tau_j\partial_{\bar z}(\overline{\Psi}-\Phi)}$$
$$+a(a+\hat a_1/\tau_j)e^{-2\tau_ji\psi}\frac12\frac{(\nu_1-i\nu_2)}{\tau_j\partial_z(\Psi-\Phi)}+\bar a(a+\hat a_1/\tau_j)\frac{\nu_2}{\tau_j\partial_{x_2}(\Psi-\bar\Phi)})d\sigma-$$
$$-\frac{1}{\tau_j}\int_\Omega(B_1(x,D)^*(q\chi_{\tau_j}a\overline{(a+\hat a_0/\tau_j)})e^{\tau_j(\overline{\Psi}-\Phi)}+B_2(x,D)^*(q\chi_{\tau_j}\bar a\overline{(a+\hat a_0/\tau_j)})e^{\tau_j(\overline{\Psi}-\bar\Phi)}$$
$$+B_3(x,D)^*(q\chi_{\tau_j}a(a+\hat a_1/\tau_j))e^{\tau_j(\Psi-\Phi)}+B_4(x,D)^*(q\chi_{\tau_j}\bar a(a+\hat a_1/\tau_j))e^{\tau_j(\Psi-\bar\Phi)})dx,$$

where

$$B_1(x,D)=\frac{\partial_{x_2}}{\partial_{x_2}(\overline{\Psi}-\Phi)},\ B_2(x,D)=\frac{\partial_{\bar z}}{\partial_{\bar z}(\overline{\Psi}-\overline{\Phi})},$$
$$B_3(x,D)=\frac{\partial_z}{\partial_z(\Psi-\Phi)},\ B_4(x,D)=\frac{\partial_{x_2}}{\partial_{x_2}(\Psi-\bar\Phi)}.$$

By (3.17) the boundary integrals in (4.11) can be estimated as $o(\frac{1}{\tau_j})$. Integrating one more time we have

$$\mathcal{K}=\int_{\Gamma_-}q\left(\frac{a\bar a}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\frac{\bar a a}{\tau_j\partial_{x_2}(\Psi-\bar\Phi)}\right)d\sigma$$
$$+\frac{1}{\tau_j}\int_\Omega(B_1(x,D)^*(q\chi_{\tau_j}(a\overline{(a+\hat a_0/\tau_j)})e^{\tau_j(\overline{\Psi}-\Phi)}+B_2(x,D)^*(q\chi_{\tau_j}\bar a\overline{(a+\hat a_0/\tau_j)})e^{\tau_j(\overline{\Psi}-\bar\Phi)}$$
$$+B_3(x,D)^*(q\chi_{\tau_j}a(a+\hat a_1/\tau_j))e^{\tau_j(\Psi-\Phi)}+B_4(x,D)^*(q\chi_{\tau_j}\bar a(a+\hat a_1/\tau_j))e^{\tau_j(\Psi-\bar\Phi)})dx=$$
$$\mathcal{K}_1+o(\frac{1}{\tau_j})+\int_{\Gamma_-}q\left(\frac{a\bar a}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\frac{\bar a a}{\tau_j\partial_{x_2}(\Psi-\bar\Phi)}\right)d\sigma$$
$$+\int_{\partial\Omega}(B_1(x,D)^*(q\chi_{\tau_j}a\overline{(a+\hat a_0/\tau_j)})\frac{\nu_2}{\tau_j^2\partial_{x_2}(\overline{\Psi}-\Phi)}+B_2(x,D)^*(q\chi_{\tau_j}\bar a\overline{(a+\hat a_0/\tau_j)})e^{2\tau_ji\psi}\frac12\frac{(\nu_1+i\nu_2)}{\tau_j^2\partial_{\bar z}(\overline{\Psi}-\overline{\Phi})}$$
$$+B_3(x,D)^*(q\chi_{\tau_j}a(a+\hat a_1/\tau_j))e^{-2\tau_ji\psi}\frac12\frac{(\nu_1-i\nu_2)}{\tau_j^2\partial_z(\Psi-\Phi)}+B_4(x,D)^*(q\chi_{\tau_j}\bar a(a+\hat a_1/\tau_j))\frac{\nu_2}{\tau_j^2\partial_{x_2}(\Psi-\bar\Phi)})d\sigma,$$

where

$$\mathcal{K}_1 = \frac{1}{{\tau_j}^2}\int_{\mathrm{supp}\,\chi_{\tau_j}\cap\mathcal{G}_-}((B_1(x,D)^*)^2(q\chi_{\tau_j}a\overline{(a+\hat a_0/\tau_j)})e^{\tau_j(\bar\Psi-\Phi)}+(B_2(x,D)^*)^2(q\chi_{\tau_j}\bar a\overline{(a+\hat a_0/\tau_j)})e^{\tau_j(\bar\Psi-\bar\Phi)}$$
$$+(B_3(x,D)^*)^2(q\chi_{\tau_j}a(a+\hat a_1/\tau_j))e^{\tau_j(\Psi-\Phi)}+(B_4(x,D)^*)^2(q\chi_{\tau_j}\bar a(a+\hat a_1/\tau_j))e^{\tau_j(\Psi-\bar\Phi)})dx.$$

Since

$$\mathrm{Re}\,(\Psi-\Phi)\le 0\quad \forall x\in\mathcal{G}_- \tag{4.9}$$

by (3.13) we have

$$|\mathcal{K}_1|\le\frac{1}{{\tau_j}^2}\int_{\mathrm{supp}\,\chi_{\tau_j}\cap\mathcal{G}_-}(|(B_1(x,D)^*)^2(q\chi_{\tau_j}a\overline{(a+\hat a_0/\tau_j)})|+|(B_2(x,D)^*)^2(q\chi_{\tau_j}\bar a\overline{(a+\hat a_0/\tau_j)})|$$
$$+|(B_3(x,D)^*)^2(q\chi_{\tau_j}a(a+\hat a_1/\tau_j))|+|(B_4(x,D)^*)^2(q\chi_{\tau_j}\bar a(a+\hat a_1/\tau_j))|)dx\le$$
$$\frac{C}{{\tau_j}^2}\int_{\mathrm{supp}\,\chi_{\tau_j}\cap\mathcal{G}_-}\frac{1}{|\ell_1(x)|^2}dx\le\frac{C{\tau_j}^{\frac{12}{80}}}{{\tau_j}^2}=o(\frac{1}{\tau_j})\quad\text{as }\tau_j\to+\infty. \tag{4.10}$$

Again, by (3.17) the boundary integrals in (4.11) can be estimated as $o(\frac{1}{\tau_j})$. By (3.27) the boundary integral in (4.12) can be estimated as $o(\frac{1}{\tau_j})$.

$$\tilde{\mathcal{K}}=\int_\Omega q(e^{\tau_j\Phi}a+e^{\tau_j\overline{\Phi}}\overline{a})v_+e^{-\tau_j\varphi}dx=\int_\Omega q(e^{\tau_j\Phi}a+e^{\tau_j\overline{\Phi}}\overline{a})\tilde\chi_{\tau_j}(e^{-\tau_j\overline{\Psi}}\overline{(a+\hat b_0/\tau_j)} \tag{4.11}$$
$$+e^{-\tau_j\Psi}(a+\hat b_1/\tau_j))dx=\int_\Omega q\tilde\chi_{\tau_j}(a\overline{(a+\hat b_0/\tau_j)}e^{-\tau_j(\overline{\Psi}-\Phi)}+\bar a\overline{(a+\hat b_0/\tau_j)}e^{-\tau_j(\overline{\Psi}-\bar\Phi)}$$
$$+a(a+\hat b_1/\tau_j)e^{-\tau_j(\Psi-\Phi)}+\bar a(a+\hat b_1/\tau_j)e^{-\tau_j(\Psi-\bar\Phi)})dx+o(\frac{1}{\tau_j})=$$
$$-\int_{\partial\Omega}q\tilde\chi_{\tau_j}(a\overline{(a+\hat b_0/\tau_j)}\frac{\nu_2}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\bar a\overline{(a+\hat b_0/\tau_j)}e^{-2\tau_j i\psi}\frac12\frac{(\nu_1+i\nu_2)}{\tau_j\partial_{\bar z}(\overline{\Psi}-\tilde\Phi)}$$
$$+a(a+\hat b_1/\tau_j)e^{2\tau_j i\psi}\frac12\frac{(\nu_1-i\nu_2)}{\tau_j\partial_z(\Psi-\Phi)}+\bar a(a+\hat b_1/\tau_j)\frac{\nu_2}{\tau_j\partial_{x_2}(\Psi-\bar\Phi)})d\sigma-$$
$$+\frac{1}{\tau_j}\int_\Omega(B_1(x,D)^*(q\tilde\chi_{\tau_j}a\overline{(a+\hat b_0/\tau_j)})e^{-\tau_j(\overline{\Psi}-\Phi)}+B_2(x,D)^*(q\tilde\chi_{\tau_j}\bar a\overline{(a+\hat b_0/\tau_j)})e^{-\tau_j(\overline{\Psi}-\bar\Phi)}$$
$$+B_3(x,D)^*(q\tilde\chi_{\tau_j}a(a+\hat b_1/\tau_j))e^{-\tau_j(\Psi-\Phi)}+B_4(x,D)^*(q\tilde\chi_{\tau_j}\bar a(a+\hat b_1/\tau_j))e^{-\tau_j(\Psi-\bar\Phi)})dx$$
$$+o(\frac{1}{\tau_j})\quad\text{as }\tau_j\to+\infty.$$

By (3.17) the boundary integrals in (4.11) can be estimated as $o(\frac{1}{\tau_j})$. Integrating one more time we have

$$\tilde{\mathcal{K}} = -\int_{\Gamma_+} q\left(\frac{a\overline{a}}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\frac{\bar a a}{\tau_j\partial_{x_2}(\Psi-\bar\Phi)}\right)d\sigma \tag{4.12}$$
$$-\frac{1}{\tau_j}\int_\Omega (B_1(x,D)^*(q\tilde\chi_{\tau_j}(a\overline{(a+\hat b_0/\tau_j)})e^{-\tau_j(\overline{\Psi}-\Phi)}+B_2(x,D)^*(q\tilde\chi_{\tau_j}\bar a\overline{(a+\hat b_0/\tau_j)})e^{-\tau_j(\overline{\Psi}-\bar\Phi)}$$
$$+B_3(x,D)^*(q\tilde\chi_{\tau_j}a(a+\hat b_1/\tau_j))e^{-\tau_j(\Psi-\Phi)}+B_4(x,D)^*(q\tilde\chi_{\tau_j}\bar a(a+\hat b_1/\tau_j))e^{-\tau_j(\Psi-\bar\Phi)})dx=$$
$$\tilde{\mathcal{K}}_1+o(\frac{1}{\tau_j})-\int_{\Gamma_+} q\left(\frac{a\overline{a}}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\frac{\bar a a}{\tau_j\partial_{x_2}(\Psi-\bar\Phi)}\right)d\sigma$$
$$-\int_{\partial\Omega}\left(B_1(x,D)^*(q\tilde\chi_{\tau_j}a\overline{(a+b_0/\tau_j)})\frac{\nu_2}{{\tau_j}^2\partial_{x_2}(\overline{\Psi}-\Phi)}+B_2(x,D)^*(q\tilde\chi_{\tau_j}\bar a\overline{(a+\hat b_0/\tau_j)})e^{-2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1+i\nu_2)}{{\tau_j}^2\partial_{\bar z}(\Psi-\Phi)}\right.$$
$$\left.+B_3(x,D)^*(q\tilde\chi_{\tau_j}a(a+\hat b_1/\tau_j))e^{2\tau_j i\psi}\frac{1}{2}\frac{(\nu_1-i\nu_2)}{{\tau_j}^2\partial_z(\Psi-\Phi)}+B_4(x,D)^*(q\tilde\chi_{\tau_j}\bar a(a+\hat b_1/\tau_j))\frac{\nu_2}{{\tau_j}^2\partial_{x_2}(\tilde\Psi-\bar\Phi)}\right)d\sigma,$$

where

$$\tilde{\mathcal{K}}_1 = \tag{4.13}$$
$$\frac{1}{{\tau_j}^2}\int_{\mathrm{supp}\,\tilde\chi_{\tau_j}\cap\mathcal{G}_+}((B_1(x,D)^*)^2(q\tilde\chi_{\tau_j}a\overline{(a+\hat b_0/\tau_j)})e^{-\tau_j(\overline{\Psi}-\Phi)}+(B_2(x,D)^*)^2(q\tilde\chi_{\tau_j}\bar a\overline{(a+\hat b_0/\tau_j)})e^{-\tau_j(\overline{\Psi}-\bar\Phi)}$$
$$+(B_3(x,D)^*)^2(q\tilde\chi_{\tau_j}a(a+\hat b_1/\tau_j))e^{-\tau_j(\Psi-\Phi)}+(B_4(x,D)^*)^2(q\tilde\chi_{\tau_j}\bar a(a+\hat b_1/\tau_j))e^{-\tau_j(\Psi-\bar\Phi)})dx.$$

Observe that

$$\mathrm{Re}\,(\Psi-\Phi)\ge 0,\quad \forall x\in\mathcal{G}_+. \tag{4.14}$$

By (4.14), (3.13) we have

$$|\tilde{\mathcal{K}}_1|\le\frac{1}{{\tau_j}^2}\int_{\mathrm{supp}\,\tilde\chi_{\tau_j}\cap\mathcal{G}_+}(|(B_1(x,D)^*)^2(q\tilde\chi_{\tau_j}a\overline{(a+\hat b_0/\tau_j)})|+|(B_2(x,D)^*)^2(q\tilde\chi_{\tau_j}\bar a\overline{(a+\hat b_0/\tau_j)})|$$
$$+|(B_3(x,D)^*)^2(q\tilde\chi_{\tau_j}a(a+\hat b_1/\tau_j))|+|(B_4(x,D)^*)^2(q\tilde\chi_{\tau_j}\bar a(a+\hat b_1/\tau_j))|)dx\le$$
$$\frac{C}{{\tau_j}^2}\int_{\mathrm{supp}\,\tilde\chi_{\tau_j}\cap\mathcal{G}_+}\frac{1}{|\ell_1(x)|^2}dx\le\frac{C{\tau_j}^{\frac{12}{80}}}{{\tau_j}^2}=o(\frac{1}{\tau_j}). \tag{4.15}$$

Using the argument similar to (4.15) we obtain the second formula in (4.8). We calculate the two remaining terms in (4.3). By (3.38) and Proposition 2.5 we have:

$$\int_\Omega qu_{11}e^{\tau_j\varphi}(ae^{-\tau_j\Phi}+\overline{a}e^{-\tau_j\overline{\Phi}})dx=o(\frac{1}{\tau_j}) \tag{4.16}$$

$$-\int_\Omega\left(\frac{e^{\tau_j\Phi}}{\tau_j}\frac{(\partial_{\overline{z}}^{-1}(aq_1)-M_1)}{4\partial_z\Phi}+\frac{e^{\tau_j\overline{\Phi}}}{\tau_j}\frac{(\partial_z^{-1}(\overline{a}q_1)-M_3)}{4\overline{\partial_z\Phi}}\right)q(ae^{-\tau_j\Phi}+\overline{a}e^{-\tau_j\overline{\Phi}})dx=$$

$$-\int_\Omega q\left(\frac{e^{\tau_j(\Phi-\overline{\Phi})}}{\tau_j}\frac{\overline{a}(\partial_{\overline{z}}^{-1}(aq_1)-M_1)}{4\partial_z\Phi}+\frac{e^{\tau_j(\overline{\Phi}-\Phi)}}{\tau_j}\frac{a(\partial_z^{-1}(\overline{a}q_1)-M_3)}{4\overline{\partial_z\Phi}}\right)dx$$

$$-\int_\Omega q\left(\frac{a}{\tau_j}\frac{(\partial_{\overline{z}}^{-1}(aq_1)-M_1)}{4\partial_z\Phi}+\frac{\overline{a}}{\tau_j}\frac{(\partial_z^{-1}(\overline{a}q_1)-M_3)}{4\overline{\partial_z\Phi}}\right)dx+o(\frac{1}{\tau_j})=$$

$$-\int_\Omega q\left(\frac{a}{\tau_j}\frac{(\partial_{\overline{z}}^{-1}(aq_1)-M_1)}{4\partial_z\Phi}+\frac{\overline{a}}{\tau_j}\frac{(\partial_z^{-1}(\overline{a}q_1)-M_3)}{4\overline{\partial_z\Phi}}\right)dx+o(\frac{1}{\tau_j})\text{ as }\tau_j\to+\infty.$$

Similarly by (3.66) and Proposition 2.5

$$\int_\Omega qv_{11}e^{-\tau_j\varphi}(ae^{\tau_j\Phi}+\overline{a}e^{\tau_j\overline{\Phi}})dx= \tag{4.17}$$

$$+\int_\Omega q\left(\frac{e^{-\tau_j\Phi}}{\tau_j}\frac{(\partial_{\overline{z}}^{-1}(aq_2)-M_2)}{4\partial_z\Phi}+\frac{e^{-\tau_j\overline{\Phi}}}{\tau_j}\frac{(\partial_z^{-1}(\overline{a}q_2)-M_4)}{4\overline{\partial_z\Phi}}\right)(ae^{\tau_j\Phi}+\overline{a}e^{\tau_j\overline{\Phi}})dx+o(\frac{1}{\tau_j})=$$

$$\int_\Omega q\left(\frac{e^{-\tau_j(\Phi-\overline{\Phi})}}{\tau_j}\frac{\overline{a}(\partial_{\overline{z}}^{-1}(aq_2)-M_2)}{4\partial_z\Phi}+\frac{e^{\tau_j(\Phi-\overline{\Phi})}}{\tau_j}\frac{a(\partial_z^{-1}(\overline{a}q_2)-M_4)}{4\overline{\partial_z\Phi}}\right)dx$$

$$+\int_\Omega q\left(\frac{a}{\tau_j}\frac{\partial_{\overline{z}}^{-1}(aq_2)-M_2}{4\partial_z\Phi}+\frac{\overline{a}}{\tau_j}\frac{\partial_z^{-1}(\overline{a}q_2)-M_4}{4\overline{\partial_z\Phi}}\right)dx+o(\frac{1}{\tau_j})=$$

$$\int_\Omega q\left(\frac{a}{\tau_j}\frac{\partial_{\overline{z}}^{-1}(aq_2)-M_2}{4\partial_z\Phi}+\frac{\overline{a}}{\tau_j}\frac{\partial_z^{-1}(\overline{a}q_2)-M_4}{4\overline{\partial_z\Phi}}\right)dx+o(\frac{1}{\tau_j})\quad\text{as }\tau_j\to+\infty$$

Therefore, applying (4.4), (4.6), (4.7), (4.8), (4.17), (4.16), in (4.3), we conclude that

$$2\frac{\pi(q|a|^2)(\widetilde{x})\mathrm{Re}\,e^{2i\tau_j\mathrm{Im}\Phi(\widetilde{x})}}{|(\det\mathrm{Im}\Phi'')(\widetilde{x})|^{\frac{1}{2}}}+\int_\Omega q(a(a_0+b_0)+\overline{a}(\overline{a}_1+\overline{b}_1))dx$$

$$+\frac{1}{4}\int_\Omega\left(qa\frac{\partial_{\overline{z}}^{-1}(aq_2)-M_2}{\partial_z\Phi}+q\overline{a}\frac{\partial_z^{-1}(q_2\overline{a})-M_4}{\overline{\partial_z\Phi}}\right)dx$$

$$-\frac{1}{4}\int_\Omega\left(qa\frac{\partial_{\overline{z}}^{-1}(q_1a)-M_1}{\partial_z\Phi}+q\overline{a}\frac{\partial_z^{-1}(q_1\overline{a})-M_3}{\overline{\partial_z\Phi}}\right)dx$$

$$+\int_{\Gamma_-}q\left(\frac{a\overline{a}}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\frac{\bar{a}a}{\tau_j\partial_{x_2}(\Psi-\bar{\Phi})}\right)d\sigma$$

$$-\int_{\Gamma_+}q\left(\frac{a\overline{a}}{\tau_j\partial_{x_2}(\overline{\Psi}-\Phi)}+\frac{\bar{a}a}{\tau_j\partial_{x_2}(\Psi-\bar{\Phi})}\right)d\sigma=o(1)$$

as $\tau_j\to+\infty$. The proof of proposition is finished. □

End of proof of Theorem 1.1. First we observe that any smooth holomorphic function $\Phi=\varphi+i\psi$ such that (2.3) holds true can be approximated by the sequence of harmonic functions constructed in Section 3. Moreover the function satisfying (2.3) has at most one internal critical point. Therefore by Proposition 4.1 the function q is zero at this critical point. Consider the set of harmonic functions ψ satisfying the following

$$\text{function } \psi \text{ equal to some constant on each connected component of the set } \Gamma_0;$$
$$\frac{\partial\psi}{\partial\vec{\tau}}|_{\Gamma_+} < 0;$$
$$\frac{\partial\psi}{\partial\vec{\tau}}|_{\Gamma_-} > 0.$$

We show that the set of critical points of harmonic functions ψ with above properties is dense in Ω. In order to do that it suffices to consider the following. Let $\partial\Omega = \overline{\cup_{k=1}^4\Gamma_k}$, where Γ_k is ark and $\Gamma_j \cap \Gamma_k = \emptyset$ for any $k \ne j$ and Ω is the unit ball centered at zero. Consider the set of harmonic functions ψ with the boundary data $\psi|_{\Gamma_k} = C_k$. We claim that for generic choice of Γ_k we can find a constants C_k such that $\nabla\psi(0) = 0$. Indeed since $\psi(x) = \frac{1}{2\pi}\int_0^{2\pi}\psi(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}dt$ we have $\partial_z\psi(0) = \frac{1}{2\pi}\int_0^{2\pi}\psi(e^{it})e^{it}dt$.

Indeed, let $C_1 = 0, C_4 = 1$ and the endpoints of the arcs Γ_k on the complex plane are given by $e^0, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}$ with $0 < \theta_1 < \theta_2 < \theta_3 < 2\pi$. Then

$$i\partial_z\psi(0) = C_2(e^{-i\theta_1} - e^{-i\theta_2}) + C_3(e^{-i\theta_2} - e^{-i\theta_3}) + (e^{-i\theta_3} - 1).$$

The equation $\partial_z\psi(0) = 0$ equivalent to

$$C_2 = -\frac{C_3(e^{-i\theta_2} - e^{-i\theta_3})(e^{i\theta_1} - e^{i\theta_2}) + (e^{-i\theta_3} - 1)(e^{i\theta_1} - e^{i\theta_2})}{|e^{-i\theta_1} - e^{-i\theta_2}|}.$$

The existence of real valued solutions C_2, C_3 to this equation is equivalent to

$$\mathrm{Im}(e^{i(\theta_1-\theta_2)} + e^{i(\theta_2-\theta_3)} - e^{i(\theta_1-\theta_3)}) \ne 0.$$

This clearly holds true for generic position of θ_j.

In the set $\Gamma_+ \cup \Gamma_-$ we make the choice of four points $\hat{x}_1, \dots, \hat{x}_4$ such that $\hat{x}_1 \in \gamma_{1,+}, \hat{x}_2 \in \Gamma_{1,-}, \hat{x}_3 \in \Gamma_{2,+}, \hat{x}_2 \in \Gamma_{2,-}$. Denote by $\hat{\Gamma}_1, \dots \hat{\Gamma}_4$ the arks connecting these points. Consider the conformal mapping Π which transforms the domain Ω into the unit ball and point $\tilde{x}$ into the center of the coordinate system. Above we show that under generic choice of the points $\hat{x}_j$ there exists a harmonic function ψ_0 which is equal to some constant on each ark $\Pi(\hat{\Gamma}_k)$. Consider the boundary data $\psi_0(\Pi)$. The corresponding harmonic function we denote as $\hat{\psi}$. The function $\hat{\psi}$ is equal to constant C_j on each ark $\hat{\Gamma}_j$ and it has only one the nondegenerate critical point located at $\tilde{x}$. Without loss of the generality we may assume that $C_0 = 0$ multiplying if this is necessary the function $\psi_0 \circ \Pi$ by nonzero constant we may assume that $C_4 = -1$. Observe that $C_2 < 0$ and $C_3 > C_2$. (Otherwise if at least one of these inequalities fail the function $\psi_0 \circ \Pi$ can not have the internal critical point.) In small neighborhood $\mathcal{F} \subset \cup_{j=1}^2\Gamma_{j,\pm}$ of the points of discontinuity of the function $\psi_0 \circ \Pi$ we approximate it by a sequence $\{\mu_k\}$ strictly monotone decreasing or strictly monotone increasing functions. Outside of $\mathcal{F}$ the function μ_k are equal to corresponding constants.

Moreover

$$\mu_k \to \psi_0 \circ \Pi \quad \text{in } L^2(\partial\Omega).$$

We claim that for all sufficiently large k the harmonic functions ψ_k such that $\psi_k|_{\partial\Omega} = \mu_k$ have a unique internal critical point which we denote as $\tilde{x}_j$. Moreover $\tilde{x}_j \to \tilde{x}$. Our proof is by contradiction. Suppose that for large j functions ψ_j do not have internal critical point or the sequence converges to some point $y \neq \tilde{x}$. Indeed for any $\Omega_0 \subset\subset \Omega$

$$\psi_k \to \psi_0 \circ \Pi \quad \text{in } C^2(\Omega_0).$$

n the other hand it is known that the number of zeros N of a holomorphic function $f(z)$ in domain G given by formula

$$N = \frac{1}{2\pi i}\int_{\partial G} \frac{\partial_z f}{f(z)} dz \tag{4.19}$$

Solving the system of Cauchy-Riemann equations we construct the the holomorphic function $\Phi_j = \varphi_j + i\psi_j$. By (**??**) for all sufficiently small positive δ and all large k $\frac{1}{2\pi i}\int_{S(\tilde{x},\delta)} \frac{\partial_z^2(\varphi_k+i\psi_k)}{\partial_z(\varphi_k+i\psi_k)} dz = 1$. This means that the function $\varphi_k + i\psi_k$ has the critical point in the ball $B(\tilde{x}, \delta)$. But this function can not have more than one critical point. So $y = \tilde{x}$. Proof of the theorem is complete. □

5. **Appendix.**

Consider the Cauchy problem for the Cauchy-Riemann equations

$$L(\phi,\psi) = (\frac{\partial\phi}{\partial x_1} - \frac{\partial\psi}{\partial x_2}, \frac{\partial\phi}{\partial x_2} + \frac{\partial\psi}{\partial x_1}) = 0 \quad \text{in } \Omega, \quad (\phi,\psi)\,|_{\Gamma_0} = (b_1(x), b_2(x)), \quad (\phi + i\psi)(\tilde{x}) = c_{0,j}. \tag{5.1}$$

Here $\hat{x}_1, \dots \hat{x}_N$ be an arbitrary fixed points in Ω. We consider the pair b_1, b_2 and complex numbers $\vec{C} = (c_{0,1}, c_{1,1}, c_{2,1}, \dots c_{0,N}, c_{1,N}, c_{2,N})$ as initial data for (5.1). The following proposition establishes the solvability of (5.1) for a dense set of Cauchy data.

Proposition 5.1. *There exists a set $\mathcal{O} \subset C^{100}(\overline{\Gamma_0})^2 \times \mathbb{C}$ such that for each $(b_1, b_2, \vec{C}) \in \mathcal{O}$, (5.1) has at least one solution $(\phi,\psi) \in (C^{100}(\overline{\Omega}))^2$ and $\overline{\mathcal{O}} = C^{100}(\overline{\Gamma_0})^2 \times \mathbb{C}$.*

Consider the Cauchy problem for the Cauchy-Riemann equations

$$L(\phi,\psi) = (\frac{\partial\phi}{\partial x_1} - \frac{\partial\psi}{\partial x_2}, \frac{\partial\phi}{\partial x_2} + \frac{\partial\psi}{\partial x_1}) = 0 \quad \text{in } \Omega, \quad (\phi,\psi)\,|_{\Gamma_0} = (b(x), 0), \quad \frac{\partial^l}{\partial z^l}(\phi + i\psi)(\hat{x}_j) = c_{0,j}, \quad \forall j \in \{1, \dots N\} \quad \text{and } \forall l \in \{0, \dots, 5\}. \tag{5.2}$$

Here $\hat{x}_1, \dots \hat{x}_N$ be an arbitrary fixed points in Ω. We consider the function b and complex numbers $\vec{C} = (c_{0,1}, c_{1,1}, c_{2,1}, c_{3,1}, c_{4,1}, c_{5,1} \dots c_{0,N}, c_{1,N}, c_{2,N}, c_{3,N}, c_{4,N}, c_{5,N})$ as initial data for (5.1). The following proposition establishes the solvability of (5.1) for a dense set of Cauchy data.

Corollary 5.1. *There exists a set $\mathcal{O} \subset C^6(\overline{\Gamma_0}) \times \mathbb{C}^{6N}$ such that for each $(b, \vec{C}) \in \mathcal{O}$, problem (5.2) has at least one solution $(\phi,\psi) \in C^6(\overline{\Omega}) \times C^6(\overline{\Omega})$ and $\overline{\mathcal{O}} = C^6(\overline{\Gamma}) \times \mathbb{C}^{6N}$.*

Now we give the proof of Proposition 2.7.

Proof. Let us introduce the space

$$H = \left\{ v \in H_0^1(\Omega) | \, \Delta v + q_0 v \in L^2(\Omega), \frac{\partial v}{\partial \nu}|_{\Gamma_+} = 0 \right\}$$

with the scalar product

$$(v_1, v_2)_H = \int_\Omega e^{2\tau\varphi}(\Delta v_1 + q_0 v_1)\overline{(\Delta v_2 + q_0 v_2)} dx.$$

By Proposition 2.1 H is a Hilbert space. Consider the linear functional on $H : v \to \int_\Omega v\overline{f}dx + \int_{\Gamma_-} g\overline{\frac{\partial v}{\partial \nu}}d\sigma$. By (2.4) this is the continuous linear functional with the norm estimated by a constant $C_{12}(\|fe^{\tau\varphi}\|_{L^2(\Omega)}/\tau^{\frac{1}{2}} + \|ge^{\tau\varphi}/\sqrt{|\partial_\nu \varphi|}\|_{L^2(\Gamma_-)})$. Therefore by the Riesz representation theorem there exists an element $\widehat{v} \in H$ so that

$$\int_\Omega v\overline{f}dx + \int_{\Gamma_-} g\overline{\frac{\partial v}{\partial \nu}}d\sigma = \int_\Omega e^{2\tau\varphi}(\Delta\widehat{v} + q_0\widehat{v})\overline{(\Delta v + q_0 v)}dx.$$

Then, as a solution to (2.8), we take the function $u = e^{2\tau\varphi}(\Delta\widehat{v} + q_0\widehat{v})$. □

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