

The geography of simply connected smooth 4-manifolds

講義内容 :

One of the fundamental problems in smooth 4-manifolds is to classify simply connected smooth 4-manifolds. The classical invariants of a simply connected smooth 4-manifold are encoded by its intersection form Q_X , a unimodular symmetric bilinear pairing on $H^2(X; \mathbb{Z})$. For example, one can determine its rank $b^2(X) = b_2^+ + b_2^-$, its signature $\text{sigma}(X) = b_2^+ - b_2^-$, and its Euler characteristic $e(X) = b_2(X) + 2$ from Q_X . M. Freedman showed that a simply connected smooth 4-manifold is determined up to homeomorphism by Q_X . The situation in the smooth category is strikingly different. Hence it is an important question to know which unimodular, symmetric, bilinear integral forms are realized as the intersection form of a simply connected smooth 4-manifold, and which simply connected smooth 4-manifolds admit more than one smooth structure. We call these geography problems of simply connected smooth 4-manifolds.

Gauge theory - Donaldson theory and Seiberg-Witten theory - has been very successful in the geography problems. For example, S. Donaldson proved that the intersection form of a simply connected, definite, smooth 4-manifold is diagonalizable, and it has been known that most known simply connected irreducible smooth 4-manifolds with b_2^+ odd and large enough admit infinitely many distinct smooth structures. Recently, there has also been progress in the case of b_2^+ small, in particular $b_2^+ = 1$.

In this lecture series I 'd like to review known results on the geography problems, in particular, the uniqueness problems of smooth structures on 4-manifolds. For this, I first survey various constructions of smooth 4-manifolds such as a logarithmic transform, fiber-sum, a knot surgery and a rational blow-down surgery. These constructions are main techniques to convert smooth structures of a given smooth 4-manifold. I also address recent developments in the geography of smooth 4-manifolds with $b_2^+ = 1$.

A tentative schedule is the following:

Lecture I: Overview of geography problems

Lecture II: Surgery 1 - A logarithmic transform and fiber-sum

Lecture III: Surgery 2 - A knot surgery (optional)

Lecture IV: Surgery 3 - A rational blow-down surgery

Lecture V: Smooth 4-manifolds with $b_2^+ = 1$