UTMS 2012–8 July 3, 2012

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by

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Abstract

In this paper we develop optimal control laws for traffic networks. A standard traffic flow model is used to formulate the traffic control in terms of "on-off" strategy for the traffic signals. It is of the form of the optimal bilinear control problem with the binary constraints. We develop a Lyapunov function based feedback law for minimizing traffic congestions and stabilizing the traffic flow. Also, a novel binary optimization method is developed to obtain a real-time optimal control law for the binary constrained optimal control problem. Both methods are tested and compared for a large scale traffic network. Our tests demonstrate that the both method provides very effective but efficient traffic control law.

1 Introduction

The rapid growth of motorization, due to the significant problems created in communities of modern cities, motivates intense research activities in the field of traffic flow ranging from empirical to theoretical study. In a reflection of complexity and nonlinearity of traffic phenomena, there are vast contributions to the study in many aspects covering mathematical modeling including numerical methods, qualitative analysis using observed data and relevant simulations with the aim of comprehending the nature and significance of traffic flow and developing advanced road networks with efficient movement of traffic and minimal traffic congestion problems. There exists vast literature on the modeling of traffic flow, for example, [18] [17] [5] [22] [19] [6] [7] [20] [2], [23] [8] [4], [1] [21] [16]. Nonetheless, as the phenomena related to traffic flow is highly complex, modeling has not yet reached a satisfactory level so that further investigation may be necessary in order to fulfill the standard demands in real world applications.

The purpose of this article is to develop two strategies (numerical methods) toward real time optimal control for traffic control. A macroscopic model (a system of ODE) is introduced and is used to describe these methods. The first strategy, which is proposed in section 3, is Lyapunov method: we develop a traffic control in terms of "on-off" strategy for the traffic signals. Our objective is to make the volume of the traffic uniformly distributed, i.e. reducing traffic congestion. Based on our traffic model we develop a Lyapunov function based control law that controls traffic flows in real time and results in stabilizing regulating traffic flows optimally. A Lyapunov function method is a feedback law for selecting the on-off signal laws based on the traffic conditions, i.e. the traffic volume coming to the junction from each traffic link.

As for the second strategy for traffic control, a simple but efficient optimization method for binary optimization problems is developed. Although several optimization methods have been developed and used for the binary optimization problems, (to mention quite a few, [9, 10], [11], [2], [3]), there has been few satisfactory optimization method available so far that can be applied to large scale

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network flow arising in real world, even with the advent of significant computer processing power. Thus, there still exists a demand for new optimization methods allowing easy implementation for solving mixed integer nonlinear programming of large-scale. We formulate a traffic control problem as an optimal control problem of controlled Markov chain in which on-off nature of traffic signals is the binary control constraint, and then apply the method to determination of the optimal traffic signal control. Our method is based on the control theoretic formulation and optimization method. The binary optimization method uses the derivative of our cost functional with respect to control variables. Such derivative is calculated efficiently by the adjoin method using the adjoint equation. The proposed binary optimization method is categorized into relaxation methods; it is a real variable optimization in which the binary constraint is treated by the exact penalty method. We develop an effective and efficient iterative method for solving the resulting optimization. Although the methods are developed and described using a simple mathematical model, it can be widely applicable to other models formulated as 0-1 integer optimal control problems.

This paper is organized as follows: In section 2, a traffic flow model is introduced. In section 3, the Lyapunov method for the traffic model is developed to stabilized the traffic volume. In section 4, we develop a novel method for binary optimization problem for the traffic control problem. In section 5, we test and demonstrate a feasibility and robustness of the proposed feedback law and the binary optimization method for a small traffic grid ($m = 2$) and a large grid ($m = 100$).

2 A Traffic Model

Traffic networks are usually modeled by graphs. Crossings such as road intersections or junctions are symbolized by nodes of the graph, and connections such as roads between nodes are represented by edges. In this section, we describe a model for traffic flow in a large traffic network proposed by Imura [12]. We consider the square grids (i, j), $i, j = 0, \ldots, m$ for the traffic network (Fig. 1(a)). At each node (junction) (i, j), we assign a traffic signal $u_{i,j}(t)$ that varies 0 or 1. If $u_{i,j}(t) = 1$ then the traffic flows "East to West" and "West to East", and if $u_{i,j}(t) = 0$ then the traffic flows "South to North" and "North to South". Throughout we assume that the signals $u_{i,j}(t)$ remain constant (0 or 1) during the time interval $\Delta t > 0$ which is a priori given, i.e.,

$$\forall i, j, \quad u_{i,j}(t) = \text{either 0 or 1 for} \quad t \in [k\Delta t, (k+1)\Delta t], \quad \forall k =\in \mathbb{N} \cup \{0\}.$$

In other words,

$$u_{i,j}(t) = u^0_{i,j}\chi_{[0,\Delta t]} + u^1_{i,j}\chi_{[\Delta t, 2\Delta t]} + \cdots, \quad u^k_{i,j} \in \{0, 1\}.$$

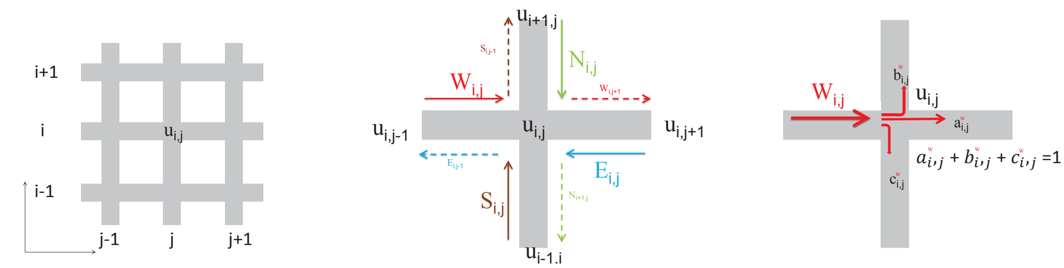
Let $E_{i,j}$, $W_{i,j}$, N, and $S_{i,j}$ denote the traffic volume entering to the node (i, j) in the direction of East, West, North and South, respectively (Fig. 1(b)). The labeling of the traffic volume is based on the labeling of the traffic signal. The subindex i, j of E, W, S, N indicates that these traffics move in the direction of the signal (i, j).

At the (i, j) junction, we assume that the turning rates $b^W_{i,j} \geq 0$ (left turn) and $c^W_{i,j} \geq 0$ (right turn) for the traffic flow $W_{i,j}$ are known a priori. We also assume there is no U-tern, i.e., the rate $a^W_{i,j}$ for traffic $W_{i,j}$ going straight is given by $a^W_{i,j} = 1 - b^W_{i,j} - c^W_{i,j}$. See Fig. 1(c). We also define $a^E, \ldots, c^N$ accordingly. Based on such assumptions, dynamic change in traffic volume $E_{i,j}$ can be represented as a balance law at the node (i, j):

$$\frac{d}{dt}E_{i,j} = -\lambda \left[u_{i,j}E_{i,j} - (a^E_{i,j+1}u_{i,j+1}E_{i,j+1} + b^S_{i,j+1}(1 - u_{i,j+1})S_{i,j+1} + c^N_{i,j+1}(1 - u_{i,j+1})N_{i,j+1})\right]$$

The first term $-\lambda u_{i,j}E_{i,j}$ is the outflow from the junction (i, j), and the remaining terms represent inflow through the junction $(i + 1, j)$. Fig. 1(d) depicts the related traffic volume to the balance law of $E_{i,j}$.

The evolution of $W_{i,j}$, N, and $S_{i,j}$ are written in a similar manner. See Fig. 1(e), 1(f) and 1(g).



(a) Signal level and Traffic nodes (b) Definition of variables (traffic volumes) (c) Junction rate

Thus we have

$$\begin{cases} \frac{d}{dt}E_{i,j} &= -\lambda\left[u_{i,j}E_{i,j} - (a^{E}_{i,j+1}u_{i,j+1}E_{i,j+1} + b^{S}_{i,j+1}(1-u_{i,j+1})S_{i,j+1} + c^{N}_{i,j+1}(1-u_{i,j+1})N_{i,j+1})\right]. \\ \frac{d}{dt}W_{i,j} &= -\lambda\left[u_{i,j}W_{i,j} - (a^{W}_{i,j-1}u_{i,j-1}W_{i,j-1} + b^{N}_{i,j-1}(1-u_{i,j-1})N_{i,j-1} + c^{S}_{i,j-1}(1-u_{i,j-1})S_{i,j-1})\right], \\ \frac{d}{dt}S_{i,j} &= -\lambda\left[(1-u_{i,j})S_{i,j} - (a^{S}_{i-1,j}(1-u_{i-1,j})S_{i-1,j} + b^{W}_{i-1,j}u_{i-1,j}W_{i-1,j} + c^{E}_{i-1,j}u_{i-1,j}E_{i-1,j})\right], \\ \frac{d}{dt}N_{i,j} &= -\lambda\left[(1-u_{i,j})N_{i,j} - (a^{N}_{i+1,j}(1-u_{i+1,j})N_{i+1,j} + b^{E}_{i+1,j}u_{i+1,j}E_{i+1,j} + c^{W}_{i+1,j}u_{i+1,j}W_{i+1,j})\right], \end{cases}, \tag{2.1}$$

for $1 \leq i,j \leq m$. The quantities $W_{i,0}$, $E_{i,m+1}$, $N_{m+1,i}$ and $S_{0,i}$ $(1 \leq i \leq m)$ represent inflows at the boundary nodes, and are given a priori. We assume that at the boundary nodes the signal is "on" for $i = 0,\ m+1$ and is "off" for $j = 0,\ m+1$ and that the traffic flow is straight, and hence, the equation for $E_{i,m}$, for example, is written by the equation

$$\frac{d}{dt}E_{i,m} = -\lambda(u_{i,m}E_{i,m} - E_{i,m+1}).$$

The locations of the boundary inflows $W_{i,0}$, $E_{i,m+1}$, $N_{m+1,i}$ and $S_{0,i}$ are displayed in Fig. 1 for the case in which there are four junctions $(m = 2)$.

Let $x(t)$ denote a $4m^2 \times 1$ column vector for traffic volume:

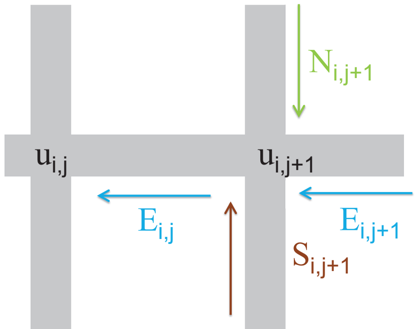
$$\begin{aligned} x(t) = [\ & E_{1,1}, E_{2,1}, \ldots, E_{m,1}, \ldots, E_{1,m}, E_{2,m}, \ldots E_{m,m}, \\ & W_{1,1}, W_{2,1}, \ldots, W_{m,1}, \ldots, W_{1,m}, W_{2,m}, \ldots W_{m,m}, \\ & S_{1,1}, S_{2,1}, \ldots, S_{m,1}, \ldots, S_{1,m}, S_{2,m}, \ldots S_{m,m}, \\ & N_{1,1}, N_{2,1}, \ldots, N_{m,1}, \ldots, N_{1,m}, N_{2,m}, \ldots N_{m,m}]^{\top}. \end{aligned}$$

Using matrix notations, the system (2.1) for the traffic volume can be expressed as a dynamical system of $x(t)$

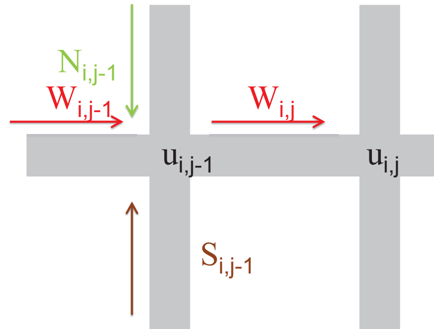
$$\frac{d}{dt}x(t) = Ax(t) + \sum_{i,j=1}^{m} u_{i,j}(t)B_{i,j}x(t) + s(t), \tag{2.2}$$

where the matrices A and $B_{i,j}$ and the vector $s(t)$ are defined as the following: Let a^{W} denote a $m \times m$ matrix whose component is $a^{W}_{i,j}$:

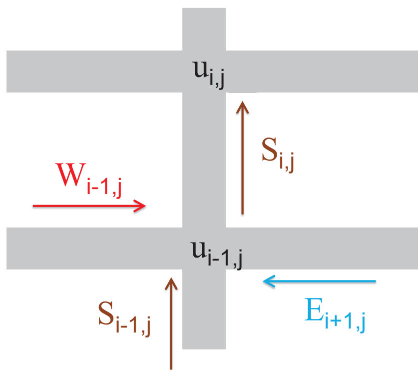
$$a^{W} = \begin{bmatrix} a^{W}_{1,1} & a^{W}_{1,2} & \cdots & a^{W}_{1,m} \\ a^{W}_{2,1} & a^{W}_{2,2} & \cdots & a^{W}_{2,m} \\ & & \ddots & \\ a^{W}_{m,1} & a^{W}_{m,2} & \cdots & a^{W}_{m,m} \end{bmatrix}.$$



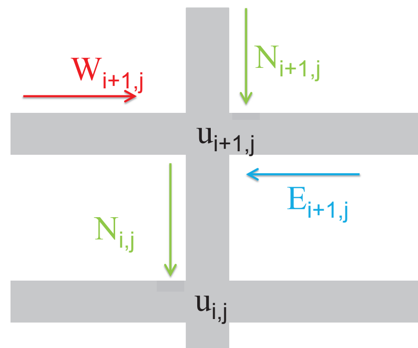
(d) Balance law for $E_{i,j}$



(e) Balance law for $W_{i,j}$



(f) Balance law for $S_{i,j}$



(g) Balance law for $N_{i,j}$

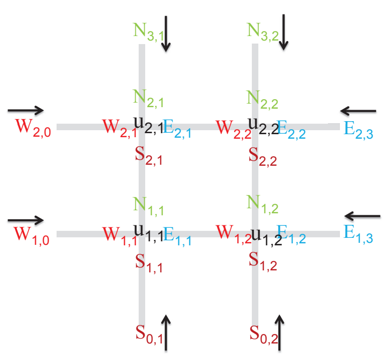


Figure 1: Inflow traffic at the boundary of the traffic network.

The matrices b^W, c^W, a^E, etc are defined analogously. Let U and L denote an upper shift matrix and a lower shift matrix, respectively: the (i,j) component of U and L are

$$U_{i,j} = \delta_{i+1,j}, \quad L_{i,j} = \delta_{i,j+1},$$

where $\delta_{i,j}$ is the Kronecker delta symbol. For example, the 3×3 shift matrices are

$$U_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We define A and $B_{i,j}$ by

$$A = \lambda \begin{bmatrix} 0 & 0 & (U \otimes I)b^S & (U \otimes I)c^N \\ 0 & 0 & (L \otimes I)c^S & (L \otimes I)b^N \\ 0 & 0 & -I \otimes I + (I \otimes L)a^S & \\ 0 & 0 & & -I \otimes I + (I \otimes U)a^N \end{bmatrix},$$

$$B_{i,j} = C(I_4 \otimes E_{i,j})$$

where

$$C = \lambda \begin{bmatrix} (U \otimes I)a^E - I \otimes I & 0 & -(U \otimes I)b^S & -(U \otimes I)c^N \\ 0 & (L \otimes I)a^W - I \otimes I & -(L \otimes I)c^S & -(L \otimes I)b^N \\ (I \otimes L)c^E & (I \otimes L)b^W & I \otimes I - (I \otimes L)a^S & 0 \\ (I \otimes U)b^E & (I \otimes U)c^W & 0 & I \otimes I - (I \otimes U)a^N \end{bmatrix},$$

and $E_{i,j}$ is a $m \times m$ matrix whose component is 0 except that its $(m(i-1)+j, m(i-1)+j)$ element is 1, i.e.,

$$E_{i,j}[\ell, \ell'] = \delta_{\ell_1, m(i-1)+j} \delta_{\ell', m(i-1)+j}, \quad \forall \ell, \ell' \in \{1, 2, \dots, m\}.$$

Lastly, $s(t)$ is defined by the sum of "inflows":

$$s(t) = \lambda \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} E_{1,m+1} \\ E_{2,m+1} \\ \vdots \\ E_{m,m+1} \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \begin{bmatrix} W_{1,0} \\ W_{2,0} \\ \vdots \\ W_{m,0} \end{bmatrix} + \lambda \begin{bmatrix} S_{0,1} \\ S_{0,2} \\ \vdots \\ S_{0,m} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} N_{m+1,1} \\ N_{m+1,2} \\ \vdots \\ N_{m+1,m} \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvalues of A consists of 0 with the $2m$ multiplicity and -1 with the $2m$ multiplicity.

3 Lyapunov stabilization method.

Our objective here is to stabilize the volume of the traffic uniformly distributed. Based on our traffic model in section 2 we develop a Lyapunov function based control law that controls traffic flows in real time and results in stabilizing regulating traffic flows optimally. A Lyapunov function method is a feedback law for selecting the on-off signal laws based on the traffic conditions, i.e. the traffic volume coming to the junction from each traffic link. It has been used in a quantum control [15] and it asymptotic property is analyzed in [15, 13].

Let us introduce a cost function

$$V(x(t)) = (Qx(t), x(t)),$$

where Q is a positive definite matrix of the size $4m \times 4m$. Then we have

$$\frac{d}{dt}V(x(t)) = (Qx(t), \dot{x}(t)) = (Qx(t), Ax(t) + \sum_{i,j} u_{i,j}(t) B_{i,j} x(t) + s(t))$$

$$= (Qx(t), Ax(t) + s(t)) + \sum_{i,j} u_{i,j}(t) c_{i,j}(t).$$

where $c_{i,j}$ is given by

$$c_{i,j}(t) = (Qx(t), B_{i,j}x(t)). \tag{3.3}$$

We determine "on" or "off" of the signal $u_{i,j}(t+\Delta t)$ at time $t+\Delta t$ by the (descent, dissipative) feedback law:

$$\begin{cases} u_{i,j}(t+\Delta t) = 1, & \text{if } c_{i,j}(t) \leq 0, \\ u_{i,j}(t+\Delta t) = 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Suppose $s(t) = 0$. If there exists $\omega > 0$ such that

$$\frac{d}{dt}V(x(t)) = (x(t), QAx(t)) + \sum_{i,j} u_{i,j}(Qx(t), B_{i,j}x(t)) \leq -\omega\,(x(t), Qx(t)) = -\omega V(x(t)), \tag{3.5}$$

for all x, then $(Qx(t), x(t))$ converges to 0 exponential with rate $\omega > 0$. If (3.5) holds, we say that (Qx, x) is a controlled Lyapunov function.

As a special case of the cost function, we consider a local cost for the traffic that flows into the (i,j)-junction:

$$V_{loc}(x_{i,j}(t)) := (Q_0 x_{i,j}(t), x_{i,j}(t))$$

Here Q_0 is a 4×4 symmetric matrix, and $x_{i,j}(t)$ is the traffic volume related to junction (i,j)

$$x_{i,j}(t) = \begin{bmatrix} E_{i,j}(t) \\ W_{i,j}(t) \\ N_{i,j}(t) \\ S_{i,j}(t) \end{bmatrix}.$$

For example, if we use Q_0 such that

$$Q_0 = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}, \tag{3.6}$$

then the local cost is the quadratic energy of the difference $x_{i,j}$:

$$\begin{aligned} V_{loc}(x_{i,j}(t)) = &(W_{i,j} - E_{i,j})^2 + (W_{i,j} - N_{i,j})^2 + (W_{i,j} - S_{i,j})^2 \\ &+ (E_{i,j} - N_{i,j})^2 + (E_{i,j} - S_{i,j})^2 + (N_{i,j} - S_{i,j})^2. \end{aligned}$$

And if we use identity matrix for Q_0, then the local cost is a local energy of $x_{i,j}(t)$:

$$V_{loc}(x_{i,j}(t)) = E_{i,j}^2 + W_{i,j}^2 + S_{i,j}^2 + N_{i,j}^2.$$

We define the total cost $V(x(t))$ as the sum of the local costs:

$$V(x(t)) = \sum_{i,j} V_{loc}(x_{i,j}(t)) = \sum_{i,j}(Q_0 x_{i,j}(t), x_{i,j}(t)) = \frac{1}{2}(x(t), Qx(t)), \quad Q = Q_0 \otimes I_{m^2}. \tag{3.7}$$

As is shown in (3.3), the quantity $c_{i,j}(t)$ can be computed using the matrices Q and $B_{i,j}$ and it is not necessary to compute the explicit representation. If $V(x)$ is composed of a simple matrix as in (3.7), we could give the explicit representation. For that purpose, we first write down the system (2.1) for the local traffic flow $x_{i,j}$ in the following form.

$$\begin{aligned} \lambda^{-1}\dot{x}_{i,j} = &- u_{i,j} d x_{i,j} + u_{i,j+1} d^1_{i,j+1} x_{i,j+1} + u_{i,j-1} d^2_{i,j-1} x_{i,j-1} + u_{i-1,j} d^3_{i-1,j} x_{i-1,j} + u_{i+1,j} d^4_{i+1,j} x_{i+1,j} \\ &- \hat{d} x_{i,j} + \hat{d}^1_{i,j+1} x_{i,j+1} + \hat{d}^2_{i,j-1} x_{i,j-1} + \hat{d}^3_{i-1,j} x_{i-1,j} + \hat{d}^4_{i+1,j} x_{i+1,j} \end{aligned}$$

where

$$d = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \quad \hat{d} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

$$d^1_{i,j} = \begin{bmatrix} a^E_{i,j} & 0 & -b^S_{i,j} & -c^N_{i,j} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad d^2_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a^W_{i,j} & -c^S_{i,j} & -b^N_{i,j} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad d^3_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c^E_{i,j} & b^W_{i,j} & -a^S_{i,j} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad d^4_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b^E_{i,j} & c^W_{i,j} & 0 & -a^N_{i,j} \end{bmatrix},$$

$$\hat{d}^1_{i,j} = \begin{bmatrix} 0 & 0 & b^S_{i,j} & c^N_{i,j} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{d}^2_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c^S_{i,j} & b^N_{i,j} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{d}^3_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a^S_{i,j} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{d}^4_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^N_{i,j} \end{bmatrix}.$$

Then one has

$$\begin{aligned} &\frac{d}{dt}V(x(t)) = \sum_{i,j}(Q_0 x_{i,j}(t), \dot{x}_{i,j}(t)) \\ =&\lambda \sum_{i,j}(Q_0 x_{i,j}(t), -u_{i,j} d x_{i,j} + u_{i,j+1} d^1_{i,j+1} x_{i,j+1} + u_{i,j-1} d^2_{i,j-1} x_{i,j-1} + u_{i-1,j} d^3_{i-1,j} x_{i-1,j} + u_{i+1,j} d^4_{i+1,j} x_{i+1,j}) \\ &+ \lambda \sum_{i,j}(Q_0 x_{i,j}(t), -\hat{d} x_{i,j} + \hat{d}^1_{i,j+1} x_{i,j+1} + \hat{d}^2_{i,j-1} x_{i,j-1} + \hat{d}^3_{i-1,j} x_{i-1,j} + \hat{d}^4_{i+1,j} x_{i+1,j}), \end{aligned}$$

By definition $c_{i,j}(t)$ is given as the coefficient of $u_{i,j}(t)$, and one obtains

$$c_{i,j}(t) = \lambda \left[(Q_0 x_{i,j}, -d x_{i,j}) + (Q_0 x_{i,j-1}, d^1_{i,j} x_{i,j}) + (Q_0 x_{i,j+1}, d^2_{i,j} x_{i,j}) + (Q_0 x_{i+1,j}, d^3_{i,j} x_{i,j}) + (Q_0 x_{i-1,j}, d^4_{i,j} x_{i,j})\right]. \tag{3.8}$$

After some works, we find that $c_{i,j}(t)$ is given by the quadratic form of $x_{i,j}(t)$,$x_{i,j+1}(t)$,$x_{i,j-1}(t)$,$x_{i+1,j}(t)$ and $x_{i-1,j}(t)$. The explicit representations of $c_{i,j}(t)$ for Q_0 in (3.6) and for $Q_0 = I_4$ are given in Appendix.

4 Binary Optimization method.

In this section, we consider the optimization problem for traffic flow:

$$\min_{u \in \{0,1\}^{Nm^2}} F(x), \tag{4.9}$$

subject to

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + \textstyle\sum_{i,j}^m u_{i,j}(t) B_{i,j} x(t) + s(t), \quad 0 \le t \le T, \\ x(0) &= x_0, \end{aligned} \tag{4.10}$$

where

$$F(x) := \int_0^T V(x(t))\,dt = \int_0^T (Qx(t), x(t))\,dt$$

and $u_{i,j}(t)$ models a signal at (i,j) junction and takes the values either 0 or 1, i.e,

$$u_{i,j}(t) = u^0_{i,j}\chi_{[0,\Delta t]}(t) + u^1_{i,j}\chi_{[\Delta t, 2\Delta t]}(t) + \cdots + u^{N-1}_{i,j}\chi_{[(N-1)\Delta t, N\Delta t]}(t), \quad u^k_{i,j} \in \{0,1\}.$$

with a given $\Delta t > 0$. The optimization problem can be seen as a binary constrained optimization problem for the signals vector

$$u := [u^0_{1,1}, u^0_{2,1}, \ldots, u^0_{m,m}, u^1_{1,1}, u^1_{2,1}, \ldots, u^2_{m,m}, \ldots, u^{N-1}_{1,1}, u^{N-1}_{2,1}, \ldots, u^{N-1}_{m,m}]^\top \in \{0,1\}^{Nm^2}.$$

4.1 Regularization of the binary constrained problem

We reformulate the problem into an *unconstrained* optimization problem: Find the control sequence $u \in \mathbb{R}^{Nm^2}$ that minimizes the cost function

$$\min \ F(x) + \psi(u),$$
$$\text{subject to} \ \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i,j}^{m} u_{i,j}(t)B_{i,j}x(t) + s(t), \quad 0 \le t \le T, \\ x(0) = x_0. \end{cases}$$

where F is a const functional for the state with a positive definite matrix;

$$F(x) = \frac{1}{2}\int_0^T (Qx(t), x(t))dt, \tag{4.11}$$

and $\psi(u)$ is the sum of sharp double well potentials W with a weight $c > 0$:

$$\psi(u) = c\sum_{k=0}^{N-1}\sum_{i,j=1}^{m} W(u_{i,j}^k).$$

Here the cost functional has been chosen so that the traffic congestion is minimized. The sharp double well function W is for the penalty formulation of the binary constraint $u_{i,j}^k \in \{0, 1\}$. That is,

$$W(s) = \begin{cases} -s & s \le 0 \\ (1-s)s & s \in [0,1] \\ s-1 & s \ge 1, \end{cases} \quad \text{and} \quad \partial W(s) = \begin{cases} -1 & s < 0 \\ 1-2s & s \in (0,1) \\ 1 & s > 1 \\ [-1,1] & s = 0, 1. \end{cases}$$

Here $\partial W(s)$ denotes the subgradient of $W(s)$ at s. Note that the subgradient becomes set if $s = 0, 1$.

For a given signal sequence $u \in \mathbb{R}^{Nm^2}$, let $x = \Phi(u)$ denote the solution of the state equation (4.10), let us define $\hat{F}(u)$ by

$$\hat{F}(u) := F(\Phi(u)).$$

Then the optimization problem is expressed in terms of this as the minimization problem

$$\min_{u \in \mathbb{R}^{Nm^2}} \ \hat{F}(u) + \psi(u).$$

The necessary optimality condition is give by the inclusion

$$\hat{F}'(u) + \partial\psi(u) \ni 0,$$

where the gradient $\hat{F}'(u)$ is identified with a row vector

$$\hat{F}'(u) = \left[\frac{\partial}{\partial u_{1,1}^0}\hat{F}'(u), \ldots, \frac{\partial}{\partial u_{m,m}^{N-1}}\hat{F}'(u)\right] \in \mathbb{R}^{1,Nm^2}$$

There are two problems, concerning the inclusion, that must be overcome for numerical treatment.

1. The subdifferential $\partial\psi(u)$ at $u_{i,j}^k = 0$ and 1 can not be determined uniquely, and it makes infeasible to handle the derivative numerically. In order to circumvent the ambiguity, we will introduce a regularized potential in the next section.
2. The computation of the gradient $\hat{F}'(u)$ will be costly due to the following reason.

The partial derivative $\frac{\partial}{\partial u_{i,j}^k}\hat{F}'(u)$ at u for $1 \le i,j \le m$ and $0 \le k \le N-1$, can be calculated by using the chain rule

$$\frac{\partial}{\partial u_{i,j}^k}\hat{F}'(u) = F'(\Phi(u))\Phi'(e_{i,j}^k)$$

where $e_{i,j}^k$ is $Nm^2 \times 1$ column vector with entries 0 except $m^2k + m(j-1) + i$ component. As can be easily verified, $y := \Phi'(e_{i,j}^k)$ is the solution of the system

$$\frac{d}{dt}y(t) = Ay(t) + e_{i,j}^k \chi_{[k\Delta t,(k+1)\Delta t]}(t) B_{i,j} y(t) + s(t), \quad y(0) = 0.$$

We must solve Nm^2 equations to obtain the gradient $\hat{F}'(u)$, which makes the computation quite costly for large scale problem ($N \gg 1$, $m \gg 1$) that arises in practice. We shall adopt the adjoint method, whose benefit is that the total work of computing $\hat{F}'(u)$ is approximately equivalent to integrating only one ODE (the adjoint equation). We will give a brief description of the adjoint method in section 4.3.

4.2 Regularized problem and the algorithm

As mentioned earlier, the subdifferential $\partial\psi(u)$ at $u_{i,j}^k = 0$ and 1 can not be determined uniquely. In order to avoid the non uniqueness of $\partial\psi$, we introduce a regularized potential $\psi_\epsilon(u)$ and consider the regularized problem:

$$\min_{u \in \mathbb{R}^{Nm^2}} \hat{F}(u) + \psi_\epsilon(u). \tag{4.12}$$

We define the regularized potential $\psi_\epsilon(u)$ as the sum of a regularized double well potential $W_\epsilon(s)$ with weight $c > 0$:

$$\psi_\epsilon(u) = c \sum_{k=0}^{N-1} \sum_{i,j=1}^{m} W_\epsilon(u_{i,j}^k).$$

The regularized double well potential $W_\epsilon(s)$ and its derivative $W'_\epsilon(s)$ are defined by

$$W_\epsilon(s) = \begin{cases} -Ls & s \le -\epsilon \\ L\dfrac{s^2}{2\epsilon} + c_2 & s \in [-\epsilon, 0] \\ \dfrac{(3-4s)s^2}{6\epsilon} + c_3 & s \in [0, \epsilon] \\ (1-s)s + c_4 & s \in [\epsilon, 1-\epsilon] \\ -\dfrac{s(6-9s+4s^2)}{6\epsilon} + c_5 & s \in [1-\epsilon, 1] \\ L\dfrac{(s-2)s}{2\epsilon} + c_6 & s \in [1, 1+\epsilon] \\ L(s-1) & s \ge 1+\epsilon \end{cases} \quad \text{and} \quad W'_\epsilon(s) = \begin{cases} -L & s \le -\epsilon \\ L\dfrac{s}{\epsilon} & s \in [-\epsilon, 0] \\ \dfrac{(1-2s)s}{\epsilon} & s \in [0, \epsilon] \\ 1-2s & s \in [\epsilon, 1-\epsilon] \\ -\dfrac{(1-2s)(s-1)}{\epsilon} & s \in [1-\epsilon, 1] \\ L\dfrac{(s-1)}{\epsilon} & s \in [1, 1+\epsilon] \\ L & s \ge 1+\epsilon \end{cases}$$

where $c_2, \ldots, c_6$ are constants defined so that $W_\epsilon(s)$ is continuous; and $L > 0$ is a slope for the W well potential; and $\epsilon > 0$ is a small parameter, which is usually chosen within a range $\epsilon \in [10^{-6}, 10^{-3}]$ in numerical computation.

The following representation of the derivative is crucial to develop our numerical algorithm. That is, we define $\Phi_\epsilon^{(1)}(s)$ and $\Phi_\epsilon^{(2)}(s)$ as Let us introduce two notations:

$$\Phi_\epsilon^{(1)}(s) = \begin{cases} L\dfrac{1}{\max\{|s|,\epsilon\}} & s \le 0 \\ \dfrac{1-2s}{\max\{|s|,\epsilon\}} & s \in [0,\frac{1}{2}] \\ \dfrac{2s-1}{\max\{|s-1|,\epsilon\}} & s \in [\frac{1}{2},1] \\ L\dfrac{1}{\max\{|s-1|,\epsilon\}} & s \ge 1 \end{cases}, \quad \Phi_\epsilon^{(2)}(s) = \begin{cases} 0 & s \le 0 \\ 0 & s \in [0,\frac{1}{2}] \\ \dfrac{2s-1}{\max\{|s-1|,\epsilon\}} & s \in [\frac{1}{2},1] \\ L\dfrac{1}{\max\{|s-1|,\epsilon\}} & s \ge 1 \end{cases}$$

Then it is easy to see that

$$W_\epsilon'(s) = \Phi_\epsilon^{(1)}(s)s - \Phi_\epsilon^{(2)}(s),$$

Let us denote the diagonal matrix and the vector depending on a vector u by A_u and b_u respectively, i.e.,

$$A(u) := c \text{ diag } [\Phi_\epsilon^{(1)}(u_{1,1}^0), \dots, \Phi_\epsilon^{(1)}(u_{m,m}^{N-1})],$$
$$b(u) := c\, [\Phi_\epsilon^{(2)}(u_{1,1}^0), \dots, \Phi_\epsilon^{(2)}(u_{m,m}^{N-1})]^\top.$$

Then $\psi_\epsilon'(u)$ is written in terms of A and b

$$\psi_\epsilon'(u) = A(u)u - b(u).$$

In view of this, the necessary optimality condition for (4.12) can be written as

$$\hat{F}'(u) + A(u)u - b(u) = 0.$$

It is not feasible to solve the optimality system directly due to the nonlinearity in $A(u)u$. We propose an iterative method that circumvents the difficulty.

(1) Initialize $: u^0$

(2) Do until converge

$$\frac{u^{n+1}-u^n}{\Delta\tau} + \hat{F}'(u^n)^\top + A(u^n)u^{n+1} - b(u^n) = 0.$$

Here $\Delta\tau$ is a time stepping width. The algorithm includes the user defined parameters c, $\Delta\tau$, ϵ and L.

4.3 Computation of the gradient $\hat{F}'(u)$ by the adjoint method

The adjoint method is explained. For a given control $u \in \mathbb{R}^{Nm^2}$, let us integrate the state equation

$$\frac{d}{dt}x(t) = \left(A + \sum_{i,j}^m u_{i,j}(t)B_{i,j}\right)x(t) + s(t), \quad x(0) = x_0. \tag{4.13}$$

to obtain $x = \Phi(u)$, and then solve the adjoint equation

$$-\frac{d}{dt}p(t) = \left(A + \sum_{i,j}^m u_{i,j}(t)B_{i,j}\right)^\top p(t) + Qx(t), \quad p(T) = 0 \tag{4.14}$$

Then, from the Lagrange calculus (e.g., see the next section) we can compute the partial derivative by

$$\frac{\partial F(u)}{\partial u_{i,j}^k} = \int_{k\Delta t}^{(k+1)\Delta t} (p(t), B_{i,j}x(t))\, dt. \tag{4.15}$$

5 Numerical tests

Numerical tests are carried out to demonstrate the effectiveness and robustness of our proposed methods.

Test 1 Lyapunov stability (Large scale)

Test 2 Optimality of the binary optimization (small scale)

Test 3 Applicability of BO and Lyapunov feed back law in Large scale

We denote the solution of (4.13) at time $t = k\Delta t$ by $x^k \in \mathbb{R}^{m^2}$, and the i th component of x^k by $x_i^k \in \mathbb{R}$.

Test 1

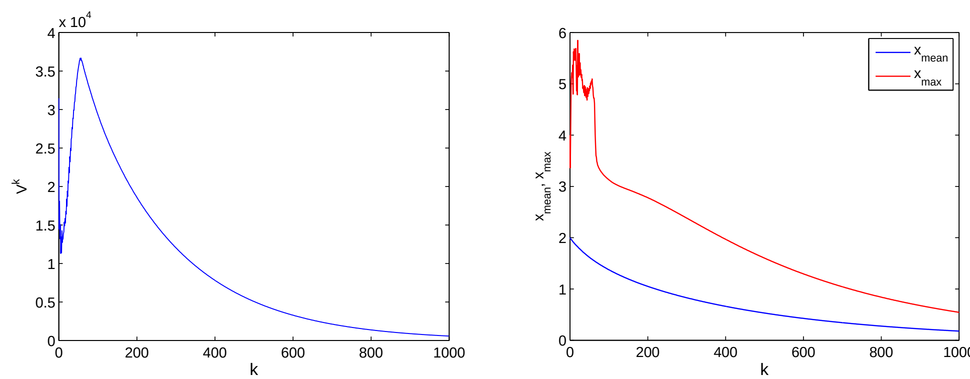
In this test, we use a large traffic network with long horizon N, i.e., $m = 100$ and $N = 1000$. The cost function $V^k := V(k\Delta t) = \frac{1}{2}(Qx^k, x^k)$ is used with Q defined as (3.7) with Q_0 in (3.6). The initial condition for the system (2.2) used in this test was

$$x^0 = x(0) = \frac{1}{4}(1, 2, \ldots, 16).$$

We used the flowing parameters in the model; $\Delta t = 0.8$, and the turning rates

$$a_{i,j}^E = a_{i,j}^W = a_{i,j}^S = a_{i,j}^N = 0.8, b_{i,j}^E = b_{i,j}^W = b_{i,j}^S = b_{i,j}^N = 0.1, c_{i,j}^E = c_{i,j}^W = c_{i,j}^S = c_{i,j}^N = 0.1.$$

for all $1 \le i, j \le 2$. The inflow $s^k = 0$ for all t. The purpose of this test is to confirm that V^k is a controlled Lyapunov function and to see the Lyapunov feedback law (3.4) can stabilize the traffic volume. The behavior of V^k is shown in Fig. 2(a). We see that V^k converges asymptotically to 0. We also computed the mean and the maximum of x^k with respect to the junction (i, j), i.e.,



(a) Test 1. V^k

(b) Test 1. x_{mean}^k and x_{max}^k

$$x_{mean}^k := \frac{\sum_{i=1}^{4m^2} x_i^k}{4m^2}, \quad x_{max}^k := \max_{1 \le i \le 4m^2} (x_i^k), \tag{5.16}$$

which are depicted in Fig. 2(b). From the figure, we find that x^k_{mean} decreases in time and asymptotically converges to 0. We can observe that x^k_{max} also converges to 0 and does not diverge, which indicates that the Lyapunov feed back law stabilizes the traffic volumes.

Test 2

In this test, we take a small traffic network with short horizon N, i.e., $m = 2$ and $N = 5$. Thus, the unknown signal vector to be determined has 20 unknowns:

$$u = (u^0_{1,1}, u^0_{1,2}, u^0_{2,1}, u^0_{2,2}, u^1_{1,1}, u^1_{1,2}, u^1_{2,1}, u^1_{2,2}, u^2_{1,1}, u^2_{1,2}, u^2_{2,1}, u^2_{2,2}, u^3_{1,1}, u^3_{1,2}, u^3_{2,1}, u^3_{2,2}, u^4_{1,1}, u^4_{1,2}, u^4_{2,1}, u^4_{2,2}) \in \mathbb{R}^{20}.$$

We used Q define by Q_0 in (3.6) for the cost functional: We consider the optimization problem (4.9) with $T = 5\Delta t$. The purpose of this test is to show the performance of the binary optimization method. Since the total number of unknowns are small, we can carry out the exhaustive search among $2^{20} \sim 10^6$ combinatorics of the switching patterns to find the optimal solution, and we can compare it to the result obtained by the binary optimization method.

The turning rates we used are

$$a^E_{i,j} = a^W_{i,j} = a^S_{i,j} = a^N_{i,j} = 0.8, b^E_{i,j} = b^W_{i,j} = b^S_{i,j} = b^N_{i,j} = 0.1, c^E_{i,j} = c^W_{i,j} = c^S_{i,j} = c^N_{i,j} = 0.1.$$

for all $1 \le i, j \le 2$. Inflow traffic at the boundary of the traffic network we used are the following:

$$E_{1,3}(t) = E_{2,3}(t) = 5, \quad W_{1,0}(t) = W_{2,0}(t) = 1, \quad S_{0,1}(t) = S_{0,2}(t) = 3, \quad N_{3,1}(t) = N_{3,2}(t) = 2,$$

for all t. See Fig. 1 for the location of the inflows. The initial condition was

$$x^0 = \frac{1}{4m^2}(1, 2, \ldots, 4m^2)^\top.$$

A period of time Δt during which signal do not change was set to be $\Delta t = 0.8$. We first run our binary optimization method to find the signal switching pattern. The parameters used in the method were set to be $\Delta\tau = 1$, $\epsilon = 10^{-5}$ and $L = 10^8$. The weight c in ψ_ϵ was adaptively chosen in time marching algorithm, i.e., $c = 10^{-3}$ was used for the first 50 iterations and was updated as $c \leftarrow 1.2c$ for every 50 iterations. The parameter choice strategy we adopted in this test is incomplete and there is room for improvement, which will be investigated in future work.

Table 1 shows the signal pattern obtained by the method. The total cost $J = \frac{1}{2}\int_0^T (Qx(t), x(t))dt$ was $J = 340.3737$, and the computation time was 8 seconds.

Next we carried out the exhaustive search to find the optimal signal switching pattern. The total computation time was 13 minutes. We found that the signal pattern was exactly the same as that obtained by the binary optimization method. This means that the binary optimization method found the optimal solution among 2^{10} signal patterns with much less computation time.

For comparison, we tested the Lyapunov method proposed in section 3. The total cost was $J = 489.2647$ and the computation time was $0.003s$. The signal pattern obtained is shown in Table 2. Although the cost was not optimal, the computed traffic volumes showed the similar trend as those computed using the optimal signal pattern.

Binary optimization method

	0	Δt	$2\Delta t$	$3\Delta t$	$4\Delta t$
$u_{1,1}(t)$	0	0	1	0	1
$u_{1,2}(t)$	0	0	1	0	1
$u_{2,1}(t)$	0	1	1	0	1
$u_{2,2}(t)$	0	1	1	0	1

Table 1: Test 2. The signal pattern obtained by the binary optimization method.

Lyapunov feedback law

	0	Δt	$2\Delta t$	$3\Delta t$	$4\Delta t$
$u_{1,1}(t)$	0	0	1	0	1
$u_{1,2}(t)$	0	1	0	1	0
$u_{2,1}(t)$	0	1	0	1	0
$u_{2,2}(t)$	0	1	1	0	1

Table 2: Test 2. The signal pattern u obtained by the binary optimization method.

Test 3-1

The numerical test was done for large scale problem $m = 100$, $N = 20$. The initial condition and inflows at boundary we used are;

$$E_{i,j}(0) = W_{i,j}(0) = S_{i,j}(0) = N_{i,j}(0) = \exp(-10((j/m - .5)^2 + (i/m - .5)^2),$$
$$s(t) = 0, \quad \forall t.$$

The cost function in this test was

$$J = \int_0^T |x(t)|^2 dt = \int_0^T |E_{i,j}(t)|^2 + |W_{i,j}(t)|^2 + |S_{i,j}(t)|^2 + |N_{i,j}(t)|^2 dt.$$

The other parameters were the same of those used in Test 2. The total cost of the Lyapunov feedback law was $J = 57761$ whereas the one of the binary optimization method was $J = 57440$, only slight difference was observed in the cost. The mean and the maximum traffic volumes (5.16) were computed and are shown in Fig. 2(d) and Fig. 2(c) respectively. One can observe that the both methods stabilize the traffic volume. The trend of the mean volume in time by both methods are almost identical, and the overall performance of the methods were satisfactory. However, this implies that the binary optimization method did not work perfectly for the problem with long horizon N. This might be due to the fact that the there were too many unknown variables to be determined, $100^2 \times 20$, and the possibly poorly chosen parameters in the algorithm, all of which should be selected adaptively. In this test c was the only parameter adaptively selected. So far, we do not have any guideline for the choice of those parameters, and this will be postponed to the future works.

To improve the performance of the binary optimization method we solved the receding horizon optimization problem sequentially [13]; Suppose $x^{kM} = x(kM\Delta t)$ is given. We solve the optimization problem to obtain the signal pattern $U^{kM} := [u^{kM}, u^{kM+1}, \ldots, u^{(k+1)M-1}]$ on the horizon $[kM\Delta t, ((k+1)M - 1)\Delta t]$;

$$U^{kM} = \arg\min \; F(x) + \psi(u),$$
$$\text{subject to} \; \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i,j}^m u_{i,j}(t) B_{i,j} x(t) + s(t), & kM\Delta t \le t \le (k+1)M\Delta t, \\ x(kM\Delta t) = x^{kM}. \end{cases}$$

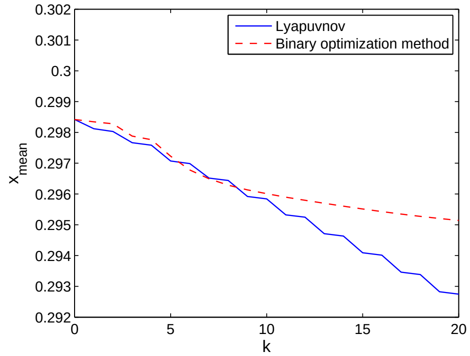
where

$$F(x) = \frac{1}{2} \int_{kM\Delta t}^{(k+1)M\Delta t} (Qx(t), x(t)) dt,$$

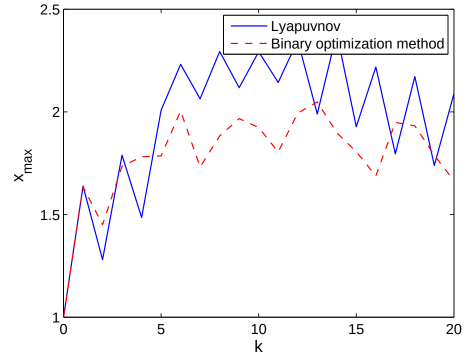
We have tested the receding horizon method with $M = 1, 2, 3, 4$ using the binary optimization algorithm. It is best when $m = 1$, i.e., instantaneous synthesis for our tests. Thus, we obtain the signal patter for $N = 20$, $u = [u^0, u^1, \ldots, u^{19}]$. The mean and the maximum of x^k are shown in Fig. 2(f) and Fig. 2(e) respectively. The total cost was reduced to $J = 52819$, and as can be clearly observed in Fig. 2(f), the significance improvement was obtained in the maximum x^k_{max}.

Test 3-2

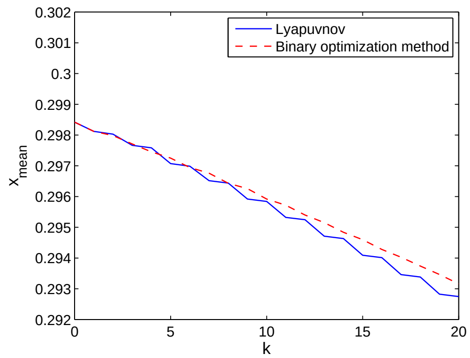
We carried out the numerical test with much larger scale, $m = 100$, $N = 200$. The sequential binary optimization method as in test 3-1 was adopted. The total costs were $J = 4.0431 \times 10^5$ for Lyapunov method and $J = 3.7833 \times 10^5$ for the sequential binary optimization method respectively. For comparison, we generated a set of 10^4 random 0-1 sequences of length Nm^2 each, and computed the total costs. We selected the 0-1 sequence among them that gave the minimum cost, which was $J = 6.1307 \times 10^5$. The mean and the maximum traffic volumes (5.16) were computed and are shown in Fig. 2(h) and Fig. 2(g) respectively. One can observe that, comparing with the result obtained by random search, the both Lyapunov and the sequential binary optimization methods stabilize the traffic volume throughout and the overall performance of the methods are satisfactory.



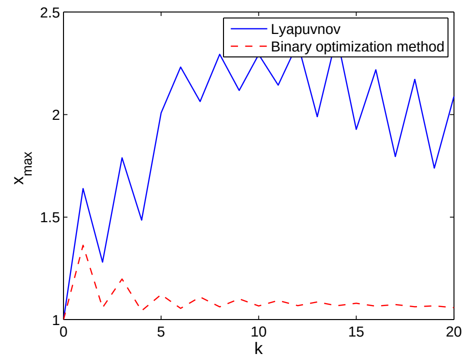
(c) Test 3-1. x^k_{mean} for Lyapunov (solid line) and the binary optimization method (dashed line).



(d) Test 3-1. x^k_{max} for Lyapunov (solid line) and the binary optimization method (dashed line).



(e) Test 3-1. x^k_{mean} for Lyapunov (solid line) and the sequential binary optimization method(dashed line).



(f) Test 3-1. x^k_{max} for Lyapunov (solid line) and the sequential binary optimization method (dashed line).

6 Hamilton Jacobi method

Consider the control problem

$$\min \quad \int_s^T Q(x(t))\,dt$$

subject to

$$\frac{d}{dt}x(t) = f(x(t), u(t)), \quad x(s) = x, \quad u(t) \in U$$

where U is the constrained set (e.g, Binary). Define the value function

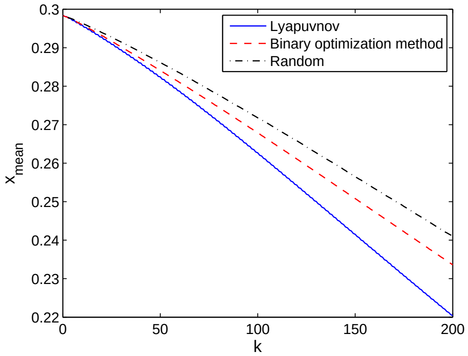
$$V(s, x) = \inf_{u \in U} \int_s^T Q(x(t))\,dt.$$

It can be proved that V is the viscosity solution to

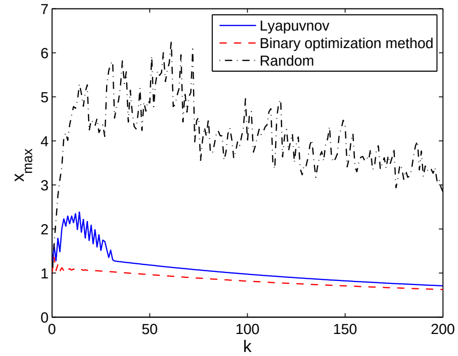
$$\frac{\partial V}{\partial s} + \min_{u \in U}\{f(x, u) \cdot \frac{\partial V}{\partial x}\} + Q(x) = 0, \quad V(T, x) = 0.$$

If we consider the infinite time horizon ($T = \infty$) the value function V satisfies

$$\min_{u \in U}\,(f(x, u), \frac{\partial V}{\partial x}) + Q(x) = 0.$$



(g) Test 3-2. x^k_{mean} for Lyapunov (solid line), the sequential binary optimization method(dashed line) and random search(dash-dot line)



(h) Test 3-2. x^k_{max} for Lyapunov (solid line), the sequential binary optimization method(dashed line) and random search(dash-dot line)

Thus,

$$\frac{d}{dt}V(x(t)) = -Q(x(t))$$

if we use

$$u(t) = \text{argmin}_{v\in U}\,(f(x(t),v), \frac{\partial V}{\partial x}).$$

That is, it is a Hamilton-Jacobi equation based Lyapunov method.

For a sufficiently large $T > 0$ we use

$$u = \text{argmin}_{u\in U}(f(x,u), \frac{\partial V(0,x)}{\partial x}) \tag{6.17}$$

to construct a feedback law but it is not feasible for a large dimension case. But, we use the following relationship between the Hamilton Jacobi solution and the corresponding two point boundary value problem to perform (6.17). If for a given initial condition, the triple $(x(t), p(t), u(t))$ satisfies the two point boundary value problem;

$$\begin{aligned}
&\tfrac{d}{dt}x(t) = f(x,u), \quad x(0) = x\\
&-\tfrac{d}{dt}p(t) = f_x(x(t),u(t))^t p(t) + Q'(x(t)), \quad p(T) = 0\\
&u(t) = \text{argmin}_{v\in U}\,(f(x(t),v), p(t))
\end{aligned} \tag{6.18}$$

then $\dfrac{\partial}{\partial t}(0,x) = p(0)$, which is a function of the initial condition x. Note that

$$\frac{d}{dt}V(0,x(t)) = f(x(t),u(t)) \cdot \frac{\partial}{\partial t}V(0,x(t))$$

Thus, our feedback law is given by

$$u(t) = \text{argmin}_{v\in U}\,(f(x(t),v), \frac{\partial}{\partial t}V(0,x(t))).$$

where $\dfrac{\partial}{\partial t}V(0,x) = p(0)$ and $p(0)$ is the solution to the two point boundary problem (6.18) given the initial condition x.

7 Conclusion

We developed a traffic network model for urban area and based on the model we developed real time optimal control laws for the signal control problems to minimize the traffic congestion. Our proposed method are based on the control theoretic formulation for bilinear control problems with binary control constraints. A Lyapunov function based feedback law is developed and analyzed and it provides very effective control law even for a large traffic network. The binary optimal control problem is also solved using an innovative optimization algorithm for binary optimization problems. The method can construct a nearly optimal solution even for a large-sale traffic network and a large control time-horizon problem. We demonstrated the applicability and effectiveness of our proposed control laws and can be applied a realistic traffic network with a large traffic links.

Our traffic control laws use the traffic volumes on traffic links and parameters at each junction (the straight and left and right turn rates). So, we will develop real-time state and parameter estimation method for determining traffic volumes and the junction parameters.

8 Appendix

8.1 The explicit representation of $c_{i,j}(t)$

Let us define a vector $X_{i,j}(t)$ by

$$X_{i,j}(t) = [E_{i,j}, E_{i,j+1}, E_{i,j-1}, E_{i+1,j}, E_{i-1,j}, W_{i,j}, \dots, W_{i-1,j}, S_{i,j}, \dots, S_{i-1,j}, N_{i,j}, \dots, N_{i-1,j}]^\top.$$

$c_{i,j}(t)$ in (3.8) is a quadratic form for $X_{i,j}(t)$:

$$c_{i,j}(t) = X_{i,j}^\top(t) H X_{i,j}(t),$$

where H is the 20×20 matrix depending on (i,j). The matrix H for Q_0 in (3.6) is given by

$$H = \left[\begin{array}{ccccc|ccccc|ccccc|ccccc}
-3 & 0 & 3a^E_{i,j} & -c^E_{i,j} & -b^E_{i,j} & 2 & 0 & -a^E_{i,j} & -c^E_{i,j} & -b^E_{i,j} & 0 & 0 & -a^E_{i,j} & 3c^E_{i,j} & -b^E_{i,j} & 0 & 0 & -a^E_{i,j} & -c^E_{i,j} & 3b^E_{i,j} \\
0 & 0 & 0 & 0 & 0 & -a^W_{i,j} & 0 & 0 & 0 & 0 & c^S_{i,j} & 0 & 0 & 0 & 0 & b^N_{i,j} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3b^S_{i,j} & 0 & 0 & 0 & 0 & -3c^N_{i,j} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -b^W_{i,j} & 0 & 0 & 0 & 0 & a^S_{i,j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -c^W_{i,j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^N_{i,j} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -3 & 3a^W_{i,j} & 0 & -b^W_{i,j} & -c^W_{i,j} & 0 & -a^W_{i,j} & 0 & 3b^W_{i,j} & -c^W_{i,j} & 0 & -a^W_{i,j} & 0 & -b^W_{i,j} & 3c^W_{i,j} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3c^S_{i,j} & 0 & 0 & 0 & 0 & -3b^N_{i,j} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^S_{i,j} & 0 & 0 & 0 & 0 & c^N_{i,j} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^S_{i,j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^N_{i,j} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & c^S_{i,j} & b^S_{i,j} & -3a^S_{i,j} & 0 & -2 & c^S_{i,j} & b^S_{i,j} & a^S_{i,j} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^N_{i,j} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c^N_{i,j} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^N_{i,j} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & b^N_{i,j} & c^N_{i,j} & 0 & -3a^N_{i,j} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],$$

and H for Q_0 being identity is given by

$$H = \left[\begin{array}{ccccc|ccccc|ccccc|ccccc}
-1 & 0 & a^E_{i,j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c^E_{i,j} & 0 & 0 & 0 & 0 & 0 & b^E_{i,j} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b^S_{i,j} & 0 & 0 & 0 & 0 & -c^N_{i,j} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -1 & a^W_{i,j} & 0 & 0 & 0 & 0 & 0 & 0 & b^W_{i,j} & 0 & 0 & 0 & 0 & 0 & c^W_{i,j} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c^S_{i,j} & 0 & 0 & 0 & 0 & -b^N_{i,j} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -a^S_{i,j} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -a^N_{i,j} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

8.2 The adjoint method.

Consider the constrained minimization

$$\min \quad J(x,u) = F(x) + H(u) \quad \text{subject to } E(x,u) = 0. \tag{8.19}$$

We use the implicit function theory for developing algorithms for (8.19).

Implicit Function Theory Let $E : X \times U \to X$ is C^1. Suppose a pair $(\bar{x}, \bar{u})$ satisfies $E(x,u) = 0$ and $E_x(\bar{x}, \bar{u})$ is bounded invertible. Then the exists a $\delta > 0$ such that for $|u - \bar{u}| < \delta$ equation $E(x,u) = 0$ has a locally defined unique solution $x = \Phi(u)$. Moreover, $\Phi : U \to X$ is continuously differentiable at $\bar{u}$ and $\dot{x} = \Phi'(\bar{x})(d)$ satisfies

$$E_x(\bar{x}, \bar{u})\dot{x} + E_u(\bar{x}, \bar{u})d = 0.$$

Theorem (Lagrange Calculus) Assume that $E(x,u) = 0$ and $E_x(x,u)$ is bounded invertible. Let $\lambda \in Y$ be the solution of the equation

$$E_x(x,u)^*\lambda + F'(x) = 0. \tag{8.20}$$

Then, for $J(u) = J(\Phi(u), u)$

$$(J'(u), d) = (H'(u), d) + (E_u(x,u)d, \lambda) \tag{8.21}$$

Proof: From the chain rule and the implicit function theorem

$$(J', d) = (F'(x), \dot{x}) + (H'(u), d),$$

and

$$E_x(x,u)\dot{x} + E_u(x,u)d = 0.$$

Thus, the claim follows from

$$(F'(x), \dot{x}) = -(E_x^*\lambda, \dot{x}) = -(\lambda, E_x\dot{x}) = (\lambda, E_u d).\square$$

In our example, the constraint $E(x,u)$ is defined by (4.10). and F is defined by (4.11) and $H(u) = \psi_\epsilon(u)$. The adjoint equation for λ (8.20) is (4.14) for $p = \lambda$. The derivative (8.21) gives (4.15). The theorem holds for a weaker assumption, e.g., see[14].

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