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# INVERSE SOURCE PROBLEM FOR THE LINERAIZED NAVIER-STOKES EQUATIONS WITH INTERIOR DATA IN ARBITRARY SUB-DOMAIN

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Abstract. We consider an inverse problem of determining a spatially varying factor in a source term in a nonstationary linearized Navier-Stokes equations by observation data in an arbitrarily fixed sub-domain over some time interval. We prove the Lipschitz stability provided that the t-dependent factor satisfies a non-degeneracy condition. Our proof based on a new Carleman estimate for the Navier-Stokes equations.

## §1. Introduction and the main result.

In a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, we consider the linearized Navier-Stokes equations for an incompressible viscous fluid flow:

$$\partial_t v - \nu\Delta v + (A\cdot\nabla)v + (v\cdot\nabla)B + \nabla p = F(x,t) \quad \text{in } Q \equiv \Omega\times(0,T), \tag{1.1}$$

$$\operatorname{div} v = 0 \quad \text{in } Q \tag{1.2}$$

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and

$$v = 0 \qquad \text{on } \Sigma \equiv \Omega \times (0,T). \tag{1.3}$$

Here $v = (v_1, v_2, v_3)^T$, $\cdot^T$ denotes the transpose of matrices, $\nu > 0$ is a constant describing the viscosity, and for simplicity we assume that the density is one. Let $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1, 2, 3$, $\Delta = \sum_{j=1}^3 \partial_j^2$, $\nabla = (\partial_1, \partial_2, \partial_3)^T$,

$$(w \cdot \nabla)v = \left( \sum_{j=1}^3 w_j \partial_j v_1, \sum_{j=1}^3 w_j \partial_j v_2, \sum_{j=1}^3 w_j \partial_j v_3 \right)^T$$

for $v = (v_1, v_2, v_3)^T$ and $w = (w_1, w_2, w_3)^T$. Henceforth let $n$ be the outward unit normal vector to $\partial\Omega$ and let $\frac{\partial u}{\partial n} = \nabla u \cdot n$. Moreover let $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{N} \cup \{0\})^3$, $\partial_x^\gamma = \partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3}$ and $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$. Throughout this paper, we assume

$$A \in W^{2,\infty}(0,T; W^{2,\infty}(\Omega)), \quad B \in W^{2,\infty}(0,T; W^{2,\infty}(\Omega)). \tag{1.4}$$

Physically $v$ denotes the velocity field of the incompressible fluid and by $R(x,t)f(x)$ we will model the density of external force causing the movement of the fluid. In this paper, we consider:

$$R(x,t) = (r_1(x,t), r_2(x,t), r_3(x,t))^T,$$

$$f = f(x),\ r_j = r_j(x,t),\ j = 1, 2, 3 : \text{real-valued}. \tag{1.5}$$

In the forward problem we are required to discuss the unique existence of solutions in suitable spaces to (1.1) - (1.3) with initial condition for a given external source term $Rf$ and there are a vast amount of works (e.g., Ladyzhenskaya [36], Temam [41] and the references therein). The forward problem is important, but any practical studies of the forward problem can be started only after suitable modeling of physical parameters such as the viscosity $\nu$, the force term $Rf$. The inverse source problems are concerned with such modeling. In our inverse problem, we mainly discuss the determination of a spatially varying function $f(x)$ for given $R(x,t)$.

**Inverse Source Problem.** *Let $\omega \subset \Omega$ be an arbitrarily given non-empty sub-domain, $0 < \theta < T$, and the velocity field $v$ satisfy (1.1) - (1.3). Then determine $f(x)$ by observation data $(v|_{\omega\times(0,T)}, v(\cdot,\theta)|_{\omega})$.*

Despite of many practical applications, the inverse problems of this type for the Navier-Stokes equations have been not studied intensively. As the relevant results, we refer to Choulli, Imanuvilov and Yamamoto [8], Fan, Di Cristo, Jiang and Nakamura [9], Fan, Jiang and Nakamura [10], Imanuvilov and Yamamoto [21].

Our main achievement is the Lipschitz stability with an arbitrary sub-domain $\omega$. On the other hand, in the existing paper [8] - [10] and [21], one has to assume that $\partial\omega \supset \partial\Omega$, that is, $\omega$ is a neighbourhood of $\partial\Omega$. Such a geometric constraint is unnatural for the equation of parabolic type. In fact, in the corresponding inverse parabolic problem (e.g., Imanuvilov and Yamamoto [20], Yamamoto [43]), we need not any geometric constraints for $\omega$.

We note that in our paper as well as [8], [9], [10], we do not assume any data of the pressure field $p$. If we assume the data $p(x,\theta)$, $x \in \Omega$, then we can argue similarly to [20] for a more general inverse problem of determining a vector-valued function $f(x)$ in $Rf$ with suitable $3\times 3$ matrix $R(x,t)$, while in this paper, we study only the case where unknown $f$ is real-valued.

As for different types of inverse problems for the Navier-Stokes equations, see Prilepko, Orlovsky and Vasin [40] and the references therein. In [40], the authors discuss inverse problems by final overdetermining observation data $u(x,T)$, $x \in \Omega$.

We introduce the following spaces:

$$\begin{cases} H^{2,1}(Q) = \{w;\, \partial_t w, \partial_x^\gamma w \in (L^2(Q))^3,\ |\gamma| \le 2\}, \\ H = \overline{\{v = (v_1, v_2, v_3) \in (C_0^\infty(\Omega))^3;\ \mathrm{div}\, v = 0\}}^{(L^2(\Omega))^3}, \\ V = H \cap (H_0^1(\Omega))^3. \end{cases}$$

Here and henceforth $A \times B$ denotes the exterior product of vectors $A, B \in \mathbb{R}^3$: $A \times B = (A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1)^T$, and $[a]_k$ denotes the $k$-th component of a vector $a$.

By Fursikov and Imanuvilov [13], Imanuvilov [15], we see that there exists a function $\eta \in C^2(\overline{\Omega})$ satisfying (i) or (ii):

(i) Case $Int(\overline{\omega} \cap \partial\Omega) = \emptyset$:

$$\eta|_{\partial\Omega} = 0, \quad \eta > 0 \text{ in } \Omega, \quad |\nabla\eta(x)| > 0 \quad \text{on } \overline{\Omega \setminus \omega}. \tag{1.6}$$

(ii) Case where $\partial\omega \cap \partial\Omega$ contains a non-empty relatively open sub-set $\Gamma$ of $\partial\Omega$:

$$\begin{aligned} &|\nabla\eta(x)| > 0 \quad \text{on } \overline{\Omega}, \quad \eta > 0 \text{ in } \Omega, \\ &\frac{\partial\eta}{\partial n} < 0 \text{ and } \eta = 0 \text{ on } \partial\Omega \setminus \Gamma_1 \text{ with some open set } \Gamma_1 \subset\subset \Gamma. \end{aligned} \tag{1.7}$$

We are ready to state our main result.

**Theorem 1.** *Let $\omega$ be an arbitrary non-empty sub-domain of $\Omega$. Let $0 < \theta < T$, and let $R(x,t) = (r_1(x,t), r_2(x,t), r_3(x,t))^T$ satisfy*

$$R(\cdot, \theta) \in C^2(\overline{\Omega}), \quad \partial_t^j R \in L^\infty(Q),\ j = 0, 1, 2 \tag{1.8}$$

*and let $f \in H_0^1(\Omega)$.*

*(i) Let $Int(\overline{\omega} \cap \partial\Omega) = \emptyset$. We assume (1.6) and*

$$R(x,\theta) \times \nabla\eta(x) \neq 0,\ x \in \overline{\Omega \setminus \omega}. \tag{1.9}$$

*Then there exists a constant $C = C(\Omega, T, \theta, R) > 0$ such that for all $v$ satisfying (1.1) - (1.3) and $\partial_t^j v \in L^2(0,T;V) \cap H^{2,1}(Q)$ with $j = 0, 1, 2$,*

$$\|f\|_{L^2(\Omega)} \le C(\|v\|_{H^2(0,T;H^1(\omega))} + \|rot\, v(\cdot,\theta)\|_{H^2(\Omega)} + \|v(\cdot,\theta)\|_{H^1(\Omega)}) \tag{1.10}$$

*provided that* $f|_\omega = 0$.

*(ii) Let* $\partial\omega \cap \partial\Omega$ *contain a non-empty relatively open sub-set* $\Gamma$ *of* $\partial\Omega$. *We assume (1.7) and*

$$R(x,\theta) \times \nabla\eta(x) \neq 0, \quad x \in \overline{\Omega}. \tag{1.11}$$

*Then estimate (1.10) holds true.*

**Example.** Let $\Omega = \{x \in \mathbb{R}^3;\, \rho_1 < |x| < \rho_2\}$ with $0 < \rho_1 < \rho_2$ and $\omega = \{x \in \mathbb{R}^3;\, \rho_2 - \delta_1 < |x| < \rho_2 - \rho_2\}$ where $\delta_1, \delta_2 > 0$ are sufficiently small and $\delta_1 > \delta_2$. Then we can directly verify that the function

$$\eta(x) = (\rho_2^{2m} - |x|^{2m})(|x|^{2m} - \rho_1^{2m})$$

with any positive $\delta_1$ and $\delta_2$, satisfies (1.6) if $m \in \mathbb{N}$ is sufficiently large.

In fact, we have

$$\nabla\eta(x) = 2mx|x|^{2m-2}(\rho_1^{2m} + \rho_2^{2m} - 2|x|^{2m})$$

and $\left(\frac{\rho_1^{2m}+\rho_2^{2m}}{2}\right)^{\frac{1}{2m}} > |x| \geq \rho_1$ implies $|\nabla\eta(x)| > 0$. Since $\lim_{m\to\infty} \left(\frac{\rho_1^{2m}+\rho_2^{2m}}{2}\right)^{\frac{1}{2m}} = \rho_2$ by $\rho_2 > \rho_1$, we see that for small $\delta_1, \delta_2 > 0$, we can choose large $m \in \mathbb{N}$ such that $|\nabla\eta(x)| > 0$ if $x \in \overline{\Omega \setminus \omega}$.

If $x \times R(x,\theta) \neq 0$, $x \in \overline{\Omega}$, then (1.9) hold true.

For determination of $f$, we have to assume the non-degeneracy condition on $R$ given by (1.9) or (1.11). By Cases (i) and (ii), we see that if we can assume $f|_\omega = 0$, then we can prove the Lipschitz stability without any geometric constraints concerning $\omega$. Since in Case (i) it may happen that $\nabla\eta(\widetilde{x}) = 0$ for some point $\widetilde{x} \in \omega$, we can not prove the stability in the inverse problem for general $\omega$ and $f$ not vanishing in $\omega$. In case (ii) where $\omega$ is an open set near $\partial\Omega$, then, without assumption $f|_\omega \equiv 0$, we can prove the same Lipschitz stability result under condition (1.11).

In Theorem 1, we note that $\theta > 0$. If $\theta = 0$, then our inverse problem is exactly an inverse problem to the forward problem, that is, the initial/ boundary value problem. However the corresponding inverse problem for a parabolic equation is open in the case of $\theta = 0$ (cf. Isakov [28], [29]), and also our inverse problem with $\theta = 0$ is an open problem.

Our main approach is based on Bukhgeim and Klibanov [7] which introduced a methodology based on Carleman estimates to inverse problems (also see Isakov [27], Klibanov [33], [34]). The key Carleman estimate is Theorem 2 in section 2, which is derived by Imanuvilov, Puel and Yamamoto [19]. Our proof is by Imanuvilov and Yamamoto [20] which modified the method in [7].

As for similar inverse problems, we refer to the following works: Amirov and Yamamoto [1], Baudouin and Puel [2], Bellassoued [3], [4], Bellassoued and Yamamoto [5], Bukhgeim [6], Imanuvilov, Isakov and Yamamoto [17], Imanuvilov and Yamamoto [22] - [26], Isakov [27] - [29], Isakov and Yamamoto [30], Khaĭdarov [31], [32], Klibanov and Timonov [35], Li [37], Li and Yamamoto [38], Yamamoto [42], [43]. This list is far from the complete and the readers can consult the references therein.

Our proof is based on a Carleman estimate for the Navier-Stokes equations. It is different from one obtained for example in Fernández-Cara, Guerrero, Imanuvilov and Puel [11], [12]. As for Carleman estimates, see further Fursikov and Imanuvilov [13], Hörmander [14], Imanuvilov [16].

The rest part of this paper is composed of two sections. In section 2, we prove a Carleman estimate for the Navier-Stokes equations which may be an independent interest. In section 3, we complete the proof of our main result.

## §2. Key Carleman estimate.

Let $\ell \in C^\infty[0,T]$ satisfy

$$\begin{cases} \ell(t) > 0, \qquad 0 < t < T, \\ \ell(t) = \begin{cases} t, & 0 \le t \le \frac{T}{4}, \\ T - t, & \frac{3}{4}T \le t \le T, \end{cases} \\ \ell(\theta) > \ell(t), \quad \forall t \in (0,T) \setminus \{\theta\}. \end{cases} \tag{2.1}$$

We set $Q_\omega = \omega \times (0,T)$,

$$\varphi(x,t) = \frac{e^{\lambda\eta(x)}}{\ell^8(t)}, \quad \alpha(x,t) = \frac{e^{\lambda\eta(x)} - e^{2\lambda\|\eta\|_{C^0(\overline{\Omega})}}}{\ell^8(t)}. \tag{2.2}$$

We further set $H^{\frac{1}{2},\frac{1}{4}}(\Sigma) = H^{\frac{1}{4}}(0,T;L^2(\partial\Omega)) \cap L^2(0,T;H^{\frac{1}{2}}(\partial\Omega))$. Hence $C, C_j$ denote generic positive constants which are dependent on $\Omega, T, R, \theta, \lambda$ but independent of $s$. Our Carleman estimate can be stated:

**Theorem 2.** *Let $\eta \in C^2(\overline{\Omega})$ satisfy (1.6) or (1.7). Let $F \in L^2(0,T;H)$, $v(\cdot,0) \in V$ and let $v \in L^2(0,T;V) \cap H^{2,1}(Q)$ satisfy (1.1) - (1.3). Then there exists a constant $\widehat{\lambda}$ such that for any $\lambda > \widehat{\lambda}$ there exist constants $C > 0$ and $\widehat{s} > 0$ independent of $s$ such that*

$$\|(\nabla v)e^{s\alpha}\|_{L^2(Q)} + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(rot\, v)e^{s\alpha}\|_{L^2(Q)} + \|s\varphi v e^{s\alpha}\|_{L^2(Q)} \le C(\|Fe^{s\alpha}\|_{L^2(Q)}$$
$$+\|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(rot\, v)e^{s\alpha}\|_{L^2(Q_\omega)} + \|s\varphi v e^{s\alpha}\|_{L^2(Q_\omega)} + \|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}(\nabla v)e^{s\alpha}\|_{L^2(Q_\omega)})$$

*for all $s \ge \widehat{s}$.*

In order to prove Theorem 2, we show the following lemmata.

**Lemma 1 ([18]).** *Let $\eta \in C^2(\overline{\Omega})$ satisfy (1.6) or (1.7). Let $y \in H^1(\Omega)$ satisfy*

$$\Delta y = f + \sum_{j=1}^{3} \partial_j f_j \qquad \textit{in } \Omega, \quad y = g \qquad \textit{on } \partial\Omega$$

*with* $f, f_j \in L^2(\Omega)$ *and* $g \in H^{\frac{1}{2}}(\partial\Omega)$, *supp* $g \subset \partial\Omega \setminus \Gamma_1$. *Then there exist a constant* $C > 0$, *independent of* $s$ *and* $\lambda$, *and parameters* $\widehat{\lambda}$ *and* $\widehat{s}$ *such that for any* $\lambda > \widehat{\lambda}$ *and* $s > \widehat{s}$,

$$\int_\Omega \left(\frac{1}{s}|\nabla y|^2 + s\lambda^2 e^{2\lambda\eta}|y|^2\right) e^{2se^{\lambda\eta}} dx \le C\Bigg( s^{-\frac{1}{2}} e^{2s}\|g\|^2_{H^{\frac{1}{2}}(\partial\Omega)} + \int_\Omega \frac{1}{s^2\lambda^2} e^{-\lambda\eta}|f|^2 e^{2se^{\lambda\eta}} dx$$
$$+ \sum_{j=1}^3 \int_\Omega e^{\lambda\eta}|f_j|^2 e^{2se^{\lambda\eta}} dx + \int_\omega \left(\frac{1}{s}|\nabla y|^2 + s\lambda^2 e^{2\lambda\eta}|y|^2\right) e^{2se^{\lambda\eta}} dx\Bigg). \tag{2.3}$$

**Lemma 2 ([19]).** *Let* $\eta \in C^2(\overline{\Omega})$ *satisfy (1.6) or (1.7). Let* $y \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ *satisfy*

$$\begin{aligned} &\partial_t y - \Delta y + \sum_{j=1}^3 b_j(x,t)\partial_j y + \sum_{j=1}^3 \partial_j(c_j(x,t)y) + d(x,t)y \\ =& f + \sum_{j=1}^3 \partial_j f_j \qquad \text{in } Q, \\ & y = g \qquad \text{on } \Sigma \end{aligned}$$

*with* $b_j, c_j, d \in L^\infty(Q)$, $f, f_j \in L^2(Q)$ *and* $g \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. *Then there exists a constant* $\widehat{\lambda}$ *such that for any* $\lambda > \widehat{\lambda}$ *there exist constants* $C > 0$ *and* $\widehat{s} = \widehat{s}(\lambda)$, *independent of* $s$, *such that*

$$\begin{aligned} &\int_Q \left(\frac{1}{s\varphi}|\nabla y|^2 + s\varphi|y|^2\right) e^{2s\alpha} dxdt \\ \le& C\Bigg( s^{-\frac{1}{2}} \left\|\varphi^{-\frac{1}{4}} g e^{s\alpha}\right\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} + s^{-\frac{1}{2}} \left\|\varphi^{-\frac{1}{8}} g e^{s\alpha}\right\|^2_{L^2(\Sigma)} + \int_Q \frac{1}{s^2\varphi^2}|f|^2 e^{2s\alpha} dxdt \\ &+ \sum_{j=1}^3 \int_Q |f_j|^2 e^{2s\alpha} dxdt + \int_{Q_\omega} s\varphi|y|^2 e^{2s\alpha} dxdt\Bigg) \end{aligned} \tag{2.4}$$

*for all* $s \ge \widehat{s}$.

Here and henceforth we set

$$\|g\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} = \|g\|_{L^2(0,T;H^{\frac{1}{2}}(\partial\Omega))} + \|g\|_{H^{\frac{1}{4}}(0,T;L^2(\partial\Omega))}.$$

**Remark.** The estimates (2.3) and (2.4) are proved in [18] and [19] under assumption (1.6). However the proof is performed locally, using a partition of unity $\{e_j\}_{j=1}^{\mathcal{N}}$. Let us choose a partition of unity such that if $\mathrm{supp}\, e_j \cap \overline{\Gamma_1} \ne \emptyset$, then $dist(\mathrm{supp}\, e_j, \partial\omega \setminus \partial\Omega) > 0$. In the case of $\mathrm{supp}\, e_j \cap \overline{\Gamma_1} \ne \emptyset$, for the function $e_j y$, we use the standard energy estimate for the parabolic or elliptic equations. In the case of $\mathrm{supp}\, e_j \cap \overline{\Gamma_1} = \emptyset$, we directly apply the Carleman estimates established in [18] and [19].

**Proof.** Set $z = \mathrm{rot}\, v$. Then, noting that $\mathrm{rot}\,\mathrm{rot}\, v = -\Delta v$ by $\mathrm{div}\, v = 0$, we have

$$\begin{cases} Lz \equiv \partial_t z - \nu\Delta z + (A\cdot\nabla)z \\ =\mathrm{rot}\, F - \sum_{j=1}^{3} \nabla A_j \times \partial_j v - \sum_{j=1}^{3} \nabla v_j \times \partial_j B - (v\cdot\nabla)\mathrm{rot}\, B, \\ \Delta v = -\mathrm{rot}\, z \quad \text{in } Q. \end{cases}$$

Let $\chi$ be a smooth function such that $\chi|_{\Omega\setminus\omega} = 1$ and $\mathrm{supp}\,\chi \cap \overline{\Gamma}_1 = \emptyset$. Then

$$L(\chi z) = [\chi, L]z + \chi\left(\mathrm{rot}\, F - \sum_{j=1}^{3} \nabla A_j \times \partial_j v - \sum_{j=1}^{3} \nabla v_j \times \partial_j B - (v\cdot\nabla)\mathrm{rot}\, B\right). \tag{2.5}$$

To equation (2.5), we apply the Carleman estimate stated in Lemma 2.

Indeed, by (1.4), we have $(\partial_k A_j)(\partial_j v_m) = \partial_j((\partial_k A_j)v_m) - (\partial_j\partial_k A_j)v_m$ etc., we rewrite the right-hand side of the first equation in (2.5) by the form $f + \sum_{j=1}^{3} \partial_j f_j$. Applying (2.4), for all sufficiently large $s$ we have

$$\begin{aligned} & s\int_Q \varphi|z|^2 e^{2s\alpha}dxdt = s\int_{\Omega\setminus\omega}\int_0^T \varphi|\chi z|^2 e^{2s\alpha}dxdt + s\int_\omega\int_0^T \varphi|\chi z|^2 e^{2s\alpha}dxdt \\ \le & C\Bigg(s^{-\frac12}||\varphi^{-\frac14} z e^{s\alpha}||^2_{H^{\frac12,\frac14}(\Sigma_1)} + s^{-\frac12}||\varphi^{-\frac18} z e^{s\alpha}||^2_{L^2(\Sigma_1)} \\ & + \int_Q |v|^2 e^{2s\alpha}dxdt + \int_Q |F|^2 e^{2s\alpha}dxdt + \int_{Q_\omega} s\varphi|z|^2 e^{2s\alpha}dxdt\Bigg) \\ & \le C\Bigg(s^{-\frac12}\left\|\varphi^{-\frac14}\frac{\partial v}{\partial n}e^{s\alpha}\right\|^2_{H^{\frac12,\frac14}(\Sigma_1)} + s^{-\frac12}\left\|\varphi^{-\frac18}\frac{\partial v}{\partial n}e^{s\alpha}\right\|^2_{L^2(\Sigma_1)} \\ & + \int_Q |v|^2 e^{2s\alpha}dxdt + \int_Q |F|^2 e^{2s\alpha}dxdt + \int_{Q_\omega} s\varphi|\nabla v|^2 e^{2s\alpha}dxdt\Bigg) \end{aligned} \tag{2.6}$$

where $\Sigma_1 = [0,T] \times (\operatorname{supp}\chi \cap \partial\Omega)$.

Here we used that $v|_{\partial\Omega} = 0$, and $z|_\Sigma$ is given by only $\frac{\partial v}{\partial n}|_\Sigma$.

To equations $\Delta(\chi v) = -\chi \mathrm{rot}\, z + [\chi, \Delta]v$ in $Q$ and $\chi v|_{\partial\Omega} = 0$, we apply estimate (2.3). Therefore, fixing $\lambda > 0$ sufficiently large, for all $s \ge \widehat{s}$ and all $t$ from the interval $(0,T)$ we have

$$\begin{aligned}
&s\int_\Omega e^{2\lambda\eta(x)}|v(x,t)|^2 e^{2se^{\lambda\eta(x)}}dx\\
\le& C\left(\int_\Omega e^{\lambda\eta(x)}|z(x,t)|^2 e^{2se^{\lambda\eta(x)}}dx + \int_\omega (|\nabla v(x,t)|^2 + se^{2\lambda\eta(x)}|v(x,t)|^2)e^{2se^{\lambda\eta(x)}}dx\right).
\end{aligned}$$

Dividing both sides of the above inequality by $\ell^8(t)$ and using $e^{2\lambda\eta(x)} \ge e^{\lambda\eta(x)}$ in $\Omega$, we have that for all $s \ge \widehat{s}$ and $t$ from interval $(0,T)$

$$\begin{aligned}
&s\int_\Omega \varphi(x,t)|v(x,t)|^2 e^{2se^{\lambda\eta(x)}}dx \le C\Bigg(\int_\Omega \varphi(x,t)|z(x,t)|^2 e^{2se^{\lambda\eta(x)}}dx\\
&+\int_\omega \left(\frac{1}{\ell^8(t)}|\nabla v(x,t)|^2 + s\ell^8(t)\varphi^2(x,t)|v(x,t)|^2\right)e^{2se^{\lambda\eta(x)}}dx\Bigg). \qquad (2.7)
\end{aligned}$$

Let $s \ge s_1 \equiv \widehat{s}\max_{0\le t\le T}\ell^8(t)$. Then $\frac{s}{\ell^8(t)} \ge \widehat{s}$ for $0 \le t \le T$. Hence substituting $\frac{s}{\ell^8(t)}$ instead of $s$ in (2.7), we obtain

$$\begin{aligned}
&\frac{s}{\ell^8(t)}\int_\Omega \varphi(x,t)|v(x,t)|^2 e^{2s\frac{e^{\lambda\eta(x)}}{\ell^8(t)}}dx \le C\Bigg(\int_\Omega \varphi(x,t)|z(x,t)|^2 e^{2s\frac{e^{\lambda\eta(x)}}{\ell^8(t)}}dx\\
&+\int_\omega \left(\frac{1}{\ell^8(t)}|\nabla v(x,t)|^2 + s\varphi^2(x,t)|v(x,t)|^2\right)e^{2s\frac{e^{\lambda\eta(x)}}{\ell^8(t)}}dx\Bigg), \quad 0 \le t \le T.
\end{aligned}$$

Multiply $e^{-2s\frac{2\|\eta\|_{C^0(\overline{\Omega})}}{\ell^8(t)}}$, and we have

$$\begin{aligned}
&\frac{s}{\ell^8(t)}\int_\Omega \varphi(x,t)|v(x,t)|^2 e^{2s\alpha(x,t)}dx \le C\left(\int_\Omega \varphi(x,t)|z(x,t)|^2 e^{2s\alpha(x,t)}dx\right.\\
&\left.+\int_\omega \left(\frac{1}{\ell^8(t)}|\nabla v(x,t)|^2 + s\varphi^2(x,t)|v(x,t)|^2\right)e^{2s\alpha(x,t)}dx\right), \quad 0 \le t \le T
\end{aligned}$$

for all sufficiently large positive $s$. Integrating over $t \in [0,T]$, we have

$$\begin{aligned}
&s\int_Q \varphi^2|v|^2 e^{2s\alpha}dxdt \le sC_1\int_Q \frac{1}{\ell^8(t)}\varphi|v|^2 e^{2s\alpha}dxdt\\
\le& C_2\left(\int_Q \varphi(x,t)|z(x,t)|^2 e^{2s\alpha}dxdt + \int_{Q_\omega}(\varphi|\nabla v|^2 + s\varphi^2|v|^2)e^{2s\alpha}dxdt\right) \qquad (2.8)
\end{aligned}$$

for all sufficiently large positive $s$. Therefore, combining (2.6) and (2.8) and absorbing the terms with lower powers of $s$ into the left-hand side, we have

$$\begin{aligned}&\int_Q (s\varphi|\mathrm{rot}\, v|^2 + s^2\varphi^2|v|^2)e^{2s\alpha}dxdt\\ \le& C_3\left( s^{-\frac{1}{2}}\left\|\varphi^{-\frac{1}{4}}\frac{\partial v}{\partial n}e^{s\widehat{\alpha}}\right\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_1)} + s^{-\frac{1}{2}}\left\|\varphi^{-\frac{1}{8}}\frac{\partial v}{\partial n}e^{s\widehat{\alpha}}\right\|^2_{L^2(\Sigma_1)} + \int_Q |F|^2e^{2s\alpha}dxdt\right.\\ &\left.+\int_{Q_\omega}(s\varphi|\mathrm{rot}\, v|^2 + s^2\varphi^2|v|^2 + s\varphi|\nabla v|^2)e^{2s\alpha}dxdt\right) \qquad (2.9)\end{aligned}$$

for all large $s > 0$. Here and henceforth, noting that $\eta = 0$ on $\partial\Omega\setminus\Gamma_1$ in both cases (1.6) and (1.7), we have

$$\alpha(x,t) = \widehat{\alpha}(t) \equiv \frac{1 - e^{2\lambda\|\eta\|_{C^0(\overline{\Omega})}}}{\ell^8(t)}, \quad x \in \partial\Omega\setminus\Gamma_1.$$

We need to estimate the first and the second terms on the right-hand side of (2.9). We set

$$W(t,x) = \ell(t)v(t,x)e^{s\widehat{\alpha}(t)}, \quad r(t,x) = \ell(t)p(t,x)e^{s\widehat{\alpha}(t)}.$$

Then we have

$$\begin{cases}\partial_t W - \nu\Delta W + (A\cdot\nabla)W + (W\cdot\nabla)B + \nabla r = \ell e^{s\widehat{\alpha}}F + \ell' v e^{s\widehat{\alpha}} + \ell s\widehat{\alpha}' e^{s\widehat{\alpha}}v,\\ \mathrm{div}\, W = 0 \quad \text{in } Q, \quad W = 0 \quad \text{on } \Sigma,\\ W(\cdot,0) = 0 \quad \text{in } \Omega.\end{cases} \qquad (2.10)$$

On the other hand,

**Lemma 3.** *Let $F \in L^2(0,T;H)$ and $Z_0 \in V$. Then for the boundary value problem*

$$\begin{cases}\partial_t Z - \nu\Delta Z + (A\cdot\nabla)Z + (Z\cdot\nabla)B + \nabla r = F \ \ in\ Q,\\ div\, Z = 0 \quad in\ Q, \qquad \int_\Omega r(t,x)dx = 0,\ \forall t\in(0,T),\\ Z = 0 \qquad on\ \Sigma,\\ Z(\cdot,0) = Z_0,\end{cases}$$

*there exists a unique solution* $Z \in H^{1,2}(Q)$ *such that*

$$\|Z\|_{H^{2,1}(Q)} + \|r\|_{L^2(0,T;H^1(\Omega))} \le C_4(\|Z_0\|_{H^1(\Omega)} + \|F\|_{L^2(Q)}).$$

We can prove the lemma similarly to Proposition 1.2 (pp.267-268) in Temam [41] where $A = B = 0$ is assumed.

We note that

$$\ell s\widehat{\alpha}' e^{s\widehat{\alpha}} v = -8\ell'\widehat{\alpha} s e^{s\widehat{\alpha}} v = 8\ell'(e^{2\lambda\|\eta\|_{C^0(\overline{\Omega})}} - 1)s\varphi v e^{s\widehat{\alpha}} \quad \text{on } \Sigma$$

and

$$|\ell s\widehat{\alpha}' e^{s\widehat{\alpha}} v| \le C_5 s|\varphi v e^{s\widehat{\alpha}}| \quad \text{on } \Sigma. \tag{2.11}$$

Applying Lemma 3 in (2.10), in view of (2.11) and $\alpha(t,x) \ge \widehat{\alpha}(t)$ for $(x,t) \in Q$ by $\eta|_\Omega > 0$, we obtain

$$\begin{aligned} &\|W\|_{H^{2,1}(Q)} + \|r\|_{L^2(0,T;H^1(\Omega))} \\ \le& C_6(\|e^{s\widehat{\alpha}} F\|_{L^2(Q)} + \|s\varphi v e^{s\widehat{\alpha}}\|_{L^2(Q)}) \le C_6(\|e^{s\alpha} F\|_{L^2(Q)} + \|s\varphi v e^{s\alpha}\|_{L^2(Q)}). \end{aligned}$$

By Theorem 2.1 (p.9) in Lions and Magenes [39], we have

$$\left\|\frac{\partial W}{\partial n}\right\|_{H^{\frac12,\frac14}(\Sigma)} \le C_7\|W\|_{H^{2,1}(Q)}$$

and so

$$\left\|\frac{\partial W}{\partial n}\right\|_{H^{\frac12,\frac14}(\Sigma)} \le C_8(\|Fe^{s\alpha}\|_{L^2(Q)} + \|s\varphi v e^{s\alpha}\|_{L^2(Q)}). \tag{2.12}$$

Estimate (2.12) yields

$$\begin{aligned} &\left\|\varphi^{-\frac14}\frac{\partial v}{\partial n} e^{s\widehat{\alpha}}\right\|_{H^{\frac12,\frac14}(\Sigma_1)} = \left\|\ell^2 \frac{\partial v}{\partial n} e^{s\widehat{\alpha}}\right\|_{H^{\frac12,\frac14}(\Sigma_1)} \\ \le& C_9\left\|\ell\frac{\partial W}{\partial n}\right\|_{H^{\frac12,\frac14}(\Sigma_1)} \le C_{10}\left\|\frac{\partial W}{\partial n}\right\|_{H^{\frac12,\frac14}(\Sigma_1)} \\ \le& C_{11}(\|Fe^{s\alpha}\|_{L^2(Q)} + \|s\varphi v e^{s\alpha}\|_{L^2(Q)}). \end{aligned} \tag{2.13}$$

Moreover

$$\begin{aligned}&\left\|\varphi^{-\frac{1}{8}}\frac{\partial v}{\partial n}e^{s\widehat{\alpha}}\right\|_{L^2(\Sigma_1)}=\left\|\ell\frac{\partial v}{\partial n}e^{s\widehat{\alpha}}\right\|_{L^2(\Sigma_1)}=\left\|\frac{\partial W}{\partial n}\right\|_{L^2(\Sigma_1)}\\ \le&C_{12}(\|Fe^{s\alpha}\|_{L^2(Q)}+\|s\varphi ve^{s\alpha}\|_{L^2(Q)}).\end{aligned}\tag{2.14}$$

Choosing $s>0$ large, in view of (2.13) and (2.14), we can absorb the first and the second terms on the right-hand side of (2.9) into the left-hand side.

Finally we have to estimate $\|(\nabla v)e^{s\alpha}\|_{L^2(Q)}$. Since $-\Delta = \mathrm{rot}\,\mathrm{rot} - \nabla\mathrm{div}$ and $\mathrm{div}\, v=0$, we have

$$-\Delta(ve^{s\alpha})=\mathrm{rot}\,\mathrm{rot}\,(ve^{s\alpha})-v\cdot\nabla e^{s\alpha}.$$

Taking the scalar product of this equation with function $ve^{s\alpha}$ in $L^2(\Omega)$, and applying $|\nabla\alpha|\le C_{13}\varphi$ in $Q$ and

$$\int_\Omega(\mathrm{rot}\,U)\cdot V dx=\int_\Omega U\cdot(\mathrm{rot}\,V)dx,\quad U,V\in H_0^1(\Omega)^3,$$

we obtain the estimate

$$\|(\nabla v)e^{s\alpha})\|^2_{L^2(Q)}\le C_{13}(\|(\mathrm{rot}\,v)e^{s\alpha}\|^2_{L^2(Q)}+\|s\varphi ve^{s\alpha}\|^2_{L^2(Q)}).$$

Thus the proof of Theorem 2 is completed. ■

We conclude this section with Carleman estimates for a first-order equation. The proof is done by integration by parts (see e.g. Lemma 3.2 in [26]).

**Lemma 4.** *Let* $\beta\in C^2(\overline{\Omega})$ *and*

$$(Lf)(x)=\sum_{j=1}^3 a_j(x)\partial_j f(x),\quad x\in\Omega,$$

*where* $a_j\in C^1(\overline{\Omega})$, $1\le j\le 3$, *and let us set*

$$\mu(x)=\sum_{j=1}^3 a_j(x)\partial_j\beta(x),\quad x\in\Omega.$$

*Then there exists a number* $\widehat{\lambda} > 0$ *such that for any* $\lambda > \widehat{\lambda}$*, we can choose* $s_3 = s_3(\lambda) > 0$ *satisfying: there exists a constant* $C = C(\Omega, \lambda) > 0$ *such that*

$$s^2 \int_\Omega \left( \mu^2(x) - \frac{C}{s} \right) |f|^2 e^{2s\beta(x)} dx \le C \int_\Omega |Lf|^2 e^{2s\beta(x)} dx$$

*for all* $s \ge s_3$ *and all* $f \in H_0^1(\Omega)$*.*

## §3. Proof of Theorem 1.

Let us set $w_1 = \partial_t v$ and $w_2 = \partial_t^2 w$. Then

$$\partial_t v - \nu \Delta v + (A \cdot \nabla) v + (v \cdot \nabla) B + \nabla p = Rf$$

and

$$\partial_t w_1 - \nu \Delta w_1 + (A \cdot \nabla) w_1 + (\partial_t A \cdot \nabla) v + (w_1 \cdot \nabla) B + (v \cdot \nabla) \partial_t B$$
$$+ \nabla \partial_t p = (\partial_t R) f$$

and

$$\partial_t w_2 - \nu \Delta w_2 + (A \cdot \nabla) w_2 + 2(\partial_t A \cdot \nabla) w_1 + (\partial_t^2 A \cdot \nabla) v$$
$$+ (w_2 \cdot \nabla) B + 2(w_1 \cdot \nabla) \partial_t B + (v \cdot \nabla) \partial_t^2 B + \nabla \partial_t^2 p = (\partial_t^2 R) f$$

and

$$\operatorname{div} v = \operatorname{div} w_1 = \operatorname{div} w_2 = 0 \quad \text{in } Q$$

and

$$v = w_1 = w_2 = 0 \qquad \text{on } \Sigma.$$

Here and henceforth we set

$$\mathcal{D} = \|v\|_{H^2(0,T;H^1(\omega))} + \|\operatorname{rot} v(\cdot, \theta)\|_{H^2(\Omega)} + \|v(\cdot, \theta)\|_{H^1(\Omega)}. \tag{3.1}$$

Therefore applications of Theorem 2 to $v, w_1, w_2$ yield

$$\begin{aligned}
&\int_Q (|\nabla v|^2 + s\varphi|\mathrm{rot}\, v|^2 + s^2\varphi^2|v|^2)e^{2s\alpha}dxdt \\
\le& C_1 \left( \int_Q |Rf|^2 e^{2s\alpha}dxdt + e^{Cs}\mathcal{D}^2 \right)
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\int_Q (|\nabla w_1|^2 + s\varphi|\mathrm{rot}\, w_1|^2 + s^2\varphi^2|w_1|^2)e^{2s\alpha}dxdt \\
\le& C_1 \Bigg( \int_Q |(\partial_t R)f|^2 e^{2s\alpha}dxdt + e^{Cs}\mathcal{D}^2 \\
&+ \int_Q (|(\partial_t A \cdot \nabla)v|^2 + |(v \cdot \nabla)\partial_t B|^2)e^{2s\alpha}dxdt \Bigg)
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\int_Q (|\nabla w_2|^2 + s\varphi|\mathrm{rot}\, w_2|^2 + s^2\varphi^2|w_2|^2)e^{2s\alpha}dxdt \le C_1 \Bigg( \int_Q |(\partial_t^2 R)f|^2 e^{2s\alpha}dxdt + e^{Cs}\mathcal{D}^2 \\
&+ \int_Q (|(\partial_t A \cdot \nabla)w_1|^2 + |(\partial_t^2 A \cdot \nabla)v|^2 + |(w_1 \cdot \nabla)\partial_t B|^2 + |(v \cdot \nabla)\partial_t^2 B|^2)e^{2s\alpha}dxdt \Bigg).
\end{aligned} \tag{3.4}$$

Hence, by (1.4) and (1.8), we apply estimate (3.2) in (3.3) and estimate (3.3) in (3.4) successively, in order to obtain

$$\begin{aligned}
&\int_Q \sum_{j=0}^{2} (|\partial_t^j \nabla v|^2 + s\varphi|\partial_t^j \mathrm{rot}\, v|^2 + s^2\varphi^2|\partial_t^j v|^2)e^{2s\alpha}dxdt \\
\le& C_2 \left( \int_Q |f|^2 e^{2s\alpha}dxdt + e^{Cs}\mathcal{D}^2 \right)
\end{aligned} \tag{3.5}$$

for all large $s > 0$. We set $\beta(x) = \alpha(x, \theta)$. Noting that $e^{2s\alpha(x,0)} = 0$ for $x \in \overline{\Omega}$, we have

$$\begin{aligned}
&C_3^{-1} \int_\Omega |\partial_t \mathrm{rot}\, v(x,\theta)|^2 e^{2s\beta(x)}dx \le \int_\Omega \varphi(x,\theta)^{-1}|\partial_t \mathrm{rot}\, v(x,\theta)|^2 e^{2s\beta(x)}dx \\
=& \int_\Omega \frac{\partial}{\partial t}\left( \int_0^\theta \varphi(x,t)^{-1}|\partial_t \mathrm{rot}\, v|^2 e^{2s\alpha}dt \right) dx \\
=& \int_\Omega \int_0^\theta \{2\varphi^{-1}(\partial_t \mathrm{rot}\, v \cdot \partial_t^2 \mathrm{rot}\, v) + 2s\varphi^{-1}(\partial_t \alpha)|\partial_t \mathrm{rot}\, v|^2 + \partial_t(\varphi^{-1})|\partial_t \mathrm{rot}\, v|^2\}e^{2s\alpha}dxdt \\
\le& C_4 \int_Q (|\partial_t \mathrm{rot}\, v|^2 + |\partial_t^2 \mathrm{rot}\, v|^2 + s\varphi|\partial_t \mathrm{rot}\, v|^2)e^{2s\alpha}dxdt.
\end{aligned}$$

Here we used

$$|\partial_t \alpha(x,t)| = \left| \frac{8\ell'(t)}{\ell^9(t)}(e^{\lambda\eta(x)} - e^{2\lambda\|\eta\|_{C^0(\overline{\Omega})}}) \right| \le C_4 \varphi^2(x,t), \quad (x,t) \in Q.$$

Hence for all large $s > 0$ inequality (3.5) implies

$$\int_\Omega |\partial_t \mathrm{rot}\, v(x,\theta)|^2 e^{2s\beta(x)} dx \le C_5 \left( \int_Q |f|^2 e^{2s\alpha} dxdt + e^{Cs}\mathcal{D}^2 \right). \tag{3.6}$$

On the other hand, applying the operator *rot* to (1.1) and noting

$$\mathrm{rot}\,(A\cdot\nabla)B = \sum_{j=1}^3 \nabla A_j \times \partial_j B + (A\cdot\nabla)\mathrm{rot}\, B$$

for $A = (A_1, A_2, A_3)^T$ and $B = (B_1, B_2, B_3)^T$, we have

$$\begin{aligned} &\mathrm{rot}\,(R(x,\theta)f(x)) = \partial_t \mathrm{rot}\, v(x,\theta) - \nu\Delta \mathrm{rot}\, v(x,\theta) \\ &+ \sum_{j=1}^3 \nabla A_j(x,\theta) \times \partial_j v(x,\theta) + (A\cdot\nabla)\mathrm{rot}\, v(x,\theta) \\ &+ \sum_{j=1}^3 \nabla v_j(x,\theta) \times \partial_j B(x,\theta) + (v\cdot\nabla)\mathrm{rot}\, B(x,\theta), \quad x \in \Omega. \end{aligned}$$

Therefore

$$|\mathrm{rot}\,(R(x,\theta)f(x))| e^{s\beta} \le |\partial_t \mathrm{rot}\, v(x,\theta)| e^{s\beta} + C_6 e^{s\beta}\mathcal{E}(x), \quad x \in \Omega,$$

where we set

$$\mathcal{E} = \sum_{|\gamma|\le 2} \|\partial_x^\gamma \mathrm{rot}\, v(\cdot,\theta)\|_{L^2(\Omega)} + \|\nabla v(\cdot,\theta)\|_{L^2(\Omega)} + \|v(\cdot,\theta)\|_{L^2(\Omega)}.$$

Hence for all sufficiently large positive $s$, inequality (3.6) implies

$$\begin{aligned} &\int_\Omega |\mathrm{rot}\,(R(x,\theta)f(x))|^2 e^{2s\beta} dx \le C_7 \left( \int_\Omega |\partial_t(\mathrm{rot}\, v)(x,\theta)|^2 e^{2s\beta} dx + e^{Cs}\int_\Omega \mathcal{E}^2(x) dx \right) \\ \le &C_8 \left( \int_Q |f|^2 e^{2s\alpha} dxdt + e^{Cs}(\mathcal{D}^2 + \mathcal{E}^2) \right). \end{aligned} \tag{3.7}$$

On the other hand, setting $R(x,\theta) = a(x) \equiv (a_1(x), a_2(x), a_3(x))^T$, we have

$$\begin{aligned}
&\mathrm{rot}\,(R(x,\theta)f(x)) = \nabla f(x) \times a(x) + f(x)\mathrm{rot}\, a(x)\\
=&((a_3\partial_2 f - a_2\partial_3 f), (a_1\partial_3 f - a_3\partial_1 f), (a_2\partial_1 f - a_1\partial_2 f))^T + f(x)\mathrm{rot}\, a(x)\\
\equiv&(L_1 f, L_2 f, L_3 f)^T + ([\mathrm{rot}\, a]_1 f, [\mathrm{rot}\, a]_2 f, [\mathrm{rot}\, a]_3 f)^T. \qquad (3.8)
\end{aligned}$$

We recall that $[a]_k$ denotes the $k$-th component of a vector $a$. Note that

$$\partial_j \beta = \frac{\lambda}{\ell^8(\theta)} e^{\lambda\eta} \partial_j \eta, \quad j = 1, 2, 3.$$

Denote that

$$\mu_1(x) = \begin{pmatrix} 0 \\ a_3 \\ -a_2 \end{pmatrix} \cdot \begin{pmatrix} \partial_1\beta \\ \partial_2\beta \\ \partial_3\beta \end{pmatrix} = \frac{\lambda}{\ell^8(\theta)} e^{\lambda\eta}(a_3\partial_2\eta - a_2\partial_3\eta),$$

$$\mu_2(x) = \begin{pmatrix} -a_3 \\ 0 \\ a_1 \end{pmatrix} \cdot \begin{pmatrix} \partial_1\beta \\ \partial_2\beta \\ \partial_3\beta \end{pmatrix} = \frac{\lambda}{\ell^8(\theta)} e^{\lambda\eta}(a_1\partial_3\eta - a_3\partial_1\eta),$$

$$\mu_3(x) = \begin{pmatrix} a_2 \\ -a_1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_1\beta \\ \partial_2\beta \\ \partial_3\beta \end{pmatrix} = \frac{\lambda}{\ell^8(\theta)} e^{\lambda\eta}(a_2\partial_1\eta - a_1\partial_2\eta),$$

that is,

$$(\mu_1(x), \mu_2(x), \mu_3(x))^T = \frac{\lambda e^{\lambda\eta}}{\ell^8(\theta)} (\nabla\eta \times a(x)).$$

We prove only for case (i) because the proof in case (ii) is very similar. Applying Lemma 4 to the first-order differential operators $L_1, L_2, L_3$ in $\Omega \setminus \overline{\omega}$, in view of $f|_\omega = 0$, we have

$$\begin{aligned}
&s^2 \int_{\Omega\setminus\overline{\omega}} \left( \mu_1(x)^2 + \mu_2(x)^2 + \mu_3(x)^2 - \frac{3C_9}{s} \right) |f(x)|^2 e^{2s\beta} dx\\
\le&C_{10} \int_{\Omega\setminus\overline{\omega}} (|\mathrm{rot}\,(R(x,\theta)f(x))|^2 + |f(x)\mathrm{rot}\, a(x)|^2) e^{2s\beta} dx.
\end{aligned}$$

Therefore (3.7) yields

$$\begin{aligned}
&s^2 \int_{\Omega\setminus\overline{\omega}} \left( \lambda^2 e^{2\lambda\eta} |\nabla\eta \times a|^2 - \frac{3C_9}{s} \right) |f(x)|^2 e^{2s\beta} dx\\
\le&C_{10} \left( \int_0^T \int_{\Omega\setminus\overline{\omega}} |f|^2 e^{2s\alpha} dx dt + \int_{\Omega\setminus\overline{\omega}} |f(x)|^2 e^{2s\beta} dx + e^{Cs}\mathcal{D}^2 \right).
\end{aligned}$$

Taking $s>0$ sufficiently large, in view of (1.9), we can absorb the second term on the right-hand side into the left-hand side. Hence

$$s^2\int_{\Omega\setminus\overline{\omega}}|f(x)|^2e^{2s\beta}dx\le C_{11}\left(\int_0^T\int_{\Omega\setminus\overline{\omega}}|f|^2e^{2s\alpha}dxdt+e^{Cs}\mathcal{D}^2\right).\qquad(3.9)$$

Since $\alpha(x,\theta)=\beta(x)\ge\alpha(x,t)$ for $(x,t)\in Q$ by the third condition in (2.1), inequality (3.9) yields

$$s^2\int_{\Omega\setminus\overline{\omega}}|f(x)|^2e^{2s\beta}dx\le C_{11}\left(\int_{\Omega\setminus\overline{\omega}}|f|^2e^{2s\beta}dx+e^{Cs}\mathcal{D}^2\right),$$

and choosing $s>0$ large, we can absorb the first term on the right-hand side into the left-hand side. Thus the proof in Case (i) is completed. ■

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The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

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