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Abstract

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega \in C^\infty$ and L be a positive number. For a three dimensional cylindrical domain $Q = \Omega \times (0, L)$, we obtain some uniqueness result of determining a complex-valued potential for the Schrödinger equation from partial Cauchy data when Dirichlet data vanish on a subboundary $(\partial\Omega \setminus \widetilde{\Gamma}) \times [0, L]$ and the corresponding Neumann data are observed on $\widetilde{\Gamma} \times [0, L]$, where $\widetilde{\Gamma}$ is an arbitrary fixed open set of $\partial\Omega$.

This article is concerned with the inverse boundary value problem of determination of a complex-valued potential for the Schrödinger equation in a cylindrical domain from partial boundary data. More precisely the problem is as follow. Let $Q = \Omega \times (0, L)$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain with $\partial\Omega \in C^\infty$. Let $\widetilde{\Gamma}$ be an arbitrary, fixed subdomain of $\partial\Omega$. Denote $\Gamma_0 = Int(\partial\Omega \setminus \widetilde{\Gamma})$, $\widetilde{\Sigma} = \widetilde{\Gamma} \times [0, L]$ and $\Sigma_0 = \Gamma_0 \times [0, L]$.

In Q, we consider the Schrödinger equation with some complex-valued potential q:

$$L_q(x, D)u = (\Delta + q)u = 0 \quad \text{in } Q. \tag{1}$$

Consider the following Dirichlet-to-Neumann map Λ_{q,Σ_0}

$$\Lambda_{q,\Sigma_0} f = \frac{\partial u}{\partial \nu}|_{\partial Q\setminus\Sigma_0}, \quad \text{where } L_q(x, D)u = 0 \quad \text{in } Q, \quad u|_{\Sigma_0} = 0, \quad u|_{\partial Q\setminus\Sigma_0} = f \tag{2}$$

with domain

$$D(\Lambda_{q,\Sigma_0}) = \{f \in H^{\frac{1}{2}}(\partial Q)|\text{supp} f \subset \partial Q \setminus \Sigma_0, \ (f, g)_{L^2(\partial Q)} = 0 \quad \forall g \in \mathcal{N}\}$$

and

$$\mathcal{N} = \{g|L_q(x, D)u = 0 \quad \text{in } Q, \quad u|_{\partial Q} = 0, \quad \frac{\partial u}{\partial \nu}|_{\partial Q} = g\}.$$

The problem (1) and (2) is the generalization of the inverse boundary value problem of recovery of the conductivity, which is also known as Calderón's problem (see [2]). It is

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related to many practical applications, for example, detecting oil or minerals by applying voltage and measuring the fluxes on earth's surface. See also Cheney, Issacson and Newell [3] for applications to medical imaging of EIT.

In case when Q is a general domain in $\mathbb{R}^n$ with $n \geq 2$, $\Sigma_0 = \emptyset$ (i.e., the case of full Dirichlet-to-Neumann map), the unique recovery of the conductivity was established in [14] and [17] in two and three dimensional cases respectively. For reconstruction of the conductivity see [15]. The assumption $\Sigma_0 = \emptyset$ means that one has to set up voltages and measure the fluxes on the whole boundary. From the practical point of view this assumption is very restrictive. In practice, this assumption does not often hold, for example because the domain Q is extremely large or we can not access to some part of ∂Q, e.g., the domain has cavities located inside. For the inverse boundary value problem with such partial Dirichlet-to-Neumann map, we refer to the following works. In [1] Bukhgeim and Uhlmann show that if voltages are applied on the boundary ∂Q_- and the corresponding fluxes are measured on some part ∂Q_+ which is approximately equal to $\partial Q \backslash \partial Q_-$, then the potential can be uniquely determined. This result and a recent improved result [10] still require the access to the whole boundary ∂Q. In [9], Isakov solves the case where voltage applied and current measured on the same set ∂Q_- provided that subboundary $\partial Q \setminus \partial Q_-$ is a part of some sphere or some plane. All the above mentioned papers treat the case where the spatial dimension more or equal 3. As for related work in slabs, see Ikehata [5], Krupchyk, Lassas and Uhlmann [11], Li and Uhlmann [13].

For general two dimensional domain, [6] proved the unique recovery of a potential for the Schrödinger equation in the case when voltage applied and flux measured both on an arbitrary open set of ∂Q. Thus [6] established the best possible uniqueness in the two dimensional case with data Λ_{q,Σ_0} defined by (2). Also see [7] which deals with the same inverse problem for more general second-order elliptic equations in the two dimensional case and [8] improves the result of [6] in terms of regularity assumption of potential for the Schrödinger equation. The conditional stability results for Calderon's problem are obtained by Novikov in [15], [16].

The purpose of this article is to establish the uniqueness with weak constraints on such subboundary in the case of three dimensional cylindrical domain Q. Our proof is based on the generalized Radon transform.

By our method we can obtain the uniqueness results for potentials in more general domains (not only cylindrical one) but we do not discuss details here.

We introduce the subset $\mathcal{O}$ of domain Ω

$$\mathcal{O} = \Omega \setminus Ch(\bar\Gamma_0), \quad Ch(\bar\Gamma_0) = \{x | x = \lambda x^1 + (1-\lambda)x^2, x^1, x^2 \in \overline{\Gamma_0}, \lambda \in (0,1)\}.$$

We have

Theorem 1 *Let q_1, q_2 be Lipschitz functions. If $\Lambda_{q_1,\Sigma_0} = \Lambda_{q_2,\Sigma_0}$ and $D(\Lambda_{q_1,\Sigma_0}) \subset D(\Lambda_{q_2,\Sigma_0})$, then $q_1 = q_2$ in $\mathcal{O} \times [0, L]$.*

From theorem 1 we obtain immediately

Corollary 2 *Let Ω be concave near Γ_0 and potentials q_1, q_2 be Lipschitz functions such that $\Lambda_{q_1,\Sigma_0} = \Lambda_{q_2,\Sigma_0}$ and $D(\Lambda_{q_1,\Sigma_0}) \subset D(\Lambda_{q_2,\Sigma_0})$. Then $q_1 = q_2$ in Q.*

First we formulate the following Carleman estimate with the linear weight function $\varphi = x_3$ for the Schrödinger operator (1). Denote $\|\cdot\|_{H^{1,\tau}(Q)} = \|\cdot\|_{H^1(Q)} + |\tau|\|\cdot\|_{L^2(Q)}$. In [1] the following theorem is proved:

Theorem 3 *Let $q \in L^\infty(Q)$. There exist τ_0 and a constant C independent of τ such that for all $\tau \geq \tau_0$*

$$\|ue^{\tau\varphi}\|_{H^{1,\tau}(Q)} \leq C(\|(L_q(x,D)u)e^{\tau\varphi}\|_{L^2(Q)} + \sqrt{\tau}\|(\frac{\partial u}{\partial \nu}e^{\tau\varphi})(\cdot, L)\|_{L^2(\Omega)}) \quad \forall u \in H^1_0(Q). \tag{3}$$

Next we formulate some known results on the generalized Radon transform:

$$(\mathcal{R}_\mu f)(\omega, p) = \int_{<\omega,x>=p} f(x)e^{\mu<\omega^\perp,x>}ds, \quad (\omega,p) \in S^1 \times \mathbb{R}, \quad \omega^\perp = (\omega_2, -\omega_1).$$

The following theorem is proved in [12]

Theorem 4 *Let f be a Lipschitz function with compact support. If $(\mathcal{R}_\mu f)(\omega,p) = 0$ for all $p > r$ then $\mathrm{supp}\, f \subset \{x \in \mathbb{R}^2 | |x| \leq r\}$.*

Similar to Corollary 2.8 of [4] p. 14 we prove

Corollary 5 *Let f be a Lipschitz function in $\mathbb{R}^2$ with compact support and E be a bounded convex set in $\mathbb{R}^2$. If $(\mathcal{R}_\mu f)(\omega,p) = 0$ for all lines $< \omega, \tilde{x} >= p$ which do not intersect E then*

$$f(x) = 0 \quad \forall x \notin E.$$

Proof of Theorem 1. Let point $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathcal{O} \times (0, L)$. Since $Ch(\bar{\Gamma}_0)$ is the convex closed set the point $(\hat{x}_1, \hat{x}_2)$ can be separated from it by some line ℓ. Then the line which is parallel to ℓ and passes through $(\hat{x}_1, \hat{x}_2)$ does not intersect $Ch(\bar{\Gamma}_0)$. After possible rotation and translation we may assume that $\hat{x}_1 = 0, \hat{x}_2 > 0$ and axis x_2 does not intersect $Ch(\bar{\Gamma}_0)$. Therefore it suffices to prove that

$$q_1 = q_2 \quad \text{on } Q \cap \{x | x_3 \in [0,L], x_1 = 0\}.$$

Without loss of generality we may assume that $\Omega \subset \mathbb{R}^1_+$. Let m be a smooth function defined on $\mathbb{R}^1$ such that $|m'| < 1$. Denote $\nabla' = (\partial_{x_1}, \partial_{x_2})$. Consider the eiconal equation

$$|\nabla'\Psi| = 1 \quad \text{in } \Omega, \quad \Psi|_{x_2=0} = m. \tag{4}$$

This equation can be integrated by the method of characteristics. The solutions, as long as they exist, have the following form

$$\Psi(x_0 + t\alpha(x_0)\vec{e}_1 + t\beta(x_0)\vec{e}_2) = m(x_0) + t \quad \forall x_0 \in \{x' | x_2 = 0\}, \quad t > 0, \tag{5}$$

where $\alpha(x_0) = m'(x_0), \beta(x_0) = \sqrt{1 - \alpha^2(x_0)}$.

Next construct the function Ψ more explicitly using the implicit function theorem. Consider the following mapping:

$$F(y) = y_2\alpha(y_1)\vec{e}_1 + y_2\beta(y_1)\vec{e}_2 + \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \quad y = (y_1, y_2).$$

Assume that

$$\alpha(0) = m'(0) = 0. \tag{6}$$

Then

$$F(0,t) = (0,t) \tag{7}$$

and

$$F'(y) = (y_2(\alpha'(y_1)\vec{e}_1 + \beta'(y_1)\vec{e}_2) + \begin{pmatrix}1\\0\end{pmatrix}, \alpha(y_1)\vec{e}_1 + \beta(y_1)\vec{e}_2) = \begin{pmatrix} y_2\alpha' + 1 & \alpha \\ y_2\beta' & \beta \end{pmatrix}.$$

In particular

$$F'(0,y_2) = \begin{pmatrix} 1 + y_2\alpha'(0) & 0 \\ 0 & 1 \end{pmatrix}.$$

As long as the function $1 + y_2\alpha'(0)$ is positive, there exists the inverse matrix

$$(F')^{-1}(0,y_2) = \begin{pmatrix} \frac{1}{1+y_2\alpha'(0)} & 0 \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Let K be a positive number such that

$$\Omega \cap \{x' | x_1 = 0\} \subset \{x' | x_1 = 0, 0 < x_2 < K\}.$$

By (8) there exists $\epsilon(K)$ such that for any $\alpha \in \mathcal{X} = \{\phi \in C^2_0(-1,1) | \phi(0) = 0\}$ satisfying $\|\alpha\|_{C^2[-1,1]} \leq \epsilon(K)$ there exists $\tilde{\delta} > 0$ such that on the set $[-\tilde{\delta}, \tilde{\delta}] \times [0, 2K]$ the matrix $(F')^{-1}$ is correctly defined

$$(F')^{-1}(y) = \frac{1}{detF'(y)} \times \begin{pmatrix} \beta & -\alpha \\ -y_2\beta' & y_2\alpha' + 1 \end{pmatrix}.$$

Then by (7) and the implicit function theorem there exists $\delta > 0$ such that the mapping $x' \to y(x')$ is correctly defined on $\mathfrak{D} = [-\delta, \delta] \times [0, K]$ and the derivative of this mapping is given by formula:

$$\frac{\partial y}{\partial x'} = (F')^{-1}(y(x')). \tag{9}$$

Differentiating the first columns on both sides of the matrix equation (9) with respect to x_1 and using (6) we have

$$\begin{pmatrix} \frac{\partial^2 y_1}{\partial x_1^2} \\ \frac{\partial^2 y_2}{\partial x_1^2} \end{pmatrix}(0,x_2) = -\frac{y_2\alpha''(0)}{(1+y_2\alpha'(0))^3}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{(1+y_2\alpha'(0))^2}\begin{pmatrix} 0 \\ -y_2(\alpha'(0))^2 \end{pmatrix}. \tag{10}$$

Then the function Ψ can be determined by formula

$$\Psi(x') = m_0(y_1(x')) + y_2(x').$$

The short computations and (8), (10) imply

$$\Delta\Psi(0,x_2) = m_0''(0)(\partial_{x_1}y_1)^2(0,x_2) + m_0'(0)(\partial^2_{x_1}y_1)(0,x_2) + \partial^2_{x_1}y_2(0,x_2) = \frac{m_0''(0)}{(1+y_2\alpha'(0))^2}$$
$$+\frac{y_2(\alpha'(0))^2}{(1+y_2\alpha'(0))^2} = \frac{\alpha'(0)}{(1+y_2\alpha'(0))^2} + \frac{y_2(\alpha'(0))^2}{(1+y_2\alpha'(0))^2} = \frac{\alpha'(0)}{(1+y_2\alpha'(0))}. \quad (11)$$

Let $a_0(x')$ be a function such that

$$2(\nabla'\Psi, \nabla' a_0) + \Delta_{x'}\Psi a_0 = 0 \quad \text{in } \mathfrak{O}, \quad (12)$$

and $a(x')$ be a smooth function such that

$$(\nabla'\Psi, \nabla' a) = 0 \quad \text{in } \mathfrak{O}. \quad (13)$$

Next we construct the functions a_0 and a.

In order to construct the function $a(x')$ we take a smooth function r

$$r \in C_0^\infty(-\epsilon, \epsilon), \quad (14)$$

where ϵ is a small parameter and we set:

$$a(x') = r(x_0) \quad \text{on the line } \{x' \in \mathbb{R}^2 | x' = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} + t\alpha(x_0)\vec{e}_1 + t\beta(x_0)\vec{e}_2, t > 0\}. \quad (15)$$

We claim that for the function a defined by these formula we have (13). Set $\vec{v}_1 = \alpha(x_0)\vec{e}_1 + \beta(x_0)\vec{e}_2, \vec{v}_2 = \alpha(x_0)\vec{e}_1 - \beta(x_0)\vec{e}_2$. Then by (5) $|\partial_{\vec{v}_1}\Psi| = 1$. Since $|\nabla'\Psi| = 1$ we have that the vector $\vec{v}_1$ is parallel to $\nabla'\Psi$. Hence vectors $\vec{v}_j$ are orthogonal. Then

$$\partial_{\vec{v}_2}\Psi = 0.$$

Therefore

$$\partial_{\nabla'\Psi} a = |\nabla'\Psi| \partial_{\frac{\nabla'\Psi}{|\nabla'\Psi|}} a = |\nabla'\Psi| \partial_{\vec{v}_1} a = 0.$$

Hence the formula (13) is proved.

We integrate equation (12) by the characteristic method. In particular using (11) we have

$$a_0(0,x_2) = e^{-\frac{1}{2}\int_0^{x_2} \frac{\alpha'(0)}{(1+y_2\alpha'(0))}dy_2} = e^{-\frac{1}{2}ln(1+x_2\alpha'(0))} = \frac{1}{\sqrt{1+x_2\alpha'(0)}}. \quad (16)$$

Next we construct the complex geometric optics solution $u_1(x,\tau)$ for the Schrödinger operator with the potential q_1. For the principal term of complex geometric optics solution we set

$$U = e^{(\tau+N)(x_3+i\Psi(x'))} a a_0. \quad (17)$$

The set $\mathcal{O}$ is closed and the axis x_2 does not intersect this set. So there exists a neighborhood of the set $\{x' | x_2 \in [0,K]\}$ such that it does not intersect $\mathcal{O}$. Thanks to (14) and (15), choosing a positive parameter ϵ sufficiently small, we obtain

$$U|_{\Sigma_0} = 0. \quad (18)$$

The simple computations imply

$$L_{q_1}(x,D)U = (\tau+N)^2(\nabla(x_3+i\Psi),\nabla(x_3+i\Psi))U$$
$$+(\tau+N)(2(\nabla'\Psi,\nabla' a_0)+\Delta_{x'}\Psi a_0)ae^{(\tau+N)(x_3+i\Psi(x'))}$$
$$+e^{(\tau+N)(x_3+i\Psi(x'))}\Delta_{x'}(a_0a)+q_1U = e^{(\tau+N)(x_3+i\Psi(x'))}\Delta_{x'}(a_0a)+q_1U. \tag{19}$$

Observe that the functions $(e^{(\tau+N)(x_3+i\Psi(x'))}\Delta_{x'}(a_0a)+q_1U)e^{-\tau x_3}$ are uniformly bounded in τ in norm of the space $L^2(Q)$. Consequently using the results of [1] we construct the last term in complex geometric optics solution- the function $u_{cor}(\cdot,\tau)$ such that

$$L_{q_1}(x,D)(e^{\tau\varphi}u_{cor}) = -L_{q_1}(x,D)U \quad \text{in} \;\; Q, \quad u_{cor}|_{\Sigma_1}=0 \tag{20}$$

and

$$\|u_{cor}\|_{L^2(Q)} = O(\frac{1}{\tau}) \quad \text{as}\; \tau\to+\infty. \tag{21}$$

Hence, by (18), (20) and (21) we have the representation

$$u_1 = U + e^{\tau x_3}O_{L^2(Q)}(\frac{1}{\tau}) \quad \text{as} \quad \tau\to+\infty, \;\; u_1|_{\Sigma_0}=0. \tag{22}$$

(By $O_{L^2(Q)}$ we mean any function $f(\cdot,\tau)$ such that $\|f(\cdot,\tau)\|_{L^2(Q)} = O(1)$ as $\tau\to+\infty$.) Similarly we set

$$V = e^{-\tau(x_3+i\Psi(x'))}a_0, \quad V|_{\Sigma_0}=0. \tag{23}$$

We multiply any smooth function a_0 satisfying (12) with a solution of equation (13) which is supported around the ray $\{x|x_2>0, x_1=0\}$ and is equal to 1 on this ray. Hence we can assume that the support of function a_0 is concentrated around this ray and (16) holds true.

Since the Dirichlet-to-Neumann maps of the Schrödinger equations with potentials q_1, q_2 are the same there exists a solution to the following boundary value problem

$$L_{q_2}(x,D)u_2 = 0 \quad \text{in}\; Q, \quad u_1=u_2 \quad on\;\partial Q, \;\text{and}\; \frac{\partial u_1}{\partial\nu} = \frac{\partial u_2}{\partial\nu} \quad on\;\partial Q\setminus\Sigma_0. \tag{24}$$

Setting $u=u_1-u_2$ and using (24) we have

$$L_{q_2}(x,D)u = (q_1-q_2)u_1 \quad \text{in}\; Q, \quad u|_{\partial Q} = \frac{\partial u}{\partial\nu}|_{\partial Q\setminus\Sigma_0}=0. \tag{25}$$

Applying to equation (25) the Carleman estimate (3) we have that there exist constants C and τ_0 independent of τ such that

$$\|ue^{-\tau\varphi}\|_{H^{1,\tau}(Q)} \le C \quad \forall\tau\ge\tau_0. \tag{26}$$

Then taking the scalar product in $L^2(Q)$ of equation (24) with V, and using (26), (23), (22) we have

$$\int_Q (q_1-q_2)u_1V dx = \int_\Omega uL_{q_2}(x,D)V dx = O(\frac{1}{\tau}) \quad \text{as}\; \tau\to+\infty.$$

This equality and (21) imply

$$\int_Q (q_1 - q_2)e^{N(x_3+i\Psi(x'))}aa_0^2dx = O(\frac{1}{\tau}) \quad \text{as} \quad \tau \to +\infty.$$

Therefore

$$\int_Q (q_1 - q_2)e^{N(x_3+i\Psi(x'))}aa_0^2dx = 0. \tag{27}$$

Setting $p_N(x') = \int_0^L (q_1 - q_2)e^{Nx_3}dx_3$ and using (16) we obtain from (27)

$$\int_0^K p_N e^{iNx_2}dx_2 = 0. \tag{28}$$

Indeed, let $r = r_h$ be a function such that it is equal to $1/2h$ on the segment $[-h, h]$ and zero otherwise. Denote the solution to equation (13) given by (15) with the initial condition r_h as $a(h)$. By (15) the function $a_0(h)$ is given by formula

$$a(h) = \begin{cases} \frac{1}{2h} & x' \in \Pi_h \\ 0 & x' \notin \Pi_h, \end{cases}$$

where $\Pi_h = \{x' \in \mathbb{R}^2_+ | x_2 \in [0, K] \; -h + \frac{\alpha(-h)}{\beta(-h)}x_2 \le x_1 \le h + \frac{\alpha(h)}{\beta(h)}x_2\}$. Therefore for any fixed x_2 from the segment $[0, K]$ the function $a(h)$ equals $\frac{1}{2h}$ on the segment $[-h + \frac{\alpha(-h)}{\beta(-h)}x_2, h + \frac{\alpha(h)}{\beta(h)}x_2]$. By (6) the length of this segment is $2h + 2\alpha'(0)x_2h + o(h)$. We rewrite (27) as

$$0 = \int_Q (q_1-q_2)e^{N(x_3+i\Psi(x'))}a(h)a_0^2dx = \frac{1}{2h}\int_{\Pi_h} p_N e^{iN\Psi(x')}a_0^2dx' = \int_0^K \int_{-h+\frac{\alpha(-h)}{\beta(-h)}x_2}^{h+\frac{\alpha(h)}{\beta(h)}x_2} \frac{p_N e^{iN\Psi(x')}a_0^2}{2h}dx_1dx_2.$$

Applying (16) and using the assumption on regularity of potentials q_j we have

$$0 = \int_0^K \int_{-h-\alpha'(0)hx_2}^{h+\alpha'(0)hx_2} \frac{p_N a_0^2 e^{iN\Psi(x')}}{2h}dx_1dx_2 + o(1) = \int_0^K \int_{-h-\alpha'(0)hx_2}^{h+\alpha'(0)hx_2} \frac{p_N(0, x_2)e^{iNx_2} + o(1)}{2h(1 + \alpha'(0)x_2)}dx_1dx_2 + o(1).$$

This proves (28).

Next we claim

$$\int_0^K p_N e^{-iNx_2}dx_2 = 0. \tag{29}$$

The proof of (29) is exactly the same as the proof of equality (28). The only difference is that instead of the function Ψ one has to use the function $\tilde{\Psi}$ defined by

$$|\nabla'\tilde{\Psi}| = 1 \quad \text{in } \Omega, \quad \tilde{\Psi}|_{x_2=0} = m, \tag{30}$$

$$\tilde{\Psi}(x_0 - t\alpha(x_0)\vec{e}_1 + t\beta(x_0)\vec{e}_2) = m(x_0) - t \quad \forall x_0 \in \{x' | x_2 = 0\} \text{ and } \forall t > 0, \tag{31}$$

where $\alpha(x_0) = m'(x_0), \beta(x_0) = \sqrt{1 - \alpha^2(x_0)}$.

From (28) and (29) setting $N = -i\gamma$ where γ is the real parameter we have

$$\mathcal{R}_\gamma(p_{-i\gamma})(\omega,p) = 0 \quad \forall(\omega,p) \in S^1 \times \mathbb{R}^1 \quad \text{such that} \quad \{x| < \omega, x >= p\} \cap Ch(\Omega \setminus \mathcal{O}) = \emptyset.$$

Applying corollary 5 we have that

$$p_{-i\gamma}(x') = 0 \quad \forall x' \in \mathcal{O} \quad \text{and} \quad \forall \gamma \in \mathbb{R}^1.$$

Therefore for any fixed $x' \in \mathcal{O}$ and any γ

$$\int_0^T (q_1 - q_2)(x', x_3) e^{i\gamma x_3} dx_3 = 0.$$

This equality implies immediately that the function $x_3 \to (q_1 - q_2)(x', x_3)$ on the segment $[0, L]$ is orthogonal to any polynomials. Therefore $(q_1 - q_2)(x)|_{\mathcal{O}\times[0,L]} = 0$. The proof of the theorem is complete.

■

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